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A Closed Formula for the Horadam Polynomials in Terms of a Tridiagonal Determinant

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Received: 19 May 2019; Accepted: 11 June 2019; Published: 12 June 2019

Abstract: In this paper, the authors present a closed formula for the Horadam polynomials in terms of a tridiagonal determinant and, as applications of the newly-established closed formula for the Horadam polynomials, derive closed formulas for the generalized Fibonacci polynomials, the Lucas polynomials, the Pell–Lucas polynomials, and the Chebyshev polynomials of the first kind in terms of tridiagonal determinants.

Keywords: closed formula; Horadam polynomial, tridiagonal determinant; generalized Fibonacci polynomial; Lucas polynomial; Pell–Lucas polynomial; Chebyshev polynomial of the first kind; Chebyshev polynomial of the second kind

MSC: 11B39; 11B83; 11C20; 11Y55; 26A06; 26A09; 26C05

1. Introduction

For $a, b, p, q \in \mathbb{Z}$, Horadam introduced in [1,2] the sequence $W_n = W_n(a, b; p, q)$ by the recurrence relation

$$W_n = pW_{n-1} + qW_{n-2}, \quad n \geq 2$$

with the initial values $W_0 = a$ and $W_1 = b$. This sequence is a generalization of several famous and known sequences such as the Fibonacci, Lucas, Pell, and Pell–Lucas sequences. These sequences in combinatorial number theory have been studied by many mathematicians for a long time. These sequences are also of great importance in many subjects such as algebra, geometry, combinatorics, approximation theory, statistics, and number theory. For more information, please refer to [1,3–5] and closely related references therein.

In [3], the Horadam polynomials $h_n(x) = h_n(x; a, b; p, q)$ were given by the recurrence relation

$$h_n(x) = pxh_{n-1}(x) + qh_{n-2}(x), \quad n \geq 3$$

with the initial values $h_1(x) = a$ and $h_2(x) = bx$. Some special cases of the Horadam polynomials $h_n(x)$ are as follows:

1. for $a = b = p = q = 1$, the Horadam polynomials $h_n(x) = h_n(x; 1, 1; 1, 1)$ are the Fibonacci polynomials $F_n(x)$;
2. for $a = 2$ and $b = p = q = 1$, the Horadam polynomials $h_n(x) = h_n(x; 2, 1; 1, 1)$ become the Lucas polynomials $L_{n-1}(x)$;

3. for $a = q = 1$ and $b = p = 2$, the Horadam polynomials $h_n(x) = h_n(x; 1, 2; 2, 1)$ reduce to the Pell polynomials $P_n(x)$;
4. for $a = b = p = 2$ and $q = 1$, the Horadam polynomials $h_n(x) = h_n(x; 2, 2; 2, 1)$ are the Pell–Lucas polynomials $Q_{n-1}(x)$;
5. for $a = b = 1, p = 2$, and $q = -1$, the Horadam polynomials $h_n(x) = h_n(x; 1, 1; 2, -1)$ are the Chebyshev polynomials of the first kind $T_{n-1}(x)$;
6. for $a = 1, b = p = 2$, and $q = -1$, the Horadam polynomials $h_n(x) = h_n(x; 1, 2; 2, -1)$ become the Chebyshev polynomials of the second kind $U_{n-1}(x)$.

The generating function of the Horadam polynomials is

$$\frac{a + xt(b - ap)}{1 - pxt - qt^2} = \sum_{n=0}^{\infty} h_n(x)t^n. \tag{1}$$

Some properties of the Horadam polynomials can be found in the papers [2,3].

It is well-known that a tridiagonal determinant a determinant whose nonzero elements locate only on the diagonal and slots horizontally or vertically adjacent the diagonal. In other words, a square determinant $H = |h_{ij}|_{n \times n}$ is called a tridiagonal determinant if $h_{ij} = 0$ for all pairs (i, j) such that $|i - j| > 1$. A determinant $H = |h_{ij}|_{n \times n}$ is called a lower (or an upper, respectively) Hessenberg determinant if $h_{ij} = 0$ for all pairs (i, j) such that $i + 1 < j$ (or $j + 1 < i$, respectively). For more details, see the papers [6–10]. There are many papers connecting the tridiagonal and Hessenberg determinants with special numbers and polynomials in combinatorial number theory. For more information, please see the papers [11–35] and closely related references therein.

In the paper, we will present a closed formula for the Horadam polynomials $h_n(x)$ in terms of a tridiagonal determinant and, as applications of this newly-established closed formula for the Horadam polynomials $h_n(x)$, derive closed formulas for the generalized Fibonacci polynomials $F_n(s, t)$, the Lucas polynomials $L_n(x)$, the Pell–Lucas polynomials $Q_n(x)$, and the Chebyshev polynomials of the first kind $T_n(x)$ in terms of tridiagonal determinants.

2. A Lemma

In order to prove our main results, we need the following lemma.

Lemma 1 ([36], p. 40, Exercise 5). *Let $u(t)$ and $v(t) \neq 0$ be differentiable functions, let $U_{(n+1) \times 1}(t)$ be an $(n + 1) \times 1$ matrix whose elements $u_{k,1}(t) = u^{(k-1)}(t)$ for $1 \leq k \leq n + 1$, let $V_{(n+1) \times n}(t)$ be an $(n + 1) \times n$ matrix whose elements*

$$v_{i,j}(t) = \begin{cases} \binom{i-1}{j-1} v^{(i-j)}(t), & i - j \geq 0; \\ 0, & i - j < 0 \end{cases}$$

for $1 \leq i \leq n + 1$ and $1 \leq j \leq n$, and let $|W_{(n+1) \times (n+1)}(t)|$ denote the lower Hessenberg determinant of the $(n + 1) \times (n + 1)$ lower Hessenberg matrix

$$W_{(n+1) \times (n+1)}(t) = \begin{bmatrix} U_{(n+1) \times 1}(t) & V_{(n+1) \times n}(t) \end{bmatrix}.$$

Then the n th derivative of the ratio $\frac{u(t)}{v(t)}$ can be computed by

$$\frac{d^n}{dx^n} \left[\frac{u(t)}{v(t)} \right] = (-1)^n \frac{|W_{(n+1) \times (n+1)}(t)|}{v^{n+1}(t)}. \tag{2}$$

This lemma has been extensively applied in the papers [13,15–29,31–35] and closely related references therein.

3. Main Results and Their Proof

Our main results can be stated as the following theorem.

Theorem 1. *The Horadam polynomials $h_n(x)$ for $n \geq 0$ can be expressed as a tridiagonal determinant*

$$h_n(x) = \frac{(-1)^n}{n!} \begin{vmatrix} a & 1 & 0 & 0 & \dots & 0 & 0 \\ x(b-ap) & -px \binom{1}{0} & 1 & 0 & \dots & 0 & 0 \\ 0 & -2q \binom{2}{0} & -px \binom{2}{1} & 1 & \dots & 0 & 0 \\ 0 & 0 & -2q \binom{3}{1} & -px \binom{3}{2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & -px \binom{n-1}{n-2} & 1 \\ 0 & 0 & 0 & 0 & \dots & -2q \binom{n}{n-2} & -px \binom{n}{n-1} \end{vmatrix}. \tag{3}$$

Consequently, the Horadam numbers $h_n = h_n(1)$ for $n \in \mathbb{N}$ can be expressed as

$$h_n = \frac{(-1)^n}{n!} \begin{vmatrix} a & 1 & 0 & 0 & \dots & 0 & 0 \\ (b-ap) & -p \binom{1}{0} & 1 & 0 & \dots & 0 & 0 \\ 0 & -2q \binom{2}{0} & -p \binom{2}{1} & 1 & \dots & 0 & 0 \\ 0 & 0 & -2q \binom{3}{1} & -p \binom{3}{2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & -p \binom{n-1}{n-2} & 1 \\ 0 & 0 & 0 & 0 & \dots & -2q \binom{n}{n-2} & -p \binom{n}{n-1} \end{vmatrix}. \tag{4}$$

Proof. Applying $u(t) = a + xt(b - ap)$ and $v(t) = 1 - pxt - qt^2$ to the formula (2) in Lemma 1 leads to

$$\frac{d^n}{dt^n} \left[\frac{a + xt(b - ap)}{1 - pxt - qt^2} \right] = \frac{(-1)^n}{(1 - pxt - qt^2)^{n+1}}$$

$$\times \begin{vmatrix} a + xt(b - ap) & 1 - pxt - qt^2 & 0 & 0 & \dots \\ x(b - ap) & -(px + 2qt) \binom{1}{0} & 1 - pxt - qt^2 & 0 & \dots \\ 0 & -2q \binom{2}{0} & -(px + 2qt) \binom{2}{1} & 1 - pxt - qt^2 & \dots \\ 0 & 0 & -2q \binom{3}{1} & -(px + 2qt) \binom{3}{2} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & \\ \dots & 0 & 0 & 0 & \\ \dots & 0 & 0 & 0 & \\ \dots & 0 & 0 & 0 & \\ \ddots & \vdots & \vdots & & \\ \dots & -(px + 2qt) \binom{n-2}{n-3} & 1 - pxt - qt^2 & \vdots & \\ \dots & -2q & -(px + 2qt) \binom{n-1}{n-2} & 1 - pxt - qt^2 & \\ \dots & 0 & -2q \binom{n}{n-2} & -(px + 2qt) \binom{n}{n-1} & \end{vmatrix}$$

$$\rightarrow (-1)^n \begin{vmatrix} a & 1 & 0 & 0 & \dots & 0 & 0 \\ x(b-ap) & -px \binom{1}{0} & 1 & 0 & \dots & 0 & 0 \\ 0 & -2q \binom{2}{0} & -px \binom{2}{1} & 1 & \dots & 0 & 0 \\ 0 & 0 & -2q \binom{3}{1} & -px \binom{3}{2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & -px \binom{n-1}{n-2} & 1 \\ 0 & 0 & 0 & 0 & \dots & -2q \binom{n}{n-2} & -px \binom{n}{n-1} \end{vmatrix}$$

as $t \rightarrow 0$ for $n \in \mathbb{N}$. By the Equation (1), we have

$$h_n(x) = \frac{1}{n!} \lim_{t \rightarrow 0} \frac{d^n}{dt^n} \left[\frac{a + xt(b-ap)}{1 - pxt - qt^2} \right]$$

$$= \frac{(-1)^n}{n!} \begin{vmatrix} a & 1 & 0 & 0 & \dots & 0 & 0 \\ x(b-ap) & -px \binom{1}{0} & 1 & 0 & \dots & 0 & 0 \\ 0 & -2q \binom{2}{0} & -px \binom{2}{1} & 1 & \dots & 0 & 0 \\ 0 & 0 & -2q \binom{3}{1} & -px \binom{3}{2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & -px \binom{n-1}{n-2} & 1 \\ 0 & 0 & 0 & 0 & \dots & -2q \binom{n}{n-2} & -px \binom{n}{n-1} \end{vmatrix}.$$

Consequently, the determinantal expression (3) is obtained.

It is easy to see that the determinantal expression 4 can be derived from setting $x \rightarrow 1$ in the Equation (3). The proof of Theorem 1 is thus complete. \square

4. Corollaries

In this section, we derive closed formulas for the generalized Fibonacci polynomials $F_n(s, t)$, the Lucas polynomials $L_n(x)$, the Pell–Lucas polynomials $Q_n(x)$, and the Chebyshev polynomials of the first kind $T_n(x)$ in terms of tridiagonal determinants.

Corollary 1 ([21], Theorem 1.1). *The generalized Fibonacci polynomials $F_n(s, t)$ for $n \geq 0$ can be expressed as*

$$F_n(s, t) = \frac{1}{n!} \begin{vmatrix} 0 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & \binom{1}{0}s & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 2 \binom{2}{0}t & \binom{2}{1}s & -1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 2 \binom{3}{1}t & \binom{3}{2}s & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \binom{n-2}{n-3}s & -1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 2 \binom{n-1}{n-3}t & \binom{n-1}{n-2}s & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 2 \binom{n}{n-2}t & \binom{n}{n-1}s \end{vmatrix}. \tag{5}$$

Consequently, the Fibonacci polynomials $F_n(s)$ and the Fibonacci numbers F_n for $n \in \mathbb{N}$ can be expressed respectively as

$$F_n(s) = \frac{1}{n!} \begin{vmatrix} 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & \binom{1}{0}s & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2\binom{2}{0} & \binom{2}{1}s & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 2\binom{3}{1} & \binom{3}{2}s & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \binom{n-2}{n-3}s & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 2\binom{n-1}{n-3} & \binom{n-1}{n-2}s & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 2\binom{n}{n-2} & \binom{n}{n-1}s \end{vmatrix}$$

and

$$F_n = \frac{1}{n!} \begin{vmatrix} 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & \binom{1}{0} & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2\binom{2}{0} & \binom{2}{1} & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 2\binom{3}{1} & \binom{3}{2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \binom{n-2}{n-3} & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 2\binom{n-1}{n-3} & \binom{n-1}{n-2} & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 2\binom{n}{n-2} & \binom{n}{n-1} \end{vmatrix}.$$

Proof. This follows from substituting $x, a = 1, q = t$, and letting $p, b \rightarrow s$ in the Equation (3). \square

Corollary 2. The Lucas polynomials $L_n(x)$ for $n \geq 0$ can be expressed as a tridiagonal determinant

$$L_n(x) = \frac{(-1)^n}{n!} \begin{vmatrix} 2 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -x & -x\binom{1}{0} & 1 & 0 & \cdots & 0 & 0 \\ 0 & -2\binom{2}{0} & -x\binom{2}{1} & 1 & \cdots & 0 & 0 \\ 0 & 0 & -2\binom{3}{1} & -x\binom{3}{2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -x\binom{n-1}{n-2} & 1 \\ 0 & 0 & 0 & 0 & \cdots & -2\binom{n}{n-2} & -x\binom{n}{n-1} \end{vmatrix}. \tag{6}$$

Proof. This follows from taking $a = 2$ and $b = p = q = 1$ in the Equation (3). \square

Corollary 3. The Pell–Lucas polynomials $Q_n(x)$ for $n \geq 0$ can be represented in terms of a tridiagonal determinant as

$$Q_n(x) = \frac{(-1)^n}{n!} \begin{vmatrix} 2 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -2x & -2x\binom{1}{0} & 1 & 0 & \cdots & 0 & 0 \\ 0 & -2\binom{2}{0} & -2x\binom{2}{1} & 1 & \cdots & 0 & 0 \\ 0 & 0 & -2\binom{3}{1} & -2x\binom{3}{2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -2x\binom{n-1}{n-2} & 1 \\ 0 & 0 & 0 & 0 & \cdots & -2\binom{n}{n-2} & -2x\binom{n}{n-1} \end{vmatrix}. \tag{7}$$

Proof. This follows from setting $a = b = p = 2$ and $q = 1$ in the Equation (3). \square

Corollary 4. The Chebyshev polynomials of the first kind $T_n(x)$ for $n \geq 0$ can be represented in terms of a tridiagonal determinant as

$$T_n(x) = \frac{(-1)^n}{n!} \begin{vmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -x & -2x \binom{1}{0} & 1 & 0 & \cdots & 0 & 0 \\ 0 & 2 \binom{2}{0} & -2x \binom{2}{1} & 1 & \cdots & 0 & 0 \\ 0 & 0 & 2 \binom{3}{1} & -2x \binom{3}{2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -2x \binom{n-1}{n-2} & 1 \\ 0 & 0 & 0 & 0 & \cdots & 2 \binom{n}{n-2} & -2x \binom{n}{n-1} \end{vmatrix}. \tag{8}$$

Proof. This follows from taking $a = b = 1$, $p = 2$, and $q = -1$ in the Equation (3). \square

5. Conclusions

The formula (2) in Lemma 1 is a very simple, direct, and effectual tool to represent a higher order derivative of a function in terms of a determinant by regarding the function as a ratio of two functions. Under some special conditions on the two functions constituting the ratio, the determinant can be a special determinant such as the tridiagonal determinant, the Hessenberg determinant, and the like.

In analytic combinatorics and analytic number theory, to express a sequence of numbers or a sequence of polynomials in terms of a special and simple determinant is an interesting and important direction and topic. However, generally, to do this is not easy, and is even very difficult. However, the formula (2) in Lemma 1 can make this work easier, simpler, and straightforward.

In this paper, by making use of the formula (2) in Lemma 1 again and considering the generating function (1) of the Horadam polynomials $h_n(x)$ as a ratio of two functions $a + xt(b - ap)$ and $1 - pxt - qt^2$, we present a closed formula (3) for the Horadam polynomials $h_n(x)$ in terms of a tridiagonal determinant and, consequently, derive closed formulas (5)–(8) for the generalized Fibonacci polynomials $F_n(s, t)$, the Lucas polynomials $L_n(x)$, the Pell–Lucas polynomials $Q_n(x)$, and the Chebyshev polynomials of the first kind $T_n(x)$ in terms of tridiagonal determinants.

Author Contributions: The authors contributed equally to this work. All authors read and approved the final manuscript.

Acknowledgments: The authors thank anonymous referees for their careful corrections to and valuable comments on the original version of this paper. The third author is supported by Grant No. MOST 107-2115-M-017-004-MY2 of the Ministry of Science and Technology of the Republic of China.

Conflicts of Interest: The authors declare no conflict of interest.

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