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A Class of Critical Magnetic Fractional Kirchhoff Problems

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Abstract: In this paper, we deal with the existence and asymptotic behavior of solutions for a fractional Kirchhoff type problem involving the electromagnetic fields and critical nonlinearity by using the classical critical point theorem. Meanwhile, an example is given to illustrate the application of the main result.

Keywords: magnetic operator; critical nonlinearities; fractional Kirchhoff type problems; variational methods

1. Introduction

The aim of this paper is to investigate the existence of solutions for a magnetic fractional Kirchhoff equation involving a critical nonlinearity:

$$\mathcal{M}([v]_{s,A}^2)(-\Delta)_A^s v = \mu v + |v|^{2_s^*-2}v + g(x,|v|)v \quad \text{in } \Omega, \quad v = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega, \tag{1}$$

where $s \in (0,1)$, $N > 2s$, Ω is an open and bounded domain with the Lipschitz boundary,

$$[v]_{s,A} = \Big(\iint_{\mathbb{R}^{2N}} \frac{|v(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})} v(y)|^2}{|x-y|^{N+2s}} dxdy \Big)^{\frac{1}{2}},$$

g is a lower order perturbation of the critical power $|v|^{2_s^*-2}v$; $\mu > 0$ is a real parameter; $2_s^* = \frac{2N}{N-2s}$ is the fractional critical Sobolev exponent; the Kirchhoff function $\mathcal{M} : \mathbb{R}^+ \to \mathbb{R}_0^+$ is a continuous function; and $(-\Delta)_A^s$ is the non-local fractional magnetic operator, defined as follows:

$$(-\triangle)_A^s \phi(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{\phi(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}\phi(y)}{|x-y|^{N+2s}} dy,$$

for $x \in \mathbb{R}^N$, along any complex valued functions $\phi \in C_0^\infty(\mathbb{R}^N, \mathbb{C})$, where $B_\varepsilon(x)$ denotes the open ball in $\mathbb{R}^N$ centered at $x \in \mathbb{R}^N$ and of radius $\varepsilon > 0$. We can consider $(-\Delta)_A^s$ as a fractional counterpart of the magnetic Laplacian $(\nabla - iA)^2$, with $A : \mathbb{R}^N \to \mathbb{R}^N$, being a L_{loc}^∞- vector potential; see Chapter 7 of [1]. For further details about this kind of operator, see [2–6] and the references therein. The equation with the fractional magnetic operator A and the critical nonlinear term $|v|^{2_s^*-2}v$ is called the fractional critical magnetic problem. For such a critical nonlinear case, Fiscella et al first proposed a bounded stationary Kirchhoff variational model. The existence and multiplicity of solutions to fractional Kirchhoff problems are obtained by using the variational method and the principle of concentrated compactness in [7]. The main difficulty about this problem is the lack of compactness of Sobolev space, and the magnetic operators in the equation make the problem more complicated. Most scholars deal with the known multiplicity, but there are few articles about the asymptotic behavior of solutions. For instance, in [8], Fiscella et al. studied bifurcation phenomena and the multiplicity of solutions for a critical magnetic fractional problem by using a classical theorem in critical point theory. Moreover, Libo et al. obtained infinitely many solutions for the degenerate magnetic fractional Kirchhoff problem by using the new version of the symmetric mountain pass theorem of Kajikiya in [9].

Clearly, when $A \equiv 0$ and ϕ is a smooth real valued function, $(-\triangle)^s_A$ becomes classical fractional Laplacian $(-\triangle)^s$, and Problem (1) becomes the fractional Kirchhoff equations involving a lower order perturbation term and a critical nonlinearity. This kind of fractional Laplacian operator has different applications in many fields, such as phase transition phenomena, continuum mechanics, game theory, and so on, as they are the typical outcome of stochastically stabilization of Lévy processes; see [10]. In particular, as $\mathcal{M} = 1$, Servadei et al. completed the study of elliptic equations with the fractional critical Sobolev exponent in [11]. For the more general case, in [12], Autuori et al. investigated the existence of solutions for a class of fractional Kirchhoff problems.

Motivated by the above works, the present paper concerns the existence and asymptotic behavior of solutions of Problem (1) and covering the degenerate case $\mathcal{M}(0) = 0$, without any monotonicity conditions on $\mathcal{M}$. In order to obtain this goal, we assume that

(M_1) There exists $\theta \in (0,1)$, such that:

$$\theta\widetilde{\mathcal{M}}(t) = \theta \int_0^t \mathcal{M}(s)ds \geq \mathcal{M}(t)t\,, \;\; \forall\, t \in \mathbb{R}_0^+;$$

(M_2) for every $\varrho > 0$, there exists $m = m(\varrho) > 0$ such that $\mathcal{M}(t) \geq m$ for all $t \geq \varrho$;

(M_3) there exists a positive a_0 such that $\mathcal{M}(t) \geq a_0 t$ for all $t \in [0,1]$.

Remark 1. *It is worth mentioning that Kirchhoff in 1883 (see [13]) presented a stationary version of the differential equation, the so-called Kirchhoff equation:*

$$\rho\frac{\partial^2 v}{\partial t^2} - \Big(\frac{p_0}{l} + \frac{e}{2L}\int_0^L |\frac{\partial v}{\partial x}|^2 dx\Big)\frac{\partial^2 v}{\partial x^2} = 0,$$

where ρ, l, e, L, p_0 are positive constants that represent the corresponding physical meanings. It is a generalization of the D'Alembert equation.

The Kirchhoff functions $\mathcal{M}$ come in many forms, and we can refer to the literature [14–16]. Among them, in [14,16], the Kirchhoff function was assumed to satisfy certain monotonicity (i.e., $\mathcal{M}(t) = a + bt^\tau$, $a, b > 0, \tau \geq 1$ for all $t \in \mathbb{R}_0^+$); we adopt the assumption of a more generalized integral form (M_1). In [15], the authors only considered the non-degenerate case, while in this paper, we consider the case including the degenerate case. Throughout the paper, Conditions (M_1)–(M_3) are very important. For this reason, we first make a brief analysis for our later use. As indicated in [17], Condition (M_3) means that:

$$\widetilde{\mathcal{M}}(t) \geq a_0\frac{t^2}{2} \tag{2}$$

for any $t \in [0,1]$. Moreover, by (M_1), we get $\widetilde{\mathcal{M}}(t) \geq \widetilde{\mathcal{M}}(1)t^\theta$ for all $t \in [0,1]$, and combining with (2) gives $\widetilde{\mathcal{M}}(t) \geq ct^\theta$ for any $t \in [0,1]$. In the same way, for any $\kappa > 0$, there exists $\gamma_\kappa = \widetilde{\mathcal{M}}(\kappa)/\kappa^\theta > 0$ such that:

$$\widetilde{\mathcal{M}}(t) \leq \gamma_\kappa t^\theta \quad \text{for any} \quad t \geq \kappa. \tag{3}$$

Thus, it follows from (3) that:

$$\lim_{t\to\infty}\frac{\widetilde{\mathcal{M}}(t)}{t} = 0, \tag{4}$$

thanks to $\theta < 1$ by (M_1).

The perturbation in problem (1) is a Carathéodory function $g : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ satisfying the following assumptions:

(g_1) There exist $\varsigma \in (2\theta, 2_s^*)$ and a nonnegative function $f(x) \in L^\tau(\Omega)$ such that $|g(x,t)| \le f(x)t^{\varsigma-2}$ for all $(x,t) \in \Omega \times \mathbb{R}^+$, where $\tau = \frac{2_s^*}{2_s^*-\varsigma}$;

(g_2) $0 < G(x,t) = \int_0^t g(x,s)sds \le \frac{1}{2}g(x,t)t^2$ a.e. $x \in \Omega,\ t \in \mathbb{R}^+$;

Remark 2. *According to* (g_1), *we can get that:*

$$G(x,t) \le f(x)t^\varsigma \text{ for any } (x,t) \in \Omega \times \mathbb{R}^+. \tag{5}$$

By (g_2), *we have that for some* $d_1, d_2 > 0$:

$$G(x,t) \ge d_1 t^2 - d_2 \text{ for any } (x,t) \in \Omega \times \mathbb{R}^+. \tag{6}$$

Now, we give the main result of this paper; our work space $X_{0,A}$ will be introduced in Section 2.

Theorem 1. *Set* $\mathcal{M}(0) = 0$ *and* $2s < N < 4s$. *If* (M_1)–(M_3) *and* (g_1)–(g_2) *hold, then there exists* $\mu_0 > 0$, *such that for any* $\mu \ge \mu_0$, *Problem* (1) *has at least a non-trivial mountain pass solution* v_μ. *Furthermore,* $\lim\limits_{\mu\to\infty} \| v_\mu \|_{X_{0,A}} = 0$.

Remark 3. *Recently, the fractional magnetic operator problem has attracted the intense interest of many scholars because of its wide application in various fields. There is a variety of problems involving magnetic operators, for example fractional magnetic Schrödinger equations, fractional magnetic Kirchhoff-type equations, critical fractional magnetic degenerate equations, and so on. For their more detailed content, please refer to the relevant theories [18–24]. To the best of our knowledge, this is the first attempt to study the existence and asymptotic behavior of solutions for this kind of equation with the critical magnetic operator.*

The paper is structured as follows. In Section 2, we recall some properties on the fractional working spaces involved and discuss the variational formulation. In Section 3, we show the validity of the structure of the mountain pass lemma and compactness criterion. In Section 4, we adapt the variational method used by Evans in [25] to prove Theorem 1.

2. Variational Setup

In this section, we first recall the basic variational frameworks. For any $v \in \mathbb{C}$, we indicate with $\mathcal{R}v$ its real part and with $\bar{v}$ its complex conjugate. Moreover, we define the magnetic Gagliardo semi-norm as:

$$[v]_{H_A^s} = \left(\iint_{\Omega\times\Omega} \frac{|v(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}v(y)|^2}{|x-y|^{N+2s}} dxdy \right)^{\frac{1}{2}},$$

according to [21]. The function space $H_A^s(\Omega, \mathbb{C})$ is endowed with the norm:

$$\|v\|_{H_A^s(\Omega,\mathbb{C})} := \left(\|v\|^2_{L^2(\Omega,\mathbb{C})} + [v]^2_{H_A^s(\Omega,\mathbb{C})} \right)^{\frac{1}{2}}, \tag{7}$$

where we denote by $L^2(\Omega, \mathbb{C})$ the space of measurable functions $v : \Omega \to \mathbb{C}$ such that:

$$\|v\|_{L^2(\Omega,\mathbb{C})} = \left(\int_\Omega |u(x)|^2 \right)^{\frac{1}{2}} = \|v\|_2 < \infty,$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{C}$.

When $A = 0$, this definition (7) becomes the usual fractional space $H^s(\Omega)$. By Proposition 2.2 in [19], it is easy to see that $C_0^\infty \subseteq H_A^s(\Omega, \mathbb{C})$.

To obtain weak solutions of Problem (1), we only need to define our workspace as in [26].

$$X_{0,A} = \{v \in H^s_A(\mathbb{R}^N, \mathbb{C}) : v(x) = 0 \ \text{ a.e. in } \mathbb{R}^N \backslash \Omega\}.$$

which extends to the magnetic structure, relative to that space introduced in [12]. We use the method in [8] to define the inner product on $X_{0,A}$:

$$\langle v, \phi \rangle_{X_{0,A}} := \mathcal{R} \iint_{\mathbb{R}^{2N}} \frac{(v(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}v(y))\overline{(\phi(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}\phi(y))}}{|x-y|^{N+2s}} dxdy.$$

The corresponding norm is:

$$\|v\|_{X_{0,A}} = \Big(\iint_{\mathbb{R}^{2N}} \frac{|v(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}v(y)|^2}{|x-y|^{N+2s}} dxdy \Big)^{\frac{1}{2}}.$$

From (Lemma 7, [27]), we know that $(X_{0,A}, \langle \cdot, \cdot \rangle_{X_{0,A}})$ is a Hilbert space and is reflexive.

Lemma 1. *(Lemma 2.2, [26]) $X_{0,A} \hookrightarrow L^2(\Omega, \mathbb{C})$ is continuous and compact.*

Remark 4. *For any $v \in X_{0,A}$, there exists $S > 0$, depending on n, s, Ω, such that:*

$$\|v\|_2 \leq S\|v\|_{X_{0,A}}. \tag{8}$$

We define the best magnetic fractional Sobolev constant given by:

$$S_A := \inf_{v \in X_{0,A} \backslash \{0\}} \frac{\iint_{\mathbb{R}^{2N}} \frac{|v(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}v(y)|^2}{|x-y|^{N+2s}} dxdy}{\Big(\int_\Omega |v(x)|^{2^*_s} dx \Big)^{\frac{2}{2^*_s}}}. \tag{9}$$

Definition 1. *We say that $v \in X_{0,A}$ is a weak solution of Problem (1), if:*

$$\mathcal{M}([v]^2_{s,A})\langle v, \phi \rangle_{X_{0,A}} = \mu\mathcal{R} \int_\Omega v(x)\overline{\phi}dx + \mathcal{R} \int_\Omega |v(x)|^{2^*_s - 2} v(x)\overline{\phi}dx + \mathcal{R} \int_\Omega g(x, |v(x)|)v(x)\overline{\phi}dx$$

for any $\phi \in X_{0,A}$.

Obviously, in order to seek the weak solutions of Problem (1), we look for critical points of the C^1-functional $I_{A,\mu} : X_{0,A} \to \mathbb{R}$ denoted by:

$$I_{A,\mu}(v) = \frac{1}{2}\widetilde{\mathcal{M}}([v]^2_{s,A}) - \mu\frac{1}{2} \int_\Omega |v(x)|^2 dx - \frac{1}{2^*_s} \int_\Omega |v(x)|^{2^*_s} dx - \int_\Omega G(x, |v(x)|)dx$$

and:

$$\langle I'_{A,\mu}(v), \phi \rangle = \mathcal{M}([v]^2_{s,A})\langle v, \phi \rangle_{X_{0,A}} - \mu\mathcal{R} \int_\Omega v(x)\overline{\phi}dx - \mathcal{R} \int_\Omega |v(x)|^{2^*_s - 2}v(x)\overline{\phi}dx - \mathcal{R} \int_\Omega g(x, |v(x)|)v(x)\overline{\phi}dx \tag{10}$$

for any $v, \phi \in X_{0,A}$.

For the convenience of the reader, we introduce the following Palais–Smale condition at suitable level c_μ.

Definition 2. *Let $I_{A,\mu} \in C^1(X_{0,A}, \mathbb{R})$; we say that $I_{A,\mu}$ satisfies the $(PS)_{c_\mu}$ condition at the level $c_\mu \in \mathbb{R}$, if any sequence $\{v_k\} \subset X_{0,A}$ such that:*

$$I_{A,\mu}(v_k) \to c_\mu, \ \ I'_{A,\mu}(v_k) \to 0 \ \ as \ \ k \to \infty, \tag{11}$$

possesses a convergent subsequence in $X_{0,A}$.

To prove Theorem 1, we apply the following classical critical point theorem for our functional $I_{A,\mu}$.

Theorem 2. *[25] Let $X_{0,A}$ be a real Banach space and $I_{A,\mu} \in C^1(X_{0,A}, \mathbb{R})$ with $I_{A,\mu}(0) = 0$. Suppose that $I_{A,\mu}$ fulfills the $(PS)_{c_\mu}$ condition and*

(i) there exist $\alpha, \rho > 0$ such that $I_{A,\mu}(v) \geq \alpha$ for all $v \in X_{0,A}$, $\|v\|_{X_{0,A}} = \rho$;
(ii) there exists $\omega \in X_{0,A}$ satisfying $\|\omega\|_{X_{0,A}} > \rho$ such that $I_{A,\mu}(\omega) < 0$. Define:

$$\Gamma = \{\gamma \in C^1([0,1], X_{0,A}) : \gamma(0) = 0, \gamma(1) = \omega\}.$$

Then:

$$c_\mu = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_{A,\mu}(\gamma(t)) \geq \alpha$$

is a critical value of $I_{A,\mu}$.

3. Proof of the Main Result

In this section, we first show that the functional $I_{A,\mu}$ meets all the structural characteristics of the mountain pass theorem, and then, we prove that the functional $I_{A,\mu}$ satisfies the Palais–Smale condition at an appropriate level c_μ for any $\mu > \mu_0$, where $\mu_0 > 0$ is a threshold. Finally, we give the proof of Theorem 1.

Lemma 2. *If (M_1)–(M_2) and (g_1) hold, then, for any $\mu > 0$, there exist $\alpha, \rho > 0$ such that $I_{A,\mu}(v) \geq \alpha$ for all $v \in X_{0,A}$, $\|v\|_{X_{0,A}} = \rho$;*

Proof. According to (M_1), we have

$$\widetilde{\mathcal{M}}(t) \geq \widetilde{\mathcal{M}}(1)t^\theta \quad \text{for all} \quad t \in [0,1]. \tag{12}$$

Fix $\mu > 0$ and let $v \in X_{0,A}$, with $\|v\|_{X_{0,A}} \leq 1$. By Equations (5) and (12), we get that

$$I_{A,\mu}(v) \geq \frac{\widetilde{\mathcal{M}}(1)}{2}\|v\|^{2\theta}_{X_{0,A}} - \mu\frac{1}{2}\|v\|_2^2 - \frac{1}{2_s^*}\|v\|^{2_s^*}_{2_s^*} - \int_\Omega f(x)|u|^\varsigma dx. \tag{13}$$

Further, it follows from Equations (8), (9) and (13), there exists $C > 0$ such that

$$\begin{aligned} I_{A,\mu}(v) \geq & \frac{\widetilde{\mathcal{M}}(1)}{2}\|v\|^{2\theta}_{X_{0,A}} - \mu\frac{1}{2}S^2\|v\|^2_{X_{0,A}} - \frac{1}{2_s^*}\cdot\frac{1}{S_A^{2_s^*/2}}\|v\|^{2_s^*}_{X_{0,A}} - \|f(x)\|_\tau\|v\|^\varsigma_{2_s^*} \\ \geq & \frac{\widetilde{\mathcal{M}}(1)}{2}\|v\|^{2\theta}_{X_{0,A}} - \mu\frac{1}{2}S^2\|v\|^2_{X_{0,A}} - \frac{1}{2_s^*}\cdot\frac{1}{S_A^{2_s^*/2}}\|v\|^{2_s^*}_{X_{0,A}} - C\frac{1}{S_A^{\varsigma/2}}\|v\|^\varsigma_{X_{0,A}}. \end{aligned}$$

Thus, choosing ρ small enough, we obtain the result, since $2\theta < 2, \varsigma < 2_s^*$. □

Lemma 3. *If (M_1)–(M_2) and (g_2) hold, then, for any $\mu > 0$, there exist ω, with $\|\omega\|_{X_{0,A}} \geq 0$, such that $I_{A,\mu}(\omega) < 0$. In particular, $\|\omega\|_{X_{0,A}} > \rho$, where ρ is given in Lemma 2.*

Proof. Let $\mu > 0$, and fix $v_0 \in X_{0,A}$ with $\|v_0\|_{X_{0,A}} = 1$. According to Equation (4), there exists $t_0 > 0$ such that $\widetilde{\mathcal{M}}(t^2) \leq 2t^2$ for any $t \geq t_0$. Hence, it follows from Equation (6) that:

$$I_{A,\mu}(tv_0) \leq t^2 - \mu\frac{1}{2}t^2\|v_0\|_2^2 - \frac{1}{2_s^*}t^{2_s^*}\|v\|^{2_s^*}_{2_s^*} - d_1t^2\|v_0\|_2^2 + d_2\|v_0\|_1$$

for any $t > t_0$. Thanks to $1 < 2 < 2_s^*$, we can conclude that $I_{A,\mu}(tv_0) \to -\infty$ as $t \to \infty$. Therefore, the assertion is true by setting $\omega = t^* v_0$, with $t^* > 0$ large enough. □

Lemma 4. *We obtain the following result:*

$$\lim_{\mu \to \infty} c_\mu = 0.$$

Proof. Fix $\mu > 0$, and take $\omega \in X_{0,A}$ to be given by Lemma 3. There exists $t_\mu > 0$ satisfying $I_{A,\mu}(t_\mu \omega) = \max_{t \geq 0} I_{A,\mu}(t\omega)$, since $I_{A,\mu}$ fulfills the mountain pass geometry. Thus, $\langle I'_{A,\mu}(t_\mu \omega), \omega \rangle = 0$, and from Equation (10):

$$t_\mu \|\omega\|^2_{X_{0,A}} \mathcal{M}(t_\mu^2 \|\omega\|^2_{X_{0,A}}) = \mu t_\mu \|\omega\|_2^2 + t_\mu^{2_s^*-1} \|\omega\|_{2_s^*}^{2_s^*} + \int_\Omega g(x, |t_\mu \omega|)\omega^2 dx \geq t_\mu^{2_s^*-1} \|\omega\|_{2_s^*}^{2_s^*} \tag{14}$$

thanks to the fact that $\mu > 0$. We assert that $\{t_\mu\}_{\mu > 0}$ is bounded. Indeed, fix $\epsilon > 0$. From Equation (4), there exists $t_0 = t_0(\epsilon) > 0$ such that $\widetilde{\mathcal{M}}(t) \leq \epsilon t$ for any $t \geq t_0$. Therefore, denoting by $\Theta = \{\mu > 0 : t_\mu^2 \|\omega\|^2_{X_{0,A}} \geq t_0\}$, we observe that:

$$t_\mu^2 \|\omega\|^2_{X_{0,A}} \mathcal{M}(t_\mu^2 \|\omega\|^2_{X_{0,A}}) \leq \theta \widetilde{\mathcal{M}}(t_\mu^2 \|\omega\|^2_{X_{0,A}}) \leq \epsilon \theta t_\mu^2 \|\omega\|^2_{X_{0,A}} \ \text{ for any } \ \mu \in \Theta \tag{15}$$

thanks to (M_1). Since $\|\omega\|_{X_{0,A}} > \rho$, it follows from Equations (14) and (15) that:

$$\epsilon \theta \|\omega\|^2_{X_{0,A}} \geq t_\mu^{2_s^*-2} \|\omega\|_{2_s^*}^{2_s^*} \ \text{ for any } \ \mu \in \Theta,$$

which means that the boundedness of $\{t_\mu\}_{\mu \in \Theta}$. Obviously, through the composition of Θ, also $\{t_\mu\}_{\mu \in \{\mathbb{R} \setminus \Theta\}}$ is bounded. This concludes the proof of the assertion.

Let a sequence $\{\mu_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^+$ such that $\mu_k \to \infty$ as $k \to \infty$. Obviously $\{t_{\mu_k}\}_{k \in \mathbb{N}}$ is bounded. Thus, there exist a constant $t_0 > 0$ and a subsequence of $\{t_{\mu_k}\}_{k \in \mathbb{N}}$, which we still write as $\{t_{\mu_k}\}_{k \in \mathbb{N}}$, such that $t_{\mu_k} \to t_0$ as $k \to \infty$. Since $\mathcal{M}$ is a continuous function, then we get that $\left\{ \mathcal{M}(t_{\mu_k}^2 \|\omega\|^2_{X_{0,A}}) \right\}_{k \in \mathbb{N}}$ is still bounded, and so, according to Equation (14), there exists $\mathbb{B} > 0$ such that for any $k \in \mathbb{N}$:

$$\mu_k t_{\mu_k} \|\omega\|_2^2 + t_{\mu_k}^{2_s^*-1} \|\omega\|_{2_s^*}^{2_s^*} + \int_\Omega g(x, |t_{\mu_k} \omega|)\omega^2 dx \leq \mathbb{B}. \tag{16}$$

We claim that $t_0 = 0$. Indeed, we can assume that $t_0 > 0$, then from (g_1) and the dominate convergence theorem:

$$\int_\Omega g(x, |t_{\mu_k} \omega|)\omega^2 dx \to \int_\Omega g(x, |t_0 \omega|)\omega^2 dx > 0 \ \text{ as } \ k \to \infty$$

thanks to (g_2). Recalling that $\mu_k \to \infty$, we have:

$$\lim_{k \to \infty} \left(\mu_k t_{\mu_k} \|\omega\|_2^2 + t_{\mu_k}^{2_s^*-1} \|\omega\|_{2_s^*}^{2_s^*} + \int_\Omega g(x, |t_{\mu_k} \omega|)\omega^2 dx \right) = \infty,$$

which contradicts Equation (16). Hence, $t_0 = 0$ and $t_\mu \to 0$ as $\mu \to \infty$, thanks to the sequence $\{\mu_k\}_{k \in \mathbb{N}}$ being arbitrary. Consider now the path $\gamma(t) = t\omega, t \in [0,1]$, $\gamma(t) \in \Gamma$. According to Lemma 2 and (g_2):

$$0 < c_\mu \leq \max_{t \in [0,1]} I_{A,\mu}(\gamma(t)) \leq I_{A,\mu}(t_\mu \omega) \leq \frac{1}{2} \widetilde{\mathcal{M}}(t_\mu^2 \|\omega\|^2_{X_{0,A}}),$$

Because $\widetilde{\mathcal{M}}$ is continuous, thus we get that $\widetilde{\mathcal{M}}(t_\mu^2 \|\omega\|^2_{X_{0,A}}) \to 0$ as $\mu \to 0$. This means that the proof of the lemma is complete. □

Lemma 5. *If* (M_1)–(M_3) *and* (g_1)–(g_2) *hold, then, there exists* $\mu_0 > 0$ *such that for any* $\mu \geq \mu_0$, *the functional* $I_{A,\mu}$ *fulfills the* $(PS)_{c_\mu}$ *condition.*

Proof. Let $\mu > 0$ and $\{v_k\}_{k\in\mathbb{N}}$ satisfy Equation (11). Thanks to the degenerate nature of Problem (1), we consider two cases.

Case 1. $\inf_{k\in\mathbb{N}} \|v_k\|_{X_{0,A}} = h_\mu > 0$. First, we show that the sequence $\{v_k\}_{k\in\mathbb{N}}$ is bounded in $X_{0,A}$. By (M_2), with $\varrho = h_\mu^2$, there exists $m = m(h_\mu^2) > 0$ such that:

$$\mathcal{M}(\|v_k\|^2_{X_{0,A}}) \geq m \quad for\ all\ \ k \in \mathbb{N}. \tag{17}$$

Moreover, according to (M_1), Equation (17), and (g_2), we get that:

$$\begin{aligned} I_{A,\mu}(v_k) - \tfrac{1}{2}\langle I'_{A,\mu}(v_k), v_k\rangle &\geq \tfrac{1}{2}\widetilde{\mathcal{M}}(\|v_k\|^2_{X_{0,A}}) - \tfrac{1}{2}\mathcal{M}(\|v_k\|^2_{X_{0,A}})\|v_k\|^2_{X_{0,A}} + \left(\tfrac{1}{2} - \tfrac{1}{2_s^*}\right)\|v_k\|^{2_s^*}_{2_s^*} \\ &\geq \left(\tfrac{1}{2\theta} - \tfrac{1}{2_s^*}\right) m\|v_k\|^2_{X_{0,A}} + \left(\tfrac{1}{2} - \tfrac{1}{2_s^*}\right)\|v_k\|^{2_s^*}_{2_s^*}, \end{aligned} \tag{18}$$

where $2\theta < 2_s^*$ and also $2 < 2_s^*$ since $N > 2S$. Thus, it follows from Equations (11), (17) and (18) that:

$$\begin{aligned} &c_\mu + o(1) \geq \left(\frac{1}{2\theta} - \frac{1}{2_s^*}\right) m\|v_k\|^2_{X_{0,A}} \geq \sigma_\mu\|v_k\|^2_{X_{0,A}}, \\ &\sigma_\mu = \left(\frac{1}{2\theta} - \frac{1}{2_s^*}\right) m > 0 \end{aligned} \tag{19}$$

as $k \to \infty$, which implies that $\{v_k\}_{k\in\mathbb{N}}$ is bounded in $X_{0,A}$, and analogously, we obtain that $\{v_k\}_{k\in\mathbb{N}}$ is bounded in $L^{2_s^*}(\Omega, \mathbb{C})$.

Considering the boundedness of $\{v_k\}_{k\in\mathbb{N}}$, going if necessary to a subsequence, still denoted by $\{v_k\}_{\{k\in\mathbb{N}\}}$, by using Lemma 1 and (Theorem 4.9, [28]), there exist $v_\mu \in X_{0,A}$, such that:

$$\begin{aligned} &v_k \rightharpoonup v_\mu \ \ in\ \ X_{0,A}, && \|v_k\|_{X_{0,A}} \to \beta_\mu, \\ &v_k \rightharpoonup v_\mu \ \ in\ \ L^{2_s^*}(\Omega, \mathbb{C}), && \|v_k - v_\mu\|_{2_s^*} \to \xi_\mu, \\ &v_k \to v_\mu \ \ in\ \ L^2(\Omega, \mathbb{C}), && v_j \to v_\mu \ \ a.e.\ \ in\ \ \mathbb{R}^n. \end{aligned} \tag{20}$$

Obviously, $\beta_\mu > 0$ since $h_\mu > 0$. Moreover, we know that zero is the unique zero of $\mathcal{M}$ by (M_2), as well as $\mathcal{M}(\|v_k\|^2_{X_{0,A}}) \to \mathcal{M}(\beta_\mu^2)$ as $k \to \infty$, thanks to continuity.

First, we claim that:

$$\lim_{\mu\to\infty} \beta_\mu = 0. \tag{21}$$

If not, $\limsup_{\mu\to\infty} \beta_\mu = \beta > 0$. Therefore, there exists a sequence, called $k \mapsto \mu_k \uparrow \infty$, such that $\beta_{\mu_k} \to \beta$ as $k \to \infty$, and taking $k \to \infty$, we obtain from the first inequality of Equation (18) along with (M_1) and Lemma 4 that:

$$0 \geq \left(\frac{1}{2\theta} - \frac{1}{2_s^*}\right)\mathcal{M}(\beta^2)\beta^2 > 0$$

by (M_2), which is impossible. Therefore, we get the result (21). Furthermore, $\|v_\mu\|_{X_{0,A}} \leq \lim_k \|v_k\|_{X_{0,A}} = \beta_\mu$ thanks to $v_k \rightharpoonup v_\mu$, so according to Equation (21) and the fractional magnetic Sobolev inequality, we have:

$$\lim_{\mu\to\infty} \|v_\mu\|_{2_s^*} = \lim_{\mu\to\infty} \|v_\mu\|_{X_{0,A}} = 0. \tag{22}$$

By the boundedness of $\{v_k\}_{k\in\mathbb{N}}$ in $L^2(\Omega,\mathbb{C})$, we get that the sequence $\{|v_k|^{2_s^*-2}v_k\}_{k\in\mathbb{N}}$ is bounded in $L^{2_s^{*'}}(\Omega,\mathbb{C})$. Therefore, by Equation (20), we obtain:

$$|v_k|^{2_s^*-2}v_k \rightharpoonup |v_\mu|^{2_s^*-2}v_\mu \;\; in \;\; L^{2_s^{*'}}(\Omega,\mathbb{C}), \tag{23}$$

where $2_s^{*'} = 2N/(N+2S)$ is the Hölder conjugate of 2_s^*. Again, by (20) and (23), and (g_1), we get:

$$\mathcal{M}(\beta_\mu^2)\langle v_\mu,\phi\rangle_{X_{0,A}} = \mu\mathcal{R}\int_\Omega v_\mu(x)\overline{\phi}dx + \mathcal{R}\int_\Omega |v_\mu(x)|^{2_s^*-2}v_\mu(x)\overline{\phi}dx + \mathcal{R}\int_\Omega g(x,|v_\mu(x)|)v_\mu(x)\overline{\phi}dx$$

for any $\phi\in X_0$. Therefore, we can easily obtain that v_μ is a critical point of the $C^1(X_{0,A})$ functional:

$$I_{A,\beta_\mu}(v) = \frac{1}{2}\mathcal{M}(\beta_\mu^2)\|v\|^2_{X_{0,A}} - \mu\frac{1}{2}\|v\|_2^2 - \frac{1}{2_s^*}\|v\|^{2_s^*}_{2_s^*} - \int_\Omega G(x,|v(x)|)dx.$$

In particular, Equations (11), (20) and (23) imply that as $k\to\infty$:

$$\begin{aligned}
o(1) = \;& \langle I'_{A,\mu}(v_k) - I'_{A,\beta_\mu}(v_\mu), v_k - v_\mu\rangle \\
= \;& \mathcal{M}(\|v_k\|^2_{X_{0,A}})\|v_k\|^2_{X_{0,A}} + \mathcal{M}(\beta_\mu^2)\|v_\mu\|^2_{X_{0,A}} - \langle v_k,v_\mu\rangle_{X_{0,A}}\Big(\mathcal{M}(\|v_k\|^2_{X_{0,A}}) + \mathcal{M}(\beta_\mu^2)\Big) \\
& -\mathcal{R}\textstyle\int_\Omega \Big(g(x,|v_k(x)|)v_k(x) - g(x,|v_\mu(x)|)v_\mu(x)\Big)\overline{(v_k(x)-v_\mu(x))}dx \\
& -\mathcal{R}\textstyle\int_\Omega \Big(|v_k(x)|^{2_s^*-2}v_k(x) - |v_\mu(x)|^{2_s^*-2}v_\mu(x)\Big)\overline{(v_k(x)-v_\mu(x))}dx \\
= \;& \mathcal{M}(\beta_\mu^2)(\beta_\mu^2 - \|v_\mu\|^2_{X_{0,A}}) - \|v_k\|^{2_s^*}_{2_s^*} + \|v_\mu\|^{2_s^*}_{2_s^*} + o(1) \\
= \;& \mathcal{M}(\beta_\mu^2)\|v_k - v_\mu\|^2_{X_{0,A}} - \|v_k - v_\mu\|^{2_s^*}_{2_s^*} + o(1).
\end{aligned} \tag{24}$$

Indeed, again from (g_1), we have the following:

$$\int_\Omega |\, g(x,|v_k(x)|)v_k(x)\overline{(v_k(x)-v_\mu(x)}\,|\, dx \le D'\|f(x)\|_\tau\|v_k - v_\mu\|_{2_s^*} \le D\|f(x)\|_\tau,$$

where $D, D' > 0$. According to $f(x)\in L^\tau(\Omega)$, we get that sequence $\{|\, g(x,|v_k(x)|)v_k(x)\overline{(v_k(x)-v_\mu(x)}\,|\}_k$ is equi-integrable in $L^1(\Omega)$. Again, by Equation (20) and the Vitali convergence theorem, we have that:

$$\lim_{k\to\infty}\int_\Omega g(x,|v_k(x)|)v_k(x)\overline{(v_k(x)-v_\mu(x)}dx = 0.$$

Similarly,

$$\lim_{k\to\infty}\int_\Omega g(x,|v_\mu(x)|)v_\mu(x)\overline{(v_k(x)-v_\mu(x)}dx = 0.$$

Furthermore, again from Equation (20) and the well known Brézis and Lieb lemma of [29], we obtain:

$$\|v_k\|^2_{X_{0,A}} = \|v_k - v_\mu\|^2_{X_{0,A}} + \|v_\mu\|^2_{X_{0,A}} + o(1), \;\; \|v_k\|^{2_s^*}_{2_s^*} = \|v_k - v_\mu\|^{2_s^*}_{2_s^*} + \|v_\mu\|^{2_s^*}_{2_s^*} + o(1)$$

as $k\to\infty$. Thus, we have proven the crucial formula:

$$\mathcal{M}(\beta_\mu^2)\lim_{k\to\infty}\|v_k - v_\mu\|^2_{X_{0,A}} = \lim_{k\to\infty}\|v_k - v_\mu\|^{2_s^*}_{2_s^*}. \tag{25}$$

If $\xi_\mu = 0$ for all $\mu \geq \mu_0$, thanks to $\beta_\mu > 0$ and $\mathcal{M}$ admitting a unique zero at zero, then Equation (25) yields at once that $v_k \to v_\mu$ in $X_{0,A}$, concluding the proof. Instead, suppose that there exists a sequence $k \mapsto \mu_k \uparrow \infty$ such that $\xi_{\mu_k} = \xi_k > 0$. Combining Equations (9), (20) and (25), we have:

$$\xi_\mu^{2_s^*} \geq S_A \mathcal{M}(\beta_\mu^2)\xi_\mu^2. \tag{26}$$

Observing Equation (24), we can get:

$$\mathcal{M}(\beta_\mu^2)(\beta_\mu^2 - \|v_\mu\|^2_{X_{0,A}}) = \xi_\mu^{2_s^*}.$$

By Equation (26), we obtain along this sequence, denoting $\beta_{\mu_k} = \beta_k, v_{\mu_k} = v_k$, that:

$$(\xi_k^{2_s^*})^{\frac{2s}{N}} = \mathcal{M}(\beta_k^2)^{\frac{2s}{N}}(\beta_k^2 - \|v_k\|^2_{X_{0,A}})^{\frac{2s}{N}} \geq S_A \mathcal{M}(\beta_k^2). \tag{27}$$

Therefore, for all k large enough, it follows from Equation (21) and (M_3) that:

$$\beta_k^{\frac{4s}{N}} \geq (\beta_k^2 - \|v_k\|^2_{X_{0,A}})^{\frac{2s}{N}} \geq S_A \mathcal{M}(\beta_k^2)^{\frac{N-2s}{N}} \geq a_0^{\frac{N-2s}{N}} S_A \beta_k^{\frac{2(N-2s)}{N}}.$$

Hence, for all k enough enough, we have:

$$\beta_k^{2\frac{4s-N}{N}} \geq a_0^{\frac{N-2s}{N}} S_A,$$

thanks to $\beta_k > 0$ for any k. According to Equation (21), we know that the above inequality is impossible, thanks to $4s > N$ by assumption. Therefore, we complete the proof of Lemma 5.

Case 2. $\inf\limits_{k\in\mathbb{N}} \|v_k\|_{X_{0,A}} = 0$. We consider two cases at zero. When zero is an accumulation point of the real sequence $\{\|v_k\|_{X_{0,A}}\}_{k\in\mathbb{N}}$, so there is a subsequence of $\{v_k\}_{k\in\mathbb{N}}$ strongly converging to $v = 0$. This means that the trivial solution is a critical point at c_μ, which is a contradiction. When zero is an isolated point of $\{v_k\}_{k\in\mathbb{N}}$, then there is a subsequence, still denoted by $\{\|v_{k_j}\|_{X_{0,A}}\}_{k\in\mathbb{N}}$, such that $\inf\limits_{k\in\mathbb{N}} \|v_{k_j}\|_{X_{0,A}} = h_\mu > 0$, and we can proceed as before. This completes the proof of the second case and of the lemma. □

Proof of Theorem 1. According to Lemmas 2, 3 and 5, we note that $I_{A,\mu}$ fulfills all conditions in Theorem 2 for any $\mu \geq \mu_0$. Thus, for any $\mu \geq \mu_0$, there exists a critical point $v_\mu \in X_{0,A}$ for $I_{A,\mu}$ at level c_μ. Clearly, $v_\mu \not\equiv 0$ thanks to the fact that $I_{A,\mu}(v_\mu) = c_\mu > 0 = I_{A,\mu}(0)$. Finally, we also obtain the asymptotic behavior thanks to Equation (22). □

Finally, we give a simple example to show a direct application of our main result.

Example 1. *Let $N > 1$ and $1 > \theta > 0$. We consider the following problem*

$$\left(1 + e^{-\iint_\Omega \frac{|v(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}v(y)|^2}{|x-y|^{N+1}}dxdy}\right)(-\triangle)_A^{\frac{1}{2}} v(x) = \mu v + |v|^{\frac{2}{N-1}}v + g(x,|v|)v \ \text{ in } \Omega, \ \ v = 0 \text{ in } \mathbb{R}^N \setminus \Omega,$$

where $g(x,v)$ satisfy

$$g(x,|v|) = b(x)|v|^{q-2},$$

where $b \in L^\infty(\Omega)$, $b(x) \geq 0$ a.e. $x \in \Omega$, and $q \in (2\theta, 2^*)$. Obviously, g satisfying Conditions (g_1)–(g_2). It is clearly that (M_1)–(M_3) hold.

Then, Theorem 1 implies that the above problem admit a non-trivial solution in $X_{0,A}$.

4. Conclusions and Further Research

In this article, we combined the fractional magnetic operator with the more generalized Kirchhoff function to consider the existence and asymptotic behavior of the solution of this new equation, which plays a fundamental role in quantum mechanics in the description of the dynamics of the particle in a non-relativistic setting; see [1]. In order to overcome the difficulties caused by the critical situation and the intervention of magnetic operators, we used special techniques to deal with this problem, and our result generalized the previous work of Autuori [12] and Servadei [11] in some aspects. In future research, we will extend multiple equations to include fractional magnetic operators; due to the operators having a certain physical background and significance, the applicability is more extensive.

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