symmetry

Article

Common Fixed Points Results on Non-Archimedean Metric Modular Spaces

Wissam Kassab [1,2]

[1] Department of Mathematics and Computer Science, University Politehnica of Bucharest, Bucharest 060042, Romania; wissamkassab@yahoo.com
[2] Department of Mathematics, International University of Beirut, Mouseitbeh Mazraa 1105, Beirut, Lebanon

Received: 2 September 2019; Accepted: 30 October 2019; Published: 2 November 2019

Abstract: This paper introduces two new contractive conditions in the setting of non-Archimedean modular spaces, via a C-class function, an altering distance function, and a control function. A non-Archimedean metric modular is shaped as a parameterized family of classical metrics; therefore, for each value of the parameter, the positivity, the symmetry, the triangle inequality, or the continuity is ensured. The main outcomes provide sufficient conditions for the existence of common fixed points for four mappings. Examples are provided in order to prove the usability of the theoretical approach. Moreover, these examples use a non-Archimedean metric modular, which is not convex, making the study of nonconvex modulars more appealing.

Keywords: non-Archimedean metric modular; common fixed point; C-class function; weak annihilator

MSC: 47H09

1. Introduction

Lately, various modular structures, viewed as alternatives to classical normed or metric spaces, have been intensely studied in connection with the fixed point theory. Many modular related research papers adopted the setting of a modular vector space (see [1–5]), while others used the more general framework of a metric modular space (see [6–11]). The notion of a metric modular, together with its stronger convex version, was firstly introduced and studied by Chistyakov in [6–9]. Although the convexity of a modular metric brings considerable advantages, the absence of the triangle inequality generates major difficulties when trying to expand some results to the modular setting. A possible solution was provided by Paknazar in [12,13] by defining the so-called non-Archimedean metric modular. In fact, the new modular proves to be a parameterized family of classical metrics; therefore, for each value of the parameter, the triangle inequality or the continuity is ensured. This makes the newly defined object a very good instrument for analyzing various contractive conditions or for using non-standard iterative procedures.

This paper uses the setting of a non-Archimedean metric modular space and defines and studies new nonlinear contractive conditions. The source for this approach is the work of Shatanawi et al. [14], who developed a similar theory, but in the framework of a complete metric space. Their work considered the almost generalized (S,T)-contractive condition introduced by Shobkolaei et al. [15] on partial metric spaces and the almost nonlinear contractive condition (via some control functions) on metric spaces introduced by Shatanawi and Postolache [16] and expanded them by means of a C-class function (see [17]). The result was a new contractive condition, called the almost nonlinear (S,T,L,F,ψ,ϕ)-convex contractive condition. In this context, this paper aims to provide an upgrade for the work of Shatanawi et al. [14]. In fact, it does not just substitute the framework of ordered metric spaces with ordered non-Archimedean metric modular spaces; it also provides two possible modular extensions for the almost nonlinear (S, T, L, F, ψ, ϕ)-convex contractive condition. Moreover, by properly including concepts as weakly compatible mappings (see Jungck [18]) or dominating and weak annihilators (see Abbas et al. [19]), several new outcomes regarding the existence of common fixed points are obtained.

2. Preliminaries

We start by recalling basic facts about metric modular spaces.

Definition 1. [6] A function $\omega\colon (0,\infty) \times X \times X \to [0,\infty]$, written as $\omega(\lambda, x, y) = \omega_\lambda(x, y)$, is known as a metric modular on X if the following axioms hold:

(i) $\omega_\lambda(x, y) = 0,\ \forall \lambda > 0$ if and only if $x = y$;
(ii) for each $x, y \in X$, $\omega_\lambda(x, y) = \omega_\lambda(y, x),\ \forall \lambda > 0$;
(iii) for each $x, y, z \in X$, $\omega_{\lambda+\mu}(x, y) \le \omega_\lambda(x, z) + \omega_\mu(z, y),\ \forall \lambda, \mu > 0$.

If (iii) is replaced with:

$$(iii') \qquad \omega_{\lambda+\mu}(x, y) \le \frac{\lambda}{\lambda+\mu}\omega_\lambda(x, z) + \frac{\mu}{\lambda+\mu}\omega_\mu(y, z),\ \forall \lambda, \mu > 0,\ \forall x, y, z \in X,$$

then the metric modular is called convex, while if (iii) is replaced with:

$$(iii'') \qquad \omega_{\max\{\lambda,\mu\}}(x, y) \le \omega_\lambda(x, z) + \omega_\mu(z, y),\ \forall \lambda, \mu > 0,\ \forall x, y, z \in X,$$

the metric modular is called non-Archimedean (see [12,13]).

Remark 1. Note that the function $\lambda \to \omega_\lambda(x, y)$ is nonincreasing on $(0, \infty)$, for each $x, y \in X$. In fact, Chistyakov called this "the essential property" of a metric modular (see [8]). Indeed, if $0 < \mu < \lambda$, then, by using the triangle property, we have:

$$\omega_\lambda(x, z) \le \omega_{\lambda-\mu}(x, x) + \omega_\mu(x, z) = \omega_\mu(x, z).$$

In addition, if ω is a convex modular, then the function $\lambda \to \lambda\omega_\lambda(x, y)$ is also nonincreasing on $(0, \infty)$ ("the main property of a convex modular"; see [7]).

Remark 2. If ω is a non-Archimedean metric modular, we notice that:

$$\omega_\lambda(x, y) = \omega_{\max\{\lambda,\lambda\}}(x, y) \le \omega_\lambda(x, z) + \omega_\lambda(z, y),\ \forall x, y, z \in X,\ \forall \lambda > 0. \tag{1}$$

Basically, Paknazar's definition includes the metric modulars for which the triangle inequality is valid. Moreover, the triangle inequality makes the metric modular continuous in the following sense: if $\lim_{n\to\infty} \omega_1(x_n, x) = 0$, then $\lim_{n\to\infty} \omega_1(x_n, y) = \omega_1(x, y),\ \forall y \in X$.

In addition, given a metric modular on X and a point $x_0 \in X$, the following two sets can be defined:

$$X_\omega(x_0) = \{x \in X : \omega_\lambda(x_0, x) \to 0 \text{ as } \lambda \to \infty\}$$

and

$$X^*_\omega(x_0) = \{x \in X : \exists \lambda = \lambda(x) > 0 \text{ such that } \omega_\lambda(x_0, x) < \infty\}.$$

They both are known as metric modular spaces (around x_0), although in general, they just satisfy the inclusion $X_\omega(x_0) \subset X^*_\omega(x_0)$. In particular, when ω is a convex metric (pseudo)modular, the two sets are equal. Throughout this paper, we shall fix a point x_0, and we shall simply denote by X_ω and X^*_ω the metric modular spaces around x_0.

The metric modular can be used to define concepts as convergence, completeness, and so on, properly.

Definition 2. [20] Let ω be a metric modular on a set X.

(i) A sequence $\{x_n\} \subset X_\omega$ (or X^*_ω if ω is convex) is called ω-convergent to a point $x \in X_\omega$ ($x \in X^*_\omega$, respectively) if $\lim_{n\to\infty} \omega_1(x_n, x) = 0$.

(ii) A sequence $\{x_n\} \subset X_\omega$ (or X^*_ω) is called ω-Cauchy if $\lim_{n,m\to\infty} \omega_1(x_n, x_m) = 0$.

(iii) The modular space X_ω (or $X^*(\omega)$ when ω is convex) is called ω-complete if each ω-Cauchy sequence $\{x_n\}$ is ω-convergent.

(iv) A subset $C \subset X_\omega$ is said to be ω-closed if the ω-limit of an ω-convergent sequence of C is in C.

The following lemma proves to be a very useful tool when dealing with non-standard contractive conditions.

Lemma 1. *Suppose that X_ω is a non-Archimedean metric modular space. Let $\{x_n\}$ be a sequence in X_ω such that $\omega_1(x_n, x_{n+1}) \to 0$ as $n \to +\infty$. If $\{x_n\}$ is not a Cauchy sequence, then there exist an $\varepsilon > 0$ and two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that:*

1. $i \le m_i < n_i$;
2. $\omega_1(x_{m_i}, x_{n_i}) \ge \varepsilon$;
3. $\omega_1(x_{m_i}, x_{n_i-1}) < \varepsilon$;
4. $\lim_{i\to+\infty} \omega_1(x_{m_i}, x_{n_i}) = \lim_{i\to+\infty} \omega_1(x_{m_i-1}, x_{n_i-1}) = \lim_{i\to+\infty} \omega_1(x_{m_i-1}, x_{m_i})$
$= \lim_{i\to+\infty} \omega_1(x_{m_i}, x_{n_i-1}) = \varepsilon$.

In addition to the above framework description, we also recall some mapping related properties. Let f and g be self-mappings of a set X. If $w = fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g. Two self-mappings f and g are said to be weakly compatible if they commute at their coincidence point, that is $fgx = gfx$ whenever $fx = gx$. For details, please see Jungck [18].

Now, consider $(X, \preceq)$ a partially ordered set. According to Abbas et al. [19], a mapping f is called a weak annihilator of g if $fgx \preceq x$, for all $x \in X$, and f is called dominating if $x \preceq fx$, for all $x \in X$.

Let us also consider the following classes of functions (see [14,16,17,21]):

- the class of altering distance functions Ψ contains all functions $\psi\colon [0, +\infty) \to [0, +\infty)$ such that:

  (1) ψ is continuous and nondecreasing;
  (2) $\psi(t) = 0$ if and only if $t = 0$.

- Φ_u denotes all functions $\varphi\colon [0, +\infty) \to [0, +\infty)$ that satisfy the following conditions:

  (1) φ is continuous on $[0, +\infty)$;
  (2) $\varphi(t) > 0$, for each $t > 0$.

- the class of control functions Φ denotes all functions $\phi\colon [0, +\infty) \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ such that:

  (1) ϕ is continuous;
  (2) $\phi(t, s, u) = 0$ if and only if $u = s = t = 0$.

- Φ_1 denotes all functions $\phi\colon [0, +\infty) \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ such that:

  (1) ϕ is continuous;

(2) $\phi(u,s,t) > 0,\ \forall (u,s,t) \neq (0,0,0)$.

- $\mathcal{C}$ denotes the set of all C-class functions (see [17]), i.e., those functions $F\colon [0,\infty)^2 \to \mathbb{R}$ with the following properties:

(1) $F(s,t) \leq s$;
(2) $F(s,t) = s$ implies that either $s = 0$ or $t = 0$;
(3) F is continuous.

By combining Ψ, Φ_1 and $\mathcal{C}$, a general nonlinear contractive condition was defined in [14], as follows.

Definition 3. [14] Let f, g, S, and T be self-mappings on a metric space (X,d). Then, f and g are said to satisfy the almost nonlinear (S,T,L,F,ψ,ϕ)-convex contractive condition if there exist $\psi \in \Psi, \varphi \in \Phi_1$, $F \in \mathcal{C}$, and $L \in [0,+\infty)$ such that:

$$\begin{aligned}
&\psi(d(fx,gy)) \leq F\Big(\psi\Big(\frac{1}{a+b+c+2e}[ad(Sx,Ty)+bd(fx,Sx)+cd(gy,Ty)\\
&+ed(Sx,gy)+ed(fx,Ty)]\Big),\phi\left(d(Sx,Ty),d(Sx,gy),d(fx,Ty)\right)\Big)\\
&+L\min\left\{d(Sx,Ty),d(Sx,gy),d(fx,Ty)\right\},
\end{aligned} \tag{2}$$

for all $x,y \in X$, where $a,b,c,e \geq 0$, with $a+b+c+2e > 0$.

The main outcome obtained in connection with the above contractive property consists of sufficiency conditions for the existence of common fixed points.

Theorem 1. *[14] Let $(X,d,\preceq)$ be a complete ordered metric space. Let f,g,T,S be self-mappings of X such that for any two comparable elements $x,y \in X$, the mappings f and g satisfy the nonlinear (S,T,L,F,ψ,ϕ)-convex contractive condition (3). Assume also the following assertions:*

1. *$fX \subseteq TX$;*
2. *$gX \subseteq SX$;*
3. *$F(\psi(a),\phi(a,a,a)) + La < \psi(a)$ for all $a > 0$;*
4. *f is dominating and a weak annihilator of T;*
5. *g is dominating and a weak annihilator of S;*
6. *$\{f,S\}$ and $\{g,T\}$ are weakly compatible;*
7. *one of fX, gX, SX, and TX is a closed subspace of X; and*
8. *X has the property (π).*

Then, f,g,S, and T have a common fixed point.

3. First Extension to Partially Ordered Non-Archimedean Metric Modular Spaces

Since for each metric $d(x,y)$, there exists a natural extension to a non-Archimedean metric modular $\omega_\lambda(x,y) = \dfrac{d(x,y)}{\lambda}$ (which means that $d(x,y) = \omega_1(x,y)$), the definition introduced in [14] inspires us to provide the following natural extension to non-Archimedean metric modular spaces.

Definition 4. Let f,g,S, and T be self-mappings on a non-Archimedean modular metric space X_ω. Then, f and g are said to satisfy the almost nonlinear (S,T,L,F,ψ,ϕ)-convex contractive condition of type I if there exist $\psi \in \Psi, \phi \in \Phi_1$, $F \in \mathcal{C}$, and $L \in [0,+\infty)$ such that:

$$\begin{aligned}&\psi(\omega_1(fx,gy)) \leq F\Big(\psi\Big(\alpha\omega_1(Sx,Ty)+\beta\omega_1(fx,Sx)+\gamma\omega_1(gy,Ty)\\&+\delta\omega_1(Sx,gy)+\delta\omega_1(fx,Ty)\Big),\phi\left(\omega_1(Sx,Ty),\omega_1(Sx,gy),\omega_1(fx,Ty)\right)\Big)\\&+L\min\left\{\omega_1(Sx,Ty),\omega_1(Sx,gy),\omega_1(fx,Ty)\right\},\end{aligned} \tag{3}$$

for all $x,y \in X$, where $\alpha,\beta,\gamma,\delta \geq 0$, with $\alpha+\beta+\gamma+2\delta = 1$.

Furthermore, it is opportune to define a modular version of property (π): Let $(X_\omega, \preceq)$ be an ordered non-Archimedean metric modular space. We say that X_ω satisfies the property (π_ω) if the following statement holds true:

(π_ω): If $\{x_n\}$ is a nondecreasing sequence in X_ω and $\{y_n\}$ is a sequence in X_ω such that $x_n \preceq y_n$ for all n, but finitely many, and y_n is ω-convergent to u, then $x_n \preceq u$ for all n, but finitely many. We mention that the ω-convergence considered here is in the sense of Definition 2.

In the following, we combine the data defined above in order to state and prove our first common fixed point result.

Theorem 2. *Let $(X_\omega, \preceq)$ be an ω-complete (in the sense of Definition 2) ordered non-Archimedean metric modular space. Let f,g,T,S be self-mappings of X_ω such that for any two comparable elements $x,y \in X_\omega$, the mappings f and g satisfy the nonlinear (S,T,L,F,ψ,ϕ)-convex contractive condition of type I (4). In addition, assume that the following assertions hold true:*

(1) $fX_\omega \subseteq TX_\omega$;
(2) $gX_\omega \subseteq SX_\omega$;
(3) $F(\psi(\beta),\phi(\alpha,\alpha,\alpha)) + L\alpha < \psi(\alpha)$ *for all* $\alpha,\beta > 0$ *with* $\beta \leq \alpha$;
(4) f *is dominating and a weak annihilator of* T;
(5) g *is dominating and a weak annihilator of* S;
(6) $\{f,S\}$ *and* $\{g,T\}$ *are weakly compatible;*
(7) *one of* $fX_\omega, gX_\omega, SX_\omega$, *and* TX_ω *is an* ω*-closed subspace of* X_ω;
(8) X_ω *has the property* (π_ω).

Then, f,g,S, and T have a common fixed point.

Proof. Let us start with an arbitrary element $x_0 \in X_\omega$. By using Hypotheses (1) and (2), we generate two sequences $\{x_n\}$, $\{y_n\} \in X_\omega$ in such a way that $y_{2t} := fx_{2t} = Tx_{2t+1}$ and $y_{2t+1} := gx_{2t+1} = Sx_{2t+2}$. Using (4) and (5), we have:

$$x_{2t} \preceq fx_{2t} = Tx_{2t+1} \preceq fTx_{2t+1} \preceq x_{2t+1} \preceq gx_{2t+1} = Sx_{2t+2} \preceq gSx_{2t+2} \preceq x_{2t+2},$$

which means that $x_n \preceq x_{n+1}$ for any nonnegative integer n; therefore, they are comparable.

Step 1. In the following, we shall focus on proving that $\{y_n\}$ is convergent.

Case I. Let us assume that there exists $n_0 \in \mathbb{N}$ such that $y_{n_0} = y_{n_0+1}$.

- If n_0 is even, that is $n_0 = 2t$, we have $y_{2t} = y_{2t+1}$. Using the fact that x_{2t+1} and x_{2t+2} are comparable and Condition (4), we have:

$$\begin{aligned}
&\psi(\omega_1(y_{2t+2}, y_{2t+1})) = \psi(\omega_1(fx_{2t+2}, gx_{2t+1})) \\
\leq\ & F\Big(\psi\Big(\alpha\omega_1(Sx_{2t+2}, Tx_{2t+1}) + \beta\omega_1(fx_{2t+2}, Sx_{2t+2}) \\
& +\gamma\omega_1(gx_{2t+1}, Tx_{2t+1}) + \delta\omega_1(Sx_{2t+2}, gx_{2t+1}) + \delta\omega_1(fx_{2t+2}, Tx_{2t+1})\Big), \\
& \phi(\omega_1(Sx_{2t+2}, Tx_{2t+1}), \omega_1(Sx_{2t+2}, gx_{2t+1}), \omega_1(fx_{2t+2}, Tx_{2t+1}))\Big) \\
& +L\min\{\omega_1(Sx_{2t+2}, Tx_{2t+1}), \omega_1(Sx_{2t+2}, gx_{2t+1}), \omega_1(fx_{2t+2}, Tx_{2t+1})\} \qquad (4)\\
=\ & F\Big(\psi\Big(\alpha\omega_1(y_{2t+1}, y_{2t}) + \beta\omega_1(y_{2t+2}, y_{2t+1}) + \gamma\omega_1(y_{2t+1}, y_{2t}) \\
& +\delta\omega_1(y_{2t+1}, y_{2t+1}) + \delta\omega_1(y_{2t+2}, y_{2t})\Big), \\
& \phi(\omega_1(y_{2t+1}, y_{2t}), \omega_1(y_{2t+1}, y_{2t+1}), \omega_1(y_{2t+2}, y_{2t}))\Big) \\
& +L\min\{\omega_1(y_{2t+1}, y_{2t}), \omega_1(y_{2t+1}, y_{2t+1}), \omega_1(y_{2t+2}, y_{2t})\} \\
=\ & F\Big(\psi\Big(\beta\omega_1(y_{2t+2}, y_{2t+1}) + \delta\omega_1(y_{2t+2}, y_{2t})\Big), \phi(0, 0, \omega_1(y_{2t+2}, y_{2t+1}))\Big).
\end{aligned}$$

Using the properties of F, we have:

$$\psi(\omega_1(y_{2t+2}, y_{2t+1}) \leq \psi\Big(\beta\omega_1(y_{2t+2}, y_{2t+1}) + \delta\omega_1(y_{2t+2}, y_{2t})\Big).$$

Since ψ is nondecreasing, then the last inequality holds only if:

$$\omega_1(y_{2t+2}, y_{2t+1}) \leq \beta\omega_1(y_{2t+2}, y_{2t+1}) + \delta\omega_1(y_{2t+2}, y_{2t}),$$

which, using the triangle inequality (1), leads to:

$$\begin{aligned}
\omega_1(y_{2t+2}, y_{2t+1}) &\leq \beta\omega_1(y_{2t+2}, y_{2t+1}) + \delta\omega_1(y_{2t+2}, y_{2t+1}) + \delta\omega_1(y_{2t+1}, y_{2t}) \\
&= \beta\omega_1(y_{2t+2}, y_{2t+1}) + \delta\omega_1(y_{2t+2}, y_{2t+1}) \qquad (5)\\
&= (\beta+\delta)\omega_1(y_{2t+2}, y_{2t+1}).
\end{aligned}$$

Moreover, the conditions $\alpha, \beta, \gamma, \delta \geq 0$, and $\alpha + \beta + \gamma + 2\delta = 1$ lead either to $\beta + \delta < 1$ or to $\beta = 1$ and $\alpha = \gamma = \delta = 0$. In the first case, we find from (5) that $\omega_1(y_{2t+2}, y_{2t+1}) = 0$; hence, $y_{2t+2} = y_{2t+1}$. In the other case, by taking a step back into the chain of inequalities (4), we find:

$$\psi(\omega_1(y_{2t+2}, y_{2t+1})) \leq F\Big(\psi\Big(\omega_1(y_{2t+2}, y_{2t+1})\Big), \phi(0, 0, \omega_1(y_{2t+2}, y_{2t+1}))\Big).$$

This tells us, in fact, that $F\Big(\psi\Big(\omega_1(y_{2t+2}, y_{2t+1})\Big), \phi(0, 0, \omega_1(y_{2t+2}, y_{2t+1}))\Big)$ is actually equal to $\psi(\omega_1(y_{2t+2}, y_{2t+1}))$. By considering the properties of F, this gives us ultimately the same conclusion as above, namely $y_{2t+2} = y_{2t+1}$.

- If n_0 is odd, that is $n_0 = 2t + 1$, by using the same technique, we find that $y_{2t+3} = y_{2t+2}$.

  Combining these two items, we may conclude that, starting with n_0, the sequence $\{y_n\}$ is a constant sequence in X_ω, and hence, it is convergent.

Case II. Let us assume now that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$. We analyze again, separately, the situation of n being even and the opposite of this.

- If n is even, then $n = 2t$ for some $t \in \mathbb{N}$. Using the comparability property of x_{2t} and x_{2t+1}, we have:

$$\begin{aligned}
& \psi(\omega_1(y_n, y_{n+1})) = \psi(\omega_1(y_{2t}, y_{2t+1})) = \psi(\omega_1(fx_{2t}, gx_{2t+1})) \\
\leq\ & F\Big(\psi\Big(\alpha\omega_1(Sx_{2t}, Tx_{2t+1}) + \beta\omega_1(fx_{2t}, Sx_{2t}) \\
& +\gamma\omega_1(gx_{2t+1}, Tx_{2t+1}) + \delta\omega_1(Sx_{2t}, gx_{2t+1}) + \delta\omega_1(fx_{2t}, Tx_{2t+1})\Big), \\
& \phi(\omega_1(Sx_{2t}, Tx_{2t+1}), \omega_1(Sx_{2t}, gx_{2t+1}), \omega_1(fx_{2t}, Tx_{2t+1}))\Big) \\
& +L\min\{\omega_1(Sx_{2t}, Tx_{2t+1}), \omega_1(Sx_{2t}, gx_{2t+1}), \omega_1(fx_{2t}, Tx_{2t+1})\} \\
=\ & F\Big(\psi\Big(\alpha\omega_1(y_{2t-1}, y_{2t}) + \beta\omega_1(y_{2t}, y_{2t-1}) + \gamma\omega_1(y_{2t+1}, y_{2t}) \\
& +\delta\omega_1(y_{2t-1}, y_{2t+1})\Big), \phi(\omega_1(y_{2t-1}, y_{2t}), \omega_1(y_{2t-1}, y_{2t+1}), 0)\Big) \\
& +L\min\{\omega_1(y_{2t-1}, y_{2t}), \omega_1(y_{2t-1}, y_{2t+1}), 0\} \\
\leq\ & F\Big(\psi\Big(\alpha\omega_1(y_{2t-1}, y_{2t}) + \beta\omega_1(y_{2t}, y_{2t-1}) + \gamma\omega_1(y_{2t+1}, y_{2t}) \\
& +\delta\omega_1(y_{2t-1}, y_{2t+1})\Big), \phi(\omega_1(y_{2t-1}, y_{2t}), \omega_1(y_{2t-1}, y_{2t+1}), 0)\Big).
\end{aligned}$$

If $\omega_1(y_{2t-1}, y_{2t}) \leq \omega_1(y_{2t}, y_{2t+1})$, then, using again the triangle inequality for the non-Archimedean metric modular, together with the nondecreasing behavior of ψ, we find:

$$\begin{aligned}
& \psi(\omega_1(y_n, y_{n+1})) = \psi(\omega_1(y_{2t}, y_{2t+1})) \\
\leq\ & F\Big(\psi\Big(\alpha\omega_1(y_{2t-1}, y_{2t}) + \beta\omega_1(y_{2t}, y_{2t-1}) + \gamma\omega_1(y_{2t+1}, y_{2t}) \\
& +\delta\omega_1(y_{2t-1}, y_{2t+1})\Big), \phi(\omega_1(y_{2t-1}, y_{2t}), \omega_1(y_{2t-1}, y_{2t+1}), 0)\Big) \\
\leq\ & \psi\Big(\alpha\omega_1(y_{2t-1}, y_{2t}) + \beta\omega_1(y_{2t}, y_{2t-1}) + \gamma\omega_1(y_{2t+1}, y_{2t}) \\
& +\delta\omega_1(y_{2t-1}, y_{2t+1})\Big) \\
\leq\ & \psi\Big(\alpha\omega_1(y_{2t-1}, y_{2t}) + \beta\omega_1(y_{2t}, y_{2t-1}) + \gamma\omega_1(y_{2t+1}, y_{2t}) \\
& +\delta\omega_1(y_{2t-1}, y_{2t}) + \delta\omega_1(y_{2t}, y_{2t+1})\Big) \\
\leq\ & \psi\Big(\ (\alpha+\beta+\delta)\,\omega_1(y_{2t-1}, y_{2t}) + (\gamma+\delta)\,\omega_1(y_{2t}, y_{2t+1})\Big) \\
\leq\ & \psi\Big(\ (\alpha+\beta+\gamma+2\delta)\,\omega_1(y_{2t}, y_{2t+1})\Big) \\
=\ & \psi(\omega_1(y_{2t}, y_{2t+1})) = \psi(\omega_1(y_n, y_{n+1})).
\end{aligned}$$

Thus,

$$\begin{aligned}
& F\Big(\psi\Big(\alpha\omega_1(y_{2t-1}, y_{2t}) + \beta\omega_1(y_{2t}, y_{2t-1}) + \gamma\omega_1(y_{2t+1}, y_{2t}) \\
& +\delta\omega_1(y_{2t-1}, y_{2t+1})\Big), \phi(\omega_1(y_{2t-1}, y_{2t}), \omega_1(y_{2t-1}, y_{2t+1}), 0)\Big) \\
& = \psi\Big(\alpha\omega_1(y_{2t-1}, y_{2t}) + \beta\omega_1(y_{2t}, y_{2t-1}) + \gamma\omega_1(y_{2t+1}, y_{2t}) + \delta\omega_1(y_{2t-1}, y_{2t+1})\Big).
\end{aligned}$$

Using the properties of F, we conclude that either:

$$\psi\Big(\alpha\omega_1(y_{2t-1}, y_{2t}) + \beta\omega_1(y_{2t}, y_{2t-1}) + \gamma\omega_1(y_{2t+1}, y_{2t}) + \delta\omega_1(y_{2t-1}, y_{2t+1})\Big) = 0$$

or

$$\phi(\omega_1(y_{2t-1}, y_{2t}), \omega_1(y_{2t-1}, y_{2t+1}), 0) = 0.$$

In both cases, we obtain that $y_{2t-1} = y_{2t}$ is necessary, leading to a contradiction. Thus,

$$\omega_1(y_{2t}, y_{2t+1}) < \omega_1(y_{2t-1}, y_{2t}), \tag{6}$$

and:

$$\begin{aligned} & \psi(\omega_1(y_{2t}, y_{2t+1})) \\ \leq\ & F\Big(\psi\Big(\alpha\omega_1(y_{2t-1}, y_{2t}) + \beta\omega_1(y_{2t}, y_{2t-1}) + \gamma\omega_1(y_{2t+1}, y_{2t}) \\ & +\delta\omega_1(y_{2t-1}, y_{2t+1})\Big), \phi(\omega_1(y_{2t-1}, y_{2t}), \omega_1(y_{2t-1}, y_{2t+1}), 0)\Big) \\ \leq\ & \psi(\omega_1(y_{2t-1}, y_{2t}). \end{aligned} \tag{7}$$

- If n is odd, then $n = 2t + 1$ for some $t \in \mathbb{N}$. Using the same arguments as in the case of an even number, we can prove that:

$$\omega_1(y_{2t+2}, y_{2t+1}) < \omega_1(y_{2t+1}, y_{2t}). \tag{8}$$

From (6) and (8), we have:

$$\omega_1(y_n, y_{n+1}) < \omega_1(y_{n-1}, y_n), \ \forall n \in \mathbb{N}.$$

Therefore, $\{\omega_1(y_{n+1}, y_n) : n \in \mathbb{N}\}$ is a nonincreasing sequence. Thus, there exists $r \geq 0$ such that:

$$\lim_{n\to+\infty} \omega_1(y_n, y_{n+1}) = r.$$

By taking lim inf in (7), we find:

$$F\left(\psi\left((\alpha+\beta+\gamma)r + \delta \cdot \liminf_{t\to+\infty} \omega_1(y_{2t-1}, y_{2t+1})\right), \phi(r, \liminf_{t\to+\infty} \omega_1(y_{2t-1}, y_{2t+1}), 0)\right) = \psi(r). \tag{9}$$

Assuming that $\delta = 0$, we find:

$$F\left(\psi(r), \phi(r, \liminf_{t\to+\infty} \omega_1(y_{2t-1}, y_{2t+1}), 0)\right) = \psi(r),$$

and since $F \in \mathcal{C}$, it follows $\psi(r) = 0$ or $\phi(r, \liminf_{t\to+\infty} \omega_1(y_{2t-1}, y_{2t+1}), 0) = 0$; both relations bring us to the conclusion that $r = 0$.

Assume now that $\delta > 0$. Equation (9) leads to:

$$\psi(r) \leq \psi\left((\alpha+\beta+\gamma)r + \delta \cdot \liminf_{t\to+\infty} \omega_1(y_{2t-1}, y_{2t+1})\right),$$

that is

$$r \leq (\alpha+\beta+\gamma)r + \delta \cdot \liminf_{t\to+\infty} \omega_1(y_{2t-1}, y_{2t+1}),$$

and even simpler, after dividing with δ,

$$2r \leq \liminf_{t\to+\infty} \omega_1(y_{2t-1}, y_{2t+1}).$$

On the other side, due to the triangle inequality, we also have:

$$\omega_1(y_{2t-1}, y_{2t+1}) \leq \omega_1(y_{2t-1}, y_{2t}) + \omega_1(y_{2t}, y_{2t+1}),$$

which gives:

$$\liminf_{t\to+\infty} \omega_1(y_{2t-1}, y_{2t+1}) \leq 2r.$$

Therefore, $\liminf_{t\to+\infty} \omega_1(y_{2t-1}, y_{2t+1}) = 2r$. Substituting this in (9) brings us to:

$$F\left(\psi\left(r\right), \phi(r, 2r, 0)\right) = \psi(r),$$

which finally leads (due to the properties of F, ψ, and φ) to the conclusion $r = 0$. Hence:

$$\lim_{n\to+\infty} \omega_1(y_n, y_{n+1}) = 0. \tag{10}$$

In the following, we take one more step closer to proving the convergence of the sequence $\{y_n\}$. For this, we show that $\{y_n\}$ is a Cauchy sequence in the complete non-Archimedean metric modular space X_ω. It is sufficient to show that $\{y_{2t}\}$ is a Cauchy sequence in X_ω. Suppose on the contrary, namely, that $\{y_{2t}\}$ is not a Cauchy sequence in X_ω. According to Lemma 1, there exist $\varepsilon > 0$ and two subsequences $\{y_{2m_i}\}$ and $\{y_{2n_i}\}$ of $\{y_{2n}\}$ such that:

1. $\omega_1(y_{2m_i}, x_{2n_i}) \geq \varepsilon$;
2. $\omega_1(y_{2m_i}, y_{2n_i-2}) < \varepsilon$;
3. $\lim_{i\to+\infty} \omega_1(x_{2m_i}, x_{2n_i}) = \varepsilon$.

Using again the triangle inequality and Relation (10), we can easily prove that:

$$\begin{aligned}\lim_{i\to+\infty} \omega_1(y_{2m_i}, y_{2n_i-1}) &= \lim_{i\to+\infty} \omega_1(y_{2m_i+1}, y_{2n_i-1}) = \lim_{i\to+\infty} \omega_1(y_{2m_i+1}, y_{2n_i}) \\ &= \lim_{i\to+\infty} \omega_1(y_{2m_i}, y_{2n_i}) = \epsilon.\end{aligned}$$

Since x_{2n_i} and x_{2m_i+1} are comparable, we have:

$$\begin{aligned}
& \psi(\omega_1(y_{2n_i}, y_{2m_i+1})) = \psi(\omega_1(fx_{2n_i}, gx_{2m_i+1})) \\
\leq\ & F\Big(\psi\Big(\alpha\omega_1(Sx_{2n_i}, Tx_{2m_i+1}) + \beta\omega_1(fx_{2n_i}, Sx_{2n_i}) \\
& +\gamma\omega_1(gx_{2m_i+1}, Tx_{2m_i+1}) + \delta\omega_1(Sx_{2n_i}, gx_{2m_i+1}) + \delta\omega_1(fx_{2n_i}, Tx_{2m_i+1})\Big), \\
& \phi(\omega_1(Sx_{2n_i}, Tx_{2m_i+1}), \omega_1(Sx_{2n_i}, gx_{2m_i+1}), \omega_1(fx_{2n_i}, Tx_{2m_i+1}))\Big) \\
& +L\min\{\omega_1(Sx_{2n_i}, Tx_{2m_i+1}), \omega_1(Sx_{2n_i}, gx_{2m_i+1}), \omega_1(fx_{2n_i}, Tx_{2m_i+1})\} \\
=\ & F\Big(\psi\Big(\alpha\omega_1(y_{2n_i-1}, y_{2m_i}) + \beta\omega_1(y_{2n_i}, y_{2n_i-1}) \\
& +\gamma\omega_1(y_{2m_i+1}, y_{2m_i}), \delta\omega_1(y_{2n_i}, y_{2m_i}) + \delta\omega_1(y_{2n_i-1}, y_{2m_i+1})\Big), \\
& \phi(\omega_1(y_{2n_i-1}, y_{2m_i}), \omega_1(y_{2n_i}, y_{2m_i}), \omega_1(y_{2n_i-1}, y_{2m_i+1}))\Big) \\
& +L\min\{\omega_1(y_{2n_i-1}, y_{2m_i}), \omega_1(y_{2n_i}, y_{2m_i}), \omega_1(y_{2n_i-1}, y_{2m_i+1})\}.
\end{aligned}$$

Letting $i \to +\infty$ and using the continuity of F, ψ, and ϕ, we get that:

$$\psi(\epsilon) \leq F\Big(\psi\Big((\alpha + 2\delta)\epsilon\Big), \phi(\epsilon, \epsilon, \epsilon)\Big) + L\epsilon.$$

By Condition (3), we also have:

$$F\Big(\psi\Big((\alpha + 2\delta)\epsilon\Big), \phi(\epsilon, \epsilon, \epsilon)\Big) + L\epsilon < \psi(\epsilon),$$

which is impossible. Therefore, our assumption that y_{2n} is not a Cauchy sequence does not hold. Moreover, because of the triangle inequality, combined with Relation (10), we may conclude that $\{y_n\}$ itself is a Cauchy sequence in X_ω.

Ultimately, by the completeness of X_ω, there exists $y \in X_\omega$ such that:

$$\lim_{n\to+\infty} \omega_1(y_n, y) = 0. \tag{11}$$

Step 2. In the next section of the proof, we shall focus on proving that y is a common fixed point of g and T.

For this, we turn our attention to Condition (7) in the hypotheses. Assume that TX_ω is closed. Since $\{y_{2t} = Tx_{2t+1}\}$ is a sequence in TX_ω ω-convergent to y, it follows that $y \in TX_\omega$; hence, there exists $u \in X_\omega$ such that $y = Tu$. Therefore,

$$\lim_{n\to+\infty} \omega_1(y_n, y) = \lim_{n\to+\infty} \omega_1(y_n, Tu) = 0.$$

Now, we show that $gu = y = Tu$.

Since, as we checked at the beginning of our proof, $\{x_n\}$ is a nondecreasing sequence and $x_n \preceq y_n$ with y_n ω-convergent to y, it follows, from property (π_ω), that $x_n \preceq y$. Since the mapping f is dominating and a weak annihilator of T, we obtain $x_{2t} \preceq y = Tu \preceq fTu \preceq u$. Thus:

$$\begin{aligned}
&\psi(\omega_1(y_{2t}, gu)) = \psi(\omega_1(fx_{2t}, gu)) \\
\leq\; & F\Big(\psi\Big(\alpha\omega_1(Sx_{2t}, Tu) + \beta\omega_1(fx_{2t}, Sx_{2t}) + \gamma\omega_1(gu, Tu) \\
& +\delta\omega_1(Sx_{2t}, gu) + \delta\omega_1(fx_{2t}, Tu)\Big), \phi(\omega_1(Sx_{2t}, Tu), \omega_1(Sx_{2t}, gu), \omega_1(fx_{2t}, Tu))\Big) \\
& +L\min\{\omega_1(Sx_{2t}, Tu), \omega_1(Sx_{2t}, gu), \omega_1(fx_{2t}, Tu)\} \\
=\; & F\Big(\psi\Big(\alpha\omega_1(y_{2t-1}, y) + \beta\omega_1(y_{2t}, y_{2t-1}) + \gamma\omega_1(gu, y) \\
& +\delta\omega_1(y_{2t-1}, gu) + \delta\omega_1(y_{2t}, y)\Big), \phi(\omega_1(y_{2t-1}, y), \omega_1(y_{2t-1}, gu), \omega_1(y_{2t}, y))\Big) \\
& +L\min\{\omega_1(y_{2t-1}, y), \omega_1(y_{2t-1}, gu), \omega_1(y_{2t}, y)\}.
\end{aligned}$$

Letting $n \to +\infty$ in the above inequalities and using (10) and (11), as well as the continuity of the metric modular stated in Remark 2, we get that:

$$\psi(\omega_1(y, gu)) \leq F\Big(\psi(\gamma\omega_1(gu, y) + \delta\omega_1(gu, y)), \phi(0, \omega_1(y, gu), 0)\Big).$$

If we are in the particular case when $\gamma = 1$ and $\alpha = \beta = \delta = 0$, the above inequality becomes:

$$\psi(\omega_1(y, gu)) \leq F\Big(\psi(\omega_1(gu, y)), \phi(0, \omega_1(y, gu), 0)\Big),$$

and since $F \in \mathcal{C}$, this ultimately leads to $\omega_1(gu, y) = 0$, that is $gu = y$.

Otherwise, $\gamma + \delta < 1$ and:

$$\psi(\omega_1(y, gu)) \leq \psi(\gamma\omega_1(gu, y) + \delta\omega_1(gu, y)),$$

that is:

$$\omega_1(y, gu) \leq \gamma\omega_1(gu, y) + \delta\omega_1(gu, y) = (\gamma + \delta)\omega_1(y, gu),$$

leading to the same conclusion. Hence, $gu = y = Tu$. Since g and T are weakly compatible, we also have:

$$gy = gTu = Tgu = Ty.$$

Now, by the comparability of x_{2t} and y, we have:

$$\begin{aligned}
&\psi(\omega_1(y_{2t}, gy)) = \psi(\omega_1(fx_{2t}, gy)) \\
\leq\ & F\Big(\psi\Big(\alpha\omega_1(Sx_{2t}, Ty) + \beta\omega_1(fx_{2t}, Sx_{2t}) + \gamma\omega_1(gy, Ty) \\
&+\delta\omega_1(Sx_{2t}, gy) + \delta\omega_1(fx_{2t}, Ty)\Big), \phi(\omega_1(Sx_{2t}, Ty), \omega_1(Sx_{2t}, gy), \omega_1(fx_{2t}, Ty))\Big) \\
&+L\min\{\omega_1(Sx_{2t}, Ty), \omega_1(Sx_{2t}, gy), \omega_1(fx_{2t}, Ty)\} \\
=\ & F\Big(\psi\Big(\alpha\omega_1(y_{2t-1}, Ty) + \beta\omega_1(y_{2t}, y_{2t-1}) + \gamma\omega_1(gy, Ty) \\
&+\delta\omega_1(y_{2t-1}, gy) + \delta\omega_1(y_{2t}, Ty)\Big), \phi(\omega_1(y_{2t-1}, Ty), \omega_1(y_{2t-1}, gy), \omega_1(y_{2t}, Ty))\Big) \\
&+L\min\{\omega_1(y_{2t-1}, Ty), \omega_1(y_{2t-1}, gy), \omega_1(y_{2t}, Ty)\}.
\end{aligned}$$

Letting $n \to +\infty$ in the above inequalities and using again (10) and (11), we obtain:

$$\begin{aligned}
&\psi(\omega_1(y, gy)) \\
&\leq F\Big(\psi\Big(\alpha\omega_1(y, Ty) + \delta\omega_1(y, Ty) + \delta\omega_1(y, Ty)\Big), \phi(\omega_1(y, Ty), \omega_1(y, Ty), \omega_1(y, Ty)\Big) \\
&+L\omega_1(y, Ty) \\
&\leq F\Big(\psi\Big(\alpha\omega_1(y, gy) + \delta\omega_1(y, gy) + \delta\omega_1(y, gy)\Big), \phi(\omega_1(y, gy), \omega_1(y, gy), \omega_1(y, gy)\Big) \\
&+L\omega_1(y, gy).
\end{aligned}$$

If assuming $\omega_1(y, gy) > 0$, we find, by considering again Condition (3) from the hypotheses, $\psi(\omega_1(y, gy)) < \psi(\omega_1(y, gy))$, which is impossible. Thus, $\omega_1(y, gy) = 0$, and hence, $gy = y = Ty$.

Step 3. Finally, we shall prove that y is a common fixed point for f and S, as well.

As $gX_\omega \subseteq SX_\omega$, we have $y = gy \in SX_\omega$, so there exists $v \in X_\omega$ such that $y = gy = Ty = Sv$. Since the mapping g is dominating and a weak annihilator of S, we have $y = gy = Sv \preceq gSv \preceq v$. Thus, y and v are comparable, and hence:

$$\begin{aligned}
&\psi(\omega_1(fv, y)) = \psi(\omega_1(fv, gy)) \\
\leq\ & F\Big(\psi\Big(\alpha\omega_1(Sv, Ty) + \beta\omega_1(fv, Sv) + \gamma\omega_1(gy, Ty) \\
&+\delta\omega_1(Sv, gy) + \delta\omega_1(fv, Ty)\Big), \phi(\omega_1(Sv, Ty), \omega_1(Sv, gy), \omega_1(fv, Ty))\Big) \\
&+L\min\{\omega_1(Sv, Ty), \omega_1(Sv, gy), \omega_1(fv, Ty)\} \\
=\ & F\Big(\psi\Big(\beta\omega_1(fv, y) + \delta\omega_1(fv, y)\Big), \phi(0, 0, \omega_1(fv, y))\Big) \\
\leq\ & \psi\Big(\beta\omega_1(fv, y) + \delta\omega_1(fv, y)\Big).
\end{aligned}$$

Analyzing again the cases $\beta = 1$ and $\beta + \delta < 1$, we find $\omega_1(fv, y) = 0$. Thus, $y = gy = Ty = Sv = fv$. Since f and S are weakly compatible, we also have:

$$fy = fSv = Sfv = Sy.$$

Finally, using the fact that y is comparable with itself, we find:

$$\begin{aligned}
&\psi(\omega_1(fy,y)) = \psi(\omega_1(fy,gy))\\
\leq\ & F\Big(\psi\Big(\alpha\omega_1(Sy,Ty) + \beta\omega_1(fy,Sy) + \gamma\omega_1(gy,Ty)\\
&+\delta\omega_1(Sy,gy) + \delta\omega_1(fy,Ty)\Big), \phi(\omega_1(Sy,Ty), \omega_1(Sy,gy), \omega_1(fy,Ty))\Big)\\
&+L\min\{\omega_1(Sy,Ty), \omega_1(Sy,gy), \omega_1(fy,Ty)\}\\
=\ & F\Big(\psi\Big(\alpha\omega_1(fy,y) + \delta\omega_1(fy,y) + \delta\omega_1(fy,y)\Big), \phi(\omega_1(fy,y), \omega_1(fy,y), \omega_1(fy,y))\Big)\\
&+L\omega_1(fy,y).
\end{aligned}$$

Hence, according to Condition (3) in the hypotheses' list, we have $\omega_1(fy,y) = 0$, that is $fy = y = Sy$. Consequently, f, g, T, and S have a common fixed point. If fX_ω is not closed, but one of the other sets in Condition (7) is closed, we follow similar arguments as above to prove the existence of a common fixed point. □

The following apparently more general result is in fact a consequence of the previous theorem.

Corollary 1. *Let $(X_\omega, \preceq)$ be an ω-complete ordered non-Archimedean metric modular space. Let f, g, T, S be self-mappings of X_ω such that for any two comparable elements $x, y \in X_\omega$, the mappings f and g satisfy the following condition: there exist $\lambda_0 > 0$, $\psi \in \Psi$, $\varphi \in \Phi_1$, $F \in \mathcal{C}$, and $L \in [0, +\infty)$ such that:*

$$\begin{aligned}
&\psi(\lambda_0\omega_1(fx,gy)) \leq F\Big(\psi\Big(\alpha\lambda_0\omega_1(Sx,Ty) + \beta\lambda_0\omega_1(fx,Sx) + \gamma\lambda_0\omega_1(gy,Ty)\\
&+\delta\lambda_0\omega_1(Sx,gy) + \delta\lambda_0\omega_1(fx,Ty)\Big), \phi\left(\omega_1(Sx,Ty), \omega_1(Sx,gy), \omega_1(fx,Ty)\right)\Big)\\
&+L\min\left\{\omega_1(Sx,Ty), \omega_1(Sx,gy), \omega_1(fx,Ty)\right\},
\end{aligned}\tag{12}$$

for all $x, y \in X$, where $\alpha, \beta, \gamma, \delta \geq 0$, with $\alpha + \beta + \gamma + 2\delta = 1$. Assume also the following assertions:

(1) $fX_\omega \subseteq TX_\omega$*;*
(2) $gX_\omega \subseteq SX_\omega$*;*
(3) $F(\psi(\beta\lambda_0), \phi(\alpha,\alpha,\alpha)) + L\alpha < \psi(\alpha\lambda_0)$ *for all $\alpha, \beta > 0$ with $\beta \leq \alpha$;*
(4) *f is dominating and a weak annihilator of T;*
(5) *g is dominating and a weak annihilator of S;*
(6) *$\{f, S\}$ and $\{g, T\}$ are weakly compatible;*
(7) *one of $fX_\omega, gX_\omega, SX_\omega$, and TX_ω is ω-closed;*
(8) *X_ω has the property (π_ω).*

Then, f, g, S, and T have a common fixed point.

Proof. By rewriting Condition (13) using the function $\bar{\psi} \in \Psi$, $\bar{\psi}(t) = \psi(\lambda_0 t)$, we arrive exactly at the hypotheses of Theorem 2, hence the conclusion. □

The following example shows the useability of our results.

Example 1. On $X = [1, +\infty)$, consider the metric modular:

$$\omega\colon (0,\infty) \times X \times X \to [0, +\infty],\ \omega_\lambda(x,y) = \frac{|x-y|}{\sqrt{\lambda}}.$$

Let us also consider $F\colon [0,\infty)^2 \to \mathbb{R}$, $F(s,t) = \frac{1}{4}s$; $\psi\colon [0,+\infty) \to [0,+\infty)$, $\psi(t) = t$; $\phi \in \Phi$ arbitrary, and $L = 0$. Define a relation on X by $x \preceq y$ if and only if $y \leq x$. Furthermore, define the mappings $f, g, S, T\colon X \to X$ by the formulas $fx = gx = \sqrt{x}$, $Tx = Sx = x^2$. Then:

1. ω is a non-Archimedean metric modular, which is not convex;
2. $X_\omega = X = [1, \infty)$; moreover, X_ω is complete in the sense defined by Abdou (see Definition 2).
3. $F \in \mathcal{C}$, $\psi \in \Psi$,
4. $F(\psi(\beta), \phi(\alpha, \alpha, \alpha)) + L\alpha < \psi(\alpha)$ for all $\alpha, \beta > 0$ with $\beta \le \alpha$;
5. $fX \subseteq TX$,
6. f is dominating and a weak annihilator of T,
7. The pair $\{f, S\}$ is weakly compatible,
8. fX is a closed subset of X_ω,
9. X_ω satisfies the property (π_ω), and
10. f and g satisfy the nonlinear (S, T, L, F, ψ, ϕ)-convex contractive condition of type I, for $\alpha = 1$ and $\beta = \gamma = \delta = 0$.

Proof. The positivity and the symmetry of ω are trivial properties. Let us focus on the last property. Assume that $\lambda \ge \mu$. We notice that:

$$\begin{aligned} \omega_{\max\{\lambda,\mu\}}(x,y) &= \omega_\lambda(x,y) \\ &= \frac{|x-y|}{\sqrt{\lambda}} \\ &\le \frac{|x-z|}{\sqrt{\lambda}} + \frac{|z-y|}{\sqrt{\lambda}} \\ &\le \frac{|x-z|}{\sqrt{\lambda}} + \frac{|z-y|}{\sqrt{\mu}} \\ &= \omega_\lambda(x,z) + \omega_\mu(z,y), \end{aligned}$$

hence ω is non-Archimedean. To prove that ω is not convex, we turn our attention to Remark 1. Indeed, in our example, the function $\lambda \to \lambda\omega_\lambda(x,y) = \sqrt{\lambda}|x-y|$ is nondecreasing, so "the main property of a convex modular" is not satisfied.

The proofs of Parts (2) to (9) are clear. The condition stated on (10) is equivalent, for the selected elements F, ψ, ϕ, and L with:

$$\omega_1(fx, gy) \le \frac{1}{4}\omega_1(Sx, Ty).$$

This holds true, since we have:

$$\begin{aligned} \omega_1(fx,gy) &= |\sqrt{x} - \sqrt{y}| \\ &= \frac{|x-y|}{|\sqrt{x}+\sqrt{y}|} \\ &= \frac{|x^2-y^2|}{(\sqrt{x}+\sqrt{y})(x+y)} \\ &\le \frac{1}{4}|x^2-y^2| \\ &= \frac{1}{4}\omega_1(Sx,Ty). \end{aligned}$$

Thus, Example 1 satisfies all the hypotheses of Theorem 2. Therefore, f, g, T, and S have a common fixed point. Here, 1 is the common fixed point of f, g, T, and S. □

Example 2. *Let us consider now the same metric modular space as in the example above and the same mappings* f, g, S, T. *In addition, take* $F\colon [0,\infty)^2 \to \mathbb{R}$, $F(s,t) = s - t$, $\psi\colon [0,+\infty) \to [0,+\infty)$, $\psi(t) = t$, *the control* *function* $\phi\colon [0,\infty)^3 \to [0,\infty)$, $\phi(t,s,u) = \frac{1}{4}(|t|+|s|+|u|)$, *and* $L = \frac{1}{2}$. *Then, the conditions (1)–(3) and (5)–(9) listed in the previous example are satisfied again. Let us now take a closer look at Condition* (4). *We have:*

$$F(\psi(\beta),\phi(\alpha,\alpha,\alpha)) + L\alpha = \beta - \frac{3\alpha}{4} + \frac{\alpha}{2} = \beta - \frac{\alpha}{4} < \alpha,$$

which holds true for all $\alpha,\beta > 0$ *with* $\beta \le \alpha$. *Finally, let us also prove that* f *and* g *satisfy the nonlinear* (S,T,L,F,ψ,ϕ)*-convex contractive condition of type I, for* $\alpha = \frac{1}{2}$, $\beta = \gamma = 0$, *and* $\delta = \frac{1}{4}$. *Indeed, for these particular choices, Inequality* (4) *in Definition* 4 *becomes:*

$$\begin{aligned} \omega_1(fx,gy) \quad &\le \quad \frac{1}{2}\omega_1(Sx,Ty) + \frac{1}{4}\omega_1(Sx,gy) + \frac{1}{4}\omega_1(fx,Ty) \\ &\quad -\frac{1}{4}\left(\omega_1(Sx,Ty) + \omega_1(Sx,gy) + \omega_1(fx,Ty)\right) \\ &\quad +\frac{1}{2}\min\left\{\omega_1(Sx,Ty),\omega_1(Sx,gy),\omega_1(fx,Ty)\right\}, \end{aligned}$$

that is

$$|fx-gy| \le \frac{1}{4}|Sx-Ty| + \frac{1}{2}\min\left\{|Sx-Ty|,|Sx-gy|,|fx-Ty|\right\},$$

or, after substituting f,g,S,T,

$$|\sqrt{x}-\sqrt{y}| \le \frac{1}{4}|x^2-y^2| + \frac{1}{2}\min\left\{|x^2-y^2|,|x^2-\sqrt{y}|,|\sqrt{x}-y^2|\right\}.$$

This condition is satisfied, as seen before. □

4. Second Extension to Partially Ordered Non-Archimedean Modular Spaces

Definition 5. Let f,g,S, and T be self-mappings on a non-Archimedean modular metric space X_ω. Then, f and g are said to satisfy the almost nonlinear (S,T,L,F,ψ,ϕ)-convex contractive condition of type II if there exist $\psi \in \Psi, \phi \in \Phi_1$, $F \in \mathcal{C}$, and $L \in [0,+\infty)$ such that:

$$\begin{aligned} &\psi(\omega_1(fx,gy)) \le F\Big(\psi\Big(\omega_{\frac{1}{\alpha}}(Sx,Ty) + \omega_{\frac{1}{\beta}}(fx,Sx) + \omega_{\frac{1}{\gamma}}(gy,Ty) \\ &+\omega_{\frac{1}{\delta}}(Sx,gy) + \omega_{\frac{1}{\delta}}(fx,Ty)\Big), \phi\left(\omega_1(Sx,Ty),\omega_1(Sx,gy),\omega_1(fx,Ty)\right)\Big) \\ &+L\min\left\{\omega_1(Sx,Ty),\omega_1(Sx,gy),\omega_1(fx,Ty)\right\}, \end{aligned} \tag{13}$$

for all $x,y \in X_\omega$, where $\alpha,\beta,\gamma,\delta > 0$, with $\alpha+\beta+\gamma+2\delta = 1$.

Now, we present the main result of this section. We emphasize the fact that it needs some stronger requirements regarding the modular than the outcome of the previous section. More precisely, we shall consider the non-Archimedean metric modular, which is also convex. In fact, the convexity of the modular interferes in our arguments, not directly, but through one of its immediate consequences, namely the following inequality (resulting from the monotonicity of $\lambda \to \lambda\omega_\lambda(x,y)$):

$$\omega_{\frac{1}{\lambda}}(x,y) \le \lambda\omega_1(x,y), \ \forall\lambda \le 1. \tag{14}$$

Theorem 3. *Let* $(X_\omega,\preceq)$ *be a complete ordered non-Archimedean metric modular space, induced by a convex modular. Let* f,g,T,S *be self-mappings of* X_ω *such that for any two comparable elements* $x,y \in X_\omega$, *the mappings* f *and* g *satisfy the nonlinear* (S,T,L,F,ψ,ϕ)*-convex contractive condition* (14). *Assume the following assertions:*

(1) $fX_\omega \subseteq TX_\omega$;
(2) $gX_\omega \subseteq SX_\omega$;
(3) $F(\psi(\beta), \phi(\alpha, \alpha, \alpha)) + L\alpha < \psi(\alpha)$ *for all* $\alpha, \beta > 0$ *with* $\beta \leq \alpha$;
(4) f *is dominating and a weak annihilator of* T;
(5) g *is dominating and a weak annihilator of* S;
(6) $\{f, S\}$ *and* $\{g, T\}$ *are weakly compatible;*
(7) *one of* $fX_\omega, gX_\omega, SX_\omega$, *and* TX_ω *is a closed subspace of* X_ω; *and*
(8) X_ω *has the property* (π_ω).

Then, f, g, S, *and* T *have a common fixed point.*

Proof. Start with $x_0 \in X_\omega$. By using Hypotheses (1) and (2), we generate two sequences $\{x_n\}, \ \{y_n\} \in X_\omega$ in such a way that $y_{2t} := fx_{2t} = Tx_{2t+1}$ and $y_{2t+1} := gx_{2t+1} = Sx_{2t+2}$. Using (4) and (5), we have:

$$x_{2t} \preceq fx_{2t} = Tx_{2t+1} \preceq fTx_{2t+1} \preceq x_{2t+1} \preceq gx_{2t+1} = Sx_{2t+2} \preceq gSx_{2t+2} \preceq x_{2t+2},$$

which means that $x_n \preceq x_{n+1}$ for any nonnegative integer n; therefore, they are comparable.

Step 1. In the following, we shall focus on proving that $\{y_n\}$ is convergent.

Case I. Let us assume that there exists $n_0 \in \mathbb{N}$ such that $y_{n_0} = y_{n_0+1}$.

- If n_0 is even, that is $n_0 = 2t$, we have $y_{2t} = y_{2t+1}$. Using the fact that x_{2t+1} and x_{2t+2} are comparable and Condition (14), we have:

$$\begin{aligned}
& \psi(\omega_1(y_{2t+2}, y_{2t+1})) = \psi(\omega_1(fx_{2t+2}, gx_{2t+1})) \\
\leq \ & F\Big(\psi\Big(\omega_{\frac{1}{\alpha}}(Sx_{2t+2}, Tx_{2t+1}) + \omega_{\frac{1}{\beta}}(fx_{2t+2}, Sx_{2t+2}) \\
& + \omega_{\frac{1}{\gamma}}(gx_{2t+1}, Tx_{2t+1}) + \omega_{\frac{1}{\delta}}(Sx_{2t+2}, gx_{2t+1}) + \omega_{\frac{1}{\delta}}(fx_{2t+2}, Tx_{2t+1})\Big), \\
& \phi(\omega_1(Sx_{2t+2}, Tx_{2t+1}), \omega_1(Sx_{2t+2}, gx_{2t+1}), \omega_1(fx_{2t+2}, Tx_{2t+1}))\Big) \\
& + L \min\{\omega_1(Sx_{2t+2}, Tx_{2t+1}), \omega_1(Sx_{2t+2}, gx_{2t+1}), \omega_1(fx_{2t+2}, Tx_{2t+1})\} \\
= \ & F\Big(\psi\Big(\omega_{\frac{1}{\alpha}}(y_{2t+1}, y_{2t}) + \omega_{\frac{1}{\beta}}(y_{2t+2}, y_{2t+1}) + \omega_{\frac{1}{\gamma}}(y_{2t+1}, y_{2t}) \\
& + \omega_{\frac{1}{\delta}}(y_{2t+1}, y_{2t+1}) + \omega_{\frac{1}{\delta}}(y_{2t+2}, y_{2t})\Big), \\
& \phi(\omega_1(y_{2t+1}, y_{2t}), \omega_1(y_{2t+1}, y_{2t+1}), \omega_1(y_{2t+2}, y_{2t}))\Big) \\
& + L \min\{\omega_1(y_{2t+1}, y_{2t}), \omega_1(y_{2t+1}, y_{2t+1}), \omega_1(y_{2t+2}, y_{2t})\} \\
= \ & F\Big(\psi\Big(\omega_{\frac{1}{\beta}}(y_{2t+2}, y_{2t+1}) + \omega_{\frac{1}{\delta}}(y_{2t+2}, y_{2t})\Big), \phi(0, 0, \omega_1(y_{2t+2}, y_{2t+1}))\Big).
\end{aligned}$$

Using the properties of F, we have:

$$\psi(\omega_1(y_{2t+2}, y_{2t+1}) \leq \psi\Big(\omega_{\frac{1}{\beta}}(y_{2t+2}, y_{2t+1}) + \omega_{\frac{1}{\delta}}(y_{2t+2}, y_{2t})\Big).$$

Since ψ is nondecreasing, then the last inequality holds only if:

$$\omega_1(y_{2t+2}, y_{2t+1}) \leq \omega_{\frac{1}{\beta}}(y_{2t+2}, y_{2t+1}) + \omega_{\frac{1}{\delta}}(y_{2t+2}, y_{2t}),$$

which, using the triangle inequality (1), leads to:

$$\begin{aligned}\omega_1(y_{2t+2}, y_{2t+1}) &\leq \omega_{\frac{1}{\beta}}(y_{2t+2}, y_{2t+1}) + \omega_{\frac{1}{\delta}}(y_{2t+2}, y_{2t+1}) + \omega_{\frac{1}{\delta}}(y_{2t+1}, y_{2t}) \\ &= \omega_{\frac{1}{\beta}}(y_{2t+2}, y_{2t+1}) + \omega_{\frac{1}{\delta}}(y_{2t+2}, y_{2t+1}).\end{aligned}$$

Moreover, the conditions $\alpha, \beta, \gamma, \delta > 0$ and $\alpha + \beta + \gamma + 2\delta = 1$ lead to $\frac{1}{\beta}, \frac{1}{\delta} > 1$, and using Inequality (14), we find $\omega_{\frac{1}{\beta}}(y_{2t+2}, y_{2t+1}) \leq \beta\omega_1(y_{2t+2}, y_{2t+1})$ and $\omega_{\frac{1}{\delta}}(y_{2t+2}, y_{2t+1}) \leq \delta\omega_1(y_{2t+2}, y_{2t+1})$; thus:

$$\omega_1(y_{2t+2}, y_{2t+1}) \leq (\beta + \gamma)\,\omega_1(y_{2t+2}, y_{2t+1}),$$

which makes sense only if $\omega_1(y_{2t+2}, y_{2t+1}) = 0$ and, hence, $y_{2t+2} = y_{2t+1}$.

- If n_0 is odd, that is $n_0 = 2t + 1$, by using the same technique, we find that $y_{2t+3} = y_{2t+2}$.

  Combining these two items, we may conclude that, starting with n_0, the sequence $\{y_n\}$ is a constant sequence in X_ω, and hence, it is convergent.

Case II. Let us assume now that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$. We analyze again, separately, the situation of n being even and the opposite of this.

- If n is even, then $n = 2t$ for some $t \in \mathbb{N}$. Using the comparability property of x_{2t} and x_{2t+1}, we have:

$$\begin{aligned}&\psi(\omega_1(y_n, y_{n+1})) = \psi(\omega_1(y_{2t}, y_{2t+1})) = \psi(\omega_1(fx_{2t}, gx_{2t+1})) \\ \leq\ & F\Big(\psi\Big(\omega_{\frac{1}{\alpha}}(Sx_{2t}, Tx_{2t+1}) + \omega_{\frac{1}{\beta}}(fx_{2t}, Sx_{2t}) \\ &+\omega_{\frac{1}{\gamma}}(gx_{2t+1}, Tx_{2t+1}) + \omega_{\frac{1}{\delta}}(Sx_{2t}, gx_{2t+1}) + \omega_{\frac{1}{\delta}}(fx_{2t}, Tx_{2t+1})\Big), \\ &\phi(\omega_1(Sx_{2t}, Tx_{2t+1}), \omega_1(Sx_{2t}, gx_{2t+1}), \omega_1(fx_{2t}, Tx_{2t+1}))\Big) \\ &+L\min\{\omega_1(Sx_{2t}, Tx_{2t+1}), \omega_1(Sx_{2t}, gx_{2t+1}), \omega_1(fx_{2t}, Tx_{2t+1})\} \\ =\ & F\Big(\psi\Big(\omega_{\frac{1}{\alpha}}(y_{2t-1}, y_{2t}) + \omega_{\frac{1}{\beta}}(y_{2t}, y_{2t-1}) + \omega_{\frac{1}{\gamma}}(y_{2t+1}, y_{2t}) \\ &+\omega_{\frac{1}{\delta}}(y_{2t-1}, y_{2t+1})\Big), \phi(\omega_1(y_{2t-1}, y_{2t}), \omega_1(y_{2t-1}, y_{2t+1}), 0)\Big) \\ &+L\min\{\omega_1(y_{2t-1}, y_{2t}), \omega_1(y_{2t-1}, y_{2t+1}), 0\} \\ \leq\ & F\Big(\psi\Big(\omega_{\frac{1}{\alpha}}(y_{2t-1}, y_{2t}) + \omega_{\frac{1}{\beta}}(y_{2t}, y_{2t-1}) + \omega_{\frac{1}{\gamma}}(y_{2t+1}, y_{2t}) \\ &+\omega_{\frac{1}{\delta}}(y_{2t-1}, y_{2t+1})\Big), \phi(\omega_1(y_{2t-1}, y_{2t}), \omega_1(y_{2t-1}, y_{2t+1}), 0)\Big).\end{aligned}$$

If $\omega_1(y_{2t-1}, y_{2t}) \leq \omega_1(y_{2t}, y_{2t+1})$, then using again the triangle inequality for the non-Archimedean metric modular and Relation (14), together with the properties of ψ and F, we find:

$$\begin{aligned}
& \psi(\omega_1(y_n, y_{n+1})) = \psi(\omega_1(y_{2t}, y_{2t+1})) \\
\leq\ & F\Big(\psi\Big(\omega_{\frac{1}{\alpha}}(y_{2t-1}, y_{2t}) + \omega_{\frac{1}{\beta}}(y_{2t}, y_{2t-1}) + \omega_{\frac{1}{\gamma}}(y_{2t+1}, y_{2t}) \\
& +\omega_{\frac{1}{\delta}}(y_{2t-1}, y_{2t+1})\Big), \phi(\omega_1(y_{2t-1}, y_{2t}), \omega_1(y_{2t-1}, y_{2t+1}), 0)\Big) \\
\leq\ & \psi\Big(\omega_{\frac{1}{\alpha}}(y_{2t-1}, y_{2t}) + \omega_{\frac{1}{\beta}}(y_{2t}, y_{2t-1}) + \omega_{\frac{1}{\gamma}}(y_{2t+1}, y_{2t}) \\
& +\omega_{\frac{1}{\delta}}(y_{2t-1}, y_{2t+1})\Big) \\
\leq\ & \psi\Big(\omega_{\frac{1}{\alpha}}(y_{2t-1}, y_{2t}) + \omega_{\frac{1}{\beta}}(y_{2t}, y_{2t-1}) + \omega_{\frac{1}{\gamma}}(y_{2t+1}, y_{2t}) \\
& +\omega_{\frac{1}{\delta}}(y_{2t-1}, y_{2t}) + \omega_{\frac{1}{\delta}}(y_{2t}, y_{2t+1})\Big) \\
\leq\ & \psi\Big(\,(\alpha+\beta+\delta)\,\omega_1(y_{2t-1}, y_{2t}) + (\gamma+\delta)\,\omega_1(y_{2t}, y_{2t+1})\Big) \\
\leq\ & \psi\Big(\,(\alpha+\beta+\gamma+2\gamma)\,\omega_1(y_{2t}, y_{2t+1})\Big) \\
=\ & \psi(\omega_1(y_{2t}, y_{2t+1})) = \psi(\omega_1(y_n, y_{n+1})).
\end{aligned}$$

Thus,

$$\begin{aligned}
& F\Big(\psi\Big(\omega_{\frac{1}{\alpha}}(y_{2t-1}, y_{2t}) + \omega_{\frac{1}{\beta}}(y_{2t}, y_{2t-1}) + \omega_{\frac{1}{\gamma}}(y_{2t+1}, y_{2t}) \\
& +\omega_{\frac{1}{\delta}}(y_{2t-1}, y_{2t+1})\Big), \phi(\omega_1(y_{2t-1}, y_{2t}), \omega_1(y_{2t-1}, y_{2t+1}), 0)\Big) \\
& = \psi\Big(\omega_{\frac{1}{\alpha}}(y_{2t-1}, y_{2t}) + \omega_{\frac{1}{\beta}}(y_{2t}, y_{2t-1}) + \omega_{\frac{1}{\gamma}}(y_{2t+1}, y_{2t}) + \omega_{\frac{1}{\delta}}(y_{2t-1}, y_{2t+1})\Big).
\end{aligned}$$

Using the properties of F, we conclude that either:

$$\psi\Big(\omega_{\frac{1}{\alpha}}(y_{2t-1}, y_{2t}) + \omega_{\frac{1}{\beta}}(y_{2t}, y_{2t-1}) + \omega_{\frac{1}{\gamma}}(y_{2t+1}, y_{2t}) + \omega_{\frac{1}{\delta}}(y_{2t-1}, y_{2t+1})\Big) = 0$$

or

$$\phi(\omega_1(y_{2t-1}, y_{2t}), \omega_1(y_{2t-1}, y_{2t+1}), 0) = 0.$$

In both cases, we have $y_{2t-1} = y_{2t}$, a contradiction. Thus,

$$\omega_1(y_{2t}, y_{2t+1}) < \omega_1(y_{2t-1}, y_{2t}), \tag{15}$$

and:

$$\begin{aligned}
& \psi(\omega_1(y_{2t}, y_{2t+1})) \\
\leq\ & F\Big(\psi\Big(\omega_{\frac{1}{\alpha}}(y_{2t-1}, y_{2t}) + \omega_{\frac{1}{\beta}}(y_{2t}, y_{2t-1}) + \omega_{\frac{1}{\gamma}}(y_{2t+1}, y_{2t}) \\
& +\omega_{\frac{1}{\delta}}(y_{2t-1}, y_{2t+1})\Big), \phi(\omega_1(y_{2t-1}, y_{2t}), \omega_1(y_{2t-1}, y_{2t+1}), 0)\Big) \\
\leq\ & \psi(\omega_1(y_{2t-1}, y_{2t}).
\end{aligned} \tag{16}$$

- If n is odd, then $n = 2t + 1$ for some $t \in \mathbb{N}$. Using the same arguments as in the case of an even number, we can prove that:

$$\omega_1(y_{2t+2}, y_{2t+1}) < \omega_1(y_{2t+1}, y_{2t}). \tag{17}$$

From (15) and (17), we have:

$$\omega_1(y_n, y_{n+1}) < \omega_1(y_{n-1}, y_n).$$

Therefore, $\{\omega_1(y_{n+1}, y_n) : n \in \mathbb{N}\}$ is a non-increasing sequence. Thus, there exists $r \geq 0$ such that:

$$\lim_{n\to+\infty} \omega_1(y_n, y_{n+1}) = r.$$

Assume that $\lambda \leq 1$, and denote $r^1_\lambda = \liminf_{n\to+\infty} \omega_{\frac{1}{\lambda}}(y_n, y_{n+1})$. Then, according to (14), $\omega_{\frac{1}{\lambda}}(y_n, y_{n+1}) \leq \lambda\omega_1(y_n, y_{n+1})$, leading to:

$$r^1_\lambda \leq \lambda r.$$

In addition, $\omega_{\frac{1}{\lambda}}(y_n, y_{n+2}) \leq \omega_{\frac{1}{\lambda}}(y_n, y_{n+1}) + \omega_{\frac{1}{\lambda}}(y_{n+1}, y_{n+2})$, leading to:

$$r^2_\lambda = \liminf_{n\to+\infty} \omega_{\frac{1}{\lambda}}(y_n, y_{n+2}) \leq 2\lambda r.$$

By taking $\liminf$ in (16), we find:

$$F(\psi(r^1_\alpha + r^1_\beta + r^1_\gamma + r^2_\delta), \phi(r, \liminf_{t\to+\infty} \omega_1(y_{2t-1}, y_{2t+1}), 0)) = \psi(r).$$

This leads, on the one hand, to the following chain of inequalities:

$$\psi(r) \leq \psi(r^1_\alpha + r^1_\beta + r^1_\gamma + r^2_\delta) \Rightarrow r \leq r^1_\alpha + r^1_\beta + r^1_\gamma + r^2_\delta \leq [\alpha + \beta + \gamma + 2\delta]\, r = r,$$

and, consequently, by turning back into the equality relation, to

$$F(\psi(r), \phi(r, \liminf_{t\to+\infty} \omega_1(y_{2t-1}, y_{2t+1}), 0)) = \psi(r),$$

which ultimately means that either $\psi(r) = 0$ or $\phi(r, \liminf_{t\to+\infty} \omega_1(y_{2t-1}, y_{2t+1}), 0) = 0$. In both cases, we find $r = 0$; hence:

$$\lim_{n\to+\infty} \omega_1(y_n, y_{n+1}) = 0. \tag{18}$$

In the following, we take one more step closer to proving the convergence of the sequence $\{y_n\}$. For this, we show that $\{y_n\}$ is a Cauchy sequence in the complete strongly non-Archimedean metric modular space X_ω. It is sufficient to show that $\{y_{2t}\}$ is a Cauchy sequence in X_ω. Suppose the contrary; that is, $\{y_{2t}\}$ is not a Cauchy sequence in X_ω. According to Lemma 1, there exist $\varepsilon > 0$ and two subsequences $\{y_{2m_i}\}$ and $\{y_{2n_i}\}$ of $\{y_{2n}\}$ such that:

1. $i \leq 2m_i < 2n_i$;
2. $\omega_1(y_{2m_i}, x_{2n_i}) \geq \varepsilon$;
3. $\omega_1(y_{2m_i}, y_{2n_i-2}) < \varepsilon$;
4. $\lim_{i\to+\infty} \omega_1(x_{2m_i}, x_{2n_i}) = \varepsilon$.

Using again the triangle inequality and Relation (18), we can easily prove that:

$$\begin{aligned}\lim_{i\to+\infty}\omega_1(y_{2m_i},y_{2n_i-1}) &= \lim_{i\to+\infty}\omega_1(y_{2m_i+1},y_{2n_i-1}) = \lim_{i\to+\infty}\omega_1(y_{2m_i+1},y_{2n_i})\\ &= \lim_{i\to+\infty}\omega_1(y_{2m_i},y_{2n_i}) = \epsilon.\end{aligned}$$

By denoting:

$$\epsilon_\lambda^{(k,l)} = \liminf_{i\to\infty}\omega_{\frac{1}{\lambda}}(y_{2m_i+k},y_{2n_i+l}),$$

for two indices $k \in \{0,1\}$ and $l \in \{0,-1\}$ and using again Inequality (14), we find:

$$\epsilon_\lambda^{(0,0)}, \epsilon_\lambda^{(1,0)}, \epsilon_\lambda^{(0,-1)}, \epsilon_\lambda^{(1,-1)} \leq \lambda\varepsilon,\ \forall\lambda \geq 1.$$

Since x_{2n_i} and x_{2m_i+1} are comparable, we have:

$$\begin{aligned}
&\psi(\omega_1(y_{2n_i},y_{2m_i+1})) = \psi(\omega_1(fx_{2n_i},gx_{2m_i+1})\\
\leq\ & F\Big(\psi\Big(\omega_{\frac{1}{\alpha}}(Sx_{2n_i},Tx_{2m_i+1}) + \omega_{\frac{1}{\beta}}(fx_{2n_i},Sx_{2n_i})\\
&+\omega_{\frac{1}{\gamma}}(gx_{2m_i+1},Tx_{2m_i+1}) + \omega_{\frac{1}{\delta}}(Sx_{2n_i},gx_{2m_i+1}) + \omega_{\frac{1}{\delta}}(fx_{2n_i},Tx_{2m_i+1})\Big),\\
&\phi(\omega_1(Sx_{2n_i},Tx_{2m_i+1}),\omega_1(Sx_{2n_i},gx_{2m_i+1}),\omega_1(fx_{2n_i},Tx_{2m_i+1}))\Big)\\
&+L\min\{\omega_1(Sx_{2n_i},Tx_{2m_i+1}),\omega_1(Sx_{2n_i},gx_{2m_i+1}),\omega_1(fx_{2n_i},Tx_{2m_i+1})\}\\
=\ & F\Big(\psi\Big(\omega_{\frac{1}{\alpha}}(y_{2n_i-1},y_{2m_i}) + \omega_{\frac{1}{\beta}}(y_{2n_i},y_{2n_i-1})\\
&+\omega_{\frac{1}{\gamma}}(y_{2m_i+1},y_{2m_i}) + \omega_{\frac{1}{\delta}}(y_{2n_i},y_{2m_i}) + \omega_{\frac{1}{\delta}}(y_{2n_i-1},y_{2m_i+1})\Big),\\
&\phi(\omega_1(y_{2n_i-1},y_{2m_i}),\omega_1(y_{2n_i},y_{2m_i}),\omega_1(y_{2n_i-1},y_{2m_i+1}))\Big)\\
&+L\min\{\omega_1(y_{2n_i-1},y_{2m_i}),\omega_1(y_{2n_i},y_{2m_i}),\omega_1(y_{2n_i-1},y_{2m_i+1})\}.
\end{aligned}$$

Letting $i \to +\infty$ and using the continuity of F, ψ, ϕ, we get that:

$$\psi(\epsilon) \leq F\Big(\psi\Big(\epsilon_\alpha^{(0,-1)} + \epsilon_\delta^{(0,0)} + \epsilon_\delta^{(1,-1)}\Big), \phi(\epsilon,\epsilon,\epsilon)\Big) + L\epsilon.$$

By Condition (3), since $\epsilon > 0$ and $\epsilon_\alpha^{(0,-1)} + \epsilon_\delta^{(0,0)} + \epsilon_\delta^{(1,-1)} \leq (\alpha+\delta+\delta)\,\epsilon < \epsilon$, we get:

$$\psi(\epsilon) \leq F\Big(\psi\Big(\epsilon_\alpha^{(0,-1)} + \epsilon_\delta^{(0,0)} + \epsilon_\delta^{(1,-1)}\Big), \phi(\epsilon,\epsilon,\epsilon)\Big) + L\epsilon < \psi(\epsilon),$$

which is impossible. Therefore, our assumption that y_{2n} is not a Cauchy sequence does not hold. Moreover, because of the triangle inequality, combined with Relation (18), we may conclude that $\{y_n\}$ itself is a Cauchy sequence in X_ω.

Ultimately, by the completeness of X_ω, there exists $y \in X_\omega$ such that:

$$\lim_{n\to+\infty}\omega_1(y_n,y) = 0. \tag{19}$$

Step 2. In the next section of the proof, we shall focus on showing that y is a common fixed point of g and T.

For this, we turn our attention to Condition (7) in the hypotheses. Assume that TX_ω is closed. Since $\{y_{2t} = Tx_{2t+1}\}$ is a sequence in TX_ω convergent to y it follows that $y \in TX_\omega$; hence, there exists $u \in X_\omega$ such that $y = Tu$. Therefore,

$$\lim_{t\to+\infty}\omega_1(y_{2t},Tu) = 0.$$

Now, we show that $gu = y = Tu$.

Since, as we checked at the beginning of our proof, $\{x_{2t}\}$ and $\{y_{2t}\}$ are nondecreasing sequences with $x_{2t} \preceq y_{2t}$ and $y_{2t} \to y$, it follows that $x_{2t} \preceq y$. Since the mapping f is dominating and a weak annihilator of T, we obtain $x_{2t} \preceq y = Tu \preceq fTu \preceq u$. Thus:

$$
\begin{aligned}
& \psi(\omega_1(y_{2t}, gu)) = \psi(\omega_1(fx_{2t}, gu)) \\
\leq\ & F\Big(\psi\Big(\omega_{\frac{1}{\alpha}}(Sx_{2t}, Tu) + \omega_{\frac{1}{\beta}}(fx_{2t}, Sx_{2t}) + \omega_{\frac{1}{\gamma}}(gu, Tu) \\
& +\omega_{\frac{1}{\delta}}(Sx_{2t}, gu) + \omega_{\frac{1}{\delta}}(fx_{2t}, Tu)\Big), \phi(\omega_1(Sx_{2t}, Tu), \omega_1(Sx_{2t}, gu), \omega_1(fx_{2t}, Tu))\Big) \\
& +L\min\{\omega_1(Sx_{2t}, Tu), \omega_1(Sx_{2t}, gu), \omega_1(fx_{2t}, Tu)\} \\
=\ & F\Big(\psi\Big(\omega_{\frac{1}{\alpha}}(y_{2t-1}, y) + \omega_{\frac{1}{\beta}}(y_{2t}, y_{2t-1}) + \omega_{\frac{1}{\gamma}}(gu, y) \\
& +\omega_{\frac{1}{\delta}}(y_{2t-1}, gu) + \omega_{\frac{1}{\delta}}(y_{2t}, y)\Big), \phi(\omega_1(y_{2t-1}, y), \omega_1(y_{2t-1}, gu), \omega_1(y_{2t}, y))\Big) \\
& +L\min\{\omega_1(y_{2t-1}, y), \omega_1(y_{2t-1}, gu), \omega_1(y_{2t}, y)\}.
\end{aligned}
$$

Letting $t \to +\infty$ in the above inequalities and using (18) and (19), we get that:

$$
\psi(\omega_1(y, gu)) \leq F\Big(\psi(\omega_{\frac{1}{\gamma}}(gu, y) + \omega_{\frac{1}{\delta}}(gu, y)), \phi(0, \omega_1(y, gu), 0)\Big).
$$

Therefore:

$$
\psi(\omega_1(y, gu)) \leq \psi(\omega_{\frac{1}{\gamma}}(gu, y) + \omega_{\frac{1}{\delta}}(gu, y)),
$$

that is:

$$
\omega_1(y, gu) \leq \omega_{\frac{1}{\gamma}}(gu, y) + \omega_{\frac{1}{\delta}}(gu, y) \leq (\gamma + \delta)\, \omega_1(y, gu),
$$

leading to the conclusion that $\omega_1(gu, y) = 0$, hence $gu = y = Tu$. Since g and T are weakly compatible, we also have:

$$
gy = gTu = Tgu = Ty.
$$

Now, by the comparability of x_{2t} and y, we have:

$$
\begin{aligned}
& \psi(\omega_1(y_{2t}, gy)) = \psi(\omega_1(fx_{2t}, gy)) \\
\leq\ & F\Big(\psi\Big(\omega_{\frac{1}{\alpha}}(Sx_{2t}, Ty) + \omega_{\frac{1}{\beta}}(fx_{2t}, Sx_{2t}) + \omega_{\frac{1}{\gamma}}(gy, Ty) \\
& +\omega_{\frac{1}{\delta}}(Sx_{2t}, gy) + \omega_{\frac{1}{\delta}}(fx_{2t}, Ty)\Big), \phi(\omega_1(Sx_{2t}, Ty), \omega_1(Sx_{2t}, gy), \omega_1(fx_{2t}, Ty))\Big) \\
& +L\min\{\omega_1(Sx_{2t}, Ty), \omega_1(Sx_{2t}, gy), \omega_1(fx_{2t}, Ty)\} \\
=\ & F\Big(\psi\Big(\omega_{\frac{1}{\alpha}}(y_{2t-1}, Ty) + \omega_{\frac{1}{\beta}}(y_{2t}, y_{2t-1}) + \omega_{\frac{1}{\gamma}}(gy, Ty) \\
& +\omega_{\frac{1}{\delta}}(y_{2t-1}, gy) + \omega_{\frac{1}{\delta}}(y_{2t}, Ty)\Big), \phi(\omega_1(y_{2t-1}, Ty), \omega_1(y_{2t-1}, gy), \omega_1(y_{2t}, Ty))\Big) \\
& +L\min\{\omega_1(y_{2t-1}, Ty), \omega_1(y_{2t-1}, gy), \omega_1(y_{2t}, Ty)\}.
\end{aligned}
$$

Letting $t \to +\infty$ in the above inequalities and using (18) and (19), we obtain:

$$
\begin{aligned}
& \psi(\omega_1(y, gy)) \\
& \leq F\Big(\psi\Big(\omega_{\frac{1}{\alpha}}(y, Ty) + \omega_{\frac{1}{\delta}}(y, Ty) + \omega_{\frac{1}{\delta}}(y, Ty)\Big), \phi(\omega_1(y, Ty), \omega_1(y, Ty), \omega_1(y, Ty)\Big) + L\omega_1(y, Ty).
\end{aligned}
$$

If assuming that $\omega_1(y, gy) > 0$, we find, by considering again Condition (3) in the hypotheses, $\psi(\omega_1(y, gy)) < \psi(\omega_1(y, gy))$, which is impossible. Thus, $\omega_1(y, gy) = 0$, and hence, $gy = y = Ty$.

Step 3. Finally, we shall prove that y is a common fixed point for f and S, as well.

As $gX_\omega \subseteq SX_\omega$, we have $y = gy \in SX_\omega$, so there exists $v \in X_\omega$ such that $y = gy = Ty = Sv$. Since the mapping g is dominating and a weak annihilator of S, we have $y = gy = Sv \preceq gSv \preceq v$. Thus, y and v are comparable, and hence:

$$\begin{aligned}
& \psi(\omega_1(fv, y)) = \psi(\omega_1(fv, gy)) \\
\leq\ & F\Big(\psi\Big(\omega_{\frac{1}{\alpha}}(Sv, Ty) + \omega_{\frac{1}{\beta}}(fv, Sv) + \omega_{\frac{1}{\gamma}}(gy, Ty) \\
& + \omega_{\frac{1}{\delta}}(Sv, gy) + \omega_{\frac{1}{\delta}}(fv, Ty)\Big), \phi(\omega_1(Sv, Ty), \omega_1(Sv, gy), \omega_1(fv, Ty))\Big) \\
& + L\min\{\omega_1(Sv, Ty), \omega_1(Sv, gy), \omega_1(fv, Ty)\} \\
=\ & F\Big(\psi\Big(\omega_{\frac{1}{\beta}}(fv, y) + \omega_{\frac{1}{\delta}}(fv, y)\Big), \phi(0, 0, \omega_1(fv, y))\Big) \\
\leq\ & \psi\Big(\omega_{\frac{1}{\beta}}(fv, y) + \omega_{\frac{1}{\delta}}(fv, y)\Big),
\end{aligned}$$

which leads to:

$$\omega_1(fv, y) \leq \omega_{\frac{1}{\beta}}(fv, y) + \omega_{\frac{1}{\delta}}(fv, y) \leq (\beta + \delta)\,\omega_1(fv, y),$$

and makes sense only if $\omega_1(fv, y) = 0$. Thus, $y = gy = Ty = Sv = fv$. Since f and S are weakly compatible, we also have:

$$fy = fSv = Sfv = Sy.$$

Finally, using the fact that y and y are comparable, we have:

$$\begin{aligned}
& \psi(\omega_1(fy, y)) = \psi(\omega_1(fy, gy)) \\
\leq\ & F\Big(\psi\Big(\omega_{\frac{1}{\alpha}}(Sy, Ty) + \omega_{\frac{1}{\beta}}(fy, Sy) + \omega_{\frac{1}{\gamma}}(gy, Ty) \\
& + \omega_{\frac{1}{\delta}}(Sy, gy) + \omega_{\frac{1}{\delta}}(fy, Ty)\Big), \phi(\omega_1(Sy, Ty), \omega_1(Sy, gy), \omega_1(fy, Ty))\Big) \\
& + L\min\{\omega_1(Sy, Ty), \omega_1(Sy, gy), \omega_1(fy, Ty)\} \\
=\ & F\Big(\psi\Big(\omega_{\frac{1}{\alpha}}(fy, y) + \omega_{\frac{1}{\delta}}(fy, y) + \omega_{\frac{1}{\delta}}(fy, y)\Big), \phi(\omega_1(fy, y), \omega_1(fy, y), \omega_1(fy, y))\Big) \\
& + L\omega_1(fy, y).
\end{aligned}$$

Hence, according to Condition (3) in the hypotheses, we have $\omega_1(fy, y) = 0$, that is $fy = y = Sy$. Consequently, f, g, T, and S have a common fixed point. If fX_ω is not closed and one of the sets in Condition (7) is closed, we follow the similar arguments as above to prove the common fixed point of the four mappings f, g, T, S. $\square$

5. Conclusions

This papers defines the notions of the almost nonlinear (S, T, L, F, ψ, ϕ)-convex contractive condition of type I and type II on a non-Archimedean modular space, as two distinct extensions for a similar contractive condition defined on metric spaces. The key elements regarding these definitions are the use of a C-class function, an altering distance function, a control function, and most importantly, the use of a complete ordered non-Archimedean metric modular space.

The main results refer to the newly defined contractive conditions and additional properties related to notions as the coincidence point, weakly compatible mappings, weak annihilator, or dominating mapping, among others. They finally state the existence of a common fixed point of four mappings.

Moreover, an example is provided to test the useability of the theoretical content. This example uses a non-Archimedean metric modular, which is not convex; this way, it becomes clear that the class of non-Archimedean modulars is not necessarily related to the class of convex modulars. While the second one was intensely studied, the former did not enjoy the same interest. Our results prove that is worth taking more interest in modulars for which the convexity is replaced by other particularities.

Funding: This research received no external funding.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Musielak, J.; Orlicz, W. On modular spaces. *Studia Math.* **1959**, *18*, 591–597.
2. Musielak J. Orlicz spaces and Modular spaces. In *Lecture Notes in Mathematics (1034)*; Springer: Berlin, Germany, 1983.
3. Abdou A.A.N.; Khamsi M.A. Fixed point theorems in modular vector spaces. *J. Nonlinear Sci. Appl.* **2017**, *10*, 4046–4057.
4. Bejenaru, A.; Postolache, M. On Suzuki mappings in modular spaces. *Symmetry* **2019**, *11*, 319.
5. Bejenaru, A.; Postolache, M. Generalized Suzuki-type mappings in modular vector spaces. *Optimization* **2019**; doi:10.1080/02331934.2019.1647202.
6. Chistyakov, V.V. Metric modulars and their application. *Doklady Math.* **2006**, *73*, 32–35.
7. Chistyakov, V.V. Modular metric spaces, I: Basic concepts. *Nonlinear Anal.* **2010**, *72*, 1–14.
8. Chistyakov, V.V. Fixed points of modular contractive maps. *Doklady Math.* **2012**, *86*, 515–518.
9. Chistyakov, V.V. Modular contractions and their application. In *Models, Algorithms, and Technologies for Network Analysis*; Springer Proceedings in Mathematics & Statistics; Springer: New York, NY, USA, 2012; Volume 32, pp. 65–92.
10. Abdou A.A.N.; Khamsi M.A. On the fixed points of nonexpansive mappings in Modular Metric Spaces. *Fixed Point Theory Appl.* **2013**, *2013*, 229.
11. Abobaker, H.; Ryan, R.A. Modular metric spaces. *Irish Math. Soc. Bull.* **2017**, *80*, 354.
12. Paknazar, M.; Kutbi, M.A.; Demma, M.; Salimi, P. On Non-Archimedean Modular Metric Space and Some Nonlinear Contraction Mappings. Available online: https://pdfs.semanticscholar.org/ (accessed on 30 October 2019).
13. Paknazar, M.; De la Sen, M. Best Proximity Point Results in Non-Archimedean Modular Metric Space. *Mathematics* **2017**, *5*, 23.
14. Shatanawi, W.; Postolache, M.; Ansari, A.H.; Kassab, W. Common fixed points of dominating and weak annihilators in ordered metric spaces via C-class functions. *J. Math. Anal.* **2017**, *8*, 54–68.
15. Shobkolaei, N.; Sedghi, S.; Roshan, J.R.; Altun, I. Common fixed point of mappings satisfying almost generalized (S,T)-contractive condition in partially ordered partial metric spaces. *Appl. Math. Comput.* **2012**, *219*, 443–452.
16. Shatanawi, W.; Postolache, M. Common fixed point theorems for dominating and weak annihilator mappings in ordered metric spaces. *Fixed Point Theory Appl.* **2013**, *2013*, 271.
17. Ansari, A.H. Note on "φ-ψ-contractive type mappings and related fixed point". In Proceedings of the 2nd Regional Conference on Mathematics and Applications, Payame Noor University, Tehran, Iran, 18–19 September 2014; pp. 377–380.
18. Jungck, G. Common fixed points for noncontinuous nonself maps on nonmetric spaces. *Far East J. Math. Sci.* **1996**, *4*, 199–215.
19. Abbas, M.; Talat, N.; Radenović, S. Common fixed points of four maps in partially ordered metric spaces. *Appl. Math. Lett.* **2011**, *24*, 1520–1526.
20. Abdou, A.A.N. Some fixed point theorems in modular metric spaces. *J. Nonlinear Sci. Appl.* **2016**, *9*, 4381–4387.
21. Khan, M.S.; Swaleh, M.; Sessa, S. Fixed point theorems by altering distances between the points. *Bull. Aust. Math. Soc.* **1984**, *30*, 1–9.