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Noether-Like Operators and First Integrals for Generalized Systems of Lane-Emden Equations

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Abstract: Coupled systems of Lane–Emden equations are of considerable interest as they model several physical phenomena, for instance population evolution, pattern formation, and chemical reactions. Assuming a complex variational structure, we classify the generalized system of Lane–Emden type equations in relation to Noether-like operators and associated first integrals. Various forms of functions appearing in the considered system are taken, and it is observed that the Noether-like operators form an Abelian algebra for the corresponding Euler–Lagrange-type systems. Interestingly, we find that in many cases, the Noether-like operators satisfy the classical Noether symmetry condition and become the Noether symmetries. Moreover, we observe that the classical Noetherian integrals and the first integrals we determine using the complex Lagrangian approach turn out to be the same for the underlying system of Lane–Emden equations.

Keywords: generalized Lane–Emden systems; Noether-like operator; conservation laws

1. Introduction

The famous Noether theorem [1] establishes an important connection between the conservation laws and symmetry properties of a system describable by a Lagrangian. From a mathematical point of view, it is the case that the essential physical explanation of a Euler–Lagrange system is hidden in its Lagrangian. The Lagrangian function, on the one hand, describes the time behavior of a mechanical system through the Euler–Lagrange equation, and on the other hand, it connects symmetries with first integrals of motion if they arise through Noether's theorem. The availability of a Noether symmetry is essential from two aspects: first, to determine conservation laws and, second, to reduce the underlying equation. A significant number of studies on Noether symmetries and first integrals have been reported in recent years. It is well known that if an equation possesses enough conserved quantities, it can be easily reduced to an integrable form.

In recent papers, the authors of [2,3] introduced the complex symmetry approach, which has been established as an appealing and elegant technique to study integrability properties of systems of ordinary differential equations (ODEs). Following the idea of [3–5], several studies have been done to view integrability properties of systems of partial differential equations (PDEs) and ODEs. For instance, the use of the complex variable technique to discuss linearization of systems of two second-order ODEs and PDEs has been presented in [6]. The procedure of converting a system of two second-order ODEs admitting Lie algebra of dimension d ($d \leq 4$) into linearizable form with the help of complex Lie point symmetries of the base equation was given in [7]. Using semi-invariants, Mahomed et al. [8] studied systems of two linear hyperbolic PDEs when they arise from a complex scalar ODE. They found that the semi-invariants under linear transformations correspond to complex semi-invariants of the $(1+1)$ linear hyperbolic equation in the complex domain. They also succeeded in linking these hyperbolic equations by introducing a complex variable structure on the manifold to the geometry of underlying differential equations. Qadir and Mahomed [9] employed the complex variable technique to study three- and four-dimensional systems of ODEs and PDEs that are transformable to a single complex ODEs. They showed that the acquired systems of ODEs are entirely different from the class that is obtained from single splitting of systems of two ODEs. Naz and Mahomed [10,11] presented a detailed analysis of the computation of Lie and Noether point symmetries of the k^{th}-order system of n ODEs by working in the complex domain. They also discussed the transonic gas flow, Maxwellian distribution, Klein–Gordon equation, dissipative wave, and Maxwellian tails by introducing complex variables. Wafo Soh and Mahomed [12] showed that by utilizing hypercomplexification, one can linearize Ermakov systems. Transforming systems of some Riccati-type equations to a single base equation, they constructed invariants of Able-type systems.

In the current study, we use the formulation of the Noether-like theorem presented in [3–5] and classify systems of Lane–Emden equations with respect to Noether-like operators they admit and related first integrals. On applying the complex symmetry approach, we see that additional insights are obtainable by utilizing the fact that a complex Lagrangian encodes information of two real Lagrangians, and it is derivable from a variational principle. As a consequence of the present study, many important symmetry properties can easily be analyzed using complex Lagrangians, and these help us to determine the invariant quantities of physically-coupled systems represented by ODEs.

The celebrated Lane–Emden (LE) equation given below is the simplest second-order ODE, which appears frequently in modeling one-dimensional problems in physics, astrophysics, and engineering, and it is still a subject of extensive analysis. A review by Wang [13], even though very selective in its list of references, covered almost all possible generalizations and qualitative properties of the LE equation.

Consider the well-known second-order LE equation:

$$y'' + \frac{n}{t}y' + f(y) = 0, \tag{1}$$

where n is a real number and $f(y)$ an arbitrary continuous function of y. The LE equation (1) has many physical applications. For instance, for fixed values of n and $f(y)$, it specifically models the thermal behavior of a spherical cloud of gas, stellar structure, an isothermal gaseous sphere, and the theory of thermionic currents [14–16]. In the literature, various techniques have been proposed concerning the solutions of Equation (1); see for example [17–20]. Several authors have proven existence and uniqueness results for the LE systems [21–24] (see also the references in these papers) and other related systems. Some other works that involve Noether symmetries and exact solutions of LE-type equations can be found in [25]. Moreover, the Noether symmetries of Equation (1) and exact solutions by taking various forms of $f(y)$ were investigated in [26].

Before going to the main discussion, it is important to recall studies in view of the Noether symmetry classification of coupled systems of LE equations. Recently, the authors of [27] took a system of LE equations given by a natural extension of (1), classified it with respect to Noether symmetries, and constructed first integrals of:

$$f'' + \frac{n}{x}f' + F_1(g) = 0, \qquad g'' + \frac{n}{x}g' + F_2(f) = 0, \tag{2}$$

where n is a real number constant and $F_1(g)$ and $F_2(f)$ are arbitrary functions. From a Noether symmetry, Muatjetjeja and Khalique [28], extended their own work and studied the classification of another system of LE equations given by:

$$f'' + \frac{n}{x}f' + h(x)g^q = 0, \qquad g'' + \frac{n}{x}g' + h(x)f^p = 0, \tag{3}$$

with respect to Noether symmetries and their first integrals. In this paper, we shall make a kind of comparison of how the complex Lagrangian formulation and the classical Noether symmetry approach generate the same first integrals for the following general class of the LE system:

$$f'' + \frac{n_1}{x} f' - \frac{n_2}{x} g' + F_1(f,g) = 0, \qquad g'' + \frac{n_2}{x} f' + \frac{n_1}{x} g' + F_2(f,g) = 0. \tag{4}$$

The famous LE system (4) has been used in modeling various physical problems such as pattern recognition, chemical reactions, and population evolution, to name a few. This system attracted the attention of many authors and has been an area of extensive research during the last couple of years (see [21–24,29,30] and the references therein).

We shall consider various forms of F_1 and F_2 to construct conserved quantities of the ensuing systems and show that reduction via quadrature can be obtained only in a few cases. We point out that the Noether-like operators we find for systems of Euler–Lagrange LE equations also satisfy the classical Noether symmetry condition for one of the known equivalent Lagrangians, emerge as Noether symmetries, and hence yield Noetherian first integrals for the subsequent systems. Thus, the Noetherian first integrals and the first integrals we obtain employing a complex Lagrangian approach turn out to be the same with respect to the Lagrangians for the underlying systems of ODEs. We shall see that many interesting insights can be obtained for systems of ODEs through the complex symmetry approach.

The layout of the paper is the following: in the next section, we briefly recall some basic definitions of Noether-like operators and the Noether-like theorem. Section 3 deals with the classification of Noether-like operators and associated first integrals for the system (4). In the last section, we present our concluding remarks.

2. Preliminaries on Noether-Like Operators and First Integrals

Before we consider the generalized system of LE equations in relation to their Noether-like operators and corresponding first integrals, it is instructive to have relevant definitions of these operators and the Noether-like theorem that will be used in our discussion. Moreover, to make the comparison, we also recall expressions for classical Noether symmetries and Noether's theorem. The contents of this section are taken from [3,4] (for more details, the reader is urged to see the references therein).

Consider the following system of nonlinear second-order ODEs:

$$f_i'' = w_i(x, f_1, f_1', f_2, f_2'), \qquad i = 1, 2. \tag{5}$$

Equation (5) represents a general class of a system of second-order ODEs and models various physical problems. However, here, we merely deal with those systems in (5) that are equivalent to a single scalar complex ODE, i.e., there exist transformations $f = f_1 + if_2$, $w = w_1 + iw_2$ that reduce the system (5) to a complex ODE, $f'' = w(x, f, f')$, which retain a variational structure. It is generally conceded that the construction of a Lagrangian for systems of nonlinear ODEs has been proven to be a complicated problem. However, we see here how one can study symmetry properties of Euler–Lagrange-type LE equations straightforwardly with the help of a complex Lagrangian, which encodes two real Lagrangians and enables us to cast the system (5) in a variational form.

Here, our aim is to determine the Noether-like operators and related first integrals of a coupled system of two LE equations. We start by assuming that the system (5) admits a complex Lagrangian $L(x, f, f')$, i.e. $L = L_1 + iL_2$. Therefore, we have two Lagrangians L_1 and L_2, which when utilized result in the following Euler–Lagrange-type system corresponding to (5):

$$\begin{aligned} &\frac{\partial L_1}{\partial f_1} + \frac{\partial L_2}{\partial f_2} - \frac{d}{dx}\left(\frac{\partial L_1}{\partial f_1'} + \frac{\partial L_2}{\partial f_2'}\right) = 0, \\ &\frac{\partial L_2}{\partial f_1} - \frac{\partial L_1}{\partial f_2} - \frac{d}{dx}\left(\frac{\partial L_2}{\partial f_1'} - \frac{\partial L_1}{\partial f_2'}\right) = 0. \end{aligned} \tag{6}$$

The operators $\mathbf{X}_1 = \varsigma_1(x,f_1,f_2)\frac{\partial}{\partial x} + \chi_1(x,f_1,f_2)\frac{\partial}{\partial f_1} + \chi_2(x,f_1,f_2)\frac{\partial}{\partial f_2}$ and $\mathbf{X}_2 = \varsigma_2(x,f_1,f_2)\frac{\partial}{\partial x} + \chi_2(x,f_1,f_2)\frac{\partial}{\partial f_1} - \chi_1(x,f_1,f_2)\frac{\partial}{\partial f_2}$ are known as Noether-like operators of (5) for the Lagrangians L_1 and L_2 if the following conditions hold:

$$\begin{aligned}\mathbf{X}_1^{(1)}L_1 - \mathbf{X}_2^{(1)}L_2 + (D_x\varsigma_1)L_1 - (D_x\varsigma_2)L_2 = D_xA_1,\\ \mathbf{X}_1^{(1)}L_2 - \mathbf{X}_2^{(1)}L_1 + (D_x\varsigma_1)L_2 + (D_x\varsigma_2)L_1 = D_xA_2,\end{aligned} \tag{7}$$

for appropriate functions A_1 and A_2. Here, $D_x = \frac{d}{dx}$.

Noether-like theorem:

If $\mathbf{X}_1$ and $\mathbf{X}_2$ are two Noether-like operators with respect to real Lagrangians L_1 and L_2, then (5) possesses the following two first integrals:

$$\begin{aligned}I_1 = \varsigma_1L_1 - \varsigma_2L_2 + \frac{\partial L_1}{\partial f_1'}(\chi_1 - f_1'\varsigma_1 - f_2'\varsigma_2) - \frac{\partial L_2}{\partial f_1'}(\chi_2 - f_1'\varsigma_2 - f_2'\varsigma_1) - A_1,\\ I_2 = \varsigma_1L_2 + \varsigma_2L_1 + \frac{\partial L_2}{\partial f_1'}(\chi_1 - f_1'\varsigma_1 - f_2'\varsigma_2) + \frac{\partial L_1}{\partial f_1'}(\chi_2 - f_1'\varsigma_2 - f_2'\varsigma_1) - A_2.\end{aligned} \tag{8}$$

Classical Noether symmetry condition:

A vector field $X = \varsigma(x,f_1,f_2)\frac{\partial}{\partial x} + \chi(x,f_1,f_2)\frac{\partial}{\partial f_1} + \eta(x,f_1,f_2)\frac{\partial}{\partial f_2}$ with its prolongation $X^{[1]} = X + (\dot{\chi} - \dot{f}_1\dot{\varsigma})\frac{\partial}{\partial \dot{f}_1} + (\dot{\eta} - \dot{f}_2\dot{\varsigma})\frac{\partial}{\partial \dot{f}_2}$ where '$\cdot$' $= \frac{d}{dx}$ is known as a Noether point symmetry corresponding to the function $L(x,f_1,f_2,f_1',f_2')$ of (5) if the following equation holds:

$$X^{[1]}(L) + D_x(\varsigma)L = D_x(A) \tag{9}$$

Noether's theorem:

For $\mathbf{X}$ to be a Noether symmetry generator for the Lagrangian $L(x,f_1,f_2,f_1',f_2')$, the following equation:

$$I = A - \left[\varsigma L + (\chi - \varsigma\dot{f}_1)\frac{\partial L}{\partial \dot{f}_1} + (\eta - \varsigma\dot{f}_2)\frac{\partial L}{\partial \dot{f}_2}\right], \tag{10}$$

provides the Noetherian first integral of (5) related to $\mathbf{X}$.

3. Noether-Like Operators and First Integrals for Different forms of F_1 and F_2 in (4)

Major computational difficulties occur when trying to classify the general nonlinear LE equation with respect to Noether symmetry operators and corresponding first integrals. We see here how the Noether-like operators play an important role in deriving conserved quantities for dynamical systems and their reduction via quadrature.

Consider the following nonlinear system, which is a generalized coupled LE-type system:

$$\begin{aligned}f_1'' + \frac{n_1f_1' - n_2f_2'}{x} + F_1(f_1,f_2) = 0,\\ f_2'' + \frac{n_1f_2' + n_2f_1'}{x} + F_2(f_1,f_2) = 0,\end{aligned} \tag{11}$$

for which we have analyzed eight cases separately. Here, n_1, n_2 are constants and F_1, F_2 are arbitrary functions of f_1 and f_2, respectively. We take different forms of F_1 and F_2 in (11) and determine Noether-like operators and conserved quantities for the subsequent systems. Therefore, for this, we proceed as: one can readily verify that the pair of Lagrangians for the system (11) when invoking (6) is given by:

$$
\begin{aligned}
L_1 &= \frac{1}{2}x^{n_1}[cos\theta(f_1'^2 - f_2'^2) - 2sin\theta f_1' f_2'] - x^{n_1}[cos\theta \int (F_1 df_1 - F_2 df_2) - sin\theta \int (F_2 df_1 + F_1 df_2)], \\
L_2 &= \frac{1}{2}x^{n_1}[2cos\theta(f_1' f_2') + sin\theta(f_1'^2 - f_2'^2)] - x^{n_1}[cos\theta \int (F_2 df_1 + F_1 df_2) + sin\theta \int (F_1 df_1 - F_2 df_2)],
\end{aligned} \tag{12}
$$

where $\theta = n_2 \ln x$.

Case 1. $F_1(f_1, f_2)$ and $F_2(f_1, f_2)$ are linear in f_1 and f_2, respectively.

In this case, we have a system of two linear ODEs. Using appropriate transformations, one can reduce the system of linear equations to a system of free particle equations, viz. $f_1'' = 0$, $f_2'' = 0$, which possesses nine Noether-like operators associated with the coupled Lagrangians (11), and they give ten first integrals. This case is well known and can be found in detail in [4].

Case 2. For $n_1, n_2 = 0$ and $F_1(f_1, f_2)$, $F_2(f_1, f_2)$ arbitrary and non-linear, as given in Case 1.

Equations (7) and (12), after some straightforward calculations, show that $\varsigma_1 = 1$, $\varsigma_2 = 0$, $\chi_1 = \chi_2 = 0$, and A_1, A_2 are constants. Therefore, we have a single Noether-like operator $\mathbf{X} = \frac{\partial}{\partial x}$. Using the pair of Lagrangians (12) and Noether-like operator $\mathbf{X}$ in (8), we obtain the following two first integrals:

$$
\begin{aligned}
I_1 &= \frac{1}{2}(f'^2 - g'^2) + \int [F_1 df - F_2 dg], \\
I_2 &= f'g' + \int [F_1 dg + F_2 df].
\end{aligned} \tag{13}
$$

Interestingly, the Noether-like operator $\mathbf{X}$ is also a Noether symmetry for each of the Lagrangians (12), and (10) generates the same first integrals as given in (13) for System (11).

Case 3. If:

$$
\begin{aligned}
F_1(f_1, f_2) &= \frac{\alpha}{2}\log(f_1^2 + f_2^2) + \gamma f_1 + \delta,\ \alpha \neq 0, \\
F_2(f_1, f_2) &= \alpha \arctan(\frac{f_2}{f_1}) + \gamma f_2,\ \alpha \neq 0
\end{aligned} \tag{14}
$$

and $n_1,\ n_2 = 0$ and $\delta = 0$, we obtain $\varsigma_1 = x,\ \varsigma = 0,\ \chi_1 = \chi_2 = 0$ with A_1, A_2 as constants. This falls into Case 2.

Case 4. For:

$$
\begin{aligned}
F_1(f_1, f_2) &= \frac{\alpha}{2}[f_1 \log(f_1^2 + f_2^2) - f_2 \arctan(f_2/f_1)] + \gamma f_1 + \delta, \alpha \neq 0, \\
F_2(f_1, f_2) &= \frac{\alpha}{2}[f_1 \arctan(f_2/f_1) + f_2 \log(f_1^2 + f_2^2)] + \gamma f_2 + \delta, \alpha \neq 0.
\end{aligned} \tag{15}
$$

If $n_1,\ n_2 = 0$, we obtain $\varsigma_1 = x,\ \varsigma_2 = 0,\ \chi_1 = \chi_2 = 0$, and $A_1 = A_2 = k$, k being a constant. This also bring us back to Case 2.

Case 5. If $F = \alpha u^r,\ \alpha \neq 0,\ r \neq 0,\ 1$.

Here, we discuss the following three cases:

Case 5.1. For $n_1 = \frac{r+3}{r-1}$ and $n_2 = 0$, the Noether-like symmetry conditions (7) result in $\varsigma_1 = x,\ \varsigma_2 = 0,\ \chi_1 = \frac{2}{1-r}f_1,\ \chi_2 = \frac{2}{1-r}f_2$, with $A_1,\ A_2$ as constants. Therefore, we get two Noether-like operators:

$$
\mathbf{X}_1 = x\frac{\partial}{\partial x} + \frac{2}{1-r}\Big(f_1\frac{\partial}{\partial f_1} + f_2\frac{\partial}{\partial f_2}\Big),\quad \mathbf{X}_2 = \frac{2}{1-r}\Big(f_2\frac{\partial}{\partial f_1} - f_1\frac{\partial}{\partial f_2}\Big). \tag{16}
$$

Utilizing (16) with (12), Equation (8) gives rise to two first integrals:

$$I_1 = \frac{1}{2}x^{n_1+1}(f_1'^2 - f_2'^2) - \frac{\alpha}{r+1}x^{n_1+1}(f_1^2+f_2^2)^{\frac{r+1}{2}}cos\theta + \frac{2}{1-r}x^{n_1}(f_1f_1' - f_2f_2') - x^{n_1+1}(f_1'^2 - f_2'^2),$$
$$I_2 = x^{n_1+1}f_1'f_2' - \frac{\alpha}{r+1}x^{n_1+1}(f_1^2+f_2^2)^{\frac{r+1}{2}}sin\theta + \frac{2}{1-r}x^{n_1}(f_1f_2' + f_1'f_2) - 2x^{n_1+1}f_1'f_2', \tag{17}$$

for (11). Here, $\theta = (r+1)\arctan(f_2/f_1)$. Utilization of transformations $f_1 = w_1 x^{\frac{v+1}{1-r}}$ and $f_2 = w_2 x^{\frac{v+1}{1-r}}$ converts the above system (17) into an integrable form as:

$$\int \frac{dw}{\pm\sqrt{4(1-r)^{-2}w^2 - 2\alpha(1+r)^{-1}f^{r+1} - C_1}} = lnxC_2, \tag{18}$$

where C_1 and C_2 are constants. Here, we can see that the Lie algebra of Noether-like operators is Abelian, i.e., $[\mathbf{X}_1, \mathbf{X}_2] = 0$.

Case 5.2. If we set $n_1 = 2$, $n_2 = 0$, and $r = 5$, Equations (6) and (12) yield the famous Emden–Fowler system [3] given by:

$$f_1'' + \frac{2}{x}f_1' + \alpha(f_1^5 - 10f_1^3f_2^2 + 5f_1f_2^4) = 0,$$
$$f_2'' + \frac{2}{x}f_2' + \alpha(f_2^5 - 10f_1^2f_2^3 + 5f_1^4f_2) = 0, \tag{19}$$

while the associated Lagrangians are:

$$L_1 = \frac{1}{2}x^2(f_1'^2 - f_2'^2) - \frac{\alpha}{6}x^2[f_1^6 - 15f_1^4f_2^2 + 15f_1^2f_2^4 - f_2^6],$$
$$L_1 = x^2f_1'f_2' - \frac{\alpha}{3}x^2[3f_1^5f_2 - 10f_1^3f_2^3 + 3f_1f_2^5]. \tag{20}$$

It is easy to see that the Emden–Fowler system (19) admits the following two Noether-like operators:

$$\mathbf{X}_1 = 2x\frac{\partial}{\partial x} - f_1\frac{\partial}{\partial f_1} - f_2\frac{\partial}{\partial f_2}, \quad \mathbf{X}_2 = f\frac{\partial}{\partial f_2} - f_2\frac{\partial}{\partial f_1}. \tag{21}$$

Utilizing these operators in Equations (8) and (20), we obtain the following constant quantities:

$$I_1 = x^3(f_1'^2 - f_2'^2) + x^2(f_1f_1' - f_2f_2') + \frac{1}{3}x^3(f_1^6 + 15f_1^2f_2^4 - 15f_1^4f_2^2 - f_2^6),$$
$$I_2 = x^3f_1'f_2' + \frac{1}{2}x^2(f_1f_2' + f_1'f_2) + x^3(f_1f_2^5 - \frac{10}{3}f_1^3f_2^3 + f_1^5f_2), \tag{22}$$

for (19). Upon checking, we see that for L_1 and L_2, the above system (19) admits $\mathbf{X}_1$ as a Noether symmetry. Therefore, from the classical Noether theorem, we can deduce the first integrals I_1 and I_2 (Noetherian integrals) for (19).

Case 5.3. If $n_1 = \frac{r+3}{r+1}$ with $r \neq -1$, we have $\varsigma_1 = x^{\frac{r-1}{r+1}}$, $\varsigma_2 = 0$, $\chi_1 = -\frac{2}{r+1}x^{\frac{-2}{r+1}}f_1$, $\chi_2 = -\frac{2}{r+1}x^{\frac{-2}{r+1}}f_2$, and $A_1 = \frac{2}{2(r+1)^2}(f_1^2 - f_2^2) + q$, $A_2 = \frac{4}{(r+1)^2}f_1f_2$, where q is constant. By invocation of the Noether-like theorem, the Noether-like operators given in (24) provide:

$$I_1 = \frac{1}{2}x^2(f_1'^2 - f_2'^2) + \frac{\alpha}{r+1}x^2(f_1^2+f_2^2)^{\frac{r+1}{2}}cos\theta + \frac{2}{r+1}x(f_1f_1' - f_2f_2') + \frac{2}{(1+r)^2}(f_1^2 - f_2^2),$$
$$I_2 = x^2f_1'f_2' + \frac{\alpha}{r+1}x^2(f_1^2+f_2^2)^{\frac{r+1}{2}}sin\theta + \frac{2}{r+1}x(f_1f_2' + f_1'f_2) + \frac{4}{(r+1)^2}f_1f_2, \tag{23}$$

where $\theta = (r+1)\arctan(f_2/f_1)$. In this case, Noether-like operators are of the form:

$$\mathbf{X}_1 = x^{\frac{r-1}{r+1}}\frac{\partial}{\partial x} - \frac{2}{r+1}x^{-\frac{2}{r+1}}\Big(f_1\frac{\partial}{\partial f_1} + f_2\frac{\partial}{\partial f_2}\Big), \quad \mathbf{X}_2 = -\frac{2}{r+1}x^{-\frac{2}{r+1}}\Big(f_2\frac{\partial}{\partial f_1} - f_1\frac{\partial}{\partial f_2}\Big). \tag{24}$$

Applying the transformations $f_1 = w_1 x^{\frac{-\nu-1}{r+1}}$ and $f_2 = w_2 x^{\frac{-\nu-1}{r+1}}$, System (23) can be converted into the variable separable form:

$$\int \frac{dw}{\pm\sqrt{-2\alpha(r+1)^{-1}w^{r+1} + C_1}} = \frac{r+1}{2}x^{\frac{2}{r+1}} + C_2, \tag{25}$$

where C_1 and C_2 are arbitrary constants.

Case 6. If F_1 and F_2 are nonlinear and are of the form $F_1(f_1, f_2) = \alpha(f_1^2 - f_2^2) + \beta f_1 + \gamma$, $F_2(f_1, f_2) = 2\alpha f_1 f_2 + \beta f_2$, $\alpha,\ \beta,\ \gamma$ are constants, and $\alpha \neq 0$.

Here, the following subcases arise:

Case 6.1. If $n_1 = 5$ and $n_2 = 0$, $\beta = 0$ and $\gamma = 0$, we obtain from (7) that $\varsigma_1 = x$, $\varsigma_2 = 0$, $\chi_1 = -2f_1$, $\chi_2 = -2f_2$, and A_1, A_2 are constants. This case falls into Case 5.1.

Case 6.2. If $n_1 = 5$, $n_2 = 0$, $\beta^2 = 4\alpha\gamma$, Equations (7) and (12) yield $\varsigma_1 = x$, $\varsigma_2 = 0$, $\chi_1 = -(2f_1 + \frac{\beta}{\alpha})$, $\chi_2 = -2f_2$, $A_1 = \frac{\beta\gamma}{6\alpha}x^6$, and $A_2 = 0$. Therefore, Noether-like operators are of the form:

$$\mathbf{X}_1 = x\frac{\partial}{\partial x} - (2f_1 + \frac{\beta}{\alpha})\frac{\partial}{\partial f_1} - 2f_2\frac{\partial}{\partial f_2}, \quad \mathbf{X}_2 = (2f_1 + \frac{\beta}{\alpha})\frac{\partial}{\partial f_2} - 2f_2\frac{\partial}{\partial f_1}. \tag{26}$$

Invocation of the Noether-like theorem (8) along with Lagrangians and Noether-like operators $\mathbf{X}_1$ and $\mathbf{X}_2$ results in two first integrals:

$$\begin{aligned} I_1 &= \frac{1}{2}x^6(f_1'^2 - f_2'^2) + \frac{1}{3}\alpha x^6(f_1^3 - 3f_1f_2^2) + \frac{1}{2}\beta x^6(f_1^2 - f_2^2) + \gamma x^6 f + 2x^5(f_1f_1' - f_2f_2') + \frac{\beta}{\alpha}x^5 f_1' \\ &+ \frac{\beta\gamma}{6\alpha}x^6 \\ I_2 &= x^6 f_1'f_2' + \frac{1}{3}\alpha x^6(3f_1^2f_2 - f_2^3) + \beta x^6 f_1 f_2 + \gamma x^6 f_2 + 2x^5(f_1f_2' + f_1'f_2) + \frac{\beta}{\alpha}x^5 f_2', \end{aligned} \tag{27}$$

for (11). Using the transformations $w_1 = x^{1+\nu}f_1 + \frac{\beta}{2\alpha}x^{\nu+1}$ and $w_1 = x^{\nu+1}f_2$, one can map the system (27) to a separable form:

$$C = 2w^2 - \frac{1}{2}x^2w'^2 - \frac{\alpha}{3}w^3, \tag{28}$$

where $w(x) = w_1 + iw_2$.

It can be verified that the Noether-like operator $\mathbf{X}_1$ in (26) is also a Noether symmetry for the Lagrangians L_1 and L_2 in Equation (12). The classical Noether's theorem generates the same Noetherian first integrals I_1 and I_2 given in Equation (27) with Lagrangians L_2 and L_1, respectively, for the resulting system of LE equations. Furthermore, we observe that $[\mathbf{X}_1, \mathbf{X}_2] = 0$, so these operators form an Abelian algebra.

Case 6.3. For $n_1 = \frac{5}{3}$, $n_2 = 0$, $\beta = 0$, and $\gamma = 0$, Equation (7) taking L_1 and L_2 from (12) with simple calculations gives $\varsigma_1 = x^{\frac{1}{3}}$, $\chi_1 = -\frac{2}{3}x^{\frac{-2}{3}}f_1$, $\chi_2 = -\frac{2}{3}x^{\frac{-2}{3}}f_2$, and $A_1 = \frac{2}{9}(f_1^2 - f_2^2) + k$, $A_2 = \frac{4}{9}(f_1f_2)$, and k is a constant. This case falls into Case 5.2.

Case 7. For $F_1(f_1, f_2) = \alpha e^{\beta f_1}cos(\beta f_2) + \gamma f_1 + \delta$, $F_2(f_1, f_2) = \alpha e^{\beta f_1}sin(\beta f_2) + \gamma f_2$, where $\alpha,\ \beta,\ \delta$ are constants and $\alpha \neq 0,\ \beta \neq 0$. Therefore, (11) takes the form:

$$\begin{aligned} f_1'' + \frac{n_1f_1' - n_2f_2'}{x} + \alpha exp(\beta f_1)cos(\beta f_2) + \gamma f_1 + \delta = 0, \\ f_2'' + \frac{n_1f_2' + n_2f_1'}{x} + \alpha exp(\beta f_1)sin(\beta f_2) + \gamma f_2 = 0, \end{aligned} \tag{29}$$

For $n_1 = 1,\ n_2 = 0,\ \gamma = 0,\ \delta = 0$, and $\beta = 1$, we obtain $\varsigma_1 = x,\ \varsigma_2 = 0, \chi_1 = -2,\ \chi_2 = 0$, and $A_1,\ A_2 = q$, where q is a constant. Therefore, the system (29) possesses the Noether-like operators:

$$\mathbf{X}_1 = x\frac{\partial}{\partial x} - \frac{2\partial}{\partial f_1}, \quad \mathbf{X}_2 = \frac{\partial}{\partial f_2}. \tag{30}$$

with the corresponding pair of Lagrangians:

$$\begin{aligned} L_1 &= \frac{1}{2}x(f_1'^2 - f_2'^2) - \alpha x e^{f_1} cos f_2, \\ L_2 &= x f_1' f_2' - \alpha x e^{f_1} sin f_2. \end{aligned} \tag{31}$$

Utilizing the Noether-like operators and Lagrangians given above, Equation (8) implies the first integrals:

$$\begin{aligned} I_1 &= \frac{1}{2}x^2(f_1'^2 - f_2'^2) + \alpha x^2 e^{f_1} cos f_2 + 2x f_1' \\ I_2 &= x^2 f_1' f_2' + \alpha x^2 e^{f_1} sin f_2 + 2x f_2'. \end{aligned} \tag{32}$$

It is important to mention here that the system (29) admits Noether-like operator $\mathbf{X}_1$ as a Noether symmetry [3], as it satisfies the classical Noether symmetry condition with Lagrangians L_1 and L_2 given in (31). Therefore, application of the classical Noether theorem remarkably generates two Noetherian first integrals, namely I_1 and I_2 given in (32). Here, again, the Lie bracket gives $[\mathbf{X}_1, \mathbf{X}_2] = 0$, which shows that the algebra of these operators is Abelian.

Case 8. Here, n_1, n_2 are nonzero, and $F_1(f_1, f_2),\ F_2(f_1, f_2)$ are arbitrary, but not of the form contained in the cases given above.

From Equation (7), after simple manipulations, we find that $\varsigma_1 = \varsigma_2 = 0, \chi_1 = \chi_2 = 0$, and A_1, A_2 are constants. We deduce that no Noether-like operators exist in this case.

4. Conclusions

In this paper, we have applied the complex Noether approach and attempted to classify a two-dimensional coupled system of LE equations that appears in physics and applied mathematics with respect to Noether-like-operators and corresponding first integrals by taking the functions F_1 and F_2 in their more general forms in Equation (11). In this study, we have observed that for some of the systems of LE equations, every pair of Noether-like operators forms an Abelian Lie algebra. We have also highlighted that for certain pairs of Lagrangians, the Noether-like operators become Noether symmetries of the Euler–Lagrange systems of LE equations and give rise to the same Noetherian first integrals as we determined from our complex approach. Therefore, the study of invariant quantities of many dynamical systems can be made with the help of complex Lagrangian formalism, which seems to be more simple and elegant.

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