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Local Convergence of Solvers with Eighth Order Having Weak Conditions

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Abstract: In particular, the problem of approximating a solution of an equation is of extreme importance in many disciplines, since numerous problems from diverse disciplines reduce to solving such equations. The solutions are found using iterative schemes since in general to find closed form solution is not possible. That is why it is important to study convergence order of solvers. We extended the applicability of an eighth-order convergent solver for solving Banach space valued equations. Earlier considerations adopting suppositions up to the ninth Fŕechet-derivative, although higher than one derivatives are not appearing on these solvers. But, we only practiced supposition on Lipschitz constants and the first-order Fŕechet-derivative. Hence, we extended the applicability of these solvers and provided the computable convergence radii of them not given in the earlier works. We only showed improvements for a certain class of solvers. But, our technique can be used to extend the applicability of other solvers in the literature in a similar fashion. We used a variety of numerical problems to show that our results are applicable to solve nonlinear problems but not earlier ones.

Keywords: order of convergence; iterative solver; Lipschitz constant; Banach space; local convergence

1. Introduction

A plethora of problems from diverse disciplines such as Applied Mathematics, Mathematical: Biology, chemistry, Economics, Physics, Environmental Sciences and also Engineering are reduced to equations on abstract spaces via mathematical modeling. The closed form solution is obtained only in rare cases. That is why it is important to develop iterates generating a sequence converging to the solution based on some suitable hypotheses on the initial information. Hence, we consider the problem of finding approximate unique solution α of

$$\Gamma(\mu) = 0, \tag{1}$$

is one of the top priorities in the field of numerical analysis. We consider that $\Gamma : \mathbb{K} \subset \mathbb{T}_1 \to \mathbb{T}_2$ is a Fréchet differentiable operator, $\mathbb{T}_1, \mathbb{T}_2$ are Banach spaces, and $\mathbb{K}$ is a convex subset of $\mathbb{T}_1$. The $L(\mathbb{T}_1, \mathbb{T}_2)$ is the space of continuous operators from $\mathbb{T}_1$ to $\mathbb{T}_2$.

We have several examples where researchers demonstrated the applicability of (1). They transformed the real life problems to (1) by adopting mathematical modeling and details can be found in [1–7]. We have to target on iterative solvers since it is not always feasible to access the solution α in an explicit pattern. We have a small number of globally convergent methods that do not require a sufficiently close starting point, e.g., Bisection method or regula falsi method. But, most of the algorithms determine one zero at a time. If the zero has been determined with sufficient accuracy, the polynomial is deflated and the algorithm is applied again on the deflated polynomial. In this way, we can determine all zeros simultaneously and also have theoretical importance [8] for the details of methods can be seen in [9–19]. Therefore, we have extended amount of iterative solvers to solve problems like expression (1). The analysis of solvers involves local convergence that stands on the knowledge around α. It also ensures the convergence of iteration procedures. One of the most significant tasks in the analysis of iterative procedures is to yield the convergence region. Hence, we suggest the radius of convergence.

We rewrite for this purpose the iterative solver suggested in [20] in the following way:

$$\begin{aligned} \nu_\tau &= \mu_\tau - A_\tau^{-1}\Gamma(\mu_\tau), \\ \mu_{\tau+1} &= \nu_\tau - 4B_\tau^{-1}\Gamma(\nu_\tau),' \end{aligned} \tag{2}$$

where $\mu_0 \in \mathbb{K}$ is an initial point, $A : \mathbb{K} \to L(\mathbb{T}_1, \mathbb{T}_2)$ given as $A(\mu_\tau) = A_\tau = \Gamma'(\mu_\tau) + Q(\mu_\tau)\Gamma(\mu_\tau)$, $B(\mu_\tau,\ \nu_\tau) = B_\tau = \Gamma'(\mu_\tau) + 4Q(\mu_\tau)\Gamma(\mu_\tau) + 2\Gamma'\left(\frac{\mu_\tau+\nu_\tau}{2}\right) + \Gamma'(\nu_\tau)$, and $Q(.\,.) : \mathbb{K}\times\mathbb{K} \to L(\mathbb{T}_1,\ \mathbb{T}_2)$ is a bilinear operator. In the special case, when $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$, $Q(\mu,\nu) = \frac{G'(\mu)}{G(\mu)}$, where $G(\mu) \neq 0$ for each $x \in \mathbb{K} - \{\alpha\}$ solver (2) reduces to a fourth-order convergent solver studied in [20]. Shah et al. [20] suggested fourth-order convergence by adopting Taylor series expansions and suppositions up to the ninth-order derivative of the involved function. Such constraints hamper the suitability of solver (2). But, only first-order derivative emerges in the solver (2). Let us assume the succeeding function Γ on $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$, $\mathbb{K} = \left[-\frac{1}{2}, \frac{3}{2}\right]$ as

$$\Gamma(\mu) = \begin{cases} \mu^3 \ln \mu^2 + \mu^5 - \mu^4, & \mu \neq 0 \\ 0, & \mu = 0 \end{cases}.$$

Then, we have that

$$\Gamma'(\mu) = 3\mu^2 \ln \mu^2 + 5\mu^4 - 4\mu^3 + 2\mu^2,$$

$$\Gamma''(\mu) = 6\mu \ln \mu^2 + 20\mu^3 - 12\mu^2 + 10\mu$$

and

$$\Gamma'''(\mu) = 6 \ln \mu^2 + 60\mu^2 - 24\mu + 22.$$

From the above derivatives, it is straightforward to see that the 3rd-order derivative of Γ is unbounded in $\mathbb{A}$. In the available literature, we have a bulk number of research articles [1–7,20–34]. In the majority of these articles, authors mention that starting guess x_0 must be adequately close to μ. But this is not offering us an idea of: how to pick x_0, how much closeness is sufficient for convergence, find radius; bounds on $\|x_n - \mu\|$ and results on uniqueness. We deal with all these questions for solver (2) in the next section.

In the present study, we adopt only conditions on the first-order derivative of Γ with generalized Lipschitz conditions. In addition, we are avoiding the Taylor series expansions because it proceeds with higher-order derivatives of Γ, but we adopt Lipschitz parameters. In this way, we are not committed to adopt higher-order derivatives for convergence order of (2). Further, we adopted the following (COC) and $(ACOC)$ for computing the convergence order:

$$\xi = \frac{ln\frac{\|\mu_{\tau+2}-\alpha\|}{\|\mu_{\tau+1}-\alpha\|}}{ln\frac{\|\mu_{\tau+1}-\alpha\|}{\|\mu_\tau-\alpha\|}}, \quad \text{for each } n = 0, 1, 2, \ldots, \tag{3}$$

or the approximate computational order of convergence $(ACOC)$ [21], defined as

$$\xi^* = \frac{ln\frac{\|\mu_{\tau+2}-\mu_{\tau+1}\|}{\|\mu_{\tau+1}-\mu_\tau\|}}{ln\frac{\|\mu_{\tau+1}-\mu_\tau\|}{\|\mu_\tau-\mu_{\tau-1}\|}}, \quad \text{for each } n = 1, 2, \ldots, \tag{4}$$

where the computational order of convergence *COC* and the approximate computational order of convergence *ACOC* [21], respectively. They do not require higher than one derivatives. It is vital to note that *ACOC* does not need the prior information of exact root μ. Finally, we investigate the applicability of our results on several numerical examples, where earlier works did not exhibit this behavior.

The remainder of this paper is coordinated in the succeeding way: We suggest the local convergence study of solver (2) in Section 2. Numerical experimentation is depicted in Section 3. Finally, we make concluding assertions in Section 4.

2. Study of Local Convergence

In this section, we suggest the local convergence study of solver (2). Therefore, we adopt some scalar functions $\Delta_0,\ \Delta,\ w_0,\ w,\ w_1$ that are non-decreasing continuous functions from $[0,\ +\infty)$ to $[0,\ +\infty)$ such that $w_0(0) = w(0) = 0$. We assume

$$p(\zeta) = 1 \tag{5}$$

has a minimal positive solution r_0 and

$$p(\zeta) = w_0(\zeta) + w_1(\zeta)\zeta \int_0^1 \Delta_0(\theta\zeta)d\theta.$$

In addition, we describe functions $g_1,\ h_1,\ q$, and h_q on $[0,\ r_0)$ as follows:

$$g_1(\zeta) = \frac{\int_0^1 w\Big((1-\theta)\zeta\Big)d\theta}{1-w_0(\zeta)} + \frac{\Big(\int_0^1 \Delta(\theta\zeta)d\theta\Big)\Big(\int_0^1 \Delta_0(\theta\zeta)d\theta\Big)w_1(\zeta)\zeta}{\Big(1-w_0(\zeta)\Big)\Big(1-p(\zeta)\Big)},$$

$$h_1(\zeta) = g_1(\zeta) - 1,$$

$$q(\zeta) = \frac{1}{2}\left[w_0(\zeta) + 4\int_0^1 \Delta_0(\theta\zeta)d\theta w_1(\zeta)\zeta + \int_0^1 \Delta\left(\frac{\theta}{2}(1+g_1(\zeta))\zeta\right)d\theta\Big(1+g_1(\zeta)\Big)\zeta + w_0\Big(g_1(\zeta)\zeta\Big)\right],$$

and

$$h_q(\zeta) = q(\zeta) - 1.$$

We have that $h_1(0) = h_q(0) = -1 < 0$ and $h_1(\zeta) \to +\infty$, $h_q(\zeta) \to +\infty$ as $\zeta \to r_0^-$. By adopting the intermediate value theorem, we can say that both function h_1 and h_q have zeros in $(0,\ r_0)$. Call as r_1 and r_q the smallest such zeros in $(0,\ r_0)$ of the functions h_1 and h_q, respectively. Further, we represent functions g_2 and h_2 on $[0,\ r_0)$ as follows:

$$g_2(\zeta) = \left[1 + \frac{2\int_0^1 \Delta\Big(\theta g_1(\zeta)\zeta\Big)d\theta}{1-q(\zeta)}\right]g_1(\zeta),$$

and

$$h_2(\zeta) = g_2(\zeta) - 1.$$

We have again that $h_2(0) = -1 < 0$ and $h_2(\zeta) \to +\infty$ as $\zeta \to r_q^-$. Let us call r_2 to be minimal zero of h_2 in $(0,\ r_q)$. Finally, we describe the convergence radius r in the following way:

$$r = \min\{r_1,\ r_2\}. \tag{6}$$

Then, we have that for each $\zeta \in [0,\ r)$

$$0 \le p(\zeta) < 1, \tag{7}$$

$$0 \le w_0(\zeta) < 1, \tag{8}$$

$$0 \le g_1(\zeta) < 1, \tag{9}$$

$$0 \le q(\zeta) < 1, \tag{10}$$

and

$$0 \le g_2(\zeta) < 1. \tag{11}$$

The $U(\lambda,\ \rho)$ and $\bar{U}(\lambda,\ \rho)$ are two open and closed balls, respectively in $\mathbb{T}_1$ centered at $\lambda \in \mathbb{T}_1$. Both have the radius $\rho > 0$.

The local convergence analysis of solver (2) is based on conditions (A):

(A_1) $\Gamma : \mathbb{K} \subseteq \mathbb{T}_1 \to \mathbb{T}_2$ is a Fréchet-differentiable operator.
(A_2) $\Delta_0,\ \Delta,\ w_0,\ w,\ w_1 : [0,\ \infty) \to [0,\ \infty)$ with $w_0(0) = w(0) = 0$ are non-decreasing continuous functions.
(A_3) There exists a zero $\alpha \in \mathbb{K}$ of Γ such that for every $\mu \in \mathbb{K}$

$$\Gamma(\alpha) = 0, \quad \Gamma'(\alpha)^{-1} \in L(\mathbb{T}_2,\ \mathbb{T}_1) \tag{12}$$

and

$$\left\| \Gamma'(\alpha)^{-1} \Big(\Gamma'(\mu) - \Gamma'(\alpha) \Big) \right\| \le w_0(\|\mu - \alpha\|). \tag{13}$$

Set $\mathbb{K}_0 := \mathbb{K} \cap U(\alpha,\ r_0)$.
(A_4)

$$\left\| \Gamma'(\alpha)^{-1} (\Gamma'(\mu) - \Gamma'(\nu)) \right\| \le w(\|\mu - \nu\|), \tag{14}$$

$$\|\Gamma'(\mu)\| \le \Delta_0(\|\mu - \alpha\|), \tag{15}$$

$$\left\| \Gamma'(\alpha)^{-1} \Gamma'(\mu) \right\| \le \Delta(\|\mu - \alpha\|), \tag{16}$$

$$\left\| \Gamma'(\alpha)^{-1} Q(\mu, \nu) \right\| \le w_1(\|\mu - \alpha\|), \text{ for every } \mu,\ \nu \in \mathbb{K}_0, \tag{17}$$

and

$$\bar{U}(\alpha, r) \subseteq \mathbb{K}, \tag{18}$$

where $Q(\mu,\ \nu) : \mathbb{K} \times \mathbb{K} \to L(\mathbb{T}_1,\ \mathbb{T}_2)$.

Then, we present the main local convergence result.

Theorem 1. *Under the conditions (A) sequence $\{\mu_\tau\}$ obtained for $\mu_0 \in U(\alpha,\ r) - \{\alpha\}$ by solver (2) exists, remains in $U(\alpha,\ r)$ for all $\tau = 0, 1, 2, \ldots$ and converges to α, so that*

$$\|\nu_\tau - \alpha\| \le g_1\Big(\|\mu_\tau - \alpha\|\Big)\|\mu_\tau - \alpha\| \le \|\mu_\tau - \alpha\| < r \tag{19}$$

and

$$\|\lambda_\tau - \alpha\| \le g_2\Big(\|\mu_\tau - \alpha\|\Big)\|\mu_\tau - \alpha\| \le \|\mu_\tau - \alpha\|. \tag{20}$$

Furthermore, if

$$\int_0^1 w_0(\theta R) d\theta < 1, \text{ for } R \ge r, \tag{21}$$

then α is the unique root of $\Gamma(\mu) = 0$ in $\mathbb{K}_1 := \mathbb{K} \cap \bar{U}(\alpha,\ R)$.

Proof. We select the mathematical induction to show expressions (19)–(21) are well defined in $U(\alpha,\ r)$. Further, they converge to required zero α. Adopting hypothesis $\mu_0 \in U(\alpha, r) - \{\alpha\}$, (5)–(7) and (13), we obtain

$$\left\| \Gamma'(\alpha)^{-1} \Big(\Gamma'(\mu_0) - \Gamma'(\alpha) \Big) \right\| \le w_0(\|\mu_0 - \alpha\|) < w_0(r) < 1. \tag{22}$$

From the expression (22) and the Banach Lemma on inverse operators [1,2] that $\Delta'(x_0)^{-1} \in L(\mathbb{T}_2,\ \mathbb{T}_1)$, $\nu_0,\ \lambda_0$ are well defined and

$$\left\|\Gamma'(\mu_0)^{-1}\Gamma'(\alpha)\right\| \le \frac{1}{1 - w_0(\|\mu_0 - \alpha\|)}. \tag{23}$$

To show that ν_0 exists, it suffices to show that $A_0^{-1} \in L(\mathbb{T}_2,\ \mathbb{T}_1)$. Using (5), (6), (8), (13) and (15), we get in turn that

$$\begin{aligned}\left\|\Gamma'(\mu)^{-1}\Big(A_0 - \Gamma'(\alpha)\Big)\right\| &\le \left\|\Gamma'(\alpha)^{-1}\Big(\Gamma'(\mu_0) - \Gamma'(\alpha)\Big)\right\| + \|\Gamma(\mu_0)\|\left\|\Gamma'(\alpha)^{-1}Q(\mu_0)\right\| \\ &\le w(\|\mu_0 - \alpha\|) + \int_0^1 \Delta_0\Big(\theta\|\mu_0 - \alpha\|\Big)d\theta w_1(\|\mu_0 - \alpha\|)\|\mu_0 - \alpha\| \\ &= p(\|\mu_0 - \alpha\|) < p(r) < 1,\end{aligned} \tag{24}$$

so $A_0^{-1} \in L(\mathbb{T}_2,\ \mathbb{T}_1)$ is well defined and

$$\left\|A_0^{-1}\Gamma'(\alpha)\right\| \le \frac{1}{1 - p(\|\mu_0 - \alpha\|)}. \tag{25}$$

By the definition of A_0 and the first substep of (2), we can write

$$\begin{aligned}\nu_0 - \alpha &= \mu_0 - \alpha - \Gamma_0'(\mu_0)^{-1}\Gamma(\mu_0) + \Big(\Gamma_0'(\mu_0)^{-1} - A_0^{-1}\Big)\Gamma(\mu_0) \\ &= \Gamma_0'(\mu_0)^{-1}\Gamma'(\alpha)\int_0^1 \Gamma'(\alpha)^{-1}\Big(\Gamma'(\alpha + \theta(\mu_0 - \alpha)) - \Gamma'(\mu_0)\Big)(\mu_0 - \alpha)d\theta \\ &+ \Gamma'(\mu_0)^{-1}\Gamma'(\alpha)\Gamma'(\alpha)^{-1}\Big(A_0 - \Gamma'(\mu_0)\Big)A_0^{-1}\Gamma'(\alpha)\Gamma'(\alpha)^{-1}\Gamma(\mu_0).\end{aligned} \tag{26}$$

We also have by (16)

$$\begin{aligned}&\Gamma(\mu_0) = \Gamma(\mu_0) - \Gamma(\alpha) = \int_0^1 \Gamma'\Big(\alpha + \theta(\mu_0 - \alpha)\Big)d\theta(\mu_0 - \alpha), \\ &\text{so} \\ &\left\|\Gamma'(\alpha)^{-1}\Gamma(\mu_0)\right\| = \left\|\int_0^1 \Gamma'(\alpha)^{-1}\Gamma'\Big(\alpha + \theta(\mu_0 - \alpha)\Big)d\theta(\mu_0 - \alpha)\right\| \le \int_0^1 \Delta\Big(\theta\|\mu_0 - \alpha\|\Big)d\theta\|\mu_0 - \alpha\|.\end{aligned} \tag{27}$$

In view of (2), (5), (6), (9), (14), (15), (22) and (24), we obtain

$$\begin{aligned}\|\nu_0 - \alpha\| &\le \left\|\Gamma'(\mu_0)^{-1}\Gamma'(\alpha)\right\|\left\|\int_0^1 \Gamma'(\alpha)^{-1}(\Gamma'(\alpha + \theta(\mu_0 - \alpha)) - \Gamma'(\mu_0))(\mu_0 - \alpha)d\theta\right\| \\ &+ \left\|\Gamma'(\mu_0)^{-1}\Gamma'(\alpha)\right\|\left\|\Gamma'(\mu_0)^{-1}\Big(A_0 - \Gamma'(\mu_0)\Big)\right\|\left\|A_0^{-1}\Gamma'(\alpha)\right\|\left\|\Gamma'(\alpha)^{-1}\Gamma(\mu_0)\right\| \\ &\le \frac{\int_0^1 w\Big((1-\theta)\|\mu_0 - \alpha\|\Big)d\theta\|\mu_0 - \alpha\|}{1 - w_0(\|\mu_0 - \alpha\|)} \\ &+ \frac{\int_0^1 \Delta_0\Big(\theta\|\mu_0 - \alpha\|\Big)d\theta \int_0^1 \Delta\Big(\theta\|\mu_0 - \alpha\|\Big)d\theta w_1\Big(\|\mu_0 - \alpha\|\Big)\|\mu_0 - \alpha\|^2}{(1 - w_0(\|\mu_0 - \alpha\|))(1 - p(\|\mu_0 - \alpha\|))} \\ &= g_1(\|\mu_0 - \alpha\|)\|\mu_0 - \alpha\| \le \|\mu_0 - \alpha\| < r,\end{aligned} \tag{28}$$

which illustrates (22) for $\tau = 0$ and $\nu_0 \in U(\alpha,\ r)$.

Next, we have to prove that $B_0^{-1} \in L(\mathbb{T}_2,\ \mathbb{T}_1)$. By (5), (6), (10), (13), (15), (16) and (28), we get

$$\left\|\left(2\Gamma'(\alpha)\right)^{-1}(B_0-2\Gamma'(\alpha))\right\| \leq \frac{1}{2}\Big[\left\|\Gamma'(\alpha)^{-1}\left(\Gamma'(\mu_0)-\Gamma'(\alpha)\right)\right\|+4\left\|\Gamma'(\alpha)^{-1}Q(\mu_0)\right\|\|\Gamma(\mu_0)\|$$
$$+2\left\|\Gamma'(\alpha)^{-1}\Gamma'\left(\frac{\mu_0+\nu_0}{2}\right)\right\|+\left\|\Gamma'(\alpha)^{-1}\left(\Gamma'(\nu_0)-\Gamma'(\alpha)\right)\right\|\Big]$$
$$\leq \frac{1}{2}\Big[w_0(\|\mu_0-\alpha\|)+4\int_0^1 \Delta_0(\theta\|\mu_0-\alpha\|)d\theta w_1(\|\mu_0-\alpha\|)\|\mu_0-\alpha\| \tag{29}$$
$$+\int_0^1 \Delta\left(\frac{\theta}{2}(\|\mu_0-\alpha\|+\|\nu_0-\alpha\|)\right)d\theta\left(\|\mu_0-\alpha\|+\|\nu_0-\alpha\|\right)+w_0(\|\mu_0-\alpha\|)\Big]$$
$$\leq q(\|\mu_0-\alpha\|)<q(r)<1.$$

Hence, $B_0^{-1}\in L(\mathbb{T}_2,\ \mathbb{T}_1)$ is valid by solver (2), and

$$\|B_0^{-1}\Gamma'(\alpha)\| \leq \frac{1}{2(1-q(\|\mu_0-\alpha\|))}. \tag{30}$$

Then, by the last sub step of solver (2), (5), (6), (11), (15), (28) and (30), we have in turn that

$$\|\mu_1-\alpha\| \leq \|\nu_0-\alpha\|+4\left\|B_0^{-1}\Gamma'(\alpha)\right\|\left\|\Gamma'(\alpha)^{-1}\Gamma(\nu_0)\right\|$$
$$\leq g_1(\|\mu_0-\alpha\|)\|\mu_0-\alpha\|+\frac{2\int_0^1\Delta\left(\theta\|\nu_0-\alpha\|\right)d\theta g_1(\|\mu_0-\alpha\|)\|\mu_0-\alpha\|}{1-q(\|\mu_0-\alpha\|)} \tag{31}$$
$$=g_2(\|\mu_0-\alpha\|)\|\mu_0-\alpha\| \leq \|\mu_0-\alpha\|<r,$$

which illustrates (20) for $\tau=0$ and $\lambda_0\in U(\alpha,\ r)$. By restoring $\mu_0,\ \nu_0,\ \mu_1$ by $\mu_\sigma,\ \nu_\sigma,\ \mu_{\sigma+1}$ in the succeeding estimates, we attain (19) and (20). Then, in view of the estimates

$$\|\mu_{\sigma+1}-\alpha\| \leq c\|\mu_\sigma-\alpha\|<r,\ c=g_2(\|\mu_0-\alpha\|)\in[0,\ 1), \tag{32}$$

that attain $\lim_{\sigma\to\infty}\mu_\sigma=\alpha$ and $\mu_{\sigma+1}\in U(\alpha,\ r)$. Finally, the uniqueness of solution is required. Therefore, we assume that $\nu^*\in D_1$ with $\Gamma(\nu^*)=0$ and $Q=\int_0^1\Gamma'(\alpha+\theta(\alpha-\nu^*))d\theta$.

By adopting (9) and (16), we yield

$$\left\|\Gamma'(\alpha)^{-1}\left(Q-\Gamma'(\alpha)\right)\right\| \leq \|\int_0^1 w_0\left(\theta\|\nu^*-\alpha\|\right)d\theta$$
$$\leq \int_0^1 w_0(\theta R)d\theta<1. \tag{33}$$

So, Q is invertible in view of

$$0=\Gamma(\alpha)-\Gamma(\nu^*)=Q(\alpha-\nu^*), \tag{34}$$

that yields $\alpha=\nu^*$. □

Remark 1. *(a) It is straightforward from the expression of* (14) *that we can drop the hypothesis* (16) *and restore as*

$$\Delta(\zeta)=1+w_0(\zeta)\ or\ \Delta(\zeta)=1+w_0(r_0), \tag{35}$$

since,

$$\left\|\Gamma'(\alpha)^{-1}\left[\left(\Gamma'(\mu)-\Gamma'(\alpha)\right)+\Gamma'(\alpha)\right]\right\| \leq 1+\left\|\Gamma'(\alpha)^{-1}\left(\Gamma'(\mu)-\Gamma'(\alpha)\right)\right\|$$
$$\leq 1+w_0(\|\mu-\alpha\|) \tag{36}$$
$$=1+w_0(\zeta)\quad for\quad \|\mu-\alpha\|\leq r_0.$$

(b) We can choose

$$r_0=w_0^{-1}(1), \tag{37}$$

instead of (5) *provided the function* w_0 *is strictly increasing.*

(c) *If* $w_0,\ w,\ \Delta$ *are constants functions, then we have*

$$r_1 = \frac{2}{2w_0 + w} \tag{38}$$

and

$$r \le r_1. \tag{39}$$

The r_1 *stands for the radius of the following Newton's solver*

$$\mu_{\tau+1} = \mu_\tau - \Gamma'(\mu_\tau)^{-1}\Gamma(\mu_\tau). \tag{40}$$

Rheindoldt [22] and Traub [5] also suggested convergence radius instead of r_1

$$r_{TR} = \frac{2}{3w_1}, \tag{41}$$

and by Argyros [1,2]

$$r_A = \frac{2}{2w_0 + w_1}, \tag{42}$$

where w_1 *is a Lipschitz parameter for* (10) *on* $\mathbb{K}$. *Hence, we have*

$$w \le w_1,\ w_0 \le w_1, \tag{43}$$

so

$$r_{TR} \le r_A \le r_1 \tag{44}$$

and

$$\frac{r_{TR}}{r_A} \to \frac{1}{3} \quad as \quad \frac{w_0}{w} \to 0. \tag{45}$$

The convergence radius q *suggested by Dennis and Schabel [1] is smaller than the radius* r_{DS}

$$q < r_{SD} = \frac{1}{2w_1} < r_{TR}. \tag{46}$$

However, q *can not be calculated by the Lipschitz conditions.*

(d) *By adopting conditions on the ninth-order derivative of operator* Γ, *the order of convergence of solver* (2) *was provided by Shah et al. [20]. But, we assume hypotheses only on first-order derivative of operator* Γ. *For obtaining the computational order of convergence* (COC), *we adopted expressions* (3) *and* (4).

(e) *Assume [1,2] satisfying the autonomous differential equation*

$$\Gamma'(\mu) = P\Big(\Gamma(\mu)\Big) \tag{47}$$

where P *is a given and continuous operator. Then,* $\Gamma'(\alpha) = P\Big(\Gamma(\alpha)\Big) = P(0)$, *our results apply. But, without knowledge of* α *and choose* $\Gamma(\mu) = e^\mu - 1$. *Hence, we select* $P(\mu) = \mu + 1$.

3. Numerical Experimentation

Here, we illustrate the theoretical consequences suggested in Section 2. Next, we choose $Q = I$ in the first four examples.

Example 1. *Let* $\mathbb{T}_1 = \mathbb{T}_2 = H$ *and* $H = C[0,\ 1]$. *We study the mixed Hammerstein-like equation [6,23], defined by*

$$\mu(s) = 1 + \int_0^1 G(s,\ \zeta) \left(\mu(\zeta)^{\frac{3}{2}} + \frac{\mu(\zeta)^2}{2} \right) d\zeta \tag{48}$$

where

$$\Gamma(s,\zeta) = \begin{cases} (1-s)\zeta,\ \zeta \le s, \\ s(1-\zeta),\ s \le \zeta, \end{cases} \tag{49}$$

defined in $[0,1] \times [0,1]$*. The solution* $\alpha(s) = 0$ *is the same as zero of* (1)*, where* $\Gamma : H \to H$*, given as:*

$$\Gamma(\mu)(s) = \mu(s) - \int_0^\zeta G(s,\ \zeta) \left(\mu(\zeta)^{\frac{3}{2}} + \frac{\mu(\zeta)^2}{2} \right) d\zeta. \tag{50}$$

But

$$\left\| \int_0^\zeta G(s,\ \zeta) d\zeta \right\| \le \frac{1}{8}. \tag{51}$$

Then, we have that

$$\Gamma'(\mu)\nu(s) = \nu(s) - \int_0^\zeta G(s,\ \zeta) \left(\frac{3}{2}\mu(\zeta)^{\frac{1}{2}} + \mu(\zeta) \right) d\zeta,$$

and since $\Gamma'(\alpha(s)) = I$*,*

$$\left\| \Gamma'(\alpha)^{-1} \big(\Gamma'(\mu) - \Gamma'(\nu) \big) \right\| \le \frac{1}{8} \left[\frac{3}{2} \|\mu - \nu\|^{\frac{1}{2}} + \|\mu - \nu\| \right]. \tag{52}$$

Therefore, we can choose

$$w_0(\zeta) = w(\zeta) = \frac{1}{8} \left[\frac{3}{2} \zeta^{\frac{1}{2}} + \zeta \right].$$

Hence, by Remark 2.2(a), we can set

$$\Delta_0(\zeta) = \Delta(\zeta) = 1 + w_0(\zeta) \text{ and } w_1(\zeta) = 1.$$

But, theorems in [20] can not be utilized to solve this problem because Γ' *is not Lipschitz. Notice though that our theorems can be utilized. We have the following radii for Example 1:*

$$r_1 = 0.321768,\ r_q = 0.284919,\ r_2 = 0.119079,$$

so

$$r = 0.119079.$$

Example 2. *Consider, setting* $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}^3$ *and* $\Omega = \Omega(0,1)$*. Then, for* $w = (\mu,\ \nu,\ \lambda)^T$ *define a function* $\Gamma : \Omega \to \mathbb{R}^3$ *as follows:*

$$\Gamma(w) = \left(e^\mu - 1,\ \frac{e-1}{2}\nu^2 + \nu,\ \lambda \right)^T. \tag{53}$$

Then, we obtain

$$\Gamma'(w) = \begin{bmatrix} e^\mu & 0 & 0 \\ 0 & (e-1)\nu + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence, for $\mu = (0,0,0)^T$ *we can choose* $w_0(\zeta) = (e-1)\zeta,\ w(\zeta) = e^{\frac{1}{e-1}}\zeta,\ \Delta_0(\zeta) = \Delta(\zeta) = e^{\frac{1}{e-1}}$*, and* $w_1(\zeta) = 1$*. By adopting these functions and parameters, we obtain the following radii for Example 2:*

$$r_1 = 0.121854,\ r_q = 0.134127,\ r_2 = 0.0370321,$$

so

$$r = 0.0370321.$$

Example 3. *Let us choose that* $\mathbb{T}_1 = \mathbb{T}_2 = H$*, facilitated by the max norm. In addition, we consider* $B(x) = \Gamma''(\mu)$ *and* $\mathbb{K} = \bar{U}(0,\ 1)$ *for every* $\mu \in \mathbb{K}$*. Choose a function* Γ *on* $\mathbb{K}$

$$\Gamma(\varphi)(\mu) = \phi(\mu) - 5\int_0^1 \mu\theta\varphi(\theta)^3 d\theta, \tag{54}$$

which yields

$$\Gamma'(\varphi(\xi))(\mu) = \xi(\mu) - 15\int_0^1 \mu\theta\varphi(\theta)^2\xi(\theta)d\theta, \text{ for each } \xi \in \Omega. \tag{55}$$

Then, we have that $w_0(\zeta) = 7.5\zeta,\ w(\zeta) = 15\zeta$ *and* $\Delta_0(\zeta) = \Delta(\zeta) = 2$*, and* $w_1(\zeta) = 1$*. We have the following radii for Example 3:*

$$r_1 = 0.0453881,\ r_q = 0.0587713,\ r_2 = 0.0133343,$$

so

$$r = 0.0133343.$$

Example 4. *By the academic problem that we considered in the introduction. We can choose* $w_0(\zeta) = w(\zeta) = 96.662907\zeta$ *and* $\Delta_0(\zeta) = 6,\ \Delta(\zeta) = 2$*, and* $w_1(\zeta) = \frac{1}{3}$*. By adopting these functions and parameters, we yield the following radii, for Example 4:*

$$r_1 = 0.00641476,\ r_q = 0.00741294,\ r_2 = 0.00247724,$$

so

$$r = 0.00247724.$$

4. Concluding Assertions

We first generalized solver (2) from functions on the real line to Banach space valued operators. Then, we presented a local convergence analysis in this setting and by using generalized-continuity conditions. Our analysis uses only the first derivative appearing in the solver. In the special case of the real line, derivatives up to the order seven were used. Notice that these high order derivatives do not appear in the solver (2) and also limit the applicability of the solver, as we saw in the introduction. Hence, the applicability of solver (2) has been significantly extended. Numerical examples and applications complete the paper.

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