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# Lie Symmetry Analysis, Explicit Solutions and Conservation Laws of a Spatially Two-Dimensional Burgers–Huxley Equation

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**Abstract:** In this paper, we investigate a spatially two-dimensional Burgers–Huxley equation that depicts the interaction between convection effects, diffusion transport, reaction gadget, nerve proliferation in neurophysics, as well as motion in liquid crystals. We have used the Lie symmetry method to study the vector fields, optimal systems of first order, symmetry reductions, and exact solutions. Furthermore, using the power series method, a set of series solutions are obtained. Finally, conservation laws are derived using optimal systems.

**Keywords:** Burgers–Huxley equation; Lie symmetry method; conservation laws; Exact solutions

## 1. introduction

Mathematical modeling of dynamical systems results in nonlinear partial differential equations (PDEs). In reality, most complicated phenomena—namely diffusion, reaction, conservation, and many more—can be illustrated by means of partial differential equations. Due to their quintessence, PDEs are studied profusely in science and engineering. Various peculiar methods are designed for obtaining their exact and approximate solutions, which, in turn, help us in quantitative and qualitative analysis of these PDEs. The interested reader can see some of these methods in [1,2].

Lie symmetry analysis is a powerful and influential tool for mathematically analyzing partial differential equations. It can be used in securing analytic solutions or in switching PDEs into solvable ordinary differential equations (ODEs). Diverse symmetry vectors are also discovered for the considered system in some cases, however, sometimes there emerges a chance of linear combination of these vectors. To avoid this, an optimal system is constructed. Each member of this system is used in lessening independent variables of the system until analytic solutions are obtained or PDEs are switched to solvable ODEs [3–7]. The system also analyzes solutions of PDEs of different kinds as well as opens many fields [8].

The generalized Burgers–Huxley equation, which has many utilizations in the fields of biology, metallurgy, chemistry, mathematics, and engineering is of the following type,

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}+au^n\frac{\partial u}{\partial x}-bu(1-u^n)(u^n-k)=0. \tag{1}$$

This is a non-linear equation that has secured much importance due to its appearance in many physical phenomena and its scientific utilization. The parameters $a, b \geq 0$ are real constants, $n$ is a positive integer and $k \in [0,1]$. When $a = 0$ and $n = 1$, Equation (1) reduces to the Huxley equation and with $b = 0,\ n = 1$, it becomes the Burgers equation. Some exact numerical and traveling wave solutions to (1) were reported in [9–11]. However, the spline collocation method for the Burgers–Huxley equation was discussed in a book by Schiesser [12]. In addition, many other analytical and numerical methods for generalized Burgers–Huxley equations have been developed in the past, see for example [13–29].

In this paper, we will analyze the Burgers–Huxley equation in two spatial dimensions, which is of the following form,

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}-u\frac{\partial u}{\partial x}-u\frac{\partial u}{\partial y}+u^3+ku-(k+1)u^2=0. \tag{2}$$

Equation (2) couples both the assets of the Burgers equation (this is one of the basic models in fluid mechanics and is used to catch some of the properties of turbulent flow in a channel, which occurs due to the interaction of the reverse outcome of convection and diffusion and also describes the format of shock waves, traffic flow, and acoustic transmission [30]) and the Huxley equation (which is used for nerve proliferation in neurophysics and wall proliferation in liquid crystals [31]). So, we will designate this equation as simply a two dimensional Burgers–Huxley equation. The combined Burgers–Huxley equation shows a prototypical imitation that specifies the interaction between the reaction gadget, diffusion transport and convection effects, nerve proliferation, and motion in liquid crystals [32].

In Section 2, the vector fields and optimal systems for (2) are obtained by using the Lie symmetry method. In Section 3, we computed the similarity diminution for one and two dimensional subalgebras and hence obtained the group invariant solutions for (2). Employing the power series method, certain power series solutions are achieved in Section 4. Finally, in Section 5, conservation laws are derived using optimal systems.

## 2. Lie Symmetry Analysis

In this section, we will study the Lie symmetries and optimal systems of the Burgers–Huxley equation. Consider the one-parameter Lie group of transformation:

$$\begin{aligned} x^* &\to x+\epsilon\xi(x,y,t;u)+O(\epsilon^2),\\ y^* &\to y+\epsilon\eta(x,y,t;u)+O(\epsilon^2),\\ t^* &\to t+\epsilon\zeta(x,y,t;u)+O(\epsilon^2),\\ u^* &\to u+\epsilon\varphi(x,y,t;u)+O(\epsilon^2), \end{aligned} \tag{3}$$

where $\epsilon$ is the group parameter. The infinitesimal operator associated with the above transformations is:

$$X=\xi(x,y,t;u)\frac{\partial}{\partial x}+\eta(x,y,t;u)\frac{\partial}{\partial y}+\zeta(x,y,t;u)\frac{\partial}{\partial t}+\varphi(x,y,t;u)\frac{\partial}{\partial u}. \tag{4}$$

The coefficient functions $\xi(x,y,t;u)$, $\eta(x,y,t;u)$, $\zeta(x,y,t;u)$, and $\varphi(x,y,t;u)$ are to be determined and the vector field $X$ satisfies the Lie symmetry condition,

$$pr^{(2)}X(\Delta)|_{\Delta=0}=0, \tag{5}$$

where

$$\Delta = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - u\frac{\partial u}{\partial x} - u\frac{\partial u}{\partial y} + u^3 + ku - (k+1)u^2 = 0.$$

The second prolongation of the infinitesimal generator is given by

$$\begin{aligned} X^{(2)} = X &+ \varphi_x(x,y,t;u)\frac{\partial}{\partial u_x} + \varphi_y(x,y,t;u)\frac{\partial}{\partial u_y} + \varphi_t(x,y,t;u)\frac{\partial}{\partial u_t} \\ &+ \varphi_{xx}(x,y,t;u)\frac{\partial}{\partial u_{xx}} + \varphi_{xy}(x,y,t;u)\frac{\partial}{\partial u_{xy}} + \varphi_{xt}(x,y,t;u)\frac{\partial}{\partial u_{xt}} \\ &+ \varphi_{yy}(x,y,t;u)\frac{\partial}{\partial u_{yy}} + \varphi_{yt}(x,y,t;u)\frac{\partial}{\partial u_{yt}} + \varphi_{tt}(x,y,t;u)\frac{\partial}{\partial u_{tt}}. \end{aligned} \tag{6}$$

with

$$\left.\begin{aligned} \varphi_t &= D_t(\varphi) - u_t D_t(\zeta) - u_x D_t(\xi) - u_y D_t(\eta), \\ \varphi_x &= D_x(\varphi) - u_t D_x(\zeta) - u_x D_x(\xi) - u_y D_x(\eta), \\ \varphi_y &= D_y(\varphi) - u_t D_y(\zeta) - u_x D_y(\xi) - u_y D_y(\eta), \\ \varphi_{xx} &= D_x(\varphi_x) - u_{xt} D_x(\zeta) - u_{xx} D_x(\xi) - u_{xy} D_x(\eta), \\ \varphi_{yy} &= D_y(\varphi_y) - u_{yt} D_y(\zeta) - u_{xy} D_y(\xi) - u_{yy} D_y(\eta), \\ \varphi_{tt} &= D_t(\varphi_t) - u_{tt} D_t(\zeta) - u_{xt} D_t(\xi) - u_{yt} D_t(\eta), \\ \varphi_{xy} &= D_x(\varphi_y) - u_{yt} D_x(\zeta) - u_{xy} D_x(\xi) - u_{yy} D_x(\eta), \\ \varphi_{xt} &= D_x(\varphi_t) - u_{tt} D_x(\zeta) - u_{xt} D_x(\xi) - u_{yt} D_x(\eta), \\ \varphi_{yt} &= D_y(\varphi_t) - u_{tt} D_y(\zeta) - u_{xt} D_y(\xi) - u_{yt} D_y(\eta), \end{aligned}\right\} \tag{7}$$

where the the operator $D_i$ is defined as:

$$D_i = \frac{\partial}{\partial x^i} + u_i\frac{\partial}{\partial u} + u_{ij}\frac{\partial}{\partial u_j} + \cdots, \quad i,j = 1,2,3.$$

Coupling Equations (5) and (6), we can obtain the compatible condition for Equation (5). Substituting extended transformations into the obtained compatible conditions and making the coefficients of several monomials in partial derivatives and numerous powers of $u$ equal, we get the following over determining system of PDEs:

$$\begin{aligned} \eta_y - \xi_x &= 0, \\ \xi_{xx} + \xi_{yy} - \xi_t - 2\varphi_{xu} - \varphi &= 0, \\ \eta_t - \eta_{xx} - \eta_{yy} + 2\varphi_{yu} + \varphi &= 0, \\ \eta_{xu} + \xi_{yu} &= 0, \\ \zeta_{xu} = 0, \ \ \eta_u = 0, \ \ \xi_u = 0, \ \ \zeta_u &= 0, \\ \zeta_{xx} + \zeta_{yy} - \zeta_t + 2\eta_y &= 0, \\ \varphi_{uu} - 2\xi_{xu} &= 0, \\ \xi_{uu} = 0, \ \ \zeta_{uu} = 0, \ \ \eta_{uu} &= 0, \\ \varphi_{uu} - 2\eta_{yu} &= 0, \\ \zeta_x + \zeta_y + k\zeta_u &= 0, \\ \xi_x + \xi_y + k\xi_u - 2\eta_y &= 0, \\ \eta_y - \eta_x &= 0, \\ \eta_x + \xi_y &= 0, \\ \zeta_x = 0, \ \ \zeta_y &= 0, \end{aligned} \tag{8}$$

$$\begin{aligned} \varphi_{xx} + \varphi_{yy} - \varphi_t - k\varphi &= 0, \\ \varphi_x + \varphi_y + k\varphi_u + 2k\varphi + 2\varphi - 2k\eta_y &= 0, \\ (k+1)\varphi_u - 3\varphi + (2k+2)\eta_y &= 0, \\ \varphi_u - 2\eta_y &= 0. \end{aligned}$$

By solving the over determining system of PDEs (8), we obtain the coefficient functions $\zeta, \xi$, and $\eta$ as:

$$\zeta = c_1,\ \xi = c_2,\ \eta = c_3,$$

where $c_1$, $c_2$, and $c_3$ are random constants. The Lie algebra of infinitesimal symmetry of Equation (2) with $k \neq 0$ is given by,

$$X_1 = \tfrac{\partial}{\partial t}, \quad X_2 = \tfrac{\partial}{\partial x}, \quad X_3 = \tfrac{\partial}{\partial y}.$$

### 2.1. Transformed Solutions

One can acquire the group transformation initiated by the Lie point symmetry operator $X_i$ $(i = 1, 2, 3)$ by solving the following ODEs

$$\begin{aligned} \frac{dx^*}{d\epsilon} &= \xi(x,t,y,u), \quad x^*|_{\epsilon=0} = x, \\ \frac{dt^*}{d\epsilon} &= \zeta(x,t,y,u), \quad t^*|_{\epsilon=0} = t, \\ \frac{dy^*}{d\epsilon} &= \varphi(x,t,y,u), \quad y^*|_{\epsilon=0} = y, \\ \frac{du^*}{d\epsilon} &= \eta(x,t,y,u), \quad u^*|_{\epsilon=0} = u. \end{aligned}$$

The one-parameter Lie symmetry groups generated by infinitesimals $X_1$, $X_2$, and $X_3$ are given by

$$r_1 : (x,y,t,u) \longrightarrow (x+\epsilon_1, y, t, u),$$

$$r_2 : (x,y,t,u) \longrightarrow (x, y+\epsilon_2, t, u),$$

$$r_3 : (x,y,t,u) \longrightarrow (x, y, t+\epsilon_3, u),$$

where $\epsilon_1$, $\epsilon_2$, and $\epsilon_3$ are group parameters.

Depending on the values of $r_1$, $r_2$, and $r_3$, if $f(x,y,t)$ is any confidential solutions of Equation (2), then the new solutions can be given by

$$\begin{aligned} r_1(\epsilon).u(x,y,t) &= f(x-\epsilon_1, y, t), \\ r_2(\epsilon).u(x,y,t) &= f(x, y-\epsilon_2, t), \\ r_3(\epsilon).u(x,y,t) &= f(x, y, t-\epsilon_3). \end{aligned} \tag{9}$$

### 2.2. Optimal System of Subalgebras

For the optimal system, we will first construct the tables for commutation relations and adjoint action of the obtained symmetries. For the sake of obtaining the adjoint representation we will use the Lie series in the form:

$$Ad(\exp(\epsilon X_i))X_j = X_j - \epsilon\,[X_i, X_j] + \frac{\epsilon^2}{2!}[X_i,[X_i,X_j]] - \cdots,$$

The commutation relations between basis elements satisfies:

$$[X_i, X_j] = 0, \quad i,j = 1,2,3,$$

whereas, the adjoint representation Table 1 is given as:

**Table 1.** Adjoint representation table.

| ***Ad*** | $X_1$ | $X_2$ | $X_3$ |
|---|---|---|---|
| $X_1$ | $X_1$ | $X_2$ | $X_3$ |
| $X_2$ | $X_1$ | $X_2$ | $X_3$ |
| $X_3$ | $X_1$ | $X_2$ | $X_3$ |

Following the method given in [8], consider the vector $X$ with random coefficients $a_1, a_2, a_3$, such that

$$X = a_1X_1 + a_2X_2 + a_3X_3,$$

suppose $a_3 \neq 0$ and set up $a_3 = 1$, so that

$$\mathrm{X} = a_1X_1 + a_2X_2 + X_3.$$

The scheme involves simplifying the coefficients as much as possible. To abolish the coefficient of $X_2$, we will use it to act on X. It is easy to see that the vector form cannot be reduced much more because commutation relations are zero.

Further, suppose $a_3 = 0$ and establish $a_2 = 1$ so that,

$$\mathrm{X} = a_1X_1 + X_2.$$

Repeating the same process and normalizing the coefficients we have the following one-dimensional optimal system of subalgebras,

$$\{X_1\ , X_2, X_3, aX_1 + X_2, aX_1 + bX_2 + X_3\},$$

where $a$ and $b$ are arbitrary constants.

## 3. Symmetry Reduction

*Symmetry Reductions for Optimal System*

Symmetry diminution and explicit solutions have numerous utilizations in the connection of differential equations. Solutions of partial differential equations asymptotically tend to the solutions of lower dimensional equations, which are obtained by symmetry diminution. Especially, exact solutions originating from symmetry technique can be utilized effectively to study properties such as asymptotics etc. In the previous section, we obtained the point symmetries and optimal system for two-dimensional Burgers–Huxley equation, now we will interrogate the symmetry reductions and exact solutions using one-dimensional subalgebras.

(i) For the linear combination $aX_1 + X_2$, the invariants can be obtained by solving the characteristic equation $\frac{dt}{a} = \frac{dx}{1} = \frac{dy}{0} = \frac{du}{0}$, giving

$$s = ax - t, \quad r = y, \quad \Psi = u. \tag{10}$$

Using,

$$u = \Psi(s, r),$$

we obtained the following PDE with two independent variables,

$$a^2\frac{\partial^2\Psi}{\partial s^2} + \frac{\partial^2\Psi}{\partial r^2} + (1 + a\Psi)\frac{\partial\Psi}{\partial s} + \Psi\frac{\partial\Psi}{\partial r} + \Psi(k - \Psi)(\Psi - 1) = 0. \tag{11}$$

By the application of the similarity transformation method on reduced Equation (11) again, we have

$$\xi = c_1, \quad \zeta = c_2, \quad \varphi = 0, \tag{12}$$

where $c_1$ and $c_2$ are random constants and the characteristic Equation for (12) is

$$\frac{ds}{1} = \frac{dr}{1} = \frac{d\Psi}{0}. \tag{13}$$

Hence, $\Psi$ can be written as:

$$\Psi = \beta(\alpha), \;\; \alpha = s - r. \tag{14}$$

Substituting Equation (14) into (11), we have the following ODE:

$$(a^2+1)\frac{d^2\beta}{d\alpha^2} + (1 + a\beta - \beta)\frac{d\beta}{d\alpha} + \beta(\beta-1)(k-\beta) = 0. \tag{15}$$

(ii) For the linear combination $aX_1 + bX_2 + X_3$, the invariants can be obtained by solving the characteristic equation

$$\frac{dt}{a} = \frac{dx}{b} = \frac{dy}{1} = \frac{du}{0},$$

giving

$$s = ax - bt, \quad r = ay - t, \quad \Psi = u. \tag{16}$$

Using,

$$u = \Psi(s, r),$$

we obtained the following PDE with two independent variables:

$$a^2\frac{\partial^2\Psi}{\partial s^2} + a^2\frac{\partial^2\Psi}{\partial r^2} + a\Psi\frac{\partial\Psi}{\partial s} + a\Psi\frac{\partial\Psi}{\partial r} + b\frac{\partial\Psi}{\partial s} + \frac{\partial\Psi}{\partial r} + \Psi(k-\Psi)(\Psi-1) = 0. \tag{17}$$

By the application of the similarity transformation method on reduced Equation (17) again, we obtain

$$\xi = c_1, \quad \zeta = c_2, \quad \varphi = 0, \tag{18}$$

where $c_1$ and $c_2$ are random constants. Choosing $c_1$ and $c_2$ equal to 1, the characteristic Equation for (18) is:

$$\frac{ds}{1} = \frac{dr}{1} = \frac{d\Psi}{0}. \tag{19}$$

Therefore, $\Psi$ can be written as:

$$\Psi = \beta(\alpha), \;\; \alpha = s - r. \tag{20}$$

Substituting Equation (20) into (17), we have following ODE:

$$2a^2\frac{d^2\beta}{d\alpha^2} + (b-1)\frac{d\beta}{d\alpha} + \beta(\beta-1)(k-\beta) = 0. \tag{21}$$

Reductions corresponding to the remaining vectors occuring in the optimal system can be obtained in a similar way, hence, we omit the details here.

## 4. Explicit Power Series Solution

After reducing partial differential equations into ordinary differential equations in the previous section, we will now explore the exact analytic solution of the reduced equations using the power series method [8].

*4.1. Series Solution of Reduced Equation (15)*

Consider the power series solution of the form:

$$\beta(\alpha) = \sum_{n=0}^{\infty} b_n \alpha^n. \tag{22}$$

Incorporating (22) into Equation (15), we get

$$\begin{aligned} 0 &= (a^2+1)\sum_{n=1}^{\infty} b_{n+2}(n+1)(n+2)\alpha^n + (k+1)\sum_{n=1}^{\infty}\sum_{m=0}^{n} b_m b_{n-m}\alpha^n \\ &- k\sum_{n=1}^{\infty} b_n\alpha^n - \sum_{n=1}^{\infty}\left(\sum_{m=0}^{n}\sum_{j=0}^{m} b_j b_{m-j} b_{n-m}\alpha^n\right) + \sum_{n=1}^{\infty} b_{n+1}(n+1)\alpha^n \\ &- kb_0 - (k+1)b_0^2 - b_0^3 + (a-1)\sum_{n=1}^{\infty}\sum_{m=0}^{n} b_m b_{n-m+1}(n-m+1)\alpha^n \\ &+ 2(a^2+1)b_2 + (a-1)b_0 b_1 + b_1. \end{aligned} \tag{23}$$

Inspecting the coefficients for $n = 0$ and $n \geq 1$, we have

$$b_2 = \frac{-(k+1)b_0^2 + b_0^3 + kb_0 - (a_1)b_0 b_1 - b_1}{2(a^2+1)}, \tag{24}$$

$$\begin{aligned} b_{n+2} &= \frac{1}{(a^2+1)(n+1)(n+2)}\Bigg( -(k+1)\sum_{m=0}^{n} b_m b_{n-m} + kb_n \\ &+ \sum_{m=0}^{n}\sum_{j=0}^{m} b_j b_{m-j} b_{n-m} - (a-1)\sum_{m=0}^{n} b_m b_{n-m+1}(n-m+1) - b_{n+1}(n+1)\Bigg). \end{aligned} \tag{25}$$

So, we find the solution of Equation (15) in the form of the power series given by

$$\beta(\alpha) = b_0 + b_1\alpha + b_2\alpha^2 + \sum_{n=1}^{\infty} b_{n+2}\alpha^{n+2}. \tag{26}$$

Back substituting the values of $\alpha$ and $\beta$, we have

$$\Psi = b_0 + b_1(s-r) + b_2(s-r)^2 + \sum_{n=1}^{\infty} b_{n+2}(s-r)^{n+2}. \tag{27}$$

Thus, in terms of original variables, we obtain the following solution of the two dimensional Burgers–Huxley equation:

$$u = b_0 + b_1(ax-t-y) + b_2(ax-t-y)^2 + \sum_{n=1}^{\infty} b_{n+2}(ax-t-y)^{n+2}, \tag{28}$$

where $b_0$ and $b_1$ are random constants and the remaining constants can be calculated using Equations (24) and (25), respectively. The convergence of Equation (28) is shown in the Appendix A.

*4.2. Series Solution of Reduced Equation (21)*

Now, consider the power series solution of (21) in the form (22). These equations together yield

$$
\begin{aligned}
&2a^2 \sum_{n=1}^{\infty} b_{n+2}(n+2)(n+1)\alpha^n + (b-1)\sum_{n=1}^{\infty} b_{n+1}(n+1)\alpha^n + (b-1)b_1 \\
&- k\sum_{n=1}^{\infty} b_n\alpha^n - b_0^3 + 4a^2 b_2 + (k+1)\sum_{n=1}^{\infty}\sum_{m=0}^{n} b_m b_{n-m}\alpha^n + (k+1)b_0^2 - kb_0 \\
&- \sum_{n=1}^{\infty}\left(\sum_{m=0}^{n}\sum_{j=0}^{m} b_j b_{m-j} b_{n-m}\alpha^n\right) = 0.
\end{aligned}
$$

Comparing the coefficients of like powers of $\alpha$, we obtain

$$
b_2 = \frac{kb_0 - (b-1)b_1 + b_0^3 - (k+1)b_0^2}{4a^2}, \tag{29}
$$

$$
\begin{aligned}
b_{n+2} = {}& \frac{1}{2a^2(n+1)(n+2)} \\
&\left(kb_n - (k+1)\sum_{m=0}^{n} b_m b_{n-m} - (b-1)(n+1)b_{n+1} + \sum_{m=0}^{n}\sum_{j=0}^{m} b_j b_{m-j} b_{n-m}\right).
\end{aligned} \tag{30}
$$

Next, we use values of $\alpha$ and $\beta$ from (20) into (22) to have

$$
\Psi = b_0 + b_1(s-r) + b_2(s-r)^2 + \sum_{n=1}^{\infty} b_{n+2}(s-r)^{n+2}, \tag{31}
$$

where $b_0$ and $b_1$ are random constants. Hence, we have the following solution in terms of original variables:

$$
\begin{aligned}
u = {}& b_0 + b_1\big(ax - ay + t(1-b)\big) + b_2\big(ax - ay + t(1-b)\big)^2 \\
&+ \sum_{n=1}^{\infty} b_{n+2}\big(ax - ay + t(1-b)\big)^{n+2},
\end{aligned} \tag{32}
$$

with $b_2$ is given in (29) and $b_{n+2}$ is given in (30).

## 5. Conservation Laws

Conservation laws are of fundamental importance in the study of partial differential equations as they provide conserved quantities for all the solutions, can detect integrability and linearization, and can be used to check the precision of numerical solution methods [33]. Conservation laws provide one of the basic rules in formulating models in mathematics and certain times a partial differential equation having a large number of conservation laws depicts a strong implication of its integrability.

*5.1. Preliminaries*

Here, we recall the Ibragimov scheme of constructing conservation laws corresponding to the given symmetries of any system of PDEs, provided that the number of equations in the system is equal to the number of dependent variables [34]. Consider a $p$-th order system of partial differential equations of $p$ independent variables and $q$ dependent variables, respectively, as $x = (x^1, x^2, ..., x^p)$ and $u = (u^1, u^2, ..., u^q)$,

$$
G_\Lambda(x, u, u_{(1)}, ..., u_{(p)}) = 0, \;\; \Lambda = 0, 1, ..., q. \tag{33}
$$

The infinitesimal generator for Equation (33) is:

$$X = \zeta^n \frac{\partial}{\partial x^n} + \beta^\Lambda \frac{\partial}{\partial u^\Lambda}, \; n = 0, 1, ..., p, \;\; \Lambda = 1, 2, ..., q.$$

The $p$-th prolongation of the point symmetry operator is:

$$X^p = X + \beta_n^{(1)\Lambda} \frac{\partial}{\partial u_n^\Lambda} + ... + \beta_{n_1 n_2 ... n_p}^{(p)\Lambda} \frac{\partial}{\partial u_{n_1 n_2 ... n_p}^\Lambda}, \; p \geqslant 1,$$

with

$$\beta_n^{(1)\Lambda} = D_n \beta^\Lambda - (D_n \zeta_m) u_m^\Lambda,$$
$$\beta_{n_1 ... n_p}^{(p)\Lambda} = D_n p \beta_{n_1 ... n_{(p-1)}}^{(p-1)\Lambda} - (D_n p \zeta_m) u_{n_1, ..., m n_{(p-1)}}^\Lambda,$$

Here

$$m, n = 1, 2, ..., p, \; \Lambda = 1, 2, ..., q,$$

and $D_n$ is the total derivative operator. A vector $T = (T^1, ..., T^n)$ is a conserved vector of (33) if

$$D_j T^j = 0, \tag{34}$$

is satisfied for all solutions of (33). Equation (34) is called a local conservation law provided that $T^i$ are free of integral terms.

**Theorem 1.** *Every Lie point symmetry of Equation (33) results in a conservation law. The conserved vector components are*

$$C^n = \zeta^n L + W^\Lambda \left[ \frac{\partial L}{\partial u_n^\Lambda} + \sum_{s \geqslant 1} D_{n_1} ... D_{n_s} (W^\Lambda) (\frac{\partial L}{\partial u_{n_1 ... n_s}^\Lambda}) \right], \tag{35}$$

*with,*

$$W^\Lambda = \eta^\Lambda - \zeta^m u_m^\Lambda.$$

*and the formal Lagrangian is defined as*

$$L = v^\Lambda G_\Lambda.$$

*In order to find the adjoint equation we have,*

$$G_\Lambda^*(x, u, v, u_1, v_1, ..., u_p, v_p) = \frac{\delta L}{\delta u^\Lambda},$$

*where* $\frac{\delta}{\delta u}$ *is the variational derivative and* $(v^\Lambda, \;\; \Lambda = 0, 1, ..., q)$ *is the adjoint variable.*

**Theorem 2.** *Equation (2) is not strictly self-adjoint.*

**Proof.** The formal Lagrangian for Equation (2) can be written as,

$$L = v(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - u\frac{\partial u}{\partial x} - u\frac{\partial u}{\partial y} - u(k-u)(u-1)). \tag{36}$$

The adjoint Equation of (2) is given by:

$$G^*(x, u, v, v_1, v_2) = \frac{\delta(vG)}{\delta u} = 0, \tag{37}$$

where $\frac{\delta}{\delta u}$ is the Euler–Lagrange operator given by,

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_i\left(\frac{\partial L}{\partial u_i}\right) + D_i D_j\left(\frac{\partial L}{\partial u_{ij}}\right). \quad (38)$$

Using the Euler–Lagrange operator in Equation (2), we have the following adjoint equation, after transforming in the original variable $u$,

$$\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + u\frac{\partial u}{\partial x} - u\frac{\partial u}{\partial y} + 2u\frac{\partial u}{\partial x} + 2u\frac{\partial u}{\partial y} + 2ku - k - 3u^2 + 2u = 0,$$

which is not strictly self-adjoint. □

### 5.2. Conservation Laws of a Spatially Two-Dimensional Burgers–Huxley Equation

Now, we will construct conservation laws for each element in the optimal system obtained in Section 2.

(i) For $X_1$, we have $W = -u_t$ and $\xi^t = 1$. Substituting these values in Equation (35), we find

$$\begin{aligned} C^t &= -vu_{xx} - vu_{yy} - vuu_x - vuu_y - kvu^2 + kuv + u^3 v - vu^2, \\ C^x &= uvu_t - v_x u_t + vu_{xt}, \\ C^y &= uvu_t - v_y u_t + vu_{yt}. \end{aligned}$$

(ii) For $X_2$, W $= -u_x$ and $\xi^x = 1$. Substituting in Equation (35), we obtain

$$\begin{aligned} C^t &= -vu_x, \\ C^x &= vu_t - vu_{yy} - uvu_y - kvu^2 + kuv + u^3 v - vu^2 - u_x v_x, \\ C^y &= uvu_x - v_y u_x + vu_{xy}. \end{aligned}$$

(iii) For the generator $X_3$, W $= -u_y$ and $\xi^y = 1$. So, in this case, we have the following conserved vector:

$$\begin{aligned} C^t &= -vu_y, \\ C^x &= u_y uv - u_y v_x + vu_{xy}, \\ C^y &= vu_t - vu_{xx} - uvu_x - kvu^2 + kuv + vu^3 - vu^2 - u_y v_y. \end{aligned}$$

(iv) For the generator $aX_1 + X_2$, W $= -au_t - u_x$ and $\xi^t =$ a, $\xi^x = 1$. Substituting into (35), we have

$$\begin{aligned} C^t &= -avu_{xx} - avu_{yy} - auvu_x - auvu_y - akvu^2 + akuv + avu^3 - avu^2 \\ &- vu_x, \\ C^x &= vu_t - vu_{yy} - uvu_y - u^2 kv + kuv + vu^3 - vu^2 + auvu_t - au_t v_x \\ &- u_x v_x + avu_{xt}, \\ C^y &= auvu_t - au_t v_y + uvu_x - u_x v_y + avu_{yt} + vu_{xy}. \end{aligned}$$

(v) For the generator $aX_1 + bX_2 + X_3$, W $= -au_t - bu_x - u_y$ and $\xi^t =$ a, $\xi^x =$ b and $\xi^y = 1$. Using these values in Equation (35), we have

$$
\begin{aligned}
C^t &= -avu_{xx} - avu_{yy} - auvu_x - auvu_y - akvu^2 + akuv + avu^3 - avu^2 \\
&- bvu_x - vu_y, \\
C^x &= vu_t - bvu_{yy} - buvu_y - bu^2kv + bkuv + bvu^3 - bvu^2 + auvu_t \\
&+ bvu_t + uvu_y - au_tv_x - bu_xv_x - v_xu_y + avu_{xt} + vuxy, \\
C^y &= vu_t - vu_{xx} - vuu_x - kvu^2 + kuv + vu^3 + auvu_t + buvu_x - av_yu_t \\
&- bu_xv_y - v_yu_y + avu_{yt} + bvu_{xy}.
\end{aligned}
$$

The conserved vectors comprise random solutions of the adjoint equation, thereby implying the interminable number of conservation laws. Conservation laws play a compelling role in the solution process of an equation or system of equations.

## 6. Concluding Remarks

In this work, we have investigated a spatially two-dimensional Burgers–Huxley equation, which combines both the properties of Burgers (convective phenomenon) and Huxley equation (nerve proliferation and motion in liquid crystals). We have computed the Lie point symmetries of (2), and constructed an optimal system of one and two dimensional subalgebras using these symmetries. Symmetry reductions are performed for one-dimensional subalgebras. Furthermore, some power series solutions are calculated using the power series method. Finally, conservation laws are derived using the Ibragimov theory and involvement of random solutions predicted that conservation laws consisted of an interminable number of conserved vectors.

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## Appendix A

In this appendix, we will discuss the convergence of the power series solution given by (28) using the implicit function theorem [35]. For Equation (25) we have

$$
\begin{aligned}
|b_{n+2}| \leq \frac{|1|}{|(a^2+1)|} \Bigg( & |k+1| \sum_{m=0}^{n} |b_m||b_{n-m}| + |k||b_n| - |b_{n+1}| \\
& |a-1| \sum_{m=0}^{n} |b_m||b_{n-m+1}| + \sum_{m=0}^{n}\sum_{j=0}^{m} |b_j||b_{m-j}||b_{n-m}| \Bigg).
\end{aligned} \tag{A1}
$$

Choosing

$$
M = \max\left( \frac{|k+1|}{|(a^2+1)|}, \frac{|k|}{|(a^2+1)|}, \frac{|1|}{|(a^2+1)|}, \frac{|a-1|}{|(a^2+1)|} \right),
$$

we have

$$
\begin{aligned}
|b_{n+2}| \leq M \Bigg( & \sum_{m=0}^{n} |b_m||b_{n-m}| + |b_n| + \sum_{m=0}^{n}\sum_{j=0}^{m} |b_j||b_{m-j}||b_{n-m}| \\
& \sum_{m=0}^{n} |b_m||b_{n-m+1}| - |b_{n+1}| \Bigg).
\end{aligned} \tag{A2}
$$

Now define a power series $\gamma(\alpha) = \sum_{n=0}^{\infty} a_n \alpha^n$ and consider $a_i = |b_i|$ for $i = 0, 1, 2, \ldots$. Then we have

$$\begin{aligned} a_{n+2} \leq M\Bigg( & \sum_{m=0}^{n} a_m a_{n-m} + a_n + \sum_{m=0}^{n} \sum_{j=0}^{m} a_j a_{m-j} a_{n-m} \\ & \sum_{m=0}^{n} a_m a_{n-m+1} - a_{n+1} \Bigg). \end{aligned} \tag{A3}$$

It can be easily seen that $|b_i| \leq a_i$ and hence the assumed series is majorant series. Now, we will show that the series $\gamma(\alpha) = \sum_{n=0}^{\infty} a_n \alpha^n$ is convergent. Through some calculations, we have

$$\begin{aligned} \gamma(\alpha) = a_0 + a_1 \alpha + M \sum_{n=0}^{\infty} \sum_{m=0}^{n} a_m a_{n-m} \alpha^{n+2} + M \sum_{n=0}^{\infty} a_n \alpha^{n+2} \\ + M \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{j=0}^{m} a_j a_{m-j} a_{n-m} \alpha^{n+2} + M \sum_{n=0}^{\infty} \sum_{m=0}^{n} a_m a_{n-m+1} \alpha^{n+2} - M \sum_{n=0}^{\infty} a_{n+1} \alpha^{n+2}. \end{aligned} \tag{A4}$$

Now, taking into account the implicit functional system with respect to $\alpha$, we have

$$\begin{aligned} \gamma(\alpha, \gamma) = \gamma - a_0 - a_1 \alpha - M(\gamma(\alpha)^2 \alpha^2 + \alpha^2 \gamma(\alpha) - \alpha(\gamma(\alpha) - a_0) \\ + (\gamma - a_0)(\gamma - a_0) + (\gamma - a_0 - a_1 \alpha)\gamma). \end{aligned} \tag{A5}$$

We consider the point $(0, a_0)$, as $\gamma$ is analytic in the neighbourhood of the point $(0, a_0)$, as $\gamma(0, a_0) = 0$ and $\frac{\partial}{\partial \gamma}(0, a_0) \neq 0$. Hence, by the implicit function theorem, the series $\gamma(\alpha) = \sum_{n=0}^{\infty} a_n \alpha^n$ is analytic in the neighbourhood of the point $(0, a_0)$ and has a positive radius of convergence. Hence, convergence is proven.

## References


1. Olver, P. *Applications of Lie Groups to Differential Equations*; Springer: New York, NY, USA, 1986.
2. Wazwaz, A.M. *Partial Differential Equations and Solitary Waves Theory*; Springer: New York, NY, USA, 2009.
3. Aliyu, A.I.; Inc, M.; Yusuf, A.; Baleanu, D. Symmetry analysis, explicit solutions, and conservation laws of a sixth-order nonlinear Ramani equation. *Symmetry* **2018**, *10*, 341. [CrossRef]
4. Rashed, A.S. Analysis of (3+1)-dimensional unsteady gas flow using optimal system of Lie symmetries. *Math. Comput. Simul.* **2019**, *156*, 327–346. [CrossRef]
5. Qin, C.Y.; Tian, S.F.; Wang, X.B.; Zou, L.; Zhang, T.T. Lie symmetry analysis, conservation laws and analytic solutions of the time fractional Kol- mogorov-Petrovskii-Piskunov equation. *Chin. J. Phys.* **2018**, *56*, 1734–1742. [CrossRef]
6. Wang, G.; Yong, X.L.; Huang, Y.; Tian, J. Symmetry, Pulson Solution, and conservation laws of the Holm-Hone equation. *Adv. Math. Phy.* **2019**. [CrossRef]
7. Wang, Y.H.; Li, L.Z. Lie symmetry analysis, analytical solution, and conservation laws of a sixth-order generalized time-fractional Sawada-Kotera equation. *Symmetry* **2019**, *11*, 1436. [CrossRef]
8. Arrigo, D.J. *Symmetry Analysis of Differential Equations*; Wiley: New York, NY, USA, 2015.
9. Yefimova, O.Y.; Kudryashov, N.A. Exact solutions of the Burgers–Huxley equation. *J. Appl. Math. Mech.* **2004**, *68*, 413–420. [CrossRef]
10. Tomasiello, S. Numerical solutions of the Burgers–Huxley equation by the IDQ method. *Int. J. Comput. Math.* **2010**, *87*, 129–140. [CrossRef]
11. Feng, Z.S.; Tian, J. Travelling wave solutions of the Burgers–Huxley equation. *IMA J. Appl. Math.* **2012**, *77*, 316–325. [CrossRef]
12. Schiesser, W.E. *Spline Collocation Methods for Partial Differential Equations: With Applications in R, BurgersHuxley Equation*; John Wiley & Sons: Hoboken, NJ, USA, 2017; Chapter 14.
13. Wang, X.Y.; Zhu, Z.S.; Lu, Y.K. Solitary wave solutions of the generalized Burgers–Huxley equation. *J. Phys. A Math. Gen.* **1999**, *23*, 271–274. [CrossRef]

14. Ismail, H.N.A; Raslan, K.; Abd-Rabboh, A.A. Adomian decomposition method for Burgers–Huxley and Burgers-Fisher equations. *Appl. Math. Comput.* **2004**, *159*, 291–301.
15. Hashim, I.; Noorani, M.S.M; Said Al-Hadidi, M.R. Solving the generalized Burgers–Huxley Equation using the Adomian decomposition method. *Math. Comput. Model.* **2006**, *43*, 1404–1411. [CrossRef]
16. Javidi, M. A numerical solution of the generalized Burger's-Huxley equation by pseudospectral method and Darvishis preconditioning. *Appl. Math. Comput.* **2006**, *175*, 1619–1628.
17. Javidi, M. A numerical solution of the generalized Burger's-Huxley equation by spectral collocation method *Appl. Math. Comput.* **2006**, *178*, 338–344.
18. Darvishi, M.T.; Kheybari, S.; Khani, F. Spectral collocation method and Darvishis preconditionings to solve the generalized Burgers–Huxley equation. *Commun. Nonlinear Sci. Numer. Simul.* **2008**, *13*, 2091–2103. [CrossRef]
19. Deng, X. Travelling wave solutions for the generalized Burgers–Huxley equation. *Appl. Math. Comput.* **2008**, *204*, 733–737. [CrossRef]
20. Batiha, B.; Noorani, M.S.M.; Hashim, I. Application of variational iteration method to the generalized Burgers–Huxley equation. *Chaos Soliton Fract.* **2008**, *36*, 660–663. [CrossRef]
21. Khattak, A.J. A computational meshless method for the generalized Burger's-Huxley equation. *Appl. Math. Model.* **2009**, *33*, 3218–3729. [CrossRef]
22. Molabahrami, A.; Khami, F. The homotopy analysis method to solve the Burgers–Huxley equation. *Nonlinear Anal. Real World Appl.* **2009**, *10*, 589–600. [CrossRef]
23. Javidi, M.; Golbabai, A. A new domain decomposition algorithm for generalized Burger's-Huxley equation based on Chebyshev polynomials and preconditioning. *Chaos Soliton Fract.* **2009**, *39*, 849–857. [CrossRef]
24. Gao, H.; Zhao, R. New exact solutions to the generalized Burgers–Huxley equation. *Appl. Math. Comput.* **2010**, *217*, 1598–1603 . [CrossRef]
25. Dehghan, M.; Saray, B.N.; Lakestani, M. Three methods based on the interpolation scaling functions and the mixed collocation finite difference schemes for the numerical solution of the nonlinear generalized Burgers–Huxley equation. *Math. Comput. Model.* **2012**, *55*, 1129–1142. [CrossRef]
26. Çelik, İ. Haar wavelet method for solving generalized Burgers–Huxley equation. *Arab J. Math. Sci.* **2012**, *18*, 25–37. [CrossRef]
27. El-Kady, M.; El-Sayed, S.M.; Fathy, H.E. Development of Galerkin method for solving the generalized Burgers Huxley equation. *Math. Probl. Eng.* **2013**. [CrossRef]
28. Çelik, İ. Chebyshev Wavelet collocation method for solving generalized Burgers–Huxley equation. *Math. Meth. Appl. Sci.* **2015**, *39*, 366–377. [CrossRef]
29. Singh, G.K.; Arora, G.; Singh, M.K. A numerical scheme for the generalized Burgers–Huxley equation. *J. Egypt. Math. Soc.* **2016**, *24*, 629–637. [CrossRef]
30. Whitham, G.B. *Linear and Nonlinear Waves*; Wiley: New York, NY, USA, 1974.
31. Wang, X.Y. Nerve propagation and wall in liquid crystals. *Phy. Lett. A* **1985**, *112*, 402–406. [CrossRef]
32. Satsuma, J. *Topics in Soliton Theory and Exactly Solvable Nonlinear Equations*; World Scientific: Singapore, 1987.
33. Anco, S.; Bluman, G. Direct construction method for conservation laws of partial differential equations Part I: Examples of conservation law classifications. *Eur. J. Appl. Math.* **2002**, *13*, 545–566. [CrossRef]
34. Ibragimov, N.H. A new conservation theorem. *J. Math. Anal. Appl.* **2007**, *333*, 311–328. [CrossRef]
35. Rudin, W. *Principles of Mathematic Analysis*; China Machine Press: Beijing, China, 2004.