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# The Principle of Differential Subordination and Its Application to Analytic and $p$-Valent Functions Defined by a Generalized Fractional Differintegral Operator

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**Abstract:** A useful family of fractional derivative and integral operators plays a crucial role on the study of mathematics and applied science. In this paper, we introduce an operator defined on the family of analytic functions in the open unit disk by using the generalized fractional derivative and integral operator with convolution. For this operator, we study the subordination-preserving properties and their dual problems. Differential sandwich-type results for this operator are also investigated.

**Keywords:** analytic function; Hadamard product; differential subordination; differential superordination; generalized fractional differintegral operator

**MSC:** 30C45; 30C50

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## 1. Introduction

Let $\mathcal{H}(\mathbb{D})$ be the family of analytic functions in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}[c,n]$ be the subfamily of $\mathcal{H}(\mathbb{D})$ consisting of functions of the form:

$$f(z) = c + b_n z^n + b_{n+1} z^{n+1} + \cdots \quad (c \in \mathbb{C};\ n \in \mathbb{N} = \{1, 2, \cdots\}).$$

Let $\mathcal{A}(p)$ denote the family of analytic functions in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} b_{p+n} z^{p+n} \ (p \in \mathbb{N};\ f^{(p+1)}(0) \neq 0). \tag{1}$$

For $f, F \in \mathcal{H}(\mathbb{D})$, the function $f(z)$ is said to be subordinate to $F(z)$ or $F(z)$ is superordinate to $f(z)$, written $f \prec F$ or $f(z) \prec F(z)$, if there exists a Schwarz function $\omega(z)$ for $z \in \mathbb{D}$ such that $f(z) = F(\omega(z))$. If $F(z)$ is univalent, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(\mathbb{D}) \subset F(\mathbb{D})$ (see [1,2]).

Let $\phi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ and $h(z)$ be univalent in $\mathbb{D}$. If $p(z)$ is analytic in $\mathbb{D}$ and satisfies

$$\phi\left(p(z), zp'(z); z\right) \prec h(z), \tag{2}$$

then $p(z)$ is solution Relation (2). The univalent function $q(z)$ is called a dominant of the solutions of Relation (2) if $p(z) \prec q(z)$ for all $p(z)$ satisfying Relation (2). A univalent dominant $\tilde{q}$ that satisfies

$\tilde{q} \prec q$ for all dominants of Relation (2) is called the best dominant. If $p(z)$ and $\phi(p(z), zp'(z); z)$ are univalent in $\mathbb{D}$ and if $p(z)$ satisfies

$$h(z) \prec \phi\left(p(z), zp'(z); z\right), \tag{3}$$

then $p(z)$ is a solution of Relation (3). An analytic function $q(z)$ is called a subordinant of the solutions of Relation (3) if $q(z) \prec p(z)$ for all $p(z)$ satisfying Relation (3). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants of Relation (3) is called the best subordinant (see [1,2]).

We now introduce the operator $S_{0,z}^{\lambda,\mu,\eta,p}$ due to Goyal and Prajapat [3] (see also [4]) as follows:

$$S_{0,z}^{\lambda,\mu,\eta,p} f(z) = \begin{cases} \dfrac{\Gamma(p+1-\mu)\Gamma(p+1-\lambda+\eta)}{\Gamma(p+1)\Gamma(p+1-\mu+\eta)} z^{\mu} J_{0,z}^{\lambda,\mu,\eta} f(z) \ (0 \leq \lambda < \eta + p + 1;\ z \in \mathbb{D}), \\[2ex] \dfrac{\Gamma(p+1-\mu)\Gamma(p+1-\lambda+\eta)}{\Gamma(p+1)\Gamma(p+1-\mu+\eta)} z^{\mu} I_{0,z}^{-\lambda,\mu,\eta} f(z) \ (-\infty < \lambda < 0;\ z \in \mathbb{D}), \end{cases} \tag{4}$$

where $J_{0,z}^{\lambda,\mu,\eta}$ and $I_{0,z}^{-\lambda,\mu,\eta}$ are the generalized fractional derivative and integral operators, respectively, due to Srivastava et al. [5] (see also [6,7]). For $f \in \mathcal{A}(p)$ of form Equation (1), we have

$$\begin{aligned} S_{0,z}^{\lambda,\mu,\eta,p} f(z) &= z^p {}_3F_2(1, 1+p, 1+p+\eta-\mu; 1+p-\mu, 1+p+\eta-\lambda; z) * f(z) \\ &= z^p + \sum_{n=1}^{\infty} \frac{(p+1)_n (p+1-\mu+\eta)_n}{(p+1-\mu)_n (p+1-\lambda+\eta)_n} b_{p+n} z^{p+n} \\ &\quad (p \in \mathbb{N};\ \mu, \eta \in \mathbb{R};\ \mu < p+1;\ -\infty < \lambda < \eta + p + 1), \end{aligned} \tag{5}$$

where ${}_qF_s$ $(q \leq s+1;\ q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ is the well-known generalized hypergeometric function (for details, see [8,9]), the symbol $*$ stands for convolution of two analytic functions [1] and $(\nu)_n$ is the Pochhammer symbol [8,10].

Setting

$$\begin{aligned} G_{p,\eta,\mu}^{\lambda}(z) &= z^p + \sum_{n=1}^{\infty} \frac{(p+1)_n (p+1-\mu+\eta)_n}{(p+1-\mu)_n (p+1-\lambda+\eta)_n} z^{p+n} \\ &\quad (p \in \mathbb{N};\ \mu, \eta \in \mathbb{R};\ \mu < \min\{p+1, p+1+\eta\};\ -\infty < \lambda < \eta + p + 1) \end{aligned} \tag{6}$$

and

$$G_{p,\eta,\mu}^{\lambda}(z) * \left[G_{p,\eta,\mu}^{\lambda,\delta}(z)\right] = \frac{z^p}{(1-z)^{\delta+p}} \ (\delta > -p;\ z \in \mathbb{D}),$$

Tang et al. [11] (see also [12]) defined the operator $H_{p,\eta,\mu}^{\lambda,\delta} : \mathcal{A}(p) \to \mathcal{A}(p)$ by

$$H_{p,\eta,\mu}^{\lambda,\delta} f(z) = \left[G_{p,\eta,\mu}^{\lambda,\delta}(z)\right] * f(z).$$

Then, for $f \in \mathcal{A}(p)$, we have

$$H_{p,\eta,\mu}^{\lambda,\delta} f(z) = z^p + \sum_{n=1}^{\infty} \frac{(\delta+p)_n (p+1-\mu)_n (p+1-\lambda+\eta)_n}{(1)_n (p+1)_n (p+1-\mu+\eta)_n} b_{p+n} z^{p+n}. \tag{7}$$

It is easy to verify that

$$z\left(H_{p,\eta,\mu}^{\lambda,\delta} f(z)\right)' = (\delta+p) H_{p,\eta,\mu}^{\lambda,\delta+1} f(z) - \delta H_{p,\eta,\mu}^{\lambda,\delta} f(z), \tag{8}$$

and

$$z\left(H_{p,\eta,\mu}^{\lambda+1,\delta} f(z)\right)' = (p+\eta-\lambda) H_{p,\eta,\mu}^{\lambda,\delta} f(z) - (\eta-\lambda) H_{p,\eta,\mu}^{\lambda+1,\delta} f(z). \tag{9}$$

Making use of the hypergeometric function in the kernel, Saigo [13] proposed generalizations of fractional calculus of both Riemann–Liouville and Weyl types. The general theory of fractional calculus thus developed was applied to the study for several multiplication properties of fractional integrals [14]. In particular, Owa et al. [15] and Srivastava et al. [5] investigated some distortion theorems involving fractional integrals, and sufficient conditions for fractional integrals of analytic functions in the open unit disk to be starlike or convex. Moreover, the theory of fractional calculus is widely applied to not only pure mathematics but also applied science. For some interesting developments in applied science such as bioengineering and applied physics, the readers may be referred to the works of (for examples) Hassan et al. [16], Magin [17], Martínez-García et al. [18] and Othman and Marin [19].

By using the principle of subordination, Miller et al. [20] investigated subordinations-preserving properties for certain integral operators. In addition, Miller and Mocanu [2] studied some important properties on superordinations as the dual problem of subordinations. Furthermore, the study of the subordinaton-preserving properties and their dual problems for various operators is a significant role in pure and applied mathematics. The aim of the present paper, motivated by the works mentioned above, is to systematically investigate the subordination- and superordination-preserving results of the generalized fractional differintegral operator defined Equation (7) with certain differential sandwich-type theorems as consequences of the results presented here. Our results give interesting new properties, and together with other papers that appeared in the last years could emphasize the perspective of the importance of differential subordinations and generalized fractional differintegral operators. We also note that, in recent years, several authors obtained many interesting results involving various linear and nonlinear operators associated with differential subordinations and their dual problrms (for details, see [21–28]).

For the proofs of our main results, we shall need some definitions and lemmas stated below.

**Definition 1** ([1])**.** *We denote by $\mathcal{Q}$ the set of all functions $q(z)$ that are analytic and injective on $\overline{\mathbb{D}}\backslash E(q)$, where*

$$E(q) = \left\{ \zeta \in \partial\mathbb{D} : \lim_{z\to\zeta} q(z) = \infty \right\},$$

*and are $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{D}\backslash E(q)$.*

**Definition 2** ([2])**.** *A function $\mathcal{I}(z,t)$ $(z \in \mathbb{D},\ t \geq 0)$ is a subordination chain if $\mathcal{I}(.,t)$ is analytic and univalent in $\mathbb{D}$ for all $t \geq 0$, $\mathcal{I}(z,.)$ is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{D}$ and $\mathcal{I}(z,s) \prec \mathcal{I}(z,t)$ for all $0 \leq s \leq t$.*

**Lemma 1** ([29])**.** *Let $H : \mathbb{C}^2 \to \mathbb{C}$ satisfy*

$$\Re\{H(i\sigma;\tau)\} \leq 0$$

*for all real $\sigma,\tau$ with $\tau \leq -n\left(1+\sigma^2\right)/2$ and $n \in \mathbb{N}$. If $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \cdots$ is analytic in $\mathbb{D}$ and*

$$\Re\left\{H\left(p(z); zp'(z)\right)\right\} > 0 \ \ (z \in \mathbb{D}),$$

*then $\Re\{p(z)\} > 0$ for $z \in \mathbb{D}$.*

**Lemma 2** ([30])**.** *Let $\kappa,\gamma \in \mathbb{C}$ with $\kappa \neq 0$ and let $h \in H(\mathbb{D})$ with $h(0) = c$. If $\Re\{\kappa h(z) + \gamma\} > 0\,(z \in \mathbb{D})$, then the solution of the differential equation:*

$$q(z) + \frac{zq'(z)}{\kappa q(z) + \gamma} = h(z) \ \ (z \in \mathbb{D};\ q(0) = c)$$

*is analytic in $\mathbb{D}$ and satisfies $\Re\{\kappa q(z) + \gamma\} > 0$ for $z \in \mathbb{D}$.*

**Lemma 3** ([1]). *Suppose that $p \in \mathcal{Q}$ with $q(0) = a$ and $q(z) = a + q_n z^n + q_{n+1} z^{n+1} + \cdots$ is analytic in $\mathbb{D}$ with $q(z) \neq a$ and $n \geq 1$. If $q(z)$ is not subordinate to $p(z)$, then there exists two points $z_0 = r_0 e^{i\theta} \in \mathbb{D}$ and $\xi_0 \in \partial\mathbb{D} \backslash E(q)$ such that*

$$q(z_0) = p(\xi_0) \text{ and } z_0 q'(z_0) = m\xi_0 p'(\xi_0) \ (m \geq n).$$

**Lemma 4** ([2]). *Let $q \in \mathcal{H}[c, 1]$ and $\varphi : \mathbb{C}^2 \to \mathbb{C}$. In addition, let $\varphi(q(z), zq'(z)) = h(z)$. If $\mathcal{I}(z, t) = \varphi(q(z), tzq'(z))$ is a subordination chain and $q \in \mathcal{H}[c, 1] \cap \mathcal{Q}$, then*

$$h(z) \prec \varphi(p(z), zp'(z)),$$

*implies that $q(z) \prec p(z)$. Moreover, if $\varphi(q(z), zq'(z)) = h(z)$ has a univalent solution $q \in \mathcal{Q}$, then $q$ is the best subordinant.*

**Lemma 5** ([31]). *The function $\mathcal{I}(z, t) : \mathbb{D} \times [0, \infty) \longrightarrow \mathbb{C}$ of the form*

$$\mathcal{I}(z, t) = a_1(t) z + \cdots (a_1(t) \neq 0; \ t \geq 0)$$

*and $\lim\limits_{t \to \infty} |a_1(t)| = \infty$ is a subordination chain if and only if*

$$\Re \left\{ \frac{z \frac{\partial \mathcal{I}(z,t)}{\partial z}}{\frac{\partial \mathcal{I}(z,t)}{\partial t}} \right\} > 0 \ (z \in \mathbb{D}; \ t \geq 0)$$

*and*

$$|\mathcal{I}(z, t)| \leq K_0 |a_1(t)| \ (t \geq 0)$$

*for constants $K_0 > 0$ and $r_0$ ($|z| < r_0 < 1$).*

## 2. Main Results

Throughout this paper, we assume that $p \in \mathbb{N}$, $\alpha, \beta > 0$, $\delta > -p$, $\mu, \eta \in \mathbb{R}$, $\mu < \min\{p+1, p+1+\eta\}$, $-\infty < \lambda < \eta + p + 1$, $H^{\lambda,\delta}_{p,\eta,\mu} f(z)/z^p \neq 0$ for $f \in \mathcal{A}(p)$ and all the powers are understood as principal values.

**Theorem 1.** *Suppose that $f, g \in \mathcal{A}(p)$ and*

$$\Re \left\{ 1 + \frac{z\phi''(z)}{\phi'(z)} \right\} > -\rho \tag{10}$$

$$\left( \phi(z) = (1-\alpha) \left[ \frac{H^{\lambda,\delta}_{p,\eta,\mu} g(z)}{z^p} \right]^\beta + \alpha \left[ \frac{H^{\lambda,\delta+1}_{p,\eta,\mu} g(z)}{H^{\lambda,\delta}_{p,\eta,\mu} g(z)} \right] \left[ \frac{H^{\lambda,\delta}_{p,\eta,\mu} g(z)}{z^p} \right]^\beta ; z \in \mathbb{D} \right),$$

*where $\rho$ is given by*

$$\rho = \frac{\alpha^2 + \beta^2 (\delta + p)^2 - \left| \alpha^2 - \beta^2 (\delta + p)^2 \right|}{4\alpha\beta(\delta + p)}. \tag{11}$$

*Then,*

$$(1-\alpha) \left[ \frac{H^{\lambda,\delta}_{p,\eta,\mu} f(z)}{z^p} \right]^\beta + \alpha \left[ \frac{H^{\lambda,\delta+1}_{p,\eta,\mu} f(z)}{H^{\lambda,\delta}_{p,\eta,\mu} f(z)} \right] \left[ \frac{H^{\lambda,\delta}_{p,\eta,\mu} f(z)}{z^p} \right]^\beta \prec \phi(z) \tag{12}$$

*implies that*

$$\left[ \frac{H^{\lambda,\delta}_{p,\eta,\mu} f(z)}{z^p} \right]^\beta \prec \left[ \frac{H^{\lambda,\delta}_{p,\eta,\mu} g(z)}{z^p} \right]^\beta \tag{13}$$

*and* $\left[\frac{H^{\lambda,\delta}_{p,\eta,\mu}g(z)}{z^p}\right]^{\beta}$ *is the best dominant.*

**Proof.** We define two functions $\Phi(z)$ and $\Psi(z)$ by

$$\Phi(z) = \left[\frac{H^{\lambda,\delta}_{p,\eta,\mu}f(z)}{z^p}\right]^{\beta} \text{ and } \Psi(z) = \left[\frac{H^{\lambda\delta}_{p,\eta,\mu}g(z)}{z^p}\right]^{\beta} \ (z \in \mathbb{D}). \tag{14}$$

Firstly, we will show that, if

$$q(z) = 1 + \frac{z\Psi''(z)}{\Psi'(z)} \ (z \in \mathbb{D}), \tag{15}$$

then

$$\Re\{q(z)\} > 0 \ (z \in \mathbb{D}).$$

From the definitions of $\Psi(z)$ and $\phi(z)$ with Equation (8), we have

$$\phi(z) = \Psi(z) + \frac{\alpha}{\beta(\delta+p)} z\Psi'(z). \tag{16}$$

Differentiation both sides of Equation (16) with respect to $z$ yields

$$\phi'(z) = \Psi'(z) + \frac{\alpha\left[z\Psi''(z) + \Psi'(z)\right]}{\beta(\delta+p)}. \tag{17}$$

From Equations (15) and (17), we easily obtain

$$1 + \frac{z\phi''(z)}{\phi'(z)} = q(z) + \frac{zq'(z)}{q(z) + \frac{\beta(\delta+p)}{\alpha}} = h(z) \ (z \in \mathbb{D}). \tag{18}$$

It follows from Relations (10) and (18) that

$$\Re\left\{h(z) + \frac{\beta(\delta+p)}{\alpha}\right\} > 0 \ (z \in \mathbb{D}). \tag{19}$$

Furthermore, by means of Lemma 2, we deduce that Equation (18) has a solution $q \in \mathcal{H}(\mathbb{D})$ with $h(0) = q(0) = 1$. Let

$$H(u, v) = u + \frac{v}{u + \frac{\beta(\delta+p)}{\alpha}} + \rho, \tag{20}$$

where $\rho$ is given by Equation (11). From Equations (18) and (19), we have

$$\Re\left\{H\left(q(z); zq'(z)\right)\right\} > 0 \ (z \in \mathbb{D}).$$

Now, we will show that

$$\Re\{H(i\sigma; \tau)\} \leq 0 \ \left(\sigma \in \mathbb{R};\ \tau \leq -\frac{1+\sigma^2}{2}\right). \tag{21}$$

From Equation (20), we obtain

$$\begin{aligned}\Re\left\{\mathcal{H}\left(i\sigma;\tau\right)\right\} &= \Re\left\{i\sigma+\frac{\tau}{\frac{\beta\left(\delta+p\right)}{\alpha}+i\sigma}+\rho\right\}\\ &= \rho+\frac{\frac{\beta\left(\delta+p\right)\tau}{\alpha}}{\left|\frac{\beta\left(\delta+p\right)}{\alpha}+i\sigma\right|^{2}}\leq-\frac{E_{\rho}\left(\sigma\right)}{2\left|\frac{\beta\left(\delta+p\right)}{\alpha}+i\sigma\right|^{2}},\end{aligned}$$

where

$$E_{\rho}\left(\sigma\right)=\left(\frac{\beta\left(\delta+p\right)}{\alpha}-2\rho\right)\sigma^{2}-2\left(\frac{\beta\left(\delta+p\right)}{\alpha}\right)^{2}\rho+\frac{\beta\left(\delta+p\right)}{\alpha}. \tag{22}$$

For $\rho$ given by Equation (11), since the coefficient of $\sigma^2$ in $E_\rho\left(\sigma\right)$ of Equation (22) is positive or equal to zero and $E_\rho\left(\sigma\right)\geq 0$, we obtain that $\Re\left\{H\left(i\sigma;\tau\right)\right\}\leq 0$ for all $\sigma\in\mathbb{R}$ and $\tau\leq-\frac{1+\sigma^2}{2}$. Thus, by applying Lemma 1, we obtain that

$$\Re\left\{q\left(z\right)\right\}>0\ \ \left(z\in\mathbb{D}\right).$$

Moreover, $\Psi'(0)\neq 0$ since $g^{(p+1)}(0)\neq 0$. Hence, $\Psi(z)$ defined by Equation (14) is convex (univalent) in $\mathbb{D}$. Next, we verify that the Condition (12) implies that

$$\Phi\left(z\right)\prec\Psi\left(z\right)$$

for $\Phi(z)$ and $\Psi(z)$ given by Equation (14). Without loss of generality, we assume that $\Psi(z)$ is analytic, univalent on $\overline{\mathbb{D}}$ and

$$\Psi^{'}\left(\xi\right)\neq 0\ \ \left(\left|\xi\right|=1\right).$$

Let us consider the function $\mathcal{I}\left(z,t\right)$ defined by

$$\mathcal{I}\left(z,t\right)=\Psi\left(z\right)+\frac{\alpha\left(1+t\right)}{\beta\left(\delta+p\right)}z\Psi'\left(z\right)\ \ \left(0\leq t<\infty;\ z\in\mathbb{D}\right). \tag{23}$$

Then, we see easily that

$$\left.\frac{\partial\mathcal{I}\left(z,t\right)}{\partial z}\right|_{z=0}=\Psi'\left(0\right)\left(1+\frac{\alpha}{\beta\left(\delta+p\right)}\left(1+t\right)\right)\neq 0\ \ \left(0\leq t<\infty;\ z\in\mathbb{D}\right).$$

This shows that

$$\mathcal{I}\left(z,t\right)=a_1\left(t\right)z+\cdots$$

satisfies the restrictions $\lim\limits_{t\to\infty}\left|a_1\left(t\right)\right|=\infty$ and $a_1\left(t\right)\neq 0\ \ \left(0\leq t<\infty\right)$. In addition, we obtain

$$\begin{aligned}\Re\left\{\frac{z\frac{\partial\mathcal{I}(z,t)}{\partial z}}{\frac{\partial\mathcal{I}(z,t)}{\partial t}}\right\} &= \Re\left\{\frac{\beta\left(\delta+p\right)}{\alpha}+\left(1+t\right)\left(1+\frac{z\Psi''\left(z\right)}{\Psi'\left(z\right)}\right)\right\}>0\\ &\qquad \left(0\leq t<\infty;\ z\in\mathbb{D}\right),\end{aligned}$$

since $\Psi\left(z\right)$ is convex and $\Re\left(\frac{\beta(\delta+p)}{\alpha}\right)>0$. Moreover, we have

$$\left|\frac{\mathcal{I}\left(z,t\right)}{a_1\left(t\right)}\right|=\left|\frac{\Psi(z)+\frac{\alpha(1+t)}{\beta(\delta+p)}z\Psi'(z)}{\Psi(0)\left(1+\frac{\alpha(1+t)}{\beta(\delta+p)}\right)}\right| \tag{24}$$

and also the function $\Psi(z)$ may be written by

$$\Psi(z) = \Psi(0) + \Psi'(0)\psi(z) \quad (z \in \mathbb{D}), \tag{25}$$

where $\psi(z)$ is a normalized univalent function in $\mathbb{D}$. We note that, for the function $\psi(z)$, we have the following sharp growth and distortion results [32]:

$$\frac{r}{(1+r)^2} \le |\psi(z)| \le \frac{r}{(1-r)^2} \quad (|z| = r < 1) \tag{26}$$

and

$$\frac{1-r}{(1+r)^3} \le \psi'(z) \le \frac{1+r}{(1-r)^3} \quad (|z| = r < 1). \tag{27}$$

Hence, by applying Equations (25), (26) and (27) to Equation (24), we can find easily an upper bound for the right-hand side of Equation (24). Thus, the function $\mathcal{I}(z,t)$ satisfies the second condition of Lemma 5, which proves that $\mathcal{I}(z,t)$ is a subordination chain. From the definition of subordination chain, we note that

$$\phi\,(z) = \Psi\,(z) + \frac{\alpha}{\beta\,(\delta+p)} z\Psi'\,(z) = \mathcal{I}\,(z,0)$$

and

$$\mathcal{I}\,(z,0) \prec \mathcal{I}\,(z,t) \;\; (0 \le t < \infty)\,,$$

which implies that

$$\mathcal{I}\,(\xi,t) \notin \mathcal{I}\,(\mathbb{D},0) = \phi\,(\mathbb{D}) \;\; (0 \le t < \infty;\; \xi \in \partial\mathbb{D})\,. \tag{28}$$

If $\Phi(z)$ is not subordinate to $\Psi(z)$, by Lemma 3, we see that there exist two points $z_0 \in \mathbb{D}$ and $\xi_0 \in \partial\mathbb{D}$ satisfying

$$\phi\,(z_0) = \Psi\,(\xi_0) \text{ and } z_0\Phi'\,(z_0) = (1+t)\,\xi_0\Psi'\,(\xi_0) \;\; (0 \le t < \infty)\,. \tag{29}$$

Hence, by using Relations (12), (14), (23) and (29), we obtain

$$\begin{aligned} \mathcal{I}\,(\xi_0,t) &= \Psi\,(\xi_0) + \frac{\alpha}{\beta\,(\delta+p)}\,(1+t)\,\xi_0\Psi'\,(\xi_0) \\ &= \Phi\,(z_0) + \frac{\alpha}{\beta\,(\delta+p)} z_0\Phi'\,(z_0) \\ &= (1-\alpha)\left[\frac{H^{\lambda,\delta}_{p,\eta,\mu} f(z_0)}{z_0^p}\right]^\beta + \alpha\left[\frac{H^{\lambda,\delta+1}_{p,\eta,\mu} f(z_0)}{H^{\lambda,\delta}_{p,\eta,\mu} f(z_0)}\right]\left[\frac{H^{\lambda,\delta}_{p,\eta,\mu} f(z_0)}{z_0^p}\right]^\beta \in \phi\,(\mathbb{D})\,. \end{aligned}$$

This Contradicts (28). Thus, we conclude that $\Phi(z) \prec \Psi(z)$. If we consider $\Phi = \Psi$, then we know that $\Psi$ is the best dominant. Therefore, we complete the proof of Theorem 1. □

**Remark 1.** *The function $\Psi'(z) \neq 0$ for $z \in \mathbb{D}$ in Theorem 1 under the assumption*

$$\Re\{q\,(z)\} = 1 + \Re\left\{\frac{z\Psi''\,(z)}{\Psi'\,(z)}\right\} > 0 \quad (z \in \mathbb{D})\,. \tag{30}$$

*In fact, if $\Psi'(z)$ has a zero of order $m$ at $z = z_1 \in \mathbb{D}\backslash\{0\}$, then we may write*

$$\Psi(z) = (z - z_1)^m \Psi_1(z) \quad (m \in \mathbb{N}),$$

*where $\Psi_1(z)$ is analytic in $\mathbb{D}\backslash\{0\}$ and $\Psi_1(z_1) \neq 0$. Then, we have*

$$q(z) = 1 + \frac{z\Psi''(z)}{\Psi'(z)} = 1 + \frac{mz}{z - z_1} + \frac{z\Psi_1'(z)}{\Psi_1(z)}. \tag{31}$$

*Thus, choosing $z \to z_1$ suitably, the real part of the right-hand side of Equation (31) can take any negative infinite values, which contradicts hypothesis Equation (30). In addition, it is obvious that $\Psi'(0) \neq 0$ since $g^{(p+1)}(0) \neq 0$.*

Using similar methods given in the proof of Theorem 1, we have the following result.

**Theorem 2.** *Suppose that $f, g \in \mathcal{A}(p)$ and*

$$\Re\left\{1 + \frac{z\psi''(z)}{\psi'(z)}\right\} > -\sigma \tag{32}$$

$$\left(\psi(z) = (1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta} g(z)}{z^p}\right]^\beta + \alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} g(z)}{H_{p,\eta,\mu}^{\lambda+1,\delta} g(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta} g(z)}{z^p}\right]^\beta ; z \in \mathbb{D}\right),$$

*where $\sigma$ is given by*

$$\sigma = \frac{\alpha^2 + \beta^2(p+\eta-\lambda)^2 - \left|\alpha^2 - \beta^2(p+\eta-\lambda)^2\right|}{4\alpha\beta(p+\eta-\lambda)}. \tag{33}$$

*Then,*

$$(1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta} f(z)}{z^p}\right]^\beta + \alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} f(z)}{H_{p,\eta,\mu}^{\lambda+1,\delta} f(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta} f(z)}{z^p}\right]^\beta \prec \psi(z) \tag{34}$$

*implies that*

$$\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta} f(z)}{z^p}\right]^\beta \prec \left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta} g(z)}{z^p}\right]^\beta \tag{35}$$

*and $\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta} g(z)}{z^p}\right]^\beta$ is the best dominant.*

Next, we derive the dual result of Theorem 1.

**Theorem 3.** *Suppose that $f, g \in \mathcal{A}(p)$ and*

$$\Re\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\rho$$

$$\left(\phi(z) = (1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} g(z)}{z^p}\right]^\beta + \alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta+1} g(z)}{H_{p,\eta,\mu}^{\lambda,\delta} g(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} g(z)}{z^p}\right]^\beta ; z \in \mathbb{D}\right),$$

*where $\rho$ is given by Equation (11). If*

$$(1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} f(z)}{z^p}\right]^\beta + \alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta+1} f(z)}{H_{p,\eta,\mu}^{\lambda,\delta} f(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} f(z)}{z^p}\right]^\beta$$

*is univalent in $\mathbb{D}$ and $\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} f(z)}{z^p}\right]^\beta \in \mathcal{H}[1,1] \cap \mathcal{Q}$, then*

$$\phi(z) \prec (1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} f(z)}{z^p}\right]^\beta + \alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta+1} f(z)}{H_{p,\eta,\mu}^{\lambda,\delta} f(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} f(z)}{z^p}\right]^\beta \tag{36}$$

*implies that*

$$\left[\frac{H^{\lambda,\delta}_{p,\eta,\mu}g(z)}{z^p}\right]^{\beta} \prec \left[\frac{H^{\lambda,\delta}_{p,\eta,\mu}f(z)}{z^p}\right]^{\beta} \tag{37}$$

*and* $\left[\frac{H^{\lambda,\delta}_{p,\eta,\mu}g(z)}{z^p}\right]^{\beta}$ *is the best subordinant.*

**Proof.** By using the functions $\Phi(z)$, $\Psi(z)$ and $q(z)$ given by Equations (14) and (15), we have

$$\phi(z) = \Psi(z) + \frac{\alpha}{\beta(\delta+p)} z\Psi'(z) = \varphi\left(\Psi(z), z\Psi'(z)\right) \tag{38}$$

and

$$\Re\{q(z)\} > 0 \ \ (z \in \mathbb{D}).$$

Next, we will show that $\Psi(z) \prec \Phi(z)$. To derive this, we consider the function $\mathcal{I}(z,t)$ defined by

$$\mathcal{I}(z,t) = \Psi(z) + \frac{\alpha}{\beta(\delta+p)} tz\Psi'(z) \quad (0 \leq t < \infty;\ z \in \mathbb{D}).$$

Then, we see that

$$\left.\frac{\partial \mathcal{I}(z,t)}{\partial z}\right|_{z=0} = \Psi'(0)\left(1 + \frac{\alpha}{\beta(\delta+p)} t\right) \neq 0 \ \ (0 \leq t < \infty;\ z \in \mathbb{D}),$$

which shows that

$$\mathcal{I}(z,t) = a_1(t)z + \cdots$$

satisfies $\lim_{t\to\infty} |a_1(t)| = \infty$ and $a_1(t) \neq 0 \ \ (0 \leq t < \infty)$. Furthermore, we obtain

$$\Re\left\{\frac{z\frac{\partial \mathcal{I}(z,t)}{\partial z}}{\frac{\partial \mathcal{I}(z,t)}{\partial t}}\right\} = \Re\left\{\frac{\beta(\delta+p)}{\alpha} + t\left(1 + \frac{z\Psi''(z)}{\Psi'(z)}\right)\right\} > 0$$
$$(0 \leq t < \infty;\ z \in \mathbb{D}).$$

By using a similar method as in the proof of Theorem 1, we can prove the second inequality of Lemma 5. Hence, $\mathcal{I}(z,t)$ is a subordination chain. Therefore, by means of Lemma 4, we see that Relation (36) must imply given by Relation (37). Moreover, since Equation (38) has a univalent solution $\Psi$, it is the best subordinant. Therefore, we complete the proof. □

Using similar techniques given in the proof of Theorem 3, we have the following result.

**Theorem 4.** *Suppose that $f, g \in \mathcal{A}(p)$ and*

$$\Re\left\{1 + \frac{z\psi''(z)}{\psi'(z)}\right\} > -\sigma$$

$$\left(\psi(z) = (1-\alpha)\left[\frac{H^{\lambda+1,\delta}_{p,\eta,\mu}g(z)}{z^p}\right]^{\beta} + \alpha\left[\frac{H^{\lambda,\delta}_{p,\eta,\mu}g(z)}{H^{\lambda+1,\delta}_{p,\eta,\mu}g(z)}\right]\left[\frac{H^{\lambda+1,\delta}_{p,\eta,\mu}g(z)}{z^p}\right]^{\beta};\ z \in \mathbb{D}\right),$$

*where $\sigma$ is given by Equation (33). If*

$$(1-\alpha)\left[\frac{H^{\lambda+1,\delta}_{p,\eta,\mu}f(z)}{z^p}\right]^{\beta} + \alpha\left[\frac{H^{\lambda,\delta}_{p,\eta,\mu}f(z)}{H^{\lambda+1,\delta}_{p,\eta,\mu}f(z)}\right]\left[\frac{H^{\lambda+1,\delta}_{p,\eta,\mu}f(z)}{z^p}\right]^{\beta}$$

*is univalent in* $\mathbb{D}$ *and* $\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta} f(z)}{z^p}\right]^{\beta} \in \mathcal{H}[1,1] \cap \mathcal{Q}$, *then*

$$\psi(z) \prec (1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta} f(z)}{z^p}\right]^{\beta} + \alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} f(z)}{H_{p,\eta,\mu}^{\lambda+1,\delta} f(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta} f(z)}{z^p}\right]^{\beta} \tag{39}$$

*implies that*

$$\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta} g(z)}{z^p}\right]^{\beta} \prec \left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta} f(z)}{z^p}\right]^{\beta} \tag{40}$$

*and* $\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta} g(z)}{z^p}\right]^{\beta}$ *is the best subordinant.*

If we combine Theorems 1 and 3, and Theorems 2 and 4, then we have the unified sandwich-type results, respectively.

**Theorem 5.** *Suppose that* $f, g_j \in \mathcal{A}(p)\ \ (j = 1,\ 2)$ *and*

$$\Re\left\{1 + \frac{z\phi_j''(z)}{\phi_j'(z)}\right\} > -\rho \tag{41}$$

$$\left(\phi_j(z) = (1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} g_j(z)}{z^p}\right]^{\beta} + \alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta+1} g_j(z)}{H_{p,\eta,\mu}^{\lambda,\delta} g_j(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} g_j(z)}{z^p}\right]^{\beta};\ z \in \mathbb{D}\right),$$

*where* $\rho$ *is given by Equation* (11). *If*

$$(1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} f(z)}{z^p}\right]^{\beta} + \alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta+1} f(z)}{H_{p,\eta,\mu}^{\lambda,\delta} f(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} f(z)}{z^p}\right]^{\beta}$$

*is univalent in* $\mathbb{D}$ *and* $\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} f(z)}{z^p}\right]^{\beta} \in \mathcal{H}[1,1] \cap \mathcal{Q}$, *then*

$$\phi_1(z) \prec (1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} f(z)}{z^p}\right]^{\beta} + \alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta+1} f(z)}{H_{p,\eta,\mu}^{\lambda,\delta} f(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} f(z)}{z^p}\right]^{\beta} \prec \phi_2(z) \tag{42}$$

*implies that*

$$\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} g_1(z)}{z^p}\right]^{\beta} \prec \left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} f(z)}{z^p}\right]^{\beta} \prec \left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} g_2(z)}{z^p}\right]^{\beta}. \tag{43}$$

*Moreover,* $\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} g_1(z)}{z^p}\right]^{\beta}$ *and* $\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} g_2(z)}{z^p}\right]^{\beta}$ *are the best subordinant and the best dominant, respectively.*

**Theorem 6.** *Suppose that* $f, g_j \in \mathcal{A}(p)\ \ (j = 1,\ 2)$ *and*

$$\Re\left\{1 + \frac{z\psi_j''(z)}{\psi_j'(z)}\right\} > -\sigma \tag{44}$$

$$\left(\psi_j(z) = (1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta} g_j(z)}{z^p}\right]^{\beta} + \alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} g_j(z)}{H_{p,\eta,\mu}^{\lambda+1,\delta} g_j(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta} g_j(z)}{z^p}\right]^{\beta};\ z \in \mathbb{D}\right),$$

*where $\sigma$ is given by Equation (33). If*

$$(1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}f(z)}{z^p}\right]^{\beta}+\alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}{H_{p,\eta,\mu}^{\lambda+1,\delta}f(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}f(z)}{z^p}\right]^{\beta}$$

*is univalent in $\mathbb{D}$ and $\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}f(z)}{z^p}\right]^{\beta}\in\mathcal{H}[1,1]\cap\mathcal{Q}$, then*

$$\psi_1(z)\prec(1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}f(z)}{z^p}\right]^{\beta}+\alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}{H_{p,\eta,\mu}^{\lambda+1,\delta}f(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}f(z)}{z^p}\right]^{\beta}\prec\psi_2(z) \tag{45}$$

*implies that*

$$\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}g_1(z)}{z^p}\right]^{\beta}\prec\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}f(z)}{z^p}\right]^{\beta}\prec\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}g_2(z)}{z^p}\right]^{\beta}. \tag{46}$$

*Moreover, $\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}g_1(z)}{z^p}\right]^{\beta}$ and $\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta}g_2(z)}{z^p}\right]^{\beta}$ are the best subordinant and the best dominant, respectively.*

We note that the assumption of Theorem 5, which states that

$$(1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}{z^p}\right]^{\beta}+\alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta+1}f(z)}{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}{z^p}\right]^{\beta}\text{ and }\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}{z^p}\right]^{\beta}$$

needs to be univalent in $\mathbb{D}$, may be exchanged by a different condition.

**Corollary 1.** *Suppose that $f, g_j\in\mathcal{A}(p)$ $(j=1,\,2)$ and*

$$\Re\left\{1+\frac{z\phi_j''(z)}{\phi_j'(z)}\right\}>-\rho$$

$$\left(\phi_j(z)=(1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}g_j(z)}{z^p}\right]^{\beta}+\alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta+1}g_j(z)}{H_{p,\eta,\mu}^{\lambda,\delta}g_j(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}g_j(z)}{z^p}\right]^{\beta};\ z\in\mathbb{D}\right)$$

*and*

$$\Re\left\{1+\frac{z\chi''(z)}{\chi'(z)}\right\}>-\rho, \tag{47}$$

$$\left(\chi(z)=(1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}{z^p}\right]^{\beta}+\alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta+1}f(z)}{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}{z^p}\right]^{\beta};\ z\in\mathbb{D}\right),$$

*where $\rho$ is given by Equation (11). Then,*

$$\phi_1(z)\prec(1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}{z^p}\right]^{\beta}+\alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta+1}f(z)}{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}{z^p}\right]^{\beta}\prec\phi_2(z)$$

*implies that*

$$\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}g_1(z)}{z^p}\right]^{\beta}\prec\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}{z^p}\right]^{\beta}\prec\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta}g_2(z)}{z^p}\right]^{\beta}.$$

**Proof.** To derive Corollary 1, we need to show that the Restriction (47) implies the univalence of $\chi(z)$. Noting that $0\le\rho<1/2$, it follows that $\chi(z)$ is close-to-convex function in $\mathbb{D}$ (see [33]) and so $\chi(z)$

is univalent in $\mathbb{D}$. In addition, by applying the similar methods given in the proof of Theorem 1, we see that the function $\Phi(z)$ defined by Equation (14) is convex (univalent) in $\mathbb{D}$. Therefore, by using Theorem 5, we get the desired result. □

Using similar methods given in the proof of Corollary 1 with Theorem 6, we obtain the following corollary.

**Corollary 2.** *Suppose that* $f, g_j \in \mathcal{A}(p)\ \ (j = 1,\ 2)$ *and*

$$\Re\left\{1 + \frac{z\psi_j''(z)}{\psi_j'(z)}\right\} > -\sigma$$

$$\left(\psi_j(z) = (1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta} g_j(z)}{z^p}\right]^{\beta} + \alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} g_j(z)}{H_{p,\eta,\mu}^{\lambda,\delta} g_j(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} g_j(z)}{z^p}\right]^{\beta};\ z \in \mathbb{D}\right)$$

*and*

$$\Re\left\{1 + \frac{z\Upsilon''(z)}{\Upsilon'(z)}\right\} > -\rho,$$

$$\left(\Upsilon(z) = (1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta} f(z)}{z^p}\right]^{\beta} + \alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} f(z)}{H_{p,\eta,\mu}^{\lambda+1,\delta} f(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta} f(z)}{z^p}\right]^{\beta};\ z \in \mathbb{D}\right),$$

*where* $\sigma$ *is given by* (33). *Then,*

$$\psi_1(z) \prec (1-\alpha)\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta} f(z)}{z^p}\right]^{\beta} + \alpha\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} f(z)}{H_{p,\eta,\mu}^{\lambda+1,\delta} f(z)}\right]\left[\frac{H_{p,\eta,\mu}^{\lambda+1,\delta} f(z)}{z^p}\right]^{\beta} \prec \psi_2(z)$$

*implies that*

$$\left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} g_1(z)}{z^p}\right]^{\beta} \prec \left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} f(z)}{z^p}\right]^{\beta} \prec \left[\frac{H_{p,\eta,\mu}^{\lambda,\delta} g_2(z)}{z^p}\right]^{\beta}.$$

## 3. Conclusions

Various applications of fractional calculus have an immense impact on the study of pure mathematic and applied science. In the present paper, we obtain new results on subordinations and superordinations for a wide class of operators defined by generalized fractional derivative operators and generalized fractional integral operators. Furthermore, the differential sandwich-type theorems are also discussed for these operators.

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