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Some General Classes of q-Starlike Functions Associated with the Janowski Functions

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Abstract: By making use of the concept of basic (or q-) calculus, various families of q-extensions of starlike functions of order α in the open unit disk $\mathbb{U}$ were introduced and studied from many different viewpoints and perspectives. In this paper, we first investigate the relationship between various known classes of q-starlike functions that are associated with the Janowski functions. We then introduce and study a new subclass of q-starlike functions that involves the Janowski functions. We also derive several properties of such families of q-starlike functions with negative coefficients including (for example) distortion theorems.

Keywords: analytic functions; univalent functions; starlike and q-starlike functions; q-derivative (or q-difference) operator; sufficient conditions; distortion theorems; Janowski functions

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1. Introduction

The basic (or q-) calculus is the ordinary classical calculus without the notion of limits, while q stands for the quantum. The application of the q-calculus was initiated by Jackson [1,2]. Later, geometrical interpretation of the q-analysis was recognized through studies on quantum groups. It also suggests a relation between integrable systems and q-analysis. Aral and Gupta [3–5] defined and studied the q-analogue of the Baskakov-Durrmeyer operator, which is based on the q-analogue of the beta function. Some other important q-generalizations and q-extensions of complex operators are the q-Picard and the q-Gauss-Weierstrass singular-integral operators, which are discussed in [6–8].

In Geometric Function Theory, several subclasses of the normalized analytic function class $\mathcal{A}$ have already been analyzed and investigated through various perspectives. The q-calculus provides valuable tools that have been extensively used in order to examine several subclasses of the normalized analytic function class $\mathcal{A}$ in the open unit disk $\mathbb{U}$. Ismail et al. [9] were the first to use the q-derivative operator D_q in order to study a certain q-analogue of the class $\mathcal{S}^*$ of starlike functions in $\mathbb{U}$ (see Definition 6 below). Mohammed and Darus [10] studied the approximation and geometric properties of these q-operators in some subclasses of analytic functions in a compact disk. These q-operators are defined by using the convolution of normalized analytic functions and q-hypergeometric functions, where several interesting results were obtained (see [11,12]). Certain basic properties of the q-close-to-convex functions were studied by Raghavendar and Swaminathan [13]. Aral et al. [14] successfully studied the applications of the q-calculus in operator theory. Kanas and Raducanu [15] used the fractional q-calculus operators in investigations of certain classes of functions, which are analytic in the open unit disk $\mathbb{U}$ by using the idea of the canonical domain. The coefficient inequality problems for q-closed-to-convex functions with respect to Janowski starlike functions were studied recently (see, for example, [16]). In the year 2016, Wongsaijai and Sukantamala [17] published a paper, in which they generalized certain subclasses of starlike functions in a systematic way. In fact, they made a very significant usage of the q-calculus basically in the context of Geometric Function Theory. Moreover, the generalized basic (or q-) hypergeometric functions were first used in Geometric Function Theory in a book chapter by Srivastava (see, for details [18], (p. 347 et seq.); see also [19]).

Motivated by the works of Wongsaijai and Sukantamala [17] and other related works cited above in this paper, we shall consider three new subfamilies of q-starlike functions with respect to Janowski functions. Several properties and characteristics, for example, sufficient conditions, inclusion results, distortion theorems, and radius problems, shall be discussed in this investigation. We shall also point out some relevant connections of our results with the existing results.

We denote by $\mathcal{H}(\mathbb{U})$ the class of functions that are analytic in the open unit disk:

$$\mathbb{U} = \{z : z \in \mathbb{C} \qquad \text{and} \qquad |z| < 1\},$$

where $\mathbb{C}$ is the set of complex numbers. Let $\mathcal{A}$ be the subclass of functions $f \in \mathcal{H}(\mathbb{U})$, which are represented by the following Taylor-Maclaurin series expansion:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{U}) \tag{1}$$

that is, which satisfy the normalization condition given by

$$f(0) = f'(0) - 1 = 0.$$

Furthermore, let $\mathcal{S}$ be the class of functions in $\mathcal{A}$, which are univalent in $\mathbb{U}$.

The familiar class of starlike functions in $\mathbb{U}$ will be denoted by $\mathcal{S}^*$, which consists of normalized functions $f \in \mathcal{S}$ that satisfy the following conditions:

$$f \in \mathcal{S} \qquad \text{and} \qquad \Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad (\forall\, z \in \mathbb{U}). \tag{2}$$

For two functions f and g, which are analytic in $\mathbb{U}$, we say that the function f is subordinate to g and write

$$f \prec g \qquad \text{or} \qquad f(z) \prec g(z),$$

if there exists a Schwarz function w, which is analytic in $\mathbb{U}$ with

$$w(0) = 0 \qquad \text{and} \qquad |w(z)| < 1$$

such that

$$f(z) = g\big(w(z)\big).$$

In particular, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence (cf., e.g., [20]; see also [21]):

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \qquad \text{and} \qquad f(\mathbb{U}) \subset g(\mathbb{U}).$$

We next denote by $\mathcal{P}$ the class of analytic functions p in $\mathbb{U}$, which are normalized by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \tag{3}$$

such that

$$\Re\{p(z)\} > 0.$$

In the next section (Section 2), we first give some basic definitions and concept details. Thereafter we will demonstrate three (presumably new) subclasses of the class $\mathcal{S}_q^*$ of q-starlike functions associated with the Janowski functions.

2. A Set of Definitions

Throughout this paper, we suppose that $0 < q < 1$ and that

$$\mathbb{N} = \{1, 2, 3, \cdots\} = \mathbb{N}_0 \setminus \{0\} \qquad (\mathbb{N}_0 := \{0, 1, 2, \cdots\}).$$

Definition 1. (See [22]) *A given function h with $h(0) = 1$ is said to belong to the class $\mathcal{P}[A, B]$ if and only if*

$$h(z) \prec \frac{1 + Az}{1 + Bz} \qquad (-1 \leqq B < A \leqq 1).$$

The analytic function class $\mathcal{P}[A, B]$ was introduced by Janowski [22], who showed that $h(z) \in \mathcal{P}[A, B]$ if and only if there exists a function $p \in \mathcal{P}$ such that

$$h(z) = \frac{(A+1)\,p(z) - (A-1)}{(B+1)\,p(z) - (B-1)} \qquad (-1 \leqq B < A \leqq 1).$$

Definition 2. *A function $f \in \mathcal{S}$ is said to belong to the class $\mathcal{S}^*[A, B]$ if and only if there exists a function $p \in \mathcal{P}$ such that*

$$\frac{zf'(z)}{f(z)} = \frac{(A+1)\,p(z) - (A-1)}{(B+1)\,p(z) - (B-1)} \qquad (-1 \leqq B < A \leqq 1). \tag{4}$$

Definition 3. *Let $q \in (0, 1)$, and define the q-number $[\lambda]_q$ by*

$$[\lambda]_q = \begin{cases} \dfrac{1 - q^{\lambda}}{1 - q} & (\lambda \in \mathbb{C}) \\[2ex] \sum\limits_{k=0}^{n-1} q^k = 1 + q + q^2 + \cdots + q^{n-1} & (\lambda = n \in \mathbb{N}). \end{cases}$$

Definition 4. *Let $q \in (0, 1)$, and define the q-factorial $[n]_q!$ by*

$$[n]_q! = \begin{cases} 1 & (n = 0) \\[2ex] \prod\limits_{k=1}^{n} [k]_q & (n \in \mathbb{N}). \end{cases}$$

Definition 5. (See [1,2]) *The q-derivative (or the q-difference) operator $D_q f$ of a function f is defined, in a given subset of $\mathbb{C}$, by*

$$(D_q f)(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1 - q)z} & (z \neq 0) \\[2ex] f'(0) & (z = 0), \end{cases} \tag{5}$$

provided that $f'(0)$ exists.

We note from Definition 5 that

$$\lim_{q\to 1-}\left(D_q f\right)(z)=\lim_{q\to 1-}\frac{f(z)-f(qz)}{(1-q)z}=f'(z)$$

for a function f, which is differentiable in a given subset of $\mathbb{C}$. It is readily deduced from (1) and (5) that

$$\left(D_q f\right)(z)=1+\sum_{n=2}^{\infty}[n]_q\,a_n z^{n-1}. \tag{6}$$

Definition 6. (See [9]) *A function $f\in\mathcal{S}$ is said to belong to the class $\mathcal{S}_q^*$ of q-starlike functions in $\mathbb{U}$ if*

$$f(0)=f'(0)-1=0 \tag{7}$$

and

$$\left|\frac{z}{f(z)}\left(D_q f\right)(z)-\frac{1}{1-q}\right|\leqq\frac{1}{1-q}\qquad(z\in\mathbb{U}). \tag{8}$$

We readily observe that, as $q\to 1-$, the closed disk:

$$\left|w-\frac{1}{1-q}\right|\leqq\frac{1}{1-q}$$

becomes the right-half complex plane, and the class $\mathcal{S}_q^*$ of q-starlike functions in $\mathbb{U}$ reduces to the familiar class $\mathcal{S}^*$ of normalized starlike functions with respect to the origin $(z=0)$. Equivalently, by using the principle of subordination between analytic functions, we can rewrite the conditions in (7) and (8) as follows (see [16]):

$$\frac{z}{f(z)}\left(D_q f\right)(z)\prec\widehat{p}(z)\qquad\left(\widehat{p}(z):=\frac{1+z}{1-qz}\right). \tag{9}$$

We now introduce three (presumably new) subclasses of the class $\mathcal{S}_q^*$ of q-starlike functions associated with the Janowski functions in the following way.

Definition 7. *A function $f\in\mathcal{A}$ is said to belong to the class $\mathcal{S}^*_{(q,1)}[A,B]$ if and only if*

$$\Re\left(\frac{(B-1)\dfrac{zD_q f(z)}{f(z)}-(A-1)}{(B+1)\dfrac{zD_q f(z)}{f(z)}-(A+1)}\right)\geqq 0.$$

We call $\mathcal{S}^*_{(q,1)}[A,B]$ the class of q-starlike functions of Type 1 associated with the Janowski functions.

Definition 8. *A function $f\in\mathcal{A}$ is said to belong to the class $f\in\mathcal{S}^*_{(q,2)}[A,B]$ if and only if*

$$\left|\frac{(B-1)\dfrac{zD_q f(z)}{f(z)}-(A-1)}{(B+1)\dfrac{zD_q f(z)}{f(z)}-(A+1)}-\frac{1}{1-q}\right|<\frac{1}{1-q}.$$

We call $\mathcal{S}^*_{(q,2)}[A,B]$ the class of q-starlike functions of Type 2 associated with the Janowski functions.

Definition 9. *A function $f \in \mathcal{A}$ is said to belong to the class $f \in \mathcal{S}^*_{(q,3)}[A,B]$ if and only if*

$$\left| \frac{(B-1)\dfrac{zD_qf(z)}{f(z)} - (A-1)}{(B+1)\dfrac{zD_qf(z)}{f(z)} - (A+1)} - 1 \right| < 1.$$

We call $\mathcal{S}^*_{(q,3)}[A,B]$ the class of q-starlike functions of Type 3 associated with the Janowski functions.

Each of the following special cases of the above-defined q-starlike functions:

$$\mathcal{S}^*_{(q,1)}[A,B], \quad \mathcal{S}^*_{(q,2)}[A,B] \qquad \text{and} \qquad \mathcal{S}^*_{(q,3)}[A,B]$$

is worthy of note.

I. If we put

$$A = 1 - 2\alpha \quad (0 \leqq \alpha < 1) \qquad \text{and} \qquad B = -1$$

in Definition 7, we get the class $\mathcal{S}^*_{(q,1)}(\alpha)$, which was introduced and studied by Wongsaijai and Sukantamala (see [17], Definition 1).

II. If we put

$$A = 1 - 2\alpha \quad (0 \leqq \alpha < 1) \qquad \text{and} \qquad B = -1,$$

in Definition 8, we are led to the class $\mathcal{S}^*_{(q,2)}(\alpha)$, which was introduced and studied by Wongsaijai and Sukantamala (see [17], Definition 2).

III. If we put

$$A = 1 - 2\alpha \quad (0 \leqq \alpha < 1) \qquad \text{and} \qquad B = -1$$

in Definition 9, we have the class $\mathcal{S}^*_{(q,3)}(\alpha)$, which was introduced and studied by Wongsaijai and Sukantamala (see [17], Definition 3).

IV. If we put

$$A = 1 - 2\alpha \quad (0 \leqq \alpha < 1) \qquad \text{and} \qquad B = -1$$

in Definition 8, we obtain the class $\mathcal{S}^*_q(\alpha)$, which was introduced and studied by Agrawal and Sahoo [23].

V. If we put

$$A = 1 \qquad \text{and} \qquad B = -1$$

in Definition 8, we get the class $\mathcal{S}^*_q$ introduced and studied by Ismail et al. [9].

VI. In Definition 8, if we let $q \to 1-$ and put $A = \lambda$ and $B = 0$, then we will arrive at the function class, studied by Ponnusamy and Singh (see [24]).

Geometrically, for $f \in \mathcal{S}^*_{(q,k)}[A,B] \quad (k = 1,2,3)$, the quotient:

$$\frac{zD_qf(z)}{f(z)}$$

lies in the domains $\Omega_j \quad (j = 1,2,3)$ given by

$$\Omega_1 = \left\{ w : w \in \mathbb{C} \quad \text{and} \quad \Re(w) > \frac{A-1}{B-1} \right\},$$

$$\Omega_2 = \left\{ w : w \in \mathbb{C} \quad \text{and} \quad \left| w - \frac{2+q(A-1)}{(B-1)q+(B+3)} \right| < \frac{A+1}{(B-1)q+(B+3)} \right\}$$

and

$$\Omega_3 = \left\{ w : w \in \mathbb{C} \quad \text{and} \quad \left| w - \frac{2}{B+3} \right| < \frac{A+1}{B+3} \right\},$$

respectively.

In this paper, many properties and characteristics, for example sufficient conditions, inclusion results, distortion theorems, and radius problems, are discussed. We also indicate relevant connections of our results with a number of other related works on this subject.

3. Main Results and Their Demonstration

We first derive the inclusion results for the following generalized q-starlike functions:

$$\mathcal{S}^*_{(q,1)}[A,B], \quad \mathcal{S}^*_{(q,2)}[A,B] \qquad \text{and} \qquad \mathcal{S}^*_{(q,3)}[A,B],$$

which are associated with the Janowski functions.

Theorem 1. *If* $-1 \leqq B < A \leqq 1$, *then*

$$\mathcal{S}^*_{(q,3)}[A,B] \subset \mathcal{S}^*_{(q,2)}[A,B] \subset \mathcal{S}^*_{(q,1)}[A,B].$$

Proof. First of all, we suppose that $f \in \mathcal{S}^*_{(q,3)}[A,B]$. Then, by Definition 9, we have

$$\left| \frac{(B-1)\frac{zD_qf(z)}{f(z)} - (A-1)}{(B+1)\frac{zD_qf(z)}{f(z)} - (A+1)} - 1 \right| < 1,$$

so that

$$\left| \frac{(B-1)\frac{zD_qf(z)}{f(z)} - (A-1)}{(B+1)\frac{zD_qf(z)}{f(z)} - (A+1)} - 1 \right| + \frac{q}{1-q} < 1 + \frac{q}{1-q}. \tag{10}$$

By using the triangle inequality and Equation (10), we find that

$$\left| \frac{(B-1)\frac{zD_qf(z)}{f(z)} - (A-1)}{(B+1)\frac{zD_qf(z)}{f(z)} - (A+1)} - \frac{1}{1-q} \right| < \frac{1}{1-q}. \tag{11}$$

The last expression in (11) now implies that $f \in \mathcal{S}^*_{(q,2)}[A,B]$, that is, that

$$\mathcal{S}^*_{(q,3)}[A,B] \subset \mathcal{S}^*_{(q,2)}[A,B].$$

Next, we let $f \in \mathcal{S}^*_{(q,2)}[A,B]$, so that

$$f \in \mathcal{S}^*_{(q,2)}[A,B] \iff \left| \frac{(B-1)\frac{zD_q f(z)}{f(z)} - (A-1)}{(B+1)\frac{zD_q f(z)}{f(z)} - (A+1)} - \frac{1}{1-q} \right| < \frac{1}{1-q},$$

by Definition 8.

Since

$$\frac{1}{1-q} > \left| \frac{(B-1)\frac{zD_q f(z)}{f(z)} - (A-1)}{(B+1)\frac{zD_q f(z)}{f(z)} - (A+1)} - \frac{1}{1-q} \right|$$

$$= \left| \frac{1}{1-q} - \frac{(B-1)\frac{zD_q f(z)}{f(z)} - (A-1)}{(B+1)\frac{zD_q f(z)}{f(z)} - (A+1)} \right|,$$

we have

$$\Re\left(\frac{(B-1)\frac{zD_q f(z)}{f(z)} - (A-1)}{(B+1)\frac{zD_q f(z)}{f(z)} - (A+1)} \right) > 0 \qquad (z \in \mathbb{U}).$$

This last equation now shows that $f \in \mathcal{S}^*_{(q,1)}[A,B]$, that is, that

$$\mathcal{S}^*_{(q,2)}[A,B] \subset \mathcal{S}^*_{(q,1)}[A,B],$$

which completes the proof of Theorem 1. □

As a special case of Theorem 1, if we put

$$A = 1 - 2\alpha \quad (0 \leqq \alpha < 1) \qquad \text{and} \qquad B = -1,$$

we get the following known result due to Wongsaijai and Sukantamala (see [17]).

Corollary 1. (See [17]) *For* $0 \leqq \alpha < 1$,

$$\mathcal{S}^*_{q,3}(\alpha) \subset \mathcal{S}^*_{q,2}(\alpha) \subset \mathcal{S}^*_{q,1}(\alpha).$$

Next, we present a remarkable simple characterization of functions in the class $\mathcal{S}^*_{(q,2)}[A,B]$ of q-starlike functions of Type 2 associated with the Janowski functions.

Theorem 2. *Let* $f \in \mathcal{A}$. *Then* $f \in \mathcal{S}^*_{(q,2)}[A,B]$ *if and only if*

$$\left| \frac{f(qz)}{f(z)} - \frac{\sigma}{(B-1)q + (B+3)} \right| \leqq \frac{(A+1)(1-q)}{(B-1)q + (B+3)},$$

where

$$\sigma = (A-1)q^2 + (B-A+2)q + B + 1.$$

Proof. The proof of Theorem 2 can be easily obtained from the fact that

$$\frac{zD_q f(z)}{f(z)} = \left(\frac{1}{1-q}\right)\left(1 - \frac{f(qz)}{f(z)}\right)$$

and Definition 8 of the class $\mathcal{S}^*_{(q,2)}[A,B]$ of q-starlike functions of Type 2 associated with the Janowski functions. □

Upon setting

$$A = 1 - 2\alpha \qquad \text{and} \qquad B = -1$$

in Theorem 2, we get the following known result.

Corollary 2. (See [23]) *Let $f \in \mathcal{A}$. Then, $f \in \mathcal{S}^*_q(\alpha)$ if and only if*

$$\left| \frac{f(qz)}{f(z)} - \alpha q \right| \leqq 1 - \alpha.$$

Our next result is directly obtained by using Theorem 1 and a known result given in [23].

Theorem 3. *The classes*

$$\mathcal{S}^*_{(q,1)}[A,B], \quad \mathcal{S}^*_{(q,2)}[A,B] \qquad \text{and} \qquad \mathcal{S}^*_{(q,3)}[A,B]$$

of the generalized q-starlike functions of Type 1, Type 2, and Type 3, respectively, satisfy the following properties:

$$\bigcap_{0<q<1} \mathcal{S}^*_{(q,1)}[A,B] = \bigcap_{0<q<1} \mathcal{S}^*_{(q,2)}[A,B] = \mathcal{S}^*[A,B]$$

and

$$\bigcap_{0<q<1} \mathcal{S}^*_{(q,1)}[A,B] = \bigcap_{0<q<1} \mathcal{S}^*_{(q,3)}[A,B] \subset \mathcal{S}^*[A,B].$$

Finally, by means of a coefficient inequality, we give a sufficient condition for the class $\mathcal{S}^*_{(q,3)}[A,B]$ of generalized q-starlike functions of Type 3, which also provides a corresponding sufficient condition for the classes $\mathcal{S}^*_{(q,1)}[A,B]$ and $\mathcal{S}^*_{(q,2)}[A,B]$ of Type 1 and Type 2, respectively.

Theorem 4. *A function $f \in \mathcal{A}$ and of the form (1) is in the class $\mathcal{S}^*_{(q,3)}[A,B]$ if it satisfies the following coefficient inequality:*

$$\sum_{n=2}^{\infty} \Big(2q\,[n-1]_q + [n]_q\,(B+1) + (A+1) \Big) |a_n| < |B - A|. \tag{12}$$

4. Analytic Functions with Negative Coefficients

In this section, we introduce new subclasses of q-starlike functions associated with the Janowski functions, which involve negative coefficients. Let $\mathcal{T}$ be a subset of $\mathcal{A}$ consisting of functions with a negative coefficient, that is,

$$f(z) = z - \sum_{n=2}^{\infty} |a_n|\, z^n \qquad (a_n \geqq 0). \tag{13}$$

We also let

$$\mathcal{TS}^*_{(q,k)}[A,B] := \mathcal{S}^*_{(q,k)}[A,B] \cap \mathcal{T} \qquad (k = 1,2,3). \tag{14}$$

Theorem 5. *If $-1 \leqq B < A \leqq 1$, then*

$$\mathcal{TS}^*_{(q,1)}[A,B] = \mathcal{TS}^*_{(q,2)}[A,B] = \mathcal{TS}^*_{(q,3)}[A,B].$$

Proof. In view of Theorem 1, it is sufficient here to show that

$$\mathcal{TS}^*_{(q,1)}\left[A,B\right] \subset \mathcal{TS}^*_{(q,3)}\left[A,B\right].$$

Indeed, if we assume that $f \in \mathcal{TS}^*_{(q,1)}\left[A,B\right]$, then we have

$$\Re\left(\frac{(B-1)\frac{zD_qf(z)}{f(z)}-(A-1)}{(B+1)\frac{zD_qf(z)}{f(z)}-(A+1)}\right) \geqq 0,$$

so that

$$\Re\left(\frac{(B-1)\frac{zD_qf(z)}{f(z)}-(A-1)}{(B+1)\frac{zD_qf(z)}{f(z)}-(A+1)}-1\right) \geqq -1.$$

After a simple calculation, we thus find that

$$\frac{2\left[f(z)-zD_qf(z)\right]}{(B+1)\,zD_qf(z)-(A+1)\,f(z)} \geqq -1,$$

that is, that

$$-\frac{2\sum\limits_{n=2}^{\infty}\left([n]_q-1\right)a_nz^n}{(A-B)+\sum\limits_{n=2}^{\infty}\left([n]_q(B+1)-(A+1)\right)a_nz^n} \geqq -1,$$

which can be written as follows:

$$\frac{2\sum\limits_{n=2}^{\infty}\left([n]_q-1\right)a_nz^n}{(A-B)+\sum\limits_{n=2}^{\infty}\left([n]_q(B+1)-(A+1)\right)a_nz^n} < 1. \tag{15}$$

The last expression in (15) implies that

$$\frac{2\sum\limits_{n=2}^{\infty}\left([n]_q-1\right)a_nz^n}{|B-A|+\sum\limits_{n=2}^{\infty}\left([n]_q(B+1)+(A+1)\right)a_nz^n} \leqq 1,$$

which satisfies (12). By Theorem 4, the proof of Theorem 5 is completed. □

In its special case, when

$$A=1-2\alpha \quad (0 \leqq \alpha < 1) \qquad \text{and} \qquad B=-1.$$

Theorem 5 reduces to the following known result.

Corollary 3. (See [17], Theorem 8) *If* $0 \leqq \alpha < 1$, *then*

$$\mathcal{TS}^*_{(q,1)}(\alpha) = \mathcal{TS}^*_{(q,2)}(\alpha) = \mathcal{TS}^*_{(q,3)}(\alpha).$$

The assertions of Theorem 5 imply that the Type 1, Type 2, and Type 3 generalized q-starlike functions associated with the Janowski functions are exactly the same. For convenience, therefore, we state the following distortion theorem by using the notation $\mathcal{TS}^*_{(q,k)}\left[A,B\right]$ in which it is tacitly assumed that $k=1,2,3$.

Theorem 6. *If* $f \in \mathcal{TS}^{*}_{(q,k)}[A,B] \quad (k=1,2,3)$, *then*

$$r-\left(\frac{|B-A|}{\Lambda(2,A,B,q)}\right)r^2 \leqq |f(z)| \leqq r+\left(\frac{|B-A|}{\Lambda(2,A,B,q)}\right)r^2$$

$$(|z|=r \quad (0<r<1)),$$

where

$$\Lambda(n,A,B,q)=2\left([n]_q-1\right)+[n]_q(B+1)+(A+1) \qquad (n\in\mathbb{N}\setminus\{1\}). \tag{16}$$

Proof. We note that the following inequality follows from Theorem 4:

$$\Lambda(2,A,B,q)\sum_{n=2}^{\infty}|a_n| \leqq \sum_{n=2}^{\infty}\Lambda(n,A,B,q)|a_n| < |B-A|,$$

which yields

$$|f(z)| \leqq r+\sum_{n=2}^{\infty}|a_n|r^n \leqq r+r^2\sum_{n=2}^{\infty}|a_n| \leqq r+\left(\frac{|B-A|}{\Lambda(2,A,B,q)}\right)r^2.$$

Similarly, we have

$$|f(z)| \geqq r-\sum_{n=2}^{\infty}|a_n|r^n \geqq r-r^2\sum_{n=2}^{\infty}|a_n| \geqq r-\left(\frac{|B-A|}{\Lambda(2,A,B,q)}\right)r^2.$$

We have thus completed the proof of Theorem 6. □

In its special case, when

$$A=1-2\alpha \quad (0\leqq\alpha<1) \qquad \text{and} \qquad B=-1,$$

if we let $q \longrightarrow 1-$, Theorem 6 reduces to the following known result.

Corollary 4. (See [25]) *If* $f \in \mathcal{TS}^{*}(\alpha)$, *then*

$$r-\left(\frac{1-\alpha}{2-\alpha}\right)r^2 \leqq |f(z)| \leqq r+\left(\frac{1-\alpha}{2-\alpha}\right)r^2 \qquad (|z|=r \quad (0<r<1)).$$

The following result (Theorem 7) can be proven by using arguments similar to those that were already presented in the proof of Theorem 6, so we choose to omit the details of our proof of Theorem 7.

Theorem 7. *If* $f \in \mathcal{TS}^{*}_{(q,k)}[A,B] \quad (k=1,2,3)$, *then*

$$1-\left(\frac{2|B-A|}{\Lambda(2,A,B,q)}\right)r \leqq |f'(z)| \leqq 1+\left(\frac{2|B-A|}{\Lambda(2,A,B,q)}\right)r$$

$$(|z|=r \quad (0<r<1)),$$

where $\Lambda(n,A,B,q)$ *is given by* (16).

In its special case, when

$$A=1-2\alpha \quad (0\leqq\alpha<1) \qquad \text{and} \qquad B=-1,$$

if we let $q \longrightarrow 1-$, Theorem 6 reduces to the following known result.

Corollary 5. (See [25]) *If* $f \in \mathcal{TS}^*(\alpha)$, *then*

$$1-\left(\frac{2(1-\alpha)}{2-\alpha}\right)r \leqq \left|f'(z)\right| \leqq 1+\left(\frac{2(1-\alpha)}{2-\alpha}\right)r \qquad (|z|=r \ \ (0<r<1)).$$

Remark 1. *By using Theorem 4, it is easy to see that the function:*

$$f_0(z) = z - \frac{|B-A|-\epsilon}{2q[n-1]_q + [n]_q(B+1)+(A+1)} z^n \in \mathcal{TS}_{(q,k)}[A,B] \tag{17}$$

where

$$0<\epsilon<\frac{n|B-A|-2q[n-1]_q+[n]_q(B+1)+(A+1)}{n}$$

and

$$2q[n-1]_q + \Big([n]_q(B+1)+(A+1)\Big) < n(|B-A|-\epsilon),$$

but

$$f_0'(z)=0$$

at

$$z_0 = \left[\frac{2q[n-1]_q + \Big([n]_q(B+1)+(A+1)\Big)}{n(|B-A|-\epsilon)}\right]^{\frac{1}{n-1}} \left(\cos\left(\frac{2k\pi}{n-1}\right)+i\sin\left(\frac{2k\pi}{n-1}\right)\right) \in \mathbb{U}.$$

That is, $f_0(z) \notin \mathcal{S}$ *and also* $f_0(z) \notin \mathcal{S}^*$. *Therefore, it is interesting to study the radius of univalency and starlikeness of class* $\mathcal{TS}_{(q,k)}[A,B]$.

Theorem 8. *Let* $f \in \mathcal{TS}_{(q,k)}[A,B]$ $(k=1,2,3)$. *Then,* f *is univalent and starlike in* $|z|<r_0$, *where*

$$r_0 = \min_{2\leq n\leq M_0}\left[\frac{2q[n-1]_q+[n]_q(B+1)+(A+1)}{n|B-A|}\right]^{\frac{1}{n-1}} \tag{18}$$

and M_0 *satisfies the following inequality:*

$$M_0 > \exp\left(1+\left|\ln\frac{(1-q)|B-A|}{(B+3)+(A-1)(1-q)}\right|\right).$$

Proof. To prove Theorem 8, it is sufficient to show that

$$\left|f'(z)-1\right|<1 \quad (|z|\leqq r_0).$$

Now, we have

$$\left|f'(z)-1\right| = \left|-\sum_{n=2}^{\infty} n a_n z^{n-1}\right| \leqq \sum_{n=2}^{\infty} n|a_n||z|^{n-1}.$$

Thus,

$$\left|f'(z)-1\right|<1$$

if

$$\sum_{n=2}^{\infty} n|a_n||z|^{n-1}<1. \tag{19}$$

In light of Theorem 4, the inequality in (19) will be true if

$$n|z|^{n-1} \leqq \frac{2q[n-1]_q+\Big([n]_q(B+1)+(A+1)\Big)}{|B-A|}. \tag{20}$$

Solving the inequality in (20) for z, we have

$$|z| \leqq \left[\frac{2q\,[n-1]_q + \left([n]_q\,(B+1) + (A+1) \right)}{n\,|B-A|} \right]^{\frac{1}{n-1}}. \tag{21}$$

Next, we need to find $M_0 \in \mathbb{N}$ satisfying (18) . Let $f : [2, \infty) \longrightarrow \mathbb{R}^+$ be the function defined by

$$f(x) = \left[\frac{2q\,[x-1]_q + [x]_q\,(B+1) + (A+1)}{x\,|B-A|} \right]^{\frac{1}{x-1}} \tag{22}$$

Differentiating on both sides of (22) logarithmically, we have

$$\begin{aligned} f'(x) = \frac{f(x)}{(x-1)^2} & \left[\ln x - \frac{(x-1)\,(B+3)\,(q^x \ln q)}{(B+3)\,(1-q^x) + (A-1)\,(1-q)} \right. \\ & \left. + \ln \frac{(1-q)\,|B-A|}{(B+3)\,(1-q^x) + (A-1)\,(1-q)} - \frac{x-1}{x} \right]. \end{aligned} \tag{23}$$

It is easy to see that the second term of (23) is positive. Since

$$\sup_{x \geqq 2} \left| \ln \frac{(1-q)\,|B-A|}{(B+3)\,(1-q^x) + (A-1)\,(1-q)} \right| = \left| \frac{(1-q)\,|B-A|}{(B+3) + (A-1)\,(1-q)} \right|$$

and

$$\sup_{x \geqq 2} \frac{x-1}{x} = 1$$

then the third and the last term in (23) can be dominated by $\ln x$ when x is sufficiently large. This implies that f is an increasing function on $[M_0, \infty]$, where

$$M_0 > \exp\left(1 + \left| \ln \frac{(1-q)\,|B-A|}{(B+3) + (A-1)\,(1-q)} \right| \right).$$

Therefore, the radius of univalence can be defined by

$$\begin{aligned} r_0 &= \inf_{n \geqq 2} \left[\frac{2q\,[n-1]_q + [n]_q\,(B+1) + (A+1)}{n\,|B-A|} \right]^{\frac{1}{n-1}} \\ &= \min_{2 \leqq n \leqq M_0} \left[\frac{2q\,[n-1]_q + [n]_q\,(B+1) + (A+1)}{n\,|B-A|} \right]^{\frac{1}{n-1}} \end{aligned} \tag{24}$$

In view of (24) , the proof of our Theorem is now completed. □

If, in Theorem 8, we let

$$B = -1 \quad \text{and} \quad A = (1-2\alpha)$$

we are led to the following known result:

Corollary 6. [17] *Let $f \in \mathcal{TS}_q$. Then, f is univalent and starlike in $|z| < r_0$, where*

$$r_0 = \min_{2 \leqq n \leqq M_0} \left[\frac{[n]_q - \alpha}{n\,(1-\alpha)} \right]^{\frac{1}{n-1}} \tag{25}$$

and M_0 satisfies the following inequality:

$$M_0 > \exp\left(1+\left|\ln\frac{(1-q)(1-\alpha)}{q+(1-q)(1-\alpha)}\right|\right).$$

Now, below, we give an example that validates Theorem 8.

Example 1. *Consider the class $\mathcal{TS}_{(0.75,k)}[0,\lambda]$ with $\lambda = 0.99$. By Theorem 8, we obtain the radius of univalency of class $\mathcal{TS}_{(0.75,k)}[0,\lambda]$, given by*

$$\begin{aligned} r_0 &= \min_{2\leqq n\leqq \exp(1+|\ln 0.08256880734|)}\left[\frac{3[n-1]_q-0.01}{(0.99)\,n}\right]^{\frac{1}{n-1}} \\ &= \min_{2\leqq n\leqq 33}\left[\frac{3[n-1]_q-0.01}{(0.99)\,n}\right]^{\frac{1}{n-1}} = 0.9691405946. \end{aligned}$$

Now, we consider the sharpness example function $f_0(z)$ defined in (17) with $n=2$ and $\epsilon = 0.001$, that is,

$$f_0(z) = z - \frac{0.989}{5.24}z^2.$$

Obviously, $f_0(z)$ is locally univalent on $\mathbb{U}_{0.9691405946}$ because $f'(z_0) = 0$ at $z_0 = 2.649140540$ outside the open disk $\mathbb{U}_{0.9691405946}$. By applying Theorem 8, function $f_0(z)$ is univalent on $\mathbb{U}_{0.9691405946}$.

The next Theorem (Theorem 9) can be derived by working in a similar way as in Theorem 8; here, we omit the proof.

Theorem 9. *Let $f \in \mathcal{TS}_{(q,k)}[A,B]$ $(k=1,2,3)$. Then f is starlike of order α in $|z| < r_1$, where*

$$r_1 = \min_{2\leqq n\leqq M_1}\left[\frac{\left(2q[n-1]_q+[n]_q(B+1)+(A+1)\right)(1-\alpha)}{(n-\alpha)|B-A|}\right]^{\frac{1}{n-1}} \tag{26}$$

and M_1 satisfies the following inequality:

$$M_1 > \exp\left(1+\left|\ln\frac{(1-q)|B-A|}{((B+3)+(A-1)(1-q))(1-\alpha)}\right|\right).$$

5. Conclusions

In our present investigation, we first defined certain new subclasses of q-starlike functions, which are associated with the Janowski function. We then discussed many properties and characteristics of each of these subclasses of q-starlike functions including, for example, sufficient conditions, inclusion results, distortion theorems, and radius problems. For the motivation and validity of our results, we have also pointed out relevant connections with those that were given in earlier works.

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