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On Central Complete and Incomplete Bell Polynomials I

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Abstract: In this paper, we introduce central complete and incomplete Bell polynomials which can be viewed as generalizations of central Bell polynomials and central factorial numbers of the second kind, and also as 'central' analogues for complete and incomplete Bell polynomials. Further, some properties and identities for these polynomials are investigated. In particular, we provide explicit formulas for the central complete and incomplete Bell polynomials related to central factorial numbers of the second kind.

Keywords: central incomplete Bell polynomials; central complete Bell polynomials; central complete Bell numbers

1. Introduction

In this paper, we introduce central incomplete Bell polynomials $T_{n,k}(x_1, x_2, \cdots, x_{n-k+1})$ given by

$$\frac{1}{k!}\Big(\sum_{m=1}^{\infty}\frac{1}{2^m}(x_m-(-1)^m x_m)\frac{t^m}{m!}\Big)^k=\sum_{n=k}^{\infty}T_{n,k}(x_1,x_2,\cdots,x_{n-k+1})\frac{t^n}{n!}$$

and central complete Bell polynomials $B_n^{(c)}(x|x_1, x_2, \cdots, x_n)$ given by

$$exp\Big(x\sum_{i=1}^{\infty}\frac{1}{2^i}(x_i-(-1)^i x_i)\frac{t^i}{i!}\Big)=\sum_{n=0}^{\infty}B_n^{(c)}(x|x_1,x_2,\cdots,x_n)\frac{t^n}{n!}$$

and investigate some properties and identities for these polynomials. They can be viewed as generalizations of central Bell polynomials and central factorial numbers of the second kind, and also as 'central' analogues for complete and incomplete Bell polynomials.

Here, we recall that the central factorial numbers $T(n,k)$ of the second kind and the central Bell polynomials $B_n^{(c)}(x)$ are given in terms of generating functions by

$$\frac{1}{k!}(e^{\frac{t}{2}}-e^{-\frac{t}{2}})^k=\sum_{n=k}^{\infty}T(n,k)\frac{t^n}{n!},\quad e^{x(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}=\sum_{n=0}^{\infty}B_n^{(c)}(x)\frac{t^n}{n!},$$

so that $T_{n,k}(1,1,\cdots,1)=T(n,k)$ and $B_n^{(c)}(x|1,1,\cdots,1)=B_n^{(c)}(x)$.

The incomplete and complete Bell polynomials have applications in such diverse areas as combinatorics, probability, algebra, modules over a *-algebra (see [1,2]), quasi local algebra and analysis. Here, we recall some applications of them and related works. The incomplete Bell polynomials $B_{n,k}(x_1, x_2, \cdots, x_{n-k+1})$ (see [3,4]) arise naturally when we want to find higher-order derivatives of composite functions. Indeed, such higher-order derivatives can be expressed in terms of incomplete Bell polynomials, which is known as Faà di Bruno formula given as in the following (see [3]):

$$\frac{d^n}{dt^n} g(f(t)) = \sum_{k=0}^{n} g^{(k)}(f(t)) B_{n,k}(f'(t), f''(t), \cdots, f^{(n-k+1)}(t)).$$

For the curious history on this formula, we let the reader refer to [5].

In addition, the number of monomials appearing in $B_{n,k} = B_{n,k}(x_1, x_2, \cdots, x_{n-k+1})$ is the number of partitioning a set with n elements into k blocks and the coefficient of each monomial is the number of partitioning a set with n elements as the corresponding k blocks. For example,

$$B_{10,7} = 3150 x_2^3 x_1^4 + 2520 x_3 x_2 x_1^5 + 210 x_4 x_1^6$$

shows that there are three ways of partitioning a set with 10 elements into seven blocks, and 3150 partitions with blocks of size 2, 2, 2, 1, 1, 1, 1, 2520 partitions with blocks of size 3, 2, 1, 1, 1, 1, 1, and 210 partitions with blocks of size 4, 1,1, 1, 1, 1, 1. This example is borrowed from [4], which gives a practical way of computing $B_{n,k}$ for any given n, k (see [4], (1.5)).

Furthermore, the incomplete Bell polynomials can be used in constructing sequences of binomial type (also called associated sequences). Indeed, for any given scalars $c_1, c_2, \cdots, c_n, \cdots$ the following form a sequence of binomial type

$$s_n(x) = \sum_{k=0}^{n} B_{n,k}(c_1, c_2, \cdots, c_{n-k+1}) x^k, \quad (n = 0, 1, 2, \cdots)$$

and, conversely, any sequence of binomial type arises in this way for some scalar sequence $c_1, c_2, \cdots, c_n \cdots$. For these, the reader may want to look at the paper [6].

There are certain connections between incomplete Bell polynomials and combinatorial Hopf algebras such as the Hopf algebra of word symmetric functions, the Hopf algebra of symmetric functions, the Faà di Bruno algebra, etc. The details can be found in [7].

The complete Bell polynomials $B_n(x_1, x_2, \cdots, x_n)$ (see [3,8–10]) have applications to probability theory. Indeed, the nth moment $\mu_n = E[X^n]$ of the random variable X is the nth complete Bell polynomial in the first n cumulants. Namely,

$$\mu_n = B_n(\kappa_1, \kappa_2, \cdots, \kappa_n).$$

For many applications to probability theory and combinatorics, the reader can refer to the Ph. D. thesis of Port [10].

Many special numbers, like Stirling numbers of both kinds, Lah numbers and idempotent numbers, appear in many combinatorial and number theoretic identities involving complete and incomplete Bell polynomials. For these, the reader refers to [3,8].

The central factorial numbers have received less attention than Stirling numbers. However, according to [11], they are at least as important as Stirling numbers, said to be "as important as Bernoulli numbers, or even more so". A systematic treatment of these important numbers was given in [11], including their properties and applications to difference calculus, spline theory, and to approximation theory, etc. For some other related references on central factorial numbers, we let the reader refer to [1,2,12–14]. Here, we note that central Bell polynomials and central factorial numbers of the second kind are respectively 'central' analogues for Bell polynomials and Stirling numbers of the second kind. They have been studied recently in [13,15].

The complete Bell polynomials and the incomplete Bell polynomials are respectively mutivariate versions for Bell polynomials and Stirling numbers of the second kind. This paper deals with central complete and incomplete Bell polynomials which are 'central' analogues for the complete and incomplete Bell polynomials. In addition, they can be viewed as generalizations of central Bell polynomials and central factorial numbers of the second kind (see [15]). The outline of the paper is as follows. After giving an introduction to the present paper in Section 1, we review some known properties and results about Bell polynomials, and incomplete and complete Bell polynomials in Section 2. We state the new and main results of this paper in Section 3, where we introduce central incomplete and complete Bell polynomials and investigate some properties and identities for them. In particular, Theorems 1 and 3 give basic formulas for computing central incomplete Bell polynomials and central complete Bell polynomials, respectively. We remark that the number of monomials appearing in $T_{n,k}(x_1, 2x_2, \cdots, 2^{n-k}x_{n-k+1})$ is the number of partitioning a set with n elements into k blocks with odd sizes and the coefficient of each monomial is the number of partitioning a set with n elements as the corresponding k blocks with odd sizes. This is illustrated by an example. Furthermore, we give expressions for the central incomplete and complete Bell polynomials with some various special arguments and also for the connection between the two Bell polynomials. We defer more detailed study of the central incomplete and complete Bell polynomials to a later paper.

2. Preliminaries

The Stirling numbers of the second kind are given in terms of generating function by (see [3,16])

$$\frac{1}{k!}(e^t - 1)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}. \tag{1}$$

The Bell polynomials are also called Tochard polynomials or exponential polynomials and defined by (see [9,13,15,17])

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}. \tag{2}$$

From Equations (1) and (2), we immediately see that (see [3,18])

$$\begin{aligned} B_n(x) &= e^{-x}\sum_{k=0}^{\infty}\frac{k^n}{k!}x^k \\ &= \sum_{k=0}^{n} x^k S_2(n,k), \quad (n \geq 0). \end{aligned} \tag{3}$$

When $x = 1$, $B_n = B_n(1)$ are called Bell numbers.

The (exponential) incomplete Bell polynomials are also called (exponential) partial Bell polynomials and defined by the generating function (see [9,15])

$$\frac{1}{k!}\Big(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\Big)^k = \sum_{n=k}^{\infty} B_{n,k}(x_1, \cdots, x_{n-k+1})\frac{t^n}{n!}, \quad (k \geq 0). \tag{4}$$

Thus, by Equation (4), we get

$$\begin{aligned} B_{n,k}(x_1, \cdots, x_{n-k+1}) = \sum \frac{n!}{i_1!i_2!\cdots i_{n-k+1}!}\Big(\frac{x_1}{1!}\Big)^{i_1}\Big(\frac{x_2}{2!}\Big)^{i_2} \times \cdots \\ \times \Big(\frac{x_{n-k+1}}{(n-k+1)!}\Big)^{i_{n-k+1}}, \end{aligned} \tag{5}$$

where the summation runs over all integers $i_1, \cdots, i_{n-k+1} \geq 0$ such that $i_1 + i_2 + \cdots + i_{n-k+1} = k$ and $i_1 + 2i_2 + \cdots + (n-k+1)i_{n-k+1} = n$.

From (1) and (4), we easily see that

$$B_{n,k}\underbrace{(1,1,\cdots,1)}_{n-k+1-times}=S_2(n,k),\quad (n,k\geq 0). \tag{6}$$

We easily deduce from (5) the next two identities:

$$B_{n,k}(\alpha x_1,\alpha x_2,\cdots,\alpha x_{n-k+1})=\alpha^k B_{n,k}(x_1,x_2,\cdots,x_{n-k+1}) \tag{7}$$

and

$$B_{n,k}(\alpha x_1,\alpha^2 x_2,\cdots,\alpha^{n-k+1}x_{n-k+1})=\alpha^n B_{n,k}(x_1,x_2,\cdots,x_{n-k+1}), \tag{8}$$

where $\alpha\in\mathbb{R}$ (see [15]).

From (4), it is not difficult to note that

$$\begin{aligned}
&\sum_{n=k}^{\infty}B_{n,k}(x,1,0,0,\cdots,0)\frac{t^n}{n!}=\frac{1}{k!}\Big(xt+\frac{t^2}{2}\Big)^k\\
&=\frac{t^k}{k!}\sum_{n=0}^{k}\binom{k}{n}\Big(\frac{t}{2}\Big)^n x^{k-n}\\
&=\sum_{n=0}^{k}\frac{(n+k)!}{k!}\binom{k}{n}\frac{1}{2^n}x^{k-n}\frac{t^{n+k}}{(n+k)!},
\end{aligned} \tag{9}$$

and

$$\sum_{n=k}^{\infty}B_{n,k}(x,1,0,0,\cdots,0)\frac{t^n}{n!}=\sum_{n=0}^{\infty}B_{n+k,k}(x,1,0,\cdots,0)\frac{t^{n+k}}{(n+k)!}. \tag{10}$$

Combining (9) with (10), we have

$$B_{n+k,k}(x,1,0,\cdots,0)=\frac{(n+k)!}{k!}\binom{k}{n}\frac{1}{2^n}x^{k-n},\quad (0\leq n\leq k). \tag{11}$$

Replacing n by $n-k$ in (11) yields the following identity

$$B_{n,k}(x,1,0,\cdots,0)=\frac{n!}{k!}\binom{k}{n-k}x^{2k-n}\Big(\frac{1}{2}\Big)^{n-k},\quad (k\leq n\leq 2k). \tag{12}$$

We recall here that the (exponential) complete Bell polynomials are defined by

$$\exp\Big(\sum_{i=1}^{\infty}x_i\frac{t^i}{i!}\Big)=\sum_{n=0}^{\infty}B_n(x_1,x_2,\cdots,x_n)\frac{t^n}{n!}. \tag{13}$$

Then, by (4) and (13), we get

$$B_n(x_1,x_2,\cdots,x_n)=\sum_{k=0}^{n}B_{n,k}(x_1,x_2,\cdots,x_{n-k+1}). \tag{14}$$

From (3), (6), (7) and (14), we have

$$\begin{aligned} B_n(x, x, \cdots, x) &= \sum_{k=0}^{n} x^k B_{n,k}(1, 1, \cdots, 1) \\ &= \sum_{k=0}^{n} x^k S_2(n,k) = B_n(x), \quad (n \geq 0). \end{aligned} \tag{15}$$

We recall that the central factorial numbers of the second kind are given by (see [19,20])

$$\frac{1}{k!}\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)^k = \sum_{n=k}^{\infty} T(n,k)\frac{t^n}{n!}, \tag{16}$$

where $k \geq 0$.

From (16), it is not difficult to derive the following expression

$$T(n,k) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\left(j - \frac{k}{2}\right)^n, \tag{17}$$

where $n, k \in \mathbb{Z}$ with $n \geq k \geq 0$, (see [16,20]).

In [20], the central Bell polynomials $B_n^{(c)}(x)$ are defined by

$$B_n^{(c)}(x) = \sum_{k=0}^{n} T(n,k)x^k, \quad (n \geq 0). \tag{18}$$

When $x = 1$, $B_n^{(c)} = B_n^{(c)}(1)$ are called the central Bell numbers.

It is not hard to derive the generating function for the central Bell polynomials from (18) as follows (see [15]):

$$e^{x\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)} = \sum_{n=0}^{\infty} B_n^{(c)}(x)\frac{t^n}{n!}. \tag{19}$$

By making use of (19), the following Dobinski-like formula was obtained earlier in [15]:

$$B_n^{(c)}(x) = \sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\binom{l+j}{j}(-1)^j\frac{1}{(l+j)!}\left(\frac{l}{2} - \frac{j}{2}\right)^n x^{l+j}, \tag{20}$$

where $n \geq 0$.

Motivated by (4) and (13), we will introduce central complete and incomplete Bell polynomials and investigate some properties and identities for these polynomials. Also, we present explicit formulas for the central complete and incomplete Bell polynomials related to central factorial numbers of the second kind.

3. On Central Complete and Incomplete Bell Polynomials

In view of (13), we may consider the *central incomplete Bell polynomials* which are given by

$$\frac{1}{k!}\left(\sum_{m=1}^{\infty}\frac{1}{2^m}\left(x_m - (-1)^m x_m\right)\frac{t^m}{m!}\right)^k = \sum_{n=k}^{\infty} T_{n,k}(x_1, x_2, \cdots, x_{n-k+1})\frac{t^n}{n!}, \tag{21}$$

where $k = 0, 1, 2, 3, \cdots$.

For $n, k \geq 0$ with $n-k \equiv 0 \pmod 2$, by (4) and (5), we get

$$T_{n,k}(x_1, x_2, \cdots, x_{n-k+1}) = \sum \frac{n!}{i_1! i_2! \cdots i_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{i_1} \left(\frac{0}{2 \cdot 2!}\right)^{i_2} \times \left(\frac{x_3}{2^2 \cdot 3!}\right)^{i_3} \cdots \left(\frac{x_{n-k+1}}{2^{n-k}(n-k+1)!}\right)^{i_{n-k+1}}, \tag{22}$$

where the summation is over all integers $i_1, i_2, \cdots, i_{n-k+1} \geq 0$ such that $i_1 + \cdots + i_{n-k+1} = k$ and $i_1 + 2i_2 + \cdots + (n-k+1)i_{n-k+1} = n$.

From (5) and (22), we note that

$$T_{n,k}(x_1, x_2, \cdots, x_{n-k+1}) = B_{n,k}\left(x_1, 0, \frac{x_3}{2^2}, 0, \cdots, \frac{x_{n-k+1}}{2^{n-k}}\right), \tag{23}$$

where $n, k \geq 0$ with $n-k \equiv 0 \pmod 2$ and $n \geq k$.
Therefore, from (22) and (23), we obtain the following theorem.

Theorem 1. *For $n, k \geq 0$ with $n \geq k$ and $n-k \equiv 0 \pmod 2$, we have*

$$\begin{aligned} T_{n,k}(x_1, x_2, \cdots, x_{n-k+1}) &= B_{n,k}\left(x_1, 0, \frac{x_3}{2^2}, 0, \cdots, \frac{x_{n-k+1}}{2^{n-k}}\right) \\ &= \sum \frac{n!}{i_1! i_3! \cdots i_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{i_1} \left(\frac{x_3}{2^2 \cdot 3!}\right)^{i_3} \times \cdots \times \left(\frac{x_{n-k+1}}{2^{n-k}(n-k+1)!}\right)^{i_{n-k+1}}, \end{aligned} \tag{24}$$

where the summation is over all integers $i_1, i_2, \cdots, i_{n-k+1} \geq 0$ such that $i_1 + i_3 + \cdots + i_{n-k+1} = k$ and $i_1 + 3i_3 + \cdots + (n-k+1)i_{n-k+1} = n$.

Remark 1. *Theorem 1 shows in particular that we have*

$$T_{n,k}(x_1, 2x_2, \cdots, 2^{n-k}x_{n-k+1}) = B_{n,k}(x_1, 0, x_3, 0, \cdots, x_{n-k+1}).$$

From this, we note that the number of monomials appearing in $T_{n,k}(x_1, 2x_2, \cdots, 2^{n-k}x_{n-k+1})$ is the number of partitioning a set with n elements into k blocks with odd sizes and the coefficient of each monomial is the number of partitioning a set with n elements as the corresponding k blocks with odd sizes. For example, from the example in Section 3 of [4], we have

$$T_{13,7}(x_1, 2x_2, 2^2x_3, 2^3x_4, 2^4x_5, 2^5x_6, 2^6x_7) = 200{,}200x_3^3x_1^4 + 72{,}072x_5x_3x_1^5 + 1716x_7x_1^6.$$

Thus, there are three ways of partitioning a set with 13 elements into seven blocks with odd sizes, and 200,200 partitions with blocks of size 3, 3, 3, 1, 1, 1, 1, 72,072 partitions with blocks of size 5, 3, 1, 1, 1, 1, 1, and 1716 partitions with blocks of size 7, 1, 1, 1, 1, 1, 1.

For $n, k \geq 0$ with $n \geq k$ and $n - k \equiv 0 \pmod 2$, by (21), we get

$$\begin{aligned}\sum_{n=k}^{\infty} T_{n,k}(x, x^2, x^3, \cdots, x^{n-k+1})\frac{t^n}{n!} &= \frac{1}{k!}\Big(xt + \frac{x^3}{2^2}\frac{t^3}{3!} + \frac{x^5}{2^4}\frac{t^5}{5!} + \cdots\Big)^k \\ &= \frac{1}{k!}\Big(e^{\frac{x}{2}t} - e^{-\frac{x}{2}t}\Big)^k = \frac{1}{k!}e^{-\frac{kx}{2}t}\Big(e^{xt} - 1\Big)^k \\ &= \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}e^{(l-\frac{k}{2})xt} \\ &= \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\sum_{n=0}^{\infty}(l - \frac{k}{2})^n x^n \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\Big(\frac{x^n}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}(l - \frac{k}{2})^n\Big)\frac{t^n}{n!}.\end{aligned} \tag{25}$$

Now, the next theorem follows by comparing the coefficients on both sides of (25).

Theorem 2. *For $n, k \geq 0$ with $n - k \equiv 0 \pmod 2$, we have*

$$\frac{x^n}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}(l - \frac{k}{2})^n = \begin{cases} T_{n,k}(x, x^2, \cdots, x^{n-k+1}), & \text{if } n \geq k, \\ 0, & \text{if } n < k. \end{cases} \tag{26}$$

In particular,

$$\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}(l - \frac{k}{2})^n = \begin{cases} T_{n,k}(1, 1, \cdots, 1), & \text{if } n \geq k, \\ 0, & \text{if } n < k. \end{cases} \tag{27}$$

For $n, k \geq 0$ with $n - k \equiv 0 \pmod 2$ and $n \geq k$, by (17) and (27), we get

$$T_{n,k}(1, 1, \cdots, 1) = T(n, k). \tag{28}$$

Therefore, by (26)–(28) and Theorem 1, we obtain the following corollary

Corollary 1. *For $n, k \geq 0$ with $n - k \equiv 0 \pmod 2$, $n \geq k$, we have*

$$T_{n,k}(x, x^2, \cdots, x^{n-k+1}) = x^n T_{n,k}(1, 1, \cdots, 1)$$

and

$$\begin{aligned}T_{n,k}(1, 1, \cdots, 1) &= T(n, k) = B_{n,k}(1, 0, \frac{1}{2^2}, \cdots, \frac{1}{2^{n-k}}) \\ &= \sum \frac{n!}{i_1! i_3! \cdots i_{n-k+1}!}\Big(\frac{1}{1!}\Big)^{i_1}\Big(\frac{1}{2^2 3!}\Big)^{i_3}\cdots\Big(\frac{1}{2^{n-k}(n-k+1)!}\Big)^{i_{n-k+1}},\end{aligned}$$

where $i_1 + i_3 + \cdots + i_{n-k+1} = k$ and $i_1 + 3i_3 + \cdots + (n-k+1)i_{n-k+1} = n$.

For $n, k \geq 0$ with $n \geq k$ and $n - k \equiv 0 \pmod 2$, we observe that

$$\sum_{n=k}^{\infty} T_{n,k}(x, 1, 0, 0, \cdots, 0)\frac{t^n}{n!} = \frac{1}{k!}(xt)^k. \tag{29}$$

Thus, we have

$$T_{n,k}(x,1,0,\cdots,0) = x^k \binom{0}{n-k}.$$

The next two identities follow easily from (24):

$$T_{n,k}(x,x,\cdots,x) = x^k T_{n,k}(1,1,\cdots,1), \tag{30}$$

and

$$T_{n,k}(\alpha x_1, \alpha x_2, \cdots, \alpha x_{n-k+1}) = \alpha^k T_{n,k}(x_1,x_2,\cdots,x_{n-k+1}),$$

where $n,k \geq 0$ with $n-k \equiv 0 \pmod 2$ and $n \geq k$.

Now, we observe that

$$\begin{aligned}
&\exp\Big(x\sum_{i=1}^{\infty}\big(\frac{1}{2}\big)^i(x_i-(-1)^i x_i)\frac{t^i}{i!}\Big)\\
&=\sum_{k=0}^{\infty}x^k\frac{1}{k!}\Big(\sum_{i=1}^{\infty}\big(\frac{1}{2}\big)^i(x_i-(-1)^i x_i)\frac{t^i}{i!}\Big)^k\\
&=1+\sum_{k=1}^{\infty}x^k\frac{1}{k!}\Big(\sum_{i=1}^{\infty}\big(\frac{1}{2}\big)^i(x_i-(-1)^i x_i)\frac{t^i}{i!}\Big)^k\\
&=1+\sum_{k=1}^{\infty}x^k\sum_{n=k}^{\infty}T_{n,k}(x_1,x_2,\cdots,x_{n-k+1})\frac{t^n}{n!}\\
&=1+\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{n}x^k T_{n,k}(x_1,x_2,\cdots,x_{n-k+1})\Big)\frac{t^n}{n!}.
\end{aligned} \tag{31}$$

In view of (13), it is natural to define the *central complete Bell polynomials* by

$$\exp\Big(x\sum_{i=1}^{\infty}\big(\frac{1}{2}\big)^i(x_i-(-1)^i x_i)\frac{t^i}{i!}\Big) = \sum_{n=0}^{\infty}B_n^{(c)}(x|x_1,x_2,\cdots,x_n)\frac{t^n}{n!}. \tag{32}$$

Thus, by (31) and (32), we get

$$B_n^{(c)}(x|x_1,x_2,\cdots,x_n) = \sum_{k=0}^{n}x^k T_{n,k}(x_1,x_2,\cdots,x_{n-k+1}). \tag{33}$$

When $x = 1$, $B_n^{(c)}(1|x_1,x_2,\cdots,x_n) = B_n^{(c)}(x_1,x_2,\cdots,x_n)$ are called the *central complete Bell numbers*.

For $n \geq 0$, we have

$$B_n^{(c)}(x_1,x_2,\cdots,x_n) = \sum_{k=0}^{n}T_{n,k}(x_1,x_2,\cdots,x_{n-k+1}) \tag{34}$$

and

$$B_0^{(c)}(x_1,x_2,\cdots,x_n) = 1.$$

By (18) and (33), we get

$$B_n^{(c)}(1,1,\cdots,1)=\sum_{k=0}^{n}T_{n,k}(1,1,\cdots,1)=\sum_{k=0}^{n}T(n,k)=B_n^{(c)}, \tag{35}$$

and

$$B_n^{(c)}(x|1,1,\cdots,1)=\sum_{k=0}^{n}x^kT_{n,k}(1,1,\cdots,1)=\sum_{k=0}^{n}x^kT(n,k)=B_n^{(c)}(x). \tag{36}$$

From (31), we note that

$$\begin{aligned}
&\exp\Big(\sum_{i=1}^{\infty}\big(\frac{1}{2}\big)^i(x_i-(-1)^ix_i)\frac{t^i}{i!}\Big)\\
&=1+\sum_{n=1}^{\infty}\frac{1}{n!}\Big(\sum_{i=1}^{\infty}\big(\frac{1}{2}\big)^i(x_i-(-1)^ix_i)\frac{t^i}{i!}\Big)^n\\
&=1+\frac{1}{1!}\sum_{i=1}^{\infty}\big(\frac{1}{2}\big)^i(x_i-(-1)^ix_i)\frac{t^i}{i!}+\frac{1}{2!}\Big(\sum_{i=1}^{\infty}\big(\frac{1}{2}\big)^i(x_i-(-1)^i\\
&\times x_i)\frac{t^i}{i!}\Big)^2+\frac{1}{3!}\Big(\sum_{i=1}^{\infty}\big(\frac{1}{2}\big)^i(x_i-(-1)^ix_i)\frac{t^i}{i!}\Big)^3+\cdots\\
&=1+\frac{1}{1!}x_1t+\frac{1}{2!}x_1^2t^2+\frac{1}{3!}\Big(x_1^3+\frac{1}{2^2}x_3\Big)t^3+\cdots\\
&=\sum_{n=0}^{\infty}\Big(\sum_{m_1+2m_2+\cdots+nm_n=n}\frac{n!}{m_1!m_2!\cdots m_n!}\Big(\frac{x_1}{1!}\Big)^{m_1}\Big(\frac{0}{2!2}\Big)^{m_2}\\
&\times\Big(\frac{x_3}{3!2^2}\Big)^{m_3}\cdots\Big(\frac{x_n\big(1-(-1)^n\big)}{n!2^n}\Big)^{m_n}\Big)\frac{t^n}{n!}.
\end{aligned} \tag{37}$$

Now, for $n\in\mathbb{N}$ with $n\equiv 1 \pmod 2$, by (32), (34) and (37), we get

$$\begin{aligned}
&B_n^{(c)}(x_1,x_2,\cdots,x_n)=\sum_{k=0}^{n}T_{n,k}(x_1,x_2,\cdots,x_{n-k+1})\\
&=\sum_{m_1+3m_3+\cdots+nm_n=n}\frac{n!}{m_1!m_3!\cdots m_n!}\Big(\frac{x_1}{1!}\Big)^{m_1}\Big(\frac{x_3}{3!2^2}\Big)^{m_3}\cdots\Big(\frac{x_n}{n!2^{n-1}}\Big)^{m_n}.
\end{aligned} \tag{38}$$

Therefore, Equation (38) yields the following theorem.

Theorem 3. *For $n\in\mathbb{N}$ with $n\equiv 1$ (mod 2), we have*

$$\begin{aligned}
&B_n^{(c)}(x_1,x_2,\cdots,x_n)=\sum_{k=0}^{n}T_{n,k}(x_1,x_2,\cdots,x_{n-k+1})\\
&=\sum_{m_1+3m_3+\cdots+nm_n=n}\frac{n!}{m_1!m_3!\cdots m_n!}\Big(\frac{x_1}{1!}\Big)^{m_1}\Big(\frac{x_3}{3!2^2}\Big)^{m_3}\cdots\Big(\frac{x_n}{n!2^{n-1}}\Big)^{m_n}.
\end{aligned}$$

Example 1. *Here, we illustrate Theorem 3 with the following example:*

$$\begin{aligned}
B_5^{(c)}(x_1,2x_2,2^2x_3,2^3x_4,2^4x_5)&=\frac{5!}{0!0!1!}\Big(\frac{x_1}{1!}\Big)^0\Big(\frac{x_3}{3!}\Big)^0\Big(\frac{x_5}{5!}\Big)^1+\frac{5!}{2!1!0!}\Big(\frac{x_1}{1!}\Big)^2\Big(\frac{x_3}{3!}\Big)^1\Big(\frac{x_5}{5!}\Big)^0\\
&+\frac{5!}{5!0!0!}\Big(\frac{x_1}{1!}\Big)^5\Big(\frac{x_3}{3!}\Big)^0\Big(\frac{x_5}{5!}\Big)^0=x_1^5+10x_1^2x_3+x_5,
\end{aligned}$$

$$T_{5,1}(x_1,2x_2,2^2x_3,2^3x_4,2^4x_5) = \frac{5!}{0!0!1!}\Big(\frac{x_1}{1!}\Big)^0\Big(\frac{x_3}{3!}\Big)^0\Big(\frac{x_5}{5!}\Big)^1 = x_5,$$
$$T_{5,3}(x_1,2x_2,2^2x_3) = \frac{5!}{2!1!}\Big(\frac{x_1}{1!}\Big)^2\Big(\frac{x_3}{3!}\Big)^1 = 10x_1^2x_3,\ T_{5,5}(x_1) = \frac{5!}{5!}\Big(\frac{x_1}{1!}\Big)^5 = x_1^5,$$
$$T_{5,0}(x_1,2x_2,2^2x_3,2^3x_4,2^4x_5,2^5x_6) = 0,\ T_{5,2}(x_1,2x_2,2^2x_3,2^3x_4) = 0,\ T_{5,4}(x_1,2x_2) = 0.$$

On the one hand, we have

$$\begin{aligned}\exp\Big(x\sum_{i=1}^{\infty}\big(\tfrac{1}{2}\big)^i(1-(-1)^i)\frac{t^i}{i!}\Big) &= 1+\sum_{k=1}^{\infty}\frac{x^k}{k!}\Big(\sum_{n=k}^{\infty}\big(\tfrac{1}{2}\big)^i(1-(-1)^i)\frac{t^i}{i!}\Big)^k \\ &= 1+\sum_{k=1}^{\infty}x^k\sum_{n=k}^{\infty}T_{n,k}(1,1,\cdots,1)\frac{t^n}{n!} \\ &= 1+\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{n}x^kT_{n,k}(1,1,\cdots,1)\Big)\frac{t^n}{n!}.\end{aligned} \tag{39}$$

On the other hand, from (19), we have

$$\begin{aligned}\exp\Big(x\sum_{i=1}^{\infty}\big(\tfrac{1}{2}\big)^i(1-(-1)^i)\frac{t^i}{i!}\Big) &= \exp\Big(x\big(t+\frac{1}{2^2}t^3+\frac{1}{2^4}t^5+\cdots\big)\Big) \\ &= \exp\Big(x\big(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\big)\Big) = \sum_{n=0}^{\infty}B_n^{(c)}(x)\frac{t^n}{n!}.\end{aligned} \tag{40}$$

Therefore, by (39) and (40), we obtain the following theorem.

Theorem 4. *For $n,k \geq 0$ with $n \geq k$, we have*

$$\sum_{k=0}^{n}x^kT_{n,k}(1,1,\cdots,1) = B_n^{(c)}(x).$$

We note from Theorem 4 the next identities:

$$\sum_{k=0}^{n}x^kT_{n,k}(1,1,\cdots,1) = \sum_{k=0}^{n}T_{n,k}(x,x,\cdots,x) = B_n^{(c)}(x,x,\cdots,x). \tag{41}$$

Thus, Theorem 4 and (41) together give us the following corollary.

Corollary 2. *For $n \geq 0$, we have*

$$B_n^{(c)}(x,x,\cdots,x) = B_n^{(c)}(x).$$

The Stirling numbers of the first kind are given in terms of the generating function by (see [3,21])

$$\frac{1}{k!}\big(\log(1+t)\big)^k = \sum_{n=k}^{\infty}S_1(n,k)\frac{t^n}{n!},\quad (k \geq 0). \tag{42}$$

In order to get the following result and using (42), we first observe that

$$\begin{aligned}\frac{1}{k!}\Big(\log\Big(1+\frac{x}{1-\frac{x}{2}}\Big)\Big)^k &= \sum_{l=k}^{\infty} S_1(l,k)\frac{1}{l!}\Big(\frac{x}{1-\frac{x}{2}}\Big)^l \\ &= \sum_{l=k}^{\infty} S_1(l,k)\frac{x^l}{l!}\big(1-\frac{x}{2}\big)^{-l} \\ &= \sum_{l=k}^{\infty}\frac{1}{l!}S_1(l,k)\sum_{n=l}^{\infty}\binom{n-1}{l-1}\big(\frac{1}{2}\big)^{n-l}x^n \\ &= \sum_{n=k}^{\infty}\Big(\sum_{l=k}^{n}\frac{1}{l!}S_1(l,k)\binom{n-1}{l-1}\big(\frac{1}{2}\big)^{n-l}\Big)x^n.\end{aligned} \tag{43}$$

The following equation can be derived from (21) and (43):

$$\begin{aligned}&\sum_{n=k}^{\infty}T_{n,k}(0!,1!,2!,\cdots,(n-k)!)\frac{t^n}{n!} \\ &= \frac{1}{k!}\Big(t+\big(\frac{1}{2}\big)^2\frac{t^3}{3}+\big(\frac{1}{2}\big)^4\frac{t^5}{5}+\big(\frac{1}{2}\big)^6\frac{t^7}{7}+\cdots\Big)^k \\ &= \frac{1}{k!}\Big(\log\big(1+\frac{t}{2}\big)-\log\big(1-\frac{t}{2}\big)\Big)^k = \frac{1}{k!}\Big(\log\Big(\frac{1+\frac{t}{2}}{1-\frac{t}{2}}\Big)\Big)^k \\ &= \frac{1}{k!}\Big(\log\big(1+\frac{t}{1-\frac{t}{2}}\big)\Big)^k = \sum_{n=k}^{\infty}\Big(\sum_{l=k}^{n}\frac{S_1(l,k)}{l!}\binom{n-1}{l-1}\big(\frac{1}{2}\big)^{n-l}\Big)t^n.\end{aligned} \tag{44}$$

Now, we obtain the following theorem by comparing the coefficients on both sides of (44).

Theorem 5. *For $n,k \geq 0$ with $n \geq k$, we have*

$$T_{n,k}(0!,1!,2!,\cdots,(n-k)!) = n!\sum_{l=k}^{n}\frac{S_1(l,k)}{l!}\binom{n-1}{l-1}\big(\frac{1}{2}\big)^{n-l}.$$

4. Conclusions

In this paper, we introduced central complete and incomplete Bell polynomials which can be viewed as generalizations of central Bell polynomials and central factorial numbers of the second kind, and also as 'central' analogues for complete and incomplete Bell polynomials. As examples and recalling some relevant works, we reminded the reader that the incomplete and complete Bell polynomials appearing in a Faà di Bruno formula, which encode integer partition information, can be used in constructing sequences of binomial type, have connections with combinatorial Hopf algebras, have applications in probability theory and arise in many combinatorial and number theoretic identities. One additional thing we want to mention here is that the Faà di Bruno formula has been proved to be very useful in finding explicit expressions for many special numbers arising from many different families of linear and nonlinear differential equations having generating functions of some special numbers and polynomials as solutions (see [22]).

The main results of the present paper are stated in Section 3, in which we introduced central incomplete and complete Bell polynomials and investigated some properties and identities. In particular, in Theorems 1 and 3, we gave basic formulas for computing central incomplete Bell polynomials and central complete Bell polynomials, respectively. We remarked that the number of monomials appearing in $T_{n,k}(x_1, 2x_2, \cdots, 2^{n-k}x_{n-k+1})$ is the number of partitioning n into k odd parts and the coefficient of each monomial is the number of partitioning n as the corresponding k odd parts. This was illustrated by an example. Furthermore, we gave expressions for the central incomplete and complete Bell polynomials with some various special arguments and also for the connection between the two Bell polynomials. In the near future, we hope to find some further properties, identities and various applications for central complete and incomplete Bell polynomials.

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