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Existence Theory for Nonlinear Third-Order Ordinary Differential Equations with Nonlocal Multi-Point and Multi-Strip Boundary Conditions

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Abstract: We investigate the solvability and Ulam stability for a nonlocal nonlinear third-order integro-multi-point boundary value problem on an arbitrary domain. The nonlinearity in the third-order ordinary differential equation involves the unknown function together with its first- and second-order derivatives. Our main results rely on the modern tools of functional analysis and are well illustrated with the aid of examples. An analogue problem involving non-separated integro-multi-point boundary conditions is also discussed.

Keywords: nonlinear boundary value problem; nonlocal; multi-point; multi-strip; existence; Ulam stability

1. Introduction

Consider a third-order ordinary differential equation of the form:

$$u'''(t) = f(t, u(t), u'(t), u''(t)), \ a < t < T, \ a, T \in \mathbb{R}, \tag{1}$$

supplemented with the boundary conditions:

$$\begin{aligned}
\int_a^T u(s)ds &= \sum_{j=1}^{m} \gamma_j u(\sigma_j) + \sum_{i=1}^{p} \xi_i \int_{\rho_i}^{\rho_{i+1}} u(s)ds, \\
\int_a^T u'(s)ds &= \sum_{j=1}^{m} \mu_j u'(\sigma_j) + \sum_{i=1}^{p} \eta_i \int_{\rho_i}^{\rho_{i+1}} u'(s)ds, \\
\int_a^T u''(s)ds &= \sum_{j=1}^{m} \nu_j u''(\sigma_j) + \sum_{i=1}^{p} \omega_i \int_{\rho_i}^{\rho_{i+1}} u''(s)ds,
\end{aligned} \tag{2}$$

where $f : [a, T] \times \mathbb{R}^3 \to \mathbb{R}$ is a continuous function, $a < \sigma_1 < \sigma_2 < \cdots < \sigma_m < \rho_1 < \rho_2 < \cdots < \rho_{p+1} < T$, and $\gamma_j, \mu_j, \nu_j \in \mathbb{R}^+ \ (j = 1, 2, \ldots, m)$, $\xi_i, \eta_i, \omega_i \in \mathbb{R}^+ \ (i = 1, 2, \ldots, p)$.

As a second problem, we study Equation (1) with the following type non-separated boundary conditions:

$$\begin{aligned} \alpha_1 u(a) + \alpha_2 u(T) &= \sum_{j=1}^{m} \gamma_j u(\sigma_j) + \sum_{i=1}^{p} \xi_i \int_{\rho_i}^{\rho_{i+1}} u(s)ds, \\ \beta_1 u'(a) + \beta_2 u'(T) &= \sum_{j=1}^{m} \mu_j u'(\sigma_j) + \sum_{i=1}^{p} \eta_i \int_{\rho_i}^{\rho_{i+1}} u'(s)ds, \\ \delta_1 u''(a) + \delta_2 u''(T) &= \sum_{j=1}^{m} \nu_j u''(\sigma_j) + \sum_{i=1}^{p} \omega_i \int_{\rho_i}^{\rho_{i+1}} u''(s)ds, \end{aligned} \tag{3}$$

where $\alpha_j, \beta_j, \delta_j \in \mathbb{R}$ $(j = 1, 2)$, while the rest of parameters are the same as fixed in the problem in Equations (1) and (2).

The subject of boundary value problems has been an interesting and important area of investigation in view of its varied application in applied sciences. One can find the examples in blood flow problems, underground water flow, chemical engineering, thermoelasticity, etc. For a detailed account of applications, see [1].

Nonlinear third-order ordinary differential equations frequently appear in the study of applied problems. In [2], the authors studied the existence of solutions for third-order nonlinear boundary value problems arising in nano-boundary layer fluid flows over stretching surfaces. In the study of magnetohydrodynamic flow of a second grade nanofluid over a nonlinear stretching sheet, the system of transformed governing equations involves a nonlinear third-order ordinary equation and is solved for local behavior of velocity distributions [3]. The investigation of the model of magnetohydrodynamic flow of second grade nanofluid over a nonlinear stretching sheet is also based on a nonlinear third-order ordinary differential equation [4].

During the last few decades, boundary value problems involving nonlocal and integral boundary conditions attracted considerable attention. In contrast to the classical boundary data, nonlocal boundary conditions help to model physical, chemical or other changes occurring within the given domain. For the study of heat conduction phenomenon in presence of nonclassical boundary condition, see [5]. The details on theoretical development of nonlocal boundary value problems can be found in the articles [6–10] and the references cited therein. On the other hand, integral boundary conditions play a key role in formulating the real world problems involving arbitrary shaped structures, for example, blood vessels in fluid flow problems [11–13]. For the recent development of the boundary value problems involving integral and multi-strip conditions, we refer the reader to the works [14–19].

In heat conduction problems, the concept of nonuniformity can be relaxed by using the boundary conditions of the form (2), which can accommodate the nonuniformities in form of points or sub-segments on the heat sources. In fact, the integro-multipoint conditions (2) can be interpreted as the sum of the values of the unknown function (e.g., temperature) at the nonlocal positions (points and sub-segments) is proportional to the value of the unknown function over the given domain. Moreover, in scattering problems, the conditions (2) can be helpful in a situation when the scattering boundary consists of finitely many sub-strips (finitely many edge-scattering problems). For details and applications in engineering problems, see [20–23].

In the present work, we derive the existence results for the problem in Equations (1) and (2) by applying Leray–Schauder nonlinear alternative and Krasnoselskii fixed-point theorem, while the uniqueness result is obtained with the aid of celebrated Banach fixed point theorem. These results are presented in Section 3. The Ulam type stability for the problem in Equations (1) and (2) is discussed in Section 4. In Section 5, we describe the outline for developing the existence theory for the problem in Equations (1) and (3). Section 2 contains the auxiliary lemmas related to the linear variants of the given problems, which lay the foundation for establishing the desired results. It is imperative to mention that the results obtained in this paper are new and yield several new results as special cases for appropriate choices of the parameters involved in the problems at hand.

2. Preliminary Result

In this section, we solve linear variants of the problems in Equations (1) and (2), and Equations (1) and (3).

Lemma 1. *For $g \in C([a,T],\mathbb{R})$ and $\Lambda \neq 0$, the unique solution of the problem consisting of the equation*

$$u'''(t) = g(t),\ t \in [a,T],$$

and the boundary condition in Equation (2) is

$$\begin{aligned} u(t) &= \int_a^t \frac{(t-s)^2}{2} g(s)ds \\ &-\frac{1}{\Lambda}\int_a^T \Big[A_1A_2\frac{(T-s)^3}{3!} + G_1(t)\frac{(T-s)^2}{2} + G_2(t)(T-s)\Big]g(s)ds \\ &+\frac{1}{\Lambda}\sum_{j=1}^{m}\int_a^{\sigma_j}\Big[\gamma_j A_1A_2\frac{(\sigma_j-s)^2}{2} + \mu_j G_1(t)(\sigma_j - s) + \nu_j G_2(t)\Big]g(s)ds \\ &+\frac{1}{\Lambda}\sum_{i=1}^{p}\int_{\rho_i}^{\rho_{i+1}}\Big[\int_a^s\Big(\xi_i A_1A_2\frac{(s-\tau)^2}{2} + \eta_i G_1(t)(s-\tau) + \omega_i G_2(t)\Big)g(\tau)d\tau\Big]ds, \end{aligned} \tag{4}$$

where

$$G_1(t) = A_1\Big(A_4(t-a) - A_5\Big),\ G_2(t) = A_3\Big(A_5 - A_4(t-a)\Big) - A_2\Big(A_6 - A_4\frac{(t-a)^2}{2}\Big), \tag{5}$$

$$\begin{cases} \Lambda = A_1A_2A_4,\ A_1 = \Big(T - a - \sum_{i=1}^{p}\omega_i(\rho_{i+1}-\rho_i) - \sum_{j=1}^{m}\nu_j\Big) \neq 0, \\ A_2 = \Big(T - a - \sum_{i=1}^{p}\eta_i(\rho_{i+1}-\rho_i) - \sum_{j=1}^{m}\mu_j\Big) \neq 0, \\ A_3 = \frac{(T-a)^2}{2} - \sum_{i=1}^{p}\eta_i\Big(\frac{(\rho_{i+1}-a)^2}{2} - \frac{(\rho_i-a)^2}{2}\Big) - \sum_{j=1}^{m}\mu_j(\sigma_j - a), \\ A_4 = \Big(T - a - \sum_{i=1}^{p}\xi_i(\rho_{i+1}-\rho_i) - \sum_{j=1}^{m}\gamma_j\Big) \neq 0, \\ A_5 = \frac{(T-a)^2}{2} - \sum_{i=1}^{p}\xi_i\Big(\frac{(\rho_{i+1}-a)^2}{2} - \frac{(\rho_i-a)^2}{2}\Big) - \sum_{j=1}^{m}\gamma_j(\sigma_j - a), \\ A_6 = \frac{(T-a)^3}{3!} - \sum_{i=1}^{p}\xi_i\Big(\frac{(\rho_{i+1}-a)^3}{3!} - \frac{(\rho_i-a)^3}{3!}\Big) - \sum_{j=1}^{m}\gamma_j\frac{(\sigma_j-a)^2}{2}. \end{cases} \tag{6}$$

Proof. Integrating $u'''(t) = g(t)$ repeatedly from a to t, we get

$$u(t) = c_0 + c_1(t-a) + c_2\frac{(t-a)^2}{2} + \int_a^t \frac{(t-s)^2}{2}g(s)ds, \tag{7}$$

where c_0, c_1 and c_2 are arbitrary unknown real constants. Moreover, from Equation (7), we have

$$u'(t) = c_1 + c_2(t-a) + \int_a^t (t-s)g(s)ds, \tag{8}$$

$$u''(t) = c_2 + \int_a^t g(s)ds. \tag{9}$$

Using the third condition of Equation (2) in Equation (9), we get

$$c_2 = \frac{1}{A_1}\Big[-\int_a^T (T-s)g(s)ds + \sum_{i=1}^{p} \omega_i \int_{\rho_i}^{\rho_{i+1}} \int_a^s g(\tau)d\tau ds + \sum_{j=1}^{m} \nu_j \int_a^{\sigma_j} g(s)ds\Big]. \tag{10}$$

Making use of the second condition of Equation (2) in Equation (8) together with Equation (10) yields

$$\begin{aligned} c_1 &= \frac{1}{A_2}\Big[-\int_a^T \frac{(T-s)^2}{2} g(s)ds + \sum_{i=1}^{p} \eta_i \int_{\rho_i}^{\rho_{i+1}} \int_a^s (s-\tau)g(\tau)d\tau ds \\ &+ \sum_{j=1}^{m} \mu_j \int_a^{\sigma_j} (\sigma_j - s)g(s)ds\Big] + \frac{A_3}{A_1A_2}\Big[-\int_a^T (T-s)g(s)ds + \sum_{i=1}^{p} \omega_i \int_{\rho_i}^{\rho_{i+1}} \int_a^s g(\tau)d\tau ds \\ &+ \sum_{j=1}^{m} \nu_j \int_a^{\sigma_j} g(s)ds\Big]. \end{aligned} \tag{11}$$

Finally, using the first condition of Equation (2) in Equation (7) together with Equations (10) and (11), we obtain

$$\begin{aligned} c_0 &= \frac{1}{\Lambda}\Big\{\Big(A_3A_5 - A_2A_6\Big)\Big[-\int_a^T (T-s)g(s)ds + \sum_{i=1}^{p} \omega_i \int_{\rho_i}^{\rho_{i+1}} \int_a^s g(\tau)d\tau ds \\ &+ \sum_{j=1}^{m} \nu_j \int_a^{\sigma_j} g(s)ds\Big] - A_1A_5\Big[-\int_a^T \frac{(T-s)^2}{2} g(s)ds \\ &+ \sum_{i=1}^{p} \eta_i \int_{\rho_i}^{\rho_{i+1}} \int_a^s (s-\tau)g(\tau)d\tau ds + \sum_{j=1}^{m} \mu_j \int_a^{\sigma_j} (\sigma_j - s)g(s)ds\Big] \\ &+ A_1A_2\Big[-\int_a^T \frac{(T-s)^3}{3!} g(s)ds + \sum_{i=1}^{p} \xi_i \int_{\rho_i}^{\rho_{i+1}} \int_a^s \frac{(s-\tau)^2}{2} g(\tau)d\tau ds \\ &+ \sum_{j=1}^{m} \gamma_j \int_a^{\sigma_j} \frac{(\sigma_j - s)^2}{2} g(s)ds\Big]\Big\}. \end{aligned} \tag{12}$$

In Equations (10)–(12), we have used the notations in Equation (6). Inserting the values of c_0, c_1 *and* c_2 in Equation (7) completes the solution to Equation (4). By direct computation, one can obtain the converse of the Lemma. □

Lemma 2. *For $h \in C([a,T],\mathbb{R})$, the problem consisting of the equation $u'''(t) = h(t), t \in [a,T]$ and non-separated boundary conditions in Equation (3) is equivalent to the integral equation*

$$\begin{aligned} u(t) &= \int_a^t \frac{(t-s)^2}{2} h(s)ds \\ &- \frac{1}{\Delta} \int_a^T \Big[\alpha_2\zeta_1\zeta_2 \frac{(T-s)^2}{2} + \beta_2 P_1(t)(T-s) + \delta_2 P_2(t)\Big] h(s)ds \\ &+ \frac{1}{\Delta} \sum_{j=1}^{m} \int_a^{\sigma_j} \Big[\gamma_j\zeta_1\zeta_2 \frac{(\sigma_j - s)^2}{2} + \mu_j P_1(t)(\sigma_j - s) + \nu_j P_2(t)\Big] h(s)ds \\ &+ \frac{1}{\Delta} \sum_{i=1}^{p} \int_{\rho_i}^{\rho_{i+1}} \Big[\int_a^s \Big(\xi_i\zeta_1\zeta_2 \frac{(s-\tau)^2}{2} + \eta_i P_1(t)(s-\tau) + \omega_i P_2(t)\Big) h(\tau)d\tau\Big] ds, \end{aligned} \tag{13}$$

where

$$P_1(t) = \zeta_1\Big(\zeta_4(t-a) - \zeta_5\Big),\ P_2(t) = \zeta_3\Big(\zeta_5 - \zeta_4(t-a)\Big) - \zeta_2\Big(\zeta_6 - \zeta_4\frac{(t-a)^2}{2}\Big), \tag{14}$$

$$\begin{cases} \Delta = \zeta_1\zeta_2\zeta_4,\ \zeta_1 = \Big(\delta_1 + \delta_2 - \sum_{i=1}^{p}\omega_i(\rho_{i+1}-\rho_i) - \sum_{j=1}^{m}\nu_j\Big) \neq 0, \\ \zeta_2 = \Big(\beta_1 + \beta_2 - \sum_{i=1}^{p}\eta_i(\rho_{i+1}-\rho_i) - \sum_{j=1}^{m}\mu_j\Big) \neq 0, \\ \zeta_3 = \beta_2(T-a) - \sum_{i=1}^{p}\eta_i\Big(\frac{(\rho_{i+1}-a)^2}{2} - \frac{(\rho_i-a)^2}{2}\Big) - \sum_{j=1}^{m}\mu_j(\sigma_j - a), \\ \zeta_4 = \Big(\alpha_1 + \alpha_2 - \sum_{i=1}^{p}\xi_i(\rho_{i+1}-\rho_i) - \sum_{j=1}^{m}\gamma_j\Big) \neq 0, \\ \zeta_5 = \alpha_2(T-a) - \sum_{i=1}^{p}\xi_i\Big(\frac{(\rho_{i+1}-a)^2}{2} - \frac{(\rho_i-a)^2}{2}\Big) - \sum_{j=1}^{m}\gamma_j(\sigma_j - a), \\ \zeta_6 = \alpha_2\frac{(T-a)^2}{2} - \sum_{i=1}^{p}\xi_i\Big(\frac{(\rho_{i+1}-a)^3}{3!} - \frac{(\rho_i-a)^3}{3!}\Big) - \sum_{j=1}^{m}\gamma_j\frac{(\sigma_j-a)^2}{2}. \end{cases} \tag{15}$$

Proof. We omit the proof as it runs parallel to that of Lemma 1. □

3. Main Results

Let us set $\widehat{f}(t) = f(t, u(t), u'(t), u''(t))$ and introduce a fixed point problem equivalent to the problem in Equations (1) and (2) via Lemma 1 as follows

$$u = \mathcal{L}u, \tag{16}$$

where the operator $\mathcal{L} : \mathcal{H} \to \mathcal{H}$ is defined by

$$\begin{aligned} (\mathcal{L}u)(t) &= \int_a^t \frac{(t-s)^2}{2}\widehat{f}(s)ds \\ &- \frac{1}{\Lambda}\int_a^T \Big[A_1A_2\frac{(T-s)^3}{3!} + G_1(t)\frac{(T-s)^2}{2} + G_2(t)(T-s)\Big]\widehat{f}(s)ds \\ &+ \frac{1}{\Lambda}\sum_{j=1}^{m}\int_a^{\sigma_j}\Big[\gamma_jA_1A_2\frac{(\sigma_j-s)^2}{2} + \mu_jG_1(t)(\sigma_j-s) + \nu_jG_2(t)\Big]\widehat{f}(s)ds \\ &+ \frac{1}{\Lambda}\sum_{i=1}^{p}\int_{\rho_i}^{\rho_{i+1}}\int_a^s \Big[\xi_iA_1A_2\frac{(s-\tau)^2}{2} + \eta_iG_1(t)(s-\tau) + \omega_iG_2(t)\Big]\widehat{f}(\tau)d\tau ds. \end{aligned} \tag{17}$$

Here, $\mathcal{H} = \{u | u,\ u',\ u'' \in C([a,T],\mathbb{R})\}$ is the Banach space equipped with the norm $\|u\|_{\mathcal{H}} = \max_{t\in[a,T]}\Big\{|u(t)| + |u'(t)| + |u''(t)|\Big\} = \|u\| + \|u'\| + \|u''\|$. From Equation (17), we have

$$\begin{aligned} (\mathcal{L}u)'(t) &= \int_a^t (t-s)\widehat{f}(s)ds - \frac{1}{A_1A_2}\int_a^T \Big[A_1\frac{(T-s)^2}{2} + G_3(t)(T-s)\Big]\widehat{f}(s)ds \\ &+ \frac{1}{A_1A_2}\sum_{j=1}^{m}\int_a^{\sigma_j}\Big[\mu_jA_1(\sigma_j-s) + \nu_jG_3(t)\Big]\widehat{f}(s)ds \\ &+ \frac{1}{A_1A_2}\sum_{i=1}^{p}\int_{\rho_i}^{\rho_{i+1}}\int_a^s \Big[\eta_iA_1(s-\tau) + \omega_iG_3(t)\Big]\widehat{f}(\tau)d\tau ds, \end{aligned} \tag{18}$$

$$
\begin{aligned}
(\mathcal{L}u)''(t) &= \int_a^t \widehat{f}(s)ds + \frac{1}{A_1}\Big[-\int_a^T (T-s)\widehat{f}(s)ds + \sum_{j=1}^{m}\int_a^{\sigma_j} \nu_j \widehat{f}(s)ds \\
&+ \sum_{i=1}^{p}\int_{\rho_i}^{\rho_{i+1}}\int_a^s \omega_i \widehat{f}(\tau)d\tau ds\Big],
\end{aligned} \tag{19}
$$

where

$$
G_3(t) = A_2(t-a) - A_3. \tag{20}
$$

Observe that the existence of the fixed points for the operator in Equation (16) implies the existence of solutions for the problem in Equations (1) and (2).

For the sake of computational convenience in the forthcoming analysis, we set

$$
Q = Q_1 + Q_2 + Q_3, \tag{21}
$$

where

$$
\begin{aligned}
Q_1 &= \frac{(T-a)^3}{3!} + \frac{1}{|A_4|}\Big[\frac{(T-a)^4}{4!} + \sum_{i=1}^{p}\xi_i\Big(\frac{(\rho_{i+1}-a)^4}{4!} - \frac{(\rho_i-a)^4}{4!}\Big) + \sum_{j=1}^{m}\gamma_j\frac{(\sigma_j-a)^3}{3!}\Big] \\
&+ \frac{b_1}{|\Lambda|}\Big[\frac{(T-a)^3}{3!} + \sum_{i=1}^{p}\eta_i\Big(\frac{(\rho_{i+1}-a)^3}{3!} - \frac{(\rho_i-a)^3}{3!}\Big) + \sum_{j=1}^{m}\mu_j\frac{(\sigma_j-a)^2}{2}\Big] \\
&+ \frac{b_2}{|\Lambda|}\Big[\frac{(T-a)^2}{2} + \sum_{i=1}^{p}\omega_i\Big(\frac{(\rho_{i+1}-a)^2}{2} - \frac{(\rho_i-a)^2}{2}\Big) + \sum_{j=1}^{m}\nu_j(\sigma_j-a)\Big],
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
Q_2 &= \frac{(T-a)^2}{2} + \frac{1}{|A_2|}\Big[\frac{(T-a)^3}{3!} + \sum_{i=1}^{p}\eta_i\Big(\frac{(\rho_{i+1}-a)^3}{3!} - \frac{(\rho_i-a)^3}{3!}\Big) + \sum_{j=1}^{m}\mu_j\frac{(\sigma_j-a)^2}{2}\Big] \\
&+ \frac{b_3}{|A_1A_2|}\Big[\frac{(T-a)^2}{2} + \sum_{i=1}^{p}\omega_i\Big(\frac{(\rho_{i+1}-a)^2}{2} - \frac{(\rho_i-a)^2}{2}\Big) + \sum_{j=1}^{m}\nu_j(\sigma_j-a)\Big],
\end{aligned} \tag{23}
$$

and

$$
Q_3 = (T-a) + \frac{1}{|A_1|}\Big[\frac{(T-a)^2}{2} + \sum_{i=1}^{p}\omega_i\Big(\frac{(\rho_{i+1}-a)^2}{2} - \frac{(\rho_i-a)^2}{2}\Big) + \sum_{j=1}^{m}\nu_j(\sigma_j-a)\Big], \tag{24}
$$

where $\max_{t\in[a,T]}|G_1(t)| = b_1$, $\max_{t\in[a,T]}|G_2(t)| = b_2$ *and* $\max_{t\in[a,T]}|G_3(t)| = b_3$ ($G_1(t)$, $G_2(t)$ are given by Equation (5) while $G_3(t)$ is defined in Equation (20)).

3.1. Existence of Solutions

In this subsection, we discuss the existence of solutions for the problem in Equations (1) and (2). In our first result, we make use of Krasnoselskii's fixed point theorem [24].

Theorem 1. *Let $f : [a,T]\times\mathbb{R}^3 \to \mathbb{R}$ be a continuous function satisfying the conditions:*

(H_1) $\Big|f(t,u,u',u'') - f(t,v,v',v'')\Big| \le \ell\Big(|u-v| + |u'-v'| + |u''-v''|\Big)$, $\forall t\in[a,T]$, $\ell > 0,\ u,v,u',v',u'',v'' \in \mathbb{R}$;

(H_2) *there exist a function $\varepsilon \in C([a,T],\mathbb{R}^+)$ with $\|\varepsilon\| = \sup_{t\in[a,T]}|\varepsilon(t)|$ such that*

$$
|\widehat{f}(t)| = |f(t,u,u',u'')| \le \varepsilon(t),\ \ \forall (t,u,u',u'') \in [a,T]\times\mathbb{R}^3;
$$

(H_3) $\ell\Big(Q - \frac{(T-a)}{6}\Big[6 + 3(T-a) + (T-a)^2\Big]\Big) < 1$, *where Q is given by Equation (21).*

Then, there exists at least one solution for the problem in Equations (1) and (2) on $[a,T]$.

Proof. Consider a closed ball $B_r = \{(u,u',u'') : \|u\|_{\mathcal{H}} \le r,\ u,\ u',\ u'' \in C([a,T],\mathbb{R})\}$ for fixed $r \ge Q\|\varepsilon\|$ and introduce the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ on B_r as follows:

$$(\mathcal{L}_1 u)(t) = \int_a^t \frac{(t-s)^2}{2}\widehat{f}(s)ds,$$

$$\begin{aligned}(\mathcal{L}_2 u)(t) &= -\frac{1}{\Lambda}\int_a^T \Big[A_1A_2\frac{(T-s)^3}{3!} + G_1(t)\frac{(T-s)^2}{2} + G_2(t)(T-s)\Big]\widehat{f}(s)ds \\ &+\frac{1}{\Lambda}\sum_{j=1}^m \int_a^{\sigma_j}\Big[\gamma_j A_1A_2\frac{(\sigma_j-s)^2}{2} + \mu_j G_1(t)(\sigma_j - s) + \nu_j G_2(t)\Big]\widehat{f}(s)ds \\ &+\frac{1}{\Lambda}\sum_{i=1}^p \int_{\rho_i}^{\rho_{i+1}}\int_a^s \Big[\xi_i A_1A_2\frac{(s-\tau)^2}{2} + \eta_i G_1(t)(s-\tau) + \omega_i G_2(t)\Big]\widehat{f}(\tau)d\tau ds.\end{aligned}$$

Moreover, we have

$$(\mathcal{L}_1 u)'(t) = \int_a^t (t-s)\widehat{f}(s)ds,\ (\mathcal{L}_1 u)''(t) = \int_a^t \widehat{f}(s)ds,$$

$$\begin{aligned}(\mathcal{L}_2 u)'(t) &= -\frac{1}{A_1A_2}\int_a^T \Big[A_1\frac{(T-s)^2}{2} + G_3(t)(T-s)\Big]\widehat{f}(s)ds \\ &+\frac{1}{A_1A_2}\sum_{j=1}^m \int_a^{\sigma_j}\Big[\mu_j A_1(\sigma_j - s) + \nu_j G_3(t)\Big]\widehat{f}(s)ds \\ &+\frac{1}{A_1A_2}\sum_{i=1}^p \int_{\rho_i}^{\rho_{i+1}}\int_a^s \Big[\eta_i A_1(s-\tau) + \omega_i G_3(t)\Big]\widehat{f}(\tau)d\tau ds,\end{aligned}$$

$$(\mathcal{L}_2 u)''(t) = \frac{1}{A_1}\Big[-\int_a^T (T-s)\widehat{f}(s)ds + \sum_{j=1}^m \int_a^{\sigma_j}\nu_j\widehat{f}(s)ds + \sum_{i=1}^p \int_{\rho_i}^{\rho_{i+1}}\int_a^s \omega_i\widehat{f}(\tau)d\tau ds\Big].$$

Notice that $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$. For $u, v \in B_r$, and $t \in [a,T]$, we have

$$\begin{aligned}&\|\mathcal{L}_1 u + \mathcal{L}_2 v\| \\ = &\sup_{t\in[a,T]}\Big\{\Big|\int_a^t \frac{(t-s)^2}{2} f(s,u(s),u'(s),u''(s))ds \\ &-\frac{1}{\Lambda}\int_a^T \Big[A_1A_2\frac{(T-s)^3}{3!} + G_1(t)\frac{(T-s)^2}{2} + G_2(t)(T-s)\Big] f(s,v(s),v'(s),v''(s))ds \\ &+\frac{1}{\Lambda}\sum_{j=1}^m \int_a^{\sigma_j}\Big[\gamma_j A_1A_2\frac{(\sigma_j-s)^2}{2} + \mu_j G_1(t)(\sigma_j-s) + \nu_j G_2(t)\Big] f(s,v(s),v'(s),v''(s))ds \\ &+\frac{1}{\Lambda}\sum_{i=1}^p \int_{\rho_i}^{\rho_{i+1}}\int_a^s \Big[\xi_i A_1A_2\frac{(s-\tau)^2}{2} + \eta_i G_1(t)(s-\tau) + \omega_i G_2(t)\Big] f(\tau,v(\tau),v'(\tau),v''(\tau))d\tau ds\Big|\Big\} \\ \le &\|\varepsilon\|\sup_{t\in[a,T]}\Big\{\frac{(t-a)^3}{3!} + \frac{1}{|A_4|}\Big[\frac{(T-a)^4}{4!} + \sum_{i=1}^p \xi_i\Big(\frac{(\rho_{i+1}-a)^4}{4!} - \frac{(\rho_i-a)^4}{4!}\Big) + \sum_{j=1}^m \gamma_j\frac{(\sigma_j-a)^3}{3!}\Big] \\ &+\frac{|G_1(t)|}{|\Lambda|}\Big[\frac{(T-a)^3}{3!} + \sum_{i=1}^p \eta_i\Big(\frac{(\rho_{i+1}-a)^3}{3!} - \frac{(\rho_i-a)^3}{3!}\Big) + \sum_{j=1}^m \mu_j\frac{(\sigma_j-a)^2}{2}\Big] \\ &+\frac{|G_2(t)|}{|\Lambda|}\Big[\frac{(T-a)^2}{2} + \sum_{i=1}^p \omega_i\Big(\frac{(\rho_{i+1}-a)^2}{2} - \frac{(\rho_i-a)^2}{2}\Big) + \sum_{j=1}^m \nu_j(\sigma_j-a)\Big]\Big\} \le \|\varepsilon\| Q_1,\end{aligned}$$

where Q_1 is given by Equation (22). In a similar manner, it can be shown that

$$\|(\mathcal{L}_1 u)' + (\mathcal{L}_2 v)'\| \leq \|\varepsilon\| Q_2, \ \|(\mathcal{L}_1 u)'' + (\mathcal{L}_2 v)''\| \leq \|\varepsilon\| Q_3,$$

where Q_2 and Q_3 are, respectively, given by Equations (23) and (24). Consequently, we obtain

$$\|\mathcal{L}_1 u + \mathcal{L}_2 v\|_{\mathcal{H}} \leq \|\varepsilon\| Q \leq r,$$

where we have used (H_2) and Equation (21). From the above inequality, it follows that $\mathcal{L}_1 u + \mathcal{L}_2 v \in B_r$. Thus, the first condition of Krasnoselskii's fixed point theorem [24] is satisfied. Next, we show that $\mathcal{L}_2$ is a contraction. For $u, v \in \mathbb{R}$, it follows by the assumption (H_1) that

$$\begin{aligned}
& \|\mathcal{L}_2 u - \mathcal{L}_2 v\| \\
\leq \ & \sup_{t \in [a,T]} \Big\{ \frac{1}{|\Lambda|} \int_a^T \Big[|A_1 A_2| \frac{(T-s)^3}{3!} + |G_1(t)| \frac{(T-s)^2}{2} + |G_2(t)|(T-s) \Big] \\
& \times \Big| f(s, u(s), u'(s), u''(s)) - f(s, v(s), v'(s), v''(s)) \Big| ds + \frac{1}{|\Lambda|} \sum_{j=1}^{m} \int_a^{\sigma_j} \Big[\gamma_j |A_1 A_2| \frac{(\sigma_j - s)^2}{2} \\
& + \mu_j |G_1(t)| (\sigma_j - s) + \nu_j |G_2(t)| \Big] \Big| f(s, u(s), u'(s), u''(s)) - f(s, v(s), v'(s), v''(s)) \Big| ds \\
& + \frac{1}{|\Lambda|} \sum_{i=1}^{p} \int_{\rho_i}^{\rho_{i+1}} \int_a^s \Big[\xi_i |A_1 A_2| \frac{(s-\tau)^2}{2} + \eta_i |G_1(t)| (s - \tau) + \omega_i |G_2(t)| \Big] \\
& \times \Big| f(\tau, u(\tau), u'(\tau), u''(\tau)) - f(\tau, v(\tau), v'(\tau), v''(\tau)) \Big| d\tau ds \Big\} \\
\leq \ & \ell \Big(\|u - v\| + \|u' - v'\| + \|u'' - v''\| \Big) \Big\{ \frac{1}{|A_4|} \Big[\frac{(T-a)^4}{4!} + \sum_{i=1}^{p} \xi_i \Big(\frac{(\rho_{i+1} - a)^4}{4!} - \frac{(\rho_i - a)^4}{4!} \Big) \\
& + \sum_{j=1}^{m} \gamma_j \frac{(\sigma_j - a)^3}{3!} \Big] + \frac{b_1}{|\Lambda|} \Big[\frac{(T-a)^3}{3!} + \sum_{i=1}^{p} \eta_i \Big(\frac{(\rho_{i+1} - a)^3}{3!} - \frac{(\rho_i - a)^3}{3!} \Big) + \sum_{j=1}^{m} \mu_j \frac{(\sigma_j - a)^2}{2} \Big] \\
& + \frac{b_2}{|\Lambda|} \Big[\frac{(T-a)^2}{2} + \sum_{i=1}^{p} \omega_i \Big(\frac{(\rho_{i+1} - a)^2}{2} - \frac{(\rho_i - a)^2}{2} \Big) + \sum_{j=1}^{m} \nu_j (\sigma_j - a) \Big] \Big\} \\
\leq \ & \ell \Big(Q_1 - \frac{(T-a)^3}{3!} \Big) \parallel u - v \parallel_{\mathcal{H}} .
\end{aligned}$$

Similarly, we can obtain

$$\|(\mathcal{L}_2 u)' - (\mathcal{L}_2 v)'\| \leq \ell \Big(Q_2 - \frac{(T-a)^2}{2} \Big) \parallel u - v \parallel_{\mathcal{H}},$$

and

$$\|(\mathcal{L}_2 u)'' - (\mathcal{L}_2 v)''\| \leq \ell \Big(Q_3 - (T-a) \Big) \parallel u - v \parallel_{\mathcal{H}} .$$

Thus, we get

$$\|\mathcal{L}_2 u - \mathcal{L}_2 v\|_{\mathcal{H}} \leq \ell \Big(Q - \frac{(T-a)}{6} \Big[6 + 3(T-a) + (T-a)^2 \Big] \Big) \parallel u - v \parallel_{\mathcal{H}},$$

which, in view of the condition (H_3), implies that $\mathcal{L}_2$ is a contraction. Thus, the second hypothesis of Krasnoselskii's fixed point theorem [24] is satisfied. Finally, we verify the third and last hypothesis of

Krasnoselskii's fixed point theorem [24] that $\mathcal{L}_1$ is compact and continuous. Observe that continuity of f implies that the operator $\mathcal{L}_1$ is continuous. In addition, $\mathcal{L}_1$ is uniformly bounded on B_r as

$$\|\mathcal{L}_1 u\|_{\mathcal{H}} \leq \|\varepsilon\| \Big[\frac{(T-a)^3}{3!} + \frac{(T-a)^2}{2} + (T-a) \Big].$$

Let us fix $\sup_{(t,u,u',u'') \in [a,T] \times B_r} |f(t,u,u',u'')| = \bar{f}$, and take $a < t_1 < t_2 < T$. Then,

$$\begin{aligned} |(\mathcal{L}_1 u)(t_2) - (\mathcal{L}_1 u)(t_1)| &= \Big| \int_a^{t_1} \Big[\frac{(t_2-s)^2}{2} - \frac{(t_1-s)^2}{2} \Big] \widehat{f}(s) ds \\ &\quad + \int_{t_1}^{t_2} \frac{(t_2-s)^2}{2} \widehat{f}(s) ds \Big| \\ &\leq \bar{f} \Big(\frac{(t_2-t_1)^3}{3} + \frac{1}{3!} \Big| (t_2-a)^3 - (t_1-a)^3 \Big| \Big) \to 0 \text{ as } t_2 \to t_1, \end{aligned}$$

independently of $u \in B_r$. In addition, we have

$$\begin{aligned} |(\mathcal{L}_1 u)'(t_2) - (\mathcal{L}_1 u)'(t_1)| &= \Big| \int_a^{t_1} [(t_2-s) - (t_1-s)] \widehat{f}(s) ds + \int_{t_1}^{t_2} (t_2-s) \widehat{f}(s) ds \Big| \\ &\leq \bar{f} \Big| (t_2-t_1)(t_1-a) + \frac{(t_2-t_1)^2}{2} \Big| \to 0 \text{ as } t_2 \to t_1, \end{aligned}$$

independently of $u \in B_r$ and

$$|(\mathcal{L}_1 u)''(t_2) - (\mathcal{L}_1 u)''(t_1)| \leq \bar{f}(t_2 - t_1) \to 0 \text{ as } t_2 \to t_1,$$

independently of $u \in B_r$. From the preceding arguments, we deduce that $\mathcal{L}_1$ is relatively compact on B_r. Hence, the operator $\mathcal{L}_1$ is compact on B_r by the Arzelá–Ascoli theorem. Since all the hypotheses of Krasnoselskii's fixed point theorem [24] are verified, its conclusion applies to the problem in Equations (1) and (2). □

Remark 1. *When the role of the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ is mutually interchanged, the condition (H_3) of Theorem 1 takes the form: $\ell \frac{(T-a)}{6} \Big[6 + 3(T-a) + (T-a)^2 \Big] < 1$.*

In the next result, we make use of Leray–Schauder nonlinear alternative for single valued maps [25].

Theorem 2. *Suppose that $f : [a,T] \times \mathbb{R}^3 \to \mathbb{R}$ is a continuous function and the following conditions hold:*

(H_4) $|\widehat{f}(t)| = |f(t,u,u',u'')| \leq p(t)\Psi(|u|)$, $\forall (t,u,u',u'') \in [a,T] \times \mathbb{R}^3$, *where* $p \in C([a,T], \mathbb{R}^+)$, *and* $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a nondecreasing function;*

(H_5) *there exists a positive constant N satisfying the inequality:*

$$\frac{N}{\|p\| \Psi(N) Q} > 1,$$

where Q is defined by Equation (21). Then, the problem in Equations (1) and (2) has at least one solution on $[a,T]$.

Proof. We verify the hypotheses of Leray–Schauder nonlinear alternative [25] in several steps. We first show that the operator $\mathcal{L} : \mathcal{H} \to \mathcal{H}$ defined by Equation (17) maps bounded sets into bounded sets in

$\mathcal{H}$. Let us consider a set $B_{\bar{r}} = \{(u,u',u'') : \|u\|_{\mathcal{H}} \le \bar{r},\ u,\ u',\ u'' \in C([a,T],\mathbb{R}),\ \bar{r} > 0\}$ and note that it is bounded in $\mathcal{H}$. Then, in view of the condition (H_4), we get

$$\begin{aligned}
\|(\mathcal{L}u)\| &= \sup_{t\in[a,T]} \Big\{ \Big| \int_a^t \frac{(t-s)^2}{2}\widehat{f}(s)ds \\
&\quad -\frac{1}{\Lambda}\int_a^T \Big[A_1A_2\frac{(T-s)^3}{3!} + G_1(t)\frac{(T-s)^2}{2} + G_2(t)(T-s)\Big]\widehat{f}(s)ds \\
&\quad +\frac{1}{\Lambda}\sum_{j=1}^{m}\int_a^{\sigma_j}\Big[\gamma_jA_1A_2\frac{(\sigma_j-s)^2}{2} + \mu_jG_1(t)(\sigma_j-s) + \nu_jG_2(t)\Big]\widehat{f}(s)ds \\
&\quad +\frac{1}{\Lambda}\sum_{i=1}^{p}\int_{\rho_i}^{\rho_{i+1}}\int_a^s\Big[\xi_iA_1A_2\frac{(s-\tau)^2}{2} + \eta_iG_1(t)(s-\tau) + \omega_iG_2(t)\Big]\widehat{f}(\tau)d\tau ds\Big|\Big\} \\
&\le \|p\|\Psi(\|u\|_{\mathcal{H}})Q_1 \le \|p\|\Psi(\bar{r})Q_1,
\end{aligned}$$

where Q_1 is given by Equation (22). Similarly, one can establish that

$$\|(\mathcal{L}u)'\| \le \|p\|\Psi(\bar{r})Q_2,\ \|(\mathcal{L}u)''\| \le \|p\|\Psi(\bar{r})Q_3,$$

where Q_2 and Q_3 are given by Equations (23) and (24), respectively. In view of the foregoing arguments, we have

$$\|(\mathcal{L}u)\|_{\mathcal{H}} \le \|p\|\Psi(\bar{r})Q,$$

where Q is given by Equation (21). Next, it is verified that the operator $\mathcal{L}$ maps bounded sets into equicontinuous sets in $\mathcal{H}$. Notice that $\mathcal{L}$ is continuous in view of the continuity of $\widehat{f}(t)$. Let $t_1, t_2 \in [a,T]$ with $t_1 < t_2$ and $u \in B_{\bar{r}}$. Then, we have

$$\begin{aligned}
&|(\mathcal{L}u)(t_2) - (\mathcal{L}u)(t_1)| \\
\le\ & \Big|\int_a^{t_1}\Big[\frac{(t_2-s)^2}{2} - \frac{(t_1-s)^2}{2}\Big]\widehat{f}(s)ds + \int_{t_1}^{t_2}\frac{(t_2-s)^2}{2}\widehat{f}(s)ds\Big| \\
&+\frac{1}{|\Lambda|}\int_a^T(t_2-t_1)\Big[|A_1A_4|\frac{(T-s)^2}{2} + \Big(|A_3A_4| + \frac{|A_2A_4|}{2}(t_2+t_1)\Big)(T-s)\Big]\widehat{f}(s)ds \\
&+\frac{1}{|\Lambda|}\sum_{j=1}^{m}\int_a^{\sigma_j}(t_2-t_1)\Big[\mu_j|A_1A_4|(\sigma_j-s) + \nu_j\Big(|A_3A_4| + \frac{|A_2A_4|}{2}(t_2+t_1^2)\Big)\Big]|\widehat{f}(s)|ds \\
&+\frac{1}{|\Lambda|}\sum_{i=1}^{p}\int_{\rho_i}^{\rho_{i+1}}\int_a^s(t_2-t_1)\Big[\eta_i|A_1A_4|(s-\tau) + \omega_i\Big(|A_3A_4| + \frac{|A_2A_4|}{2}(t_2+t_1)\Big)\Big]|\widehat{f}(\tau)|d\tau ds \\
\le\ & \|p\|\Psi(\bar{r})\Big\{\frac{(t_2-t_1)^3}{3} + \frac{1}{3!}\Big|(t_2-a)^3 - (t_1-a)^3\Big| \\
&+\frac{(t_2-t_1)}{|A_2|}\Big[\frac{(T-a)^3}{3!} + \sum_{j=1}^{m}\mu_j\frac{(\sigma_j-a)^2}{2} + \sum_{i=1}^{p}\eta_i\Big(\frac{(\rho_{i+1}-a)^3}{3!} - \frac{(\rho_i-a)^3}{3!}\Big)\Big] \\
&+\frac{1}{|\Lambda|}\Big(|A_3A_4|(t_2-t_1) + \frac{|A_2A_4|}{2}(t_2^2-t_1^2)\Big)\Big[\frac{(T-a)^2}{2} + \sum_{j=1}^{m}\nu_j(\sigma_j-a) \\
&+\sum_{i=1}^{p}\omega_i\Big(\frac{(\rho_{i+1}-a)^2}{2} - \frac{(\rho_i-a)^2}{2}\Big)\Big]\Big\} \to 0 \text{ as } (t_2-t_1)\to 0,
\end{aligned}$$

independently of $u \in B_{\bar{r}}$. Moreover, we have

$$\begin{aligned} |(\mathcal{L}u)'(t_2) - (\mathcal{L}u)'(t_1)| &\leq \|p\|\Psi(\bar{r})\Big\{\Big|(t_2 - t_1)(t_1 - a) + \frac{(t_2 - t_1)^2}{2}\Big| \\ &\quad + \frac{(t_2 - t_1)}{|A_1|}\Big[\frac{(T - a)^2}{2} + \sum_{j=1}^{m} \nu_j(\sigma_j - a) \\ &\quad + \sum_{i=1}^{p} \omega_i\Big(\frac{(\rho_{i+1} - a)^2}{2} - \frac{(\rho_i - a)^2}{2}\Big)\Big]\Big\} \to 0 \text{ as } (t_2 - t_1) \to 0, \end{aligned}$$

independently of $u \in B_{\bar{r}}$ and

$$\begin{aligned} |(\mathcal{L}u)''(t_2) - (\mathcal{L}u)''(t_1)| &\leq \Big|\int_{t_1}^{t_2} \widehat{f}(s)ds\Big| \\ &\leq \|p\|\Psi(\bar{r})(t_2 - t_1) \to 0 \text{ as } (t_2 - t_1) \to 0, \end{aligned}$$

independently of $u \in B_{\bar{r}}$. In view of the foregoing arguments, the Arzelá–Ascoli theorem applies and hence the operator $\mathcal{L} : \mathcal{H} \to \mathcal{H}$ is completely continuous. The conclusion of Leray–Schauder nonlinear alternative [25] is applicable once we establish the boundedness of all solutions to the equation $u = \lambda \mathcal{L}u$ for $\lambda \in [0, 1]$. Let u be a solution of the problem in Equations (1) and (2). Then, as before, one can find that

$$|u(t)| = |\lambda(\mathcal{L}u)(t)| \leq \|p\|\Psi(\|u\|_{\mathcal{H}})Q,$$

which can alternatively be written in the following form after taking the norm for $t \in [a, T]$:

$$\frac{\|u\|_{\mathcal{H}}}{\|p\|\Psi(\|u\|_{\mathcal{H}})Q} \leq 1.$$

By the assumption (H_5), we can find a positive number N such that $\|u\|_{\mathcal{H}} \neq N$. Introduce a set $U = \{u \in C([a, T], \mathbb{R}) : \|u\|_{\mathcal{H}} < N\}$ such that the operator $\mathcal{L} : \overline{U} \to C([a, T], \mathbb{R})$ is continuous and completely continuous. In view of the the choice of U, there does not exist any $u \in \partial U$ satisfying $u = \lambda \mathcal{L}(u)$ for some $\lambda \in (0, 1)$. Thus, it follows from the nonlinear alternative of Leray–Schauder nonlinear alternative [25] that $\mathcal{L}$ has a fixed point $u \in \overline{U}$ which corresponds a solution of the problem in Equations (1) and (2). □

3.2. Uniqueness of Solutions

In this subsection, the uniqueness of solutions for the problem in Equations (1) and (2) is established by means of contraction mapping principle due to Banach.

Theorem 3. *Let $f : [a, T] \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function satisfying the assumption (H_1) with $\ell < Q^{-1}$, where Q is given by Equation (21). Then, there exists a unique solution for the problem in Equations (1) and (2) on $[a, T]$.*

Proof. Let us define a set $B_w = \{u, u', u'' \in C([a, T], \mathbb{R}) : \|u\|_{\mathcal{H}} \leq w\}$, where $w \geq \dfrac{QM}{1 - \ell Q}$, $\sup_{t \in [a,T]} |f(t, 0, 0, 0)| = M$, and show that $\mathcal{L}B_w \subset B_w$, where the operator $\mathcal{L}$ is defined by

Equation (17). For any $u \in B_w, t \in [a,T]$, one can find with the aid of the condition (H_1) that $|\widehat{f}(t)| \leq \|u\|_{\mathcal{H}} + M \leq \ell w + M$. Then, for $u \in B_w$, we have

$$\begin{aligned}
\|(\mathcal{L}u)\| &= \sup_{t\in[a,T]}\Bigg|\int_a^t \frac{(t-s)^2}{2}\widehat{f}(s)ds \\
&\quad -\frac{1}{\Lambda}\int_a^T \Big[A_1A_2\frac{(T-s)^3}{3!} + G_1(t)\frac{(T-s)^2}{2} + G_2(t)(T-s)\Big]\widehat{f}(s)ds \\
&\quad +\frac{1}{\Lambda}\sum_{j=1}^m\int_a^{\sigma_j}\Big[\gamma_jA_1A_2\frac{(\sigma_j-s)^2}{2} + \mu_jG_1(t)(\sigma_j-s) + \nu_jG_2(t)\Big]\widehat{f}(s)ds \\
&\quad +\frac{1}{\Lambda}\sum_{i=1}^p\int_{\rho_i}^{\rho_{i+1}}\int_a^s\Big[\xi_iA_1A_2\frac{(s-\tau)^2}{2} + \eta_iG_1(t)(s-\tau) + \omega_iG_2(t)\Big]\widehat{f}(\tau)d\tau ds\Bigg| \\
&\leq \sup_{t\in[a,T]}\Big\{\frac{(t-a)^3}{3!} + \frac{1}{|A_4|}\Big[\frac{(T-a)^4}{4!} + \sum_{i=1}^p\xi_i\Big(\frac{(\rho_{i+1}-a)^4}{4!} - \frac{(\rho_i-a)^4}{4!}\Big) + \sum_{j=1}^m\gamma_j\frac{(\sigma_j-a)^3}{3!}\Big] \\
&\quad +\frac{|G_1(t)|}{|\Lambda|}\Big[\frac{(T-a)^3}{3!} + \sum_{i=1}^p\eta_i\Big(\frac{(\rho_{i+1}-a)^3}{3!} - \frac{(\rho_i-a)^3}{3!}\Big) + \sum_{j=1}^m\mu_j\frac{(\sigma_j-a)^2}{2}\Big] \\
&\quad +\frac{|G_2(t)|}{|\Lambda|}\Big[\frac{(T-a)^2}{2} + \sum_{i=1}^p\omega_i\Big(\frac{(\rho_{i+1}-a)^2}{2} - \frac{(\rho_i-a)^2}{2}\Big) + \sum_{j=1}^m\nu_j(\sigma_j-a)\Big]\Big\}(\ell w + M) \\
&\leq Q_1(\ell w + M),
\end{aligned}$$

where Q_1 is given by Equation (22). In addition,

$$\|(\mathcal{L}u)'\| \leq (\ell w + M)Q_2 \text{ and } \|(\mathcal{L}u)''\| \leq (\ell w + M)Q_3,$$

where Q_2 and Q_3 are, respectively, given by Equations (23) and (24). Consequently, we have

$$\|(\mathcal{L}u)\|_{\mathcal{H}} \leq (\ell w + M)Q \leq w,$$

where Q is given by Equation (21). This shows that $\mathcal{L}B_w \subset B_w$. Next, it is shown that the operator $\mathcal{L}$ is a contraction. For that, let $u, v \in \mathcal{H}$. Then, we have

$$\begin{aligned}
\|\mathcal{L}u - \mathcal{L}v\| &= \sup_{t\in[0,T]}\Big|\mathcal{L}u(t) - \mathcal{L}v(t)\Big| \\
&\leq \sup_{t\in[a,T]}\Big\{\int_a^t\frac{(t-s)^2}{2}\Big|f(s,u(s),u'(s),u''(s)) - f(s,v(s),v'(s),v''(s))\Big|ds \\
&\quad +\frac{1}{|\Lambda|}\int_a^T\Big[|A_1A_2|\frac{(T-s)^3}{3!} + |G_1(t)|\frac{(T-s)^2}{2} + |G_2(t)|(T-s)\Big] \\
&\quad \times\Big|f(s,u(s),u'(s),u''(s)) - f(s,v(s),v'(s),v''(s))\Big|ds \\
&\quad +\frac{1}{|\Lambda|}\sum_{j=1}^m\int_a^{\sigma_j}\Big[\gamma_j|A_1A_2|\frac{(\sigma_j-s)^2}{2} + \mu_j|G_1(t)|(\sigma_j-s) + \nu_j|G_2(t)|\Big] \\
&\quad \times\Big|f(s,u(s),u'(s),u''(s)) - f(s,v(s),v'(s),v''(s))\Big|ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{|\Lambda|}\sum_{i=1}^{p}\int_{\rho_i}^{\rho_{i+1}}\int_a^s\Big[\xi_i|A_1A_2|\frac{(s-\tau)^2}{2}+\eta_i|G_1(t)|(s-\tau)+\omega_i|G_2(t)|\Big]\\
&\times\Big|f(\tau,u(\tau),u'(\tau),u''(\tau))-f(\tau,v(\tau),v'(\tau),v''(\tau))\Big|d\tau ds\Big\}\\
\leq\;& \ell\Big(\|u-v\|+\|u'-v'\|+\|u''-v''\|\Big)\Big\{\frac{(T-a)^3}{3!}\\
&+\frac{1}{|A_4|}\Big[\frac{(T-a)^4}{4!}+\sum_{i=1}^{p}\xi_i\Big(\frac{(\rho_{i+1}-a)^4}{4!}-\frac{(\rho_i-a)^4}{4!}\Big)+\sum_{j=1}^{m}\gamma_j\frac{(\sigma_j-a)^3}{3!}\Big]\\
&+\frac{b_1}{|\Lambda|}\Big[\frac{(T-a)^3}{3!}+\sum_{i=1}^{p}\eta_i\Big(\frac{(\rho_{i+1}-a)^3}{3!}-\frac{(\rho_i-a)^3}{3!}\Big)+\sum_{j=1}^{m}\mu_j\frac{(\sigma_j-a)^2}{2}\Big]\\
&+\frac{b_2}{|\Lambda|}\Big[\frac{(T-a)^2}{2}+\sum_{i=1}^{p}\omega_i\Big(\frac{(\rho_{i+1}-a)^2}{2}-\frac{(\rho_i-a)^2}{2}\Big)+\sum_{j=1}^{m}\nu_j(\sigma_j-a)\Big]\Big\}\\
\leq\;& \ell Q_1\parallel u-v\parallel_{\mathcal{H}}.
\end{aligned}
$$

In a similar manner, one can obtain

$$\|(\mathcal{L}u)'-(\mathcal{L}v)'\|\leq \ell Q_2\parallel u-v\parallel_{\mathcal{H}},\ \|(\mathcal{L}u)''-(\mathcal{L}v)''\|\leq \ell Q_3\parallel u-v\parallel_{\mathcal{H}}.$$

Consequently, we deduce that

$$\|\mathcal{L}u-\mathcal{L}v\|_{\mathcal{H}}\leq \ell Q\parallel u-v\parallel_{\mathcal{H}},$$

which, in view of the given condition ($\ell < Q^{-1}$), shows that the operator $\mathcal{L}$ is a contraction. Thus, by the conclusion of Banach contraction mapping principle, the operator $\mathcal{L}$ has a unique fixed point, which implies that the problem in Equations (1) and (2) has a unique solution on $[a,T]$. □

3.3. Examples

Here, we illustrate the results obtained in the last subsections with the aid of examples.

Example 1. *Consider the following integral multi-point and multi-strip boundary value problem:*

$$u'''(t)=\frac{1}{45\sqrt{t^2+3}}\tan^{-1}u(t)+\frac{1}{162}\frac{|u'|}{(|u'|+1)}+\frac{1}{270t}\frac{|u''|^2}{(|u''|^2+1)}+\cos(t-1),\ t\in[1,4], \tag{25}$$

$$\begin{cases}\displaystyle\int_1^4 u(s)ds=\sum_{i=1}^{4}\xi_i\int_{\rho_i}^{\rho_{i+1}}u(s)ds+\sum_{j=1}^{3}\gamma_j u(\sigma_j),\\ \displaystyle\int_1^4 u'(s)ds=\sum_{i=1}^{4}\eta_i\int_{\rho_i}^{\rho_{i+1}}u'(s)ds+\sum_{j=1}^{3}\mu_j u'(\sigma_j),\\ \displaystyle\int_1^4 u''(s)ds=\sum_{i=1}^{4}\omega_i\int_{\rho_i}^{\rho_{i+1}}u''(s)ds+\sum_{j=1}^{3}\nu_j u''(\sigma_j),\end{cases} \tag{26}$$

where $a=1$, $T=4$, $m=3$, $p=4$, $\gamma_1=1/2$, $\gamma_2=7/10$, $\gamma_3=9/10$, $\mu_1=1/4$, $\mu_2=5/12$, $\mu_3=7/12$, $\nu_1=2/5$, $\nu_2=13/20$, $\nu_3=9/10$, $\sigma_1=7/4$, $\sigma_2=15/8$, $\sigma_3=16/8$, $\rho_1=5/2$, $\rho_2=8/3$, $\rho_3=17/6$, $\rho_4=18/6$, $\rho_5=19/6$, $\xi_1=3/4$, $\xi_2=25/28$, $\xi_3=29/28$, $\xi_4=33/28$, $\eta_1=2/7$, $\eta_2=$

$23/56$, $\eta_3 = 15/28$, $\eta_4 = 37/56$, $\omega_1 = 1/5$, $\omega_2 = 2/5$, $\omega_3 = 3/5$, $\omega_4 = 4/5$. Clearly, $|f(t,u,u',u'')| \leq \frac{\pi}{90\sqrt{t^2+3}} + \frac{1}{270t} + \frac{163}{162}$ and

$$\Big|f(t,u,u',u'') - f(t,v,v',v'')\Big| \leq \ell\Big(|u-v| + |u'-v'| + |u''-v''|\Big)$$

with $\ell = 1/90$. Using the given data, it is found that $A_1 \approx 0.716667 \neq 0$, $A_2 \approx 1.434524 \neq 0$, $A_3 \approx 2.768849$, $A_4 \approx 0.257143 \neq 0$, $A_5 \approx 1.414087$, $A_6 \approx 2.512768$, and $|\Lambda| \approx 0.264363$ (Λ and A_i $(i = 1,\ldots,6)$ are defined by Equation (6)), $Q_1 \approx 35.810002$, $Q_2 \approx 18.708093$, $Q_3 \approx 12.638560$ and $Q \approx 67.156655$ (Q_1, Q_2, Q_3 and Q are given by Equations (22), (23), (24) and (21), respectively). Furthermore, we note that all the conditions of Theorem 1 are satisfied with

$$\ell\Big(Q - \frac{(T-a)}{6}\Big[6 + 3(T-a) + (T-a)^2\Big]\Big) \approx 0.612852 < 1.$$

Hence, the problem in Equations (25) and (26) has a solution on $[1,4]$ by Theorem 1.

Since $\ell Q \approx 0.746185 < 1$, therefore the conclusion of Theorem 3 also applies to Equation (26).

Example 2. *Consider the third-order ordinary differential equation*

$$u'''(t) = \frac{1}{18\sqrt{t+24}}\Big[\frac{1}{21\pi}\sin(3\pi u) + \frac{3}{4}u'(t) + \frac{|u''|}{|u''|+1}\Big],\ t \in [1,4] \tag{27}$$

supplemented with the boundary conditions in Equation (26). Evidently,

$$|f(t,u,u',u'')| \leq \frac{1}{18\sqrt{t+24}}\Big(\frac{|u|}{7} + \frac{3}{4}|u'(t)| + 1\Big).$$

Let us set $\Psi(\|u\|) = \frac{\|u\|}{7} + \frac{3}{4}\|u'\| + 1$, $p(t) = \frac{1}{18\sqrt{t+24}}$, $(\|p\| = \frac{1}{90})$. The condition (H_5) implies that $N > 2.235673$. In consequence, it follows by the conclusion of Theorem 2 that the problem (27) and (26) has at least one solution on $[1,4]$.

4. Ulam Stability

This section is concerned with the Ulam stability of the problem in Equations (1) and (2) by considering its equivalent integral equation:

$$\begin{aligned} v(t) \;&= \int_a^t \frac{(t-s)^2}{2}\widehat{f}(s)ds \\ &-\frac{1}{\Lambda}\int_a^T \Big[A_1A_2\frac{(T-s)^3}{3!} + G_1(t)\frac{(T-s)^2}{2} + G_2(t)(T-s)\Big]\widehat{f}(s)ds \\ &+\frac{1}{\Lambda}\sum_{j=1}^{m}\int_a^{\sigma_j}\Big[\gamma_jA_1A_2\frac{(\sigma_j-s)^2}{2} + \mu_jG_1(t)(\sigma_j-s) + \nu_jG_2(t)\Big]\widehat{f}(s)ds \\ &+\frac{1}{\Lambda}\sum_{i=1}^{p}\int_{\rho_i}^{\rho_{i+1}}\int_a^s \Big[\xi_iA_1A_2\frac{(s-\tau)^2}{2} \\ &+\eta_iG_1(t)(s-\tau) + \omega_iG_2(t)\Big]\widehat{f}(\tau)d\tau ds. \end{aligned} \tag{28}$$

Let us introduce a continuous nonlinear operator $\chi : \mathcal{H} \to \mathcal{H}$ given by

$$\chi v(t) = v'''(t) - \widehat{f}(t).$$

Definition 1. *For each $\epsilon > 0$ and for each solution $v \in \mathcal{H}$, we call the problem in Equations (1) and (2) Ulam–Hyers stable provided that*

$$\|\chi v\| \leq \epsilon, \tag{29}$$

and there exists a solution $v_1 \in \mathcal{H}$ of Equation (1) such that $\|v_1 - v\| \leq \varrho\epsilon_1$ for positive real numbers ϱ and $\epsilon_1(\epsilon)$.

Definition 2. *Let there exist a function $\kappa \in C(\mathbb{R}^+, \mathbb{R}^+)$ and a solution $v_1 \in \mathcal{H}$ of Equation (1) with $|v_1(t) - v(t)| \leq \kappa(\epsilon), t \in [a, T]$ for each solution $v \in \mathcal{H}$ of Equation (1). Then, the problem in Equations (1) and (2) is called generalized Ulam–Hyers stable.*

Definition 3. *The problem in Equations (1) and (2) is said to be Ulam–Hyers–Rassias stable with respect to $\varphi \in C([a, T], \mathbb{R}^+)$ if*

$$|\chi v(t)| \leq \epsilon\varphi(t),\ t \in [a, T], \tag{30}$$

and there exists a solution $v_1 \in \mathcal{H}$ of Equation (1) such that

$$|v_1(t) - v(t)| \leq \varrho\epsilon_1\varphi(t),\ t \in [a, T],$$

where $\epsilon, \varrho, \epsilon_1$ are the same as defined in Definition 1.

Theorem 4. *If (H_1) and the condition $\ell < Q^{-1}$ (see Theorem 3) are satisfied, then the problem in Equations (1) and (2) is both Ulam–Hyers and generalized Ulam–Hyers stable.*

Proof. Recall that $v_1 \in \mathcal{H}$ is a unique solution of Equation (1) by Theorem 3.6. Let $v \in \mathcal{H}$ be an other solution of (1) which satisfies Equation (29). For every solution $v \in \mathcal{H}$ (given by Equation (28)) of Equation (1), it is easy to see that χ and $\mathcal{L} - I$ are equivalent operators. Therefore, it follows from Equations (16) and (29) and the fixed point property of the operator $\mathcal{L}$ given by Equation (17) that

$$\begin{aligned} |v_1(t) - v(t)| &= |\mathcal{L}v_1(t) - \mathcal{L}v(t) + \mathcal{L}v(t) - v(t)| \leq |\mathcal{L}v_1(t) - \mathcal{L}v(t)| + |\mathcal{L}v(t) - v(t)| \\ &\leq \ell Q \|v_1 - v\|_{\mathcal{H}} + \epsilon, \end{aligned}$$

which, on taking the norm for $t \in [a, T]$ and solving for $\|v_1 - v\|_{\mathcal{H}}$, yields

$$\|v_1 - v\|_{\mathcal{H}} \leq \frac{\epsilon}{1 - \ell Q},$$

where $\epsilon > 0$ and $\ell Q < 1$ (given condition).

Letting $\epsilon_1 = \frac{\epsilon}{1-\ell Q}$, and $\varrho = 1$, the Ulam–Hyers stability condition holds true. Furthermore, one can notice that the generalized Ulam–Hyers stability condition also holds valid if we set $\kappa(\epsilon) = \frac{\epsilon}{1-\ell Q}$. □

Theorem 5. *Let the assumptions of Theorem 4 be satisfied and that there exists a function $\varphi \in C([a, T], \mathbb{R}^+)$ satisfying the condition in Equation (30). Then, the problem in Equations (1) and (2) is Ulam–Hyers–Rassias stable with respect to φ.*

Proof. As argued in the proof of Theorem 4, we can get

$$\|v_1 - v\|_{\mathcal{H}} \leq \epsilon_1\|\varphi\|,$$

with $\epsilon_1 = \frac{\epsilon}{1-\ell Q}$. □

5. Existence Results for the Problem in Equations (1) and (3)

We only outline the idea for obtaining the existence and uniqueness results for the problem in Equations (1) and (3). In relation to the problem in Equations (1) and (3), we introduce an operator $\mathcal{S}:\mathcal{H}\to\mathcal{H}$ by Lemma 2 as

$$\begin{aligned}(\mathcal{S}u)(t) &= \int_a^t \frac{(t-s)^2}{2}\widehat{f}(s)ds \\ &-\frac{1}{\Delta}\int_a^T \Big[\alpha_2\zeta_1\zeta_2\frac{(T-s)^2}{2}+\beta_2P_1(t)(T-s)+\delta_2P_2(t)\Big]\widehat{f}(s)ds \\ &+\frac{1}{\Delta}\sum_{j=1}^m\int_a^{\sigma_j}\Big[\gamma_j\zeta_1\zeta_2\frac{(\sigma_j-s)^2}{2}+\mu_jP_1(t)(\sigma_j-s)+\nu_jP_2(t)\Big]\widehat{f}(s)ds \\ &+\frac{1}{\Delta}\sum_{i=1}^p\int_{\rho_i}^{\rho_{i+1}}\int_a^s\Big[\xi_i\zeta_1\zeta_2\frac{(s-\tau)^2}{2}+\eta_iP_1(t)(s-\tau)+\omega_iP_2(t)\Big]\widehat{f}(\tau)d\tau ds,\end{aligned} \tag{31}$$

where

$$P_1(t)=\zeta_1\Big(\zeta_4(t-a)-\zeta_5\Big),\ P_2(t)=\zeta_3\Big(\zeta_5-\zeta_4(t-a)\Big)-\zeta_2\Big(\zeta_6-\zeta_4\frac{(t-a)^2}{2}\Big),$$

and $\zeta_i (i=1,\ldots,6)$ are given by Equation (15).

Moreover, we set

$$\begin{aligned}\Theta_1 &= \frac{(T-a)^3}{3!}+\frac{1}{|\zeta_4|}\Big[|\alpha_2|\frac{(T-a)^3}{3!}+\sum_{i=1}^p\xi_i\Big(\frac{(\rho_{i+1}-a)^4}{4!}-\frac{(\rho_i-a)^4}{4!}\Big)+\sum_{j=1}^m\gamma_j\frac{(\sigma_j-a)^3}{3!}\Big] \\ &+\frac{p_1}{|\Delta|}\Big[|\beta_2|\frac{(T-a)^2}{2}+\sum_{i=1}^p\eta_i\Big(\frac{(\rho_{i+1}-a)^3}{3!}-\frac{(\rho_i-a)^3}{3!}\Big)+\sum_{j=1}^m\mu_j\frac{(\sigma_j-a)^2}{2}\Big] \\ &+\frac{p_2}{|\Delta|}\Big[|\delta_2|(T-a)+\sum_{i=1}^p\omega_i\Big(\frac{(\rho_{i+1}-a)^2}{2}-\frac{(\rho_i-a)^2}{2}\Big)+\sum_{j=1}^m\nu_j(\sigma_j-a)\Big], \\ \Theta_2 &= \frac{(T-a)^2}{2}+\frac{1}{|\zeta_2|}\Big[|\beta_2|\frac{(T-a)^2}{2}+\sum_{i=1}^p\eta_i\Big(\frac{(\rho_{i+1}-a)^3}{3!}-\frac{(\rho_i-a)^3}{3!}\Big)+\sum_{j=1}^m\mu_j\frac{(\sigma_j-a)^2}{2}\Big] \\ &+\frac{p_3}{|\zeta_1\zeta_2|}\Big[|\delta_2|(T-a)+\sum_{i=1}^p\omega_i\Big(\frac{(\rho_{i+1}-a)^2}{2}-\frac{(\rho_i-a)^2}{2}\Big)+\sum_{j=1}^m\nu_j(\sigma_j-a)\Big], \\ \Theta_3 &= (T-a)+\frac{1}{|\zeta_1|}\Big[|\delta_2|(T-a)+\sum_{i=1}^p\omega_i\Big(\frac{(\rho_{i+1}-a)^2}{2}-\frac{(\rho_i-a)^2}{2}\Big)+\sum_{j=1}^m\nu_j(\sigma_j-a)\Big],\end{aligned} \tag{32}$$

where $\max_{t\in[a,T]}|P_1(t)|=p_1, \max_{t\in[a,T]}|P_2(t)|=p_2$ and $\max_{t\in[a,T]}|\zeta_2(t-a)-\zeta_3|=p_3$ ($P_1(t)$ and $P_2(t)$ are given by Equation (14)). With the aid of the operator $\mathcal{S}$ defined by Equation (31) and the notations in Equation (32), we can obtain the existence results (analog to the ones derived in Section 3) for the problem in Equations (1) and (3). As an example, we formulate the uniqueness result for the problem in Equations (1) and (3) as follows.

Theorem 6. *Let $f:[a,T]\times\mathbb{R}^3\to\mathbb{R}$ be a continuous function satisfying the Lipschitz condition (H_1) with the Lipschitz constant ℓ_1 (instead of ℓ in (H_1)) such that $\ell_1(\Theta_1+\Theta_2+\Theta_3)<1$, where Θ_1,Θ_2 and Θ_3 are given by (32). Then, the problem in Equations (1) and (3) has a unique solution on $[a,T]$.*

Now, we present an example illustrating Theorem 6.

Example 3. *Consider the following problem:*

$$\begin{cases} u'''(t) = \dfrac{1}{210}\sin u + \dfrac{1}{4\sqrt{t+440}}u'(t) + \dfrac{1}{168}\dfrac{|u''|}{(|u''|+1)} + e^{-t}, \ t \in [1,4], \\ \alpha_1 u(a) + \alpha_2 u(T) = \sum_{i=1}^{4} \xi_i \int_{\rho_i}^{\rho_{i+1}} u(s)ds + \sum_{j=1}^{3} \gamma_j u(\sigma_j), \\ \beta_1 u'(a) + \beta_2 u'(T) = \sum_{i=1}^{4} \eta_i \int_{\rho_i}^{\rho_{i+1}} u'(s)ds + \sum_{j=1}^{3} \mu_j u'(\sigma_j), \\ \delta_1 u''(a) + \delta_2 u''(T) = \sum_{i=1}^{4} \omega_i \int_{\rho_i}^{\rho_{i+1}} u''(s)ds + \sum_{j=1}^{3} \nu_j u''(\sigma_j), \end{cases} \tag{33}$$

where $\alpha_1 = 1/4$, $\alpha_2 = 1/2$, $\beta_1 = 1/5$, $\beta_2 = 3/8$, $\delta_1 = 1/3$, $\delta_2 = 2/3$. *The other constants are the same as chosen in example* 3.7. *Clearly,* $|f(t,u,u',u'') - f(t,v,v',v'')| \leq \ell_1(|u-v| + |u'-v'| + |u''-v''|)$, *with* $\ell_1 = 1/84$. *Using the given data, we find that* $|\zeta_1| \approx 1.283333 \neq 0, |\zeta_2| \approx 0.990476 \neq 0, |\zeta_3| \approx 0.606151, |\zeta_4| \approx 1.992857 \neq 0,\ |\zeta_5| \approx 1.585913, |\zeta_6| \approx 0.262769$, *and* $|\Delta| \approx 2.533142$ *(*Δ *and* ζ_i $(i = 1,\ldots,6)$ *are given by Equation (15)),* $\Theta_1 \approx 23.050129, \Theta_2 \approx 15.505245, \Theta_3 \approx 6.434525$ *(*Θ_1*,* Θ_2 *and* Θ_3 *are given by Equation (32)) and* $\ell_1(\Theta_1 + \Theta_2 + \Theta_3) \approx 0.535594 < 1$. *Obviously, all the conditions of Theorem 6 hold and therefore Theorem 6 applies to the problem in Equation (33).*

6. Conclusions

We developed the existence theory and Ulam stability for a third-order nonlinear ordinary differential equation equipped with: (i) nonlocal integral multi-point and multi-strip; and (ii) non-separated integro-multi-point boundary conditions. The results obtained in this paper are new and quite general, and lead to several new ones for appropriate choices of the parameters involved in the problems at hand. For example, letting $\gamma_j = \rho_j = \nu_j = 0, \forall j$ and $\xi_i = \eta_i = \omega_i = 0, \forall i$ in Equation (2), the results for the problem in Equations (1) and (2), respectively, correspond to the ones for: (i) nonlocal integral multi-strip boundary conditions; and (ii) nonlocal integral multi-point boundary conditions. Likewise, by fixing $\alpha_k = \beta_k = \delta_k = 0, k = 1,2$ in the results of this paper, we obtain the ones for a third-order differential equation with purely nonlocal multi-point and multi-strip boundary conditions. Setting $\gamma_j = \rho_j = \nu_j = \xi_i = \eta_i = \omega_i = 0, \forall j,i$ and $\alpha_k = \beta_k = \delta_k = 1, k = 1,2$, the results obtained for the problem in Equations (1) and (3) reduce to the ones for anti-periodic boundary conditions. In the nutshell, the work presented in this paper significantly contributes to the existing literature on the topic.

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References

1. Zheng, L.; Zhang, X. *Modeling and Analysis of Modern Fluid Problems. Mathematics in Science and Engineering*; Elsevier/Academic Press: London, UK, 2017.
2. Akyildiz, F.T.; Bellout, H.; Vajravelu, K.; van Gorder, R.A. Van Gorder, Existence results for third order nonlinear boundary value problems arising in nano boundary layer fluid flows over stretching surfaces. *Nonlinear Anal. Real World Appl.* **2011**, *12*, 2919–2930. [CrossRef]
3. Hayat, T.; Aziz, A.; Muhammad, T.; Ahmad, B. On magnetohydrodynamic flow of second grade nanofluid over a nonlinear stretching sheet. *J. Magn. Magn. Mater.* **2016**, *408*, 99–106. [CrossRef]
4. Hayat, T.; Kiyani, M.Z.; Ahmad, I.; Ahmad, B. On analysis of magneto Maxwell nano-material by surface with variable thickness. *Int. J. Mech. Sci.* **2017**, *131–132*, 1016–1025. [CrossRef]

5. Ionkin, N.I. The solution of a certain boundary value problem of the theory of heat conduction with a nonclassical boundary condition. *Differ. Uravn.* **1977**, *13*, 294–304. (In Russian)
6. Il'in, V.A.; Moiseev, E.I. Nonlocal boundary-value problem of the first kind for a Sturm-Liouville operator in its differential and finite difference aspects. *Diff. Equa.* **1987**, *23* , 803–810.
7. Eloe, P.W.; Ahmad, B. Positive solutions of a nonlinear nth order boundary value problem with nonlocal conditions. *Appl. Math. Lett.* **2005**, *18* , 521–527. [CrossRef]
8. Webb, J.R.L.; Infante, G. Positive solutions of nonlocal boundary value problems: A unified approach. *J. London Math. Soc.* **2006**, *74* , 673–693. [CrossRef]
9. Clark, S.; Henderson, J. Uniqueness implies existence and uniqueness criterion for non local boundary value problems for third-order differential equations. *Proc. Am. Math. Soc.* **2006**, *134*, 3363–3372. [CrossRef]
10. Graef, J.R.; Webb, J.R.L. Third order boundary value problems with nonlocal boundary conditions. *Nonlinear Anal.* **2009**, *71*, 1542–1551. [CrossRef]
11. Taylor, C.; Hughes, T.; Zarins, C. Finite element modeling of blood flow in arteries. *Comput. Methods Appl. Mech. Eng.* **1998**, *158*, 155–196. [CrossRef]
12. Nicoud, F.; Schfonfeld, T. Integral boundary conditions for unsteady biomedical CFD applications. *Int. J. Numer. Meth. Fluids* **2002**, *40*, 457–465. [CrossRef]
13. Ahmad, B.; Alsaedi, A.; Alghamdi, B.S. Analytic approximation of solutions of the forced Duffing equation with integral boundary conditions. *Nonlinear Anal. Real World Appl.* **2008**, *9*, 1727–1740. [CrossRef]
14. Ahmad, B.; Ntouyas, S.K. A study of higher-order nonlinear ordinary differential equations with four-point nonlocal integral boundary conditions. *J. Appl. Math. Comput.* **2012**, *39*, 1–2, 97–108. [CrossRef]
15. Henderson, J. Smoothness of solutions with respect to multi-strip integral boundary conditions for nth order ordinary differential equations. *Nonlinear Anal. Model. Control* **2014**, *19*, 396–412. [CrossRef]
16. Ahmad, B.; Ntouyas, S.K.; Alsulami, H. Existence of solutions or nonlinear nth-order differential equations and inclusions with nonlocal and integral boundary conditions via fixed point theory. *Filomat* **2014**, *28*, 2149–2162. [CrossRef]
17. Karaca, I.Y.; Fen, F.T. Positive solutions of nth-order boundary value problems with integral boundary conditions. *Math. Model. Anal.* **2015**, *20*, 188–204. [CrossRef]
18. Ahmad, B.; Alsaedi, A.; Al-Malki, N. On higher-order nonlinear boundary value problems with nonlocal multipoint integral boundary conditions. *Lith. Math. J.* **2016**, *56*, 143–163. [CrossRef]
19. Boukrouche, M.; Tarzia, D.A. A family of singular ordinary differential equations of the third order with an integral boundary condition. *Bound. Value Probl.* **2018**, *32*. [CrossRef]
20. Ahmad, B.; Hayat, T. Diffraction of a plane wave by an elastic knife-edge adjacent to a rigid strip. *Canad. Appl. Math. Quart.* **2001**, *9*, 303–316.
21. Wang, G.S.; Blom, A.F. A strip model for fatigue crack growth predictions under general load conditions. *Eng. Fract. Mech.* **1991**, *40*, 507–533. [CrossRef]
22. Yusufoglu, E.; Turhan, I. A mixed boundary value problem in orthotropic strip containing a crack. *J. Franklin Inst.* **2012**, *349*, 2750–2769. [CrossRef]
23. Renterghem, T.V.; Botteldooren, D.; Verheyen, K. Road traffic noise shielding by vegetation belts of limited depth. *J. Sound Vib.* **2012**, *331*, 2404–2425. [CrossRef]
24. Smart, D.R. *Fixed Point Theorems*; Cambridge Tracts in Mathematics: London, UK, 2005.
25. Granas, A.; Dugundji, J. *Fixed Point Theory*; Springer: New York, NY, USA, 2005.