


Article

Oscillatory Properties of Solutions of Even-Order Differential Equations

Elmetwally M. Elabbasy ^{1,†}, Rami Ahmad El-Nabulsi ^{2,*,†} and Osama Moaaz ^{1,†} 
and Omar Bazighifan ^{3,†} 

¹ Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt; emelabbasy@mans.edu.eg (E.M.E.); o_moaaz@mans.edu.eg (O.M.)

² Athens Institute for Education and Research, Mathematics and Physics Divisions, 10671 Athens, Greece

³ Department of Mathematics, Faculty of Science, Hadhramout University, Hadhramout 50512, Yemen; o.bazighifan@gmail.com

* Correspondence: nabulsiahmadrami@yahoo.fr

† These authors contributed equally to this work.

Received: 24 December 2019; Accepted: 19 January 2020; Published: 2 February 2020



Abstract: This work is concerned with the oscillatory behavior of solutions of even-order neutral differential equations. By using Riccati transformation and the integral averaging technique, we obtain a new oscillation criteria. This new theorem complements and improves some known results from the literature. An example is provided to illustrate the main results.

Keywords: oscillatory solutions; even-order; neutral delay differential equations

1. Introduction

Neutral differential equations are used in numerous applications in technology and natural science. For instance, they are frequently used for the study of distributed networks containing lossless transmission lines; see Hale [1], and therefore their qualitative properties are important.

Very recently, some scholars have been attracted by the problems of the oscillations of differential equations and made relative advances therein, as in [2–11].

Delay differential equations are often studied in one of two cases

$$\int_{t_0}^{\infty} r^{-1/\alpha_1}(s) ds = \infty \quad (1)$$

or

$$\int_{t_0}^{\infty} r^{-1/\alpha_1}(s) ds < \infty,$$

which are said to be in the canonical or noncanonical form, respectively, see [12]. For the canonical form, many authors in [13–16] studied the asymptotic behavior of the solutions of equation

$$\left(r(t) \left(z^{(n-1)}(t) \right)^\alpha \right)' + q(t) x^\beta(\tau(t)) = 0. \quad (2)$$

In the noncanonical form, Li and Rogovchenko [17] studied the asymptotic properties of solutions of higher-order neutral differential (2) under the assumptions that allow applications to even- and odd-order equations with delayed and advanced arguments.

This paper is motivated by several recent studies [3,7,9,10] of such higher order equations. Using the integral averaging technique and the Riccati transformation, we study the asymptotic properties of solutions of even order neutral delay differential equations of the form

$$\left(r(t) \left| z^{(n-1)}(t) \right|^{\alpha_1-1} z^{(n-1)}(t) \right)' + \sum_{i=1}^m q_i(t) |y(\sigma_i(t))|^{\alpha_i-1} y(\sigma_i(t)) = 0, \quad (3)$$

where n is an even natural number and m is a natural number. In this paper, we assume that α_i are positive integers, $\alpha_{i+1} > \alpha_i$, $r \in C^1([t_0, \infty), \mathbb{R})$, $r(t) > 0$, $r'(t) \geq 0$, $a \in C([t_0, \infty), [0, 1))$, $\lim_{t \rightarrow \infty} a(t) = \infty$, $q_i, \sigma_i \in C([t_0, \infty), \mathbb{R})$, $q_i > 0$, $\sigma_i(t) \leq t$, and $\lim_{t \rightarrow \infty} \sigma_i(t) = \infty$ for all $i = 1, \dots, m$. During the following results, for clarity of presentation, we study only the case where $m = 3$. Moreover, we denote, for convenience, that

$$\begin{aligned} z(t) &: = y(t) + a(t) y(\tau(t)), \\ \sigma(t) &: = \min \{ \sigma_i(t), i = 1, 2, 3 \} \\ B(t) &: = \int_{t_0}^t \frac{1}{r^{1/\alpha_1}(t)} dt, \quad F_+(t) := \max \{ F(t), 0 \}, \\ A_i(t) &: = q_i(t) (1 - a(\sigma_i(t)))^{\alpha_i}, \text{ for all } i = 1, 2, 3. \\ m_1 &: = \frac{\alpha_3 + \alpha_2 - 2\alpha_1}{\alpha_2 - \alpha_1}, \quad m_2 := \frac{\alpha_3 + \alpha_2 - 2\alpha_1}{\alpha_3 - \alpha_1} \end{aligned}$$

and

$$A(t) = A_1(t) + (m_1 A_2(t))^{1/m_1} (m_2 A_3(t))^{1/m_2}.$$

We say that a function, y , is a solution of (3), we mean a non-trivial real function $z(t) \in C^{n-1}([t_y, \infty))$, $t_y \geq t_0$, satisfying (3) on $[t_y, \infty)$ and which has the property $r(t) \left(z^{(n-1)}(t) \right)^{\alpha_1} \in C([t_y, \infty))$. We consider only those solutions y of (3) which satisfy $\sup \{ |y(t)| : t \geq T \} > 0$, for any $T \geq t_y$. A solution of (3) is called oscillatory if it has arbitrary large zeros, otherwise it is called nonoscillatory. Studying the functional differential equations, the continuity of all functions, and $r(t) > 0$ are sufficient conditions for the existence of one or more solutions of the equation.

To establish our main results, we make use of the following lemmas:

Lemma 1 ([18]). *Let C and D nonnegative real numbers. Then*

$$C^\mu + (\mu - 1) D^\mu - \mu C D^{(\mu-1)} \geq 0, \quad \mu > 1,$$

where the equality holds, if and only if, $C = D$.

Lemma 2 ([19]). *Let $h \in C^n([t_0, \infty), (0, \infty))$. If $h^{(n)}(t)$ is eventually of one sign for all large t , then there exist a $t_y > t_1$ for some $t_1 > t_0$ and an integer m , $0 \leq m \leq n$ with $n + m$ even for $h^{(n)}(t) \geq 0$ or $n + m$ odd for $h^{(n)}(t) \leq 0$ such that $m > 0$ implies that $h^{(k)}(t) > 0$ for $t > t_y$, $k = 0, 1, \dots, m - 1$ and $m \leq n - 1$ implies that $(-1)^{m+k} h^{(k)}(t) > 0$ for $t > t_y$, $k = m, m + 1, \dots, n - 1$.*

Lemma 3 ([20]). *Let $h(t) \in C^n([t_0, \infty), (0, \infty))$. If $h^{(n-1)}(t) h^{(n)}(t) \leq 0$ for all $t \geq t_y$, then for every $\theta \in (0, 1)$, there exists a constant $M > 0$ such that*

$$h'(\theta t) \geq M t^{n-2} h^{(n-1)}(t),$$

for all sufficiently large t .

This paper is concerned with the oscillatory behavior of a class of even-order neutral differential equations with multi-delays. Firstly, by using the Riccati transformations, we obtain a new oscillation

criteria for this equation. Secondly, using the integral averaging technique, we establish a Philos type oscillation criterion. This new theorem complements and improves some known results in the literature. Finally, an example is provided to illustrate the main results.

2. Oscillation Criteria

In this section, we establish new oscillation results for Equation (3) using the Riccati transformation.

Theorem 1. Assume that (1) holds. If there exists a positive function $\rho \in C^1([t_0, \infty), (0, \infty))$ such that

$$\int_{t_0}^{\infty} \left(\rho(s) A(s) - \frac{1}{(\alpha_1 + 1)^{\alpha_1 + 1}} \frac{r^{\alpha_1 + 1}(t) (\rho'_+(t))^{\alpha_1 + 1}}{(\theta M \sigma'(s) \sigma^{n-2}(s) \rho(s) r(s))^{\alpha_1}} \right) ds = \infty, \tag{4}$$

then all solutions of (3) are oscillatory.

Proof. Assume that (3) has a nonoscillatory solution y . Without loss of generality, we may assume that there exists a $t_1 \in [t_0, \infty)$ such that $y(t) > 0, y(\tau(t)) > 0$ and $y(\sigma_i(t)) > 0$ for all $i = 1, 2, 3$ and $t \in [t_1, \infty)$. It follows from Lemma 2 that

$$z(t) > 0, z'(t) > 0, z^{(n-1)}(t) > 0 \text{ and } z^{(n)}(t) < 0,$$

for $t \geq t_1$. Since $\tau(t) \leq t$ and $z'(t) > 0$, we find

$$\begin{aligned} y(t) &= z(t) - a(t)y(\tau(t)) \geq z(t) - a(t)z(\tau(t)) \geq z(t) - a(t)z(t) \\ &\geq (1 - a(t))z(t). \end{aligned}$$

Hence,

$$y(\sigma_i(t)) \geq (1 - a(\sigma_i(t)))z(\sigma_i(t)), \quad i = 1, 2, 3,$$

which, with (3), gives

$$\begin{aligned} \left(r(t) \left(z^{(n-1)}(t) \right)^{\alpha_1} \right)' &= -q_1(t)y^{\alpha_1}(\sigma_1(t)) - q_2(t)y^{\alpha_2}(\sigma_2(t)) - q_3(t)y^{\alpha_3}(\sigma_3(t)) \\ &\leq -A_1(t)z^{\alpha_1}(\sigma_1(t)) - A_2(t)z^{\alpha_2}(\sigma_2(t)) - A_3(t)z^{\alpha_3}(\sigma_3(t)). \end{aligned} \tag{5}$$

Using Lemma 3, we obtain

$$z'(\theta\sigma(t)) \geq M\sigma^{n-2}(t)z^{(n-1)}(\sigma(t)) \geq M\sigma^{n-2}(t)z^{(n-1)}(t). \tag{6}$$

Now, we define a generalized Riccati substitution ω by

$$\omega(t) := \rho(t)r(t) \frac{\left(z^{(n-1)}(t) \right)^{\alpha_1}}{z^{\alpha_1}(\theta\sigma(t))}. \tag{7}$$

Then, $\omega(t) > 0$. By differentiating (7), we obtain

$$\begin{aligned} \omega'(t) &= \rho'(t)r(t) \frac{\left(z^{(n-1)}(t) \right)^{\alpha_1}}{z^{\alpha_1}(\theta\sigma(t))} + \rho(t) \frac{\left(r(t) \left(z^{(n-1)}(t) \right)^{\alpha_1} \right)'}{z^{\alpha_1}(\theta\sigma(t))} \\ &\quad - \alpha_1 \theta \rho(t)r(t) \frac{\left(z^{(n-1)}(t) \right)^{\alpha_1} z'(\theta\sigma(t)) \sigma'(t)}{z^{\alpha_1 + 1}(\theta\sigma(t))}. \end{aligned} \tag{8}$$

Since $z'(t) > 0$ and $\sigma(t) \leq \sigma_i(t)$, we have

$$z(\sigma_i(t)) \geq z(\sigma(t)), \quad i = 1, 2, 3.$$

Hence, from (5), (6), and (8), we see that

$$\begin{aligned} \omega'(t) \leq & \frac{\rho'(t)}{\rho(t)} \omega(t) - \rho(t) (A_1(t) + A_2(t) z^{\alpha_2 - \alpha_1}(\sigma_2(t)) + A_3(t) z^{\alpha_3 - \alpha_1}(\sigma_3(t))) \\ & - \alpha_1 \theta M \sigma'(t) \sigma^{n-2}(t) \rho(t) r(t) \frac{(z^{(n-1)}(t))^{\alpha_1 + 1}}{z^{\alpha_1 + 1}(\theta \sigma(t))}. \end{aligned}$$

This implies that

$$\begin{aligned} \omega'(t) \leq & \frac{\rho'(t)}{\rho(t)} \omega(t) - \rho(t) (A_1(t) + A_2(t) z^{\alpha_2 - \alpha_1}(\sigma_2(t)) + A_3(t) z^{\alpha_3 - \alpha_1}(\sigma_3(t))) \\ & - \frac{\alpha_1 \theta M \sigma'(t) \sigma^{n-2}(t)}{(\rho(t) r(t))^{\frac{1}{\alpha_1}}} \omega^{(\alpha_1 + 1)/\alpha_1}(t). \end{aligned} \quad (9)$$

Using Youngs inequality

$$|uv| \leq \frac{1}{c_1} |u|^{c_1} + \frac{1}{c_2} |v|^{c_2}, \quad c_1 > 1, \quad c_2 > 1, \quad \frac{1}{c_1} + \frac{1}{c_2} = 1,$$

with $u = (m_1 A_2(t) z^{\alpha_2 - \alpha_1}(\sigma(t)))^{\frac{1}{m_1}}$, $v = (m_2 A_3(t) z^{\alpha_3 - \alpha_1}(\sigma(t)))^{\frac{1}{m_2}}$ and $c_i = m_i$, we obtain

$$A_2(t) z^{\alpha_2 - \alpha_1}(\sigma(t)) + A_3(t) z^{\alpha_3 - \alpha_1}(\sigma(t)) \geq (m_1 A_2(t))^{\frac{1}{m_1}} (m_2 A_3(t))^{\frac{1}{m_2}}. \quad (10)$$

Combining (9) and (10), we have

$$\omega'(t) \leq \frac{\rho'(t)}{\rho(t)} \omega(t) - \rho(t) A(t) - \frac{\alpha_1 \theta M \sigma'(t) \sigma^{n-2}(t)}{(\rho(t) r(t))^{\frac{1}{\alpha_1}}} \omega^{(\alpha_1 + 1)/\alpha_1}(t).$$

If we set $\mu = (\alpha_1 + 1) / \alpha_1$, then we find

$$\omega'(t) \leq \frac{\rho'(t)}{\rho(t)} \omega(t) - \rho(t) A(t) - \frac{\alpha_1 \theta M \sigma'(t) \sigma^{n-2}(t)}{(\rho(t) r(t))^{\mu-1}} \omega^\mu(t). \quad (11)$$

Now, using Lemma 1 with

$$C = \left(\frac{\alpha_1 \theta M \sigma'(t) \sigma^{n-2}(t)}{(\rho(t) r(t))^{\mu-1}} \right)^{1/\mu} \omega(t)$$

and

$$D = \left(\frac{r(t) \rho'_+(t)}{\mu} \left(\alpha_1 \theta M \sigma'(t) \sigma^{n-2}(t) \right)^{-1/\mu} \right)^{1/(\mu-1)},$$

we obtain

$$\frac{\rho'(t)}{\rho(t)} \omega(t) - \frac{\alpha_1 \theta M \sigma'(t) \sigma^{n-2}(t)}{(\rho(t) r(t))^{\mu-1}} \omega^\mu(t) \leq \frac{1}{(\alpha_1 + 1)^{\alpha_1 + 1}} \frac{r^{\alpha_1 + 1}(t) (\rho'_+(t))^{\alpha_1 + 1}}{(\theta M \sigma'(t) \sigma^{n-2}(t) \rho(t) r(t))^{\alpha_1}}. \quad (12)$$

Hence, from (11) and (12), we have

$$\omega'(t) \leq \frac{1}{(\alpha_1 + 1)^{\alpha_1 + 1}} \frac{r^{\alpha_1 + 1}(t) (\rho'_+(t))^{\alpha_1 + 1}}{(\theta M \sigma'(t) \sigma^{n-2}(t) \rho(t) r(t))^{\alpha_1}} - \rho(t) A(t).$$

Integrating from t_1 to t we find

$$\int_{t_1}^t \left(\rho(s) A(s) - \frac{1}{(\alpha_1 + 1)^{\alpha_1 + 1}} \frac{r^{\alpha_1 + 1}(s) (\rho'_+(s))^{\alpha_1 + 1}}{(\theta M \sigma'(s) \sigma^{n-2}(s) \rho(s) r(s))^{\alpha_1}} \right) ds \leq \omega(t_1) - \omega(t) < \omega(t_1).$$

which contradicts (4).

This completes the proof. \square

In Theorem 1, we can obtain different conditions for oscillation of all solutions of Equation (3) with different choices of $\rho(t)$. If we set $\rho(t) := B^{\alpha_1}(\sigma(t))$, then we obtain the following corollary.

Corollary 1. Assume that (1) holds. If

$$\limsup_{t \rightarrow \infty} \int_{t_1}^t \left(B^{\alpha_1}(\sigma(s)) A(s) - \frac{1}{(\alpha_1 + 1)^{\alpha_1 + 1}} \frac{|\alpha_1 r(s) \sigma'(s) r^{-1/\alpha_1}(\sigma(s))|^{\alpha_1 + 1}}{B(\sigma(s)) (\theta M r(s) \sigma'(s) \sigma^{n-2}(s))^{\alpha_1}} \right) ds = \infty,$$

then all solutions of (3) are oscillatory.

3. Kamenev-Type Criteria

In the section theorem, we establish new oscillation results for Equation (3) using the integral averaging technique to establish the Philos-type.

Definition 1. Let

$$D = \{(t, s) \in \mathbb{R}^2 : t \geq s \geq t_0\} \text{ and } D_0 = \{(t, s) \in \mathbb{R}^2 : t > s \geq t_0\}.$$

A kernel function $H \in C(D, \mathbb{R})$ is said to belong to the function class \mathfrak{S} , written by $H \in \mathfrak{S}$, if

- (i₁) $H(t, t) = 0$ and $H(t, s) > 0$, $(t, s) \in D_0$ for $t \geq t_0$,
- (i₂) H has a nonpositive continuous partial derivative $\partial H / \partial s$ on D_0 with respect to the second variable, and there exist functions $h \in C(D_0, \mathbb{R})$ and $\delta \in C^1([t_0, \infty), (0, \infty))$ such that

$$-\frac{\partial}{\partial s} (H(t, s) \delta(s)) = H(t, s) A(s) \frac{\rho'(t)}{\rho(t)} + h(t, s). \tag{13}$$

Theorem 2. Assume that (1) holds. If there exist functions $\rho, \delta \in C^1([t_0, \infty), (0, \infty))$ such that (13) and

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_1)} \int_{t_1}^t (H(t, s) \delta(s) \rho(s) A(s) - \Theta(s)) ds = \infty, \tag{14}$$

hold, where

$$\Theta(s) := \left(\frac{h(t, s)}{\alpha_1 + 1} \right)^{\alpha_1 + 1} \frac{r(s) \rho(s)}{(\theta M H(t, s) \delta(s) \sigma'(s) \sigma^{n-2}(s))^{\alpha_1}},$$

then every solution of (3) is oscillatory.

Proof. Proceeding as in the proof of Lemma 1, we obtain (11). Multiplying (11) by $H(t, s) \delta(s)$ and integrating from t_1 to t , we find

$$\begin{aligned} \int_{t_1}^t H(t, s) \delta(s) \rho(s) A(s) ds &\leq \int_{t_1}^t H(t, s) \delta(s) \frac{\rho'(t)}{\rho(t)} \omega(s) ds - \int_{t_1}^t H(t, s) \delta(s) \omega'(s) ds \\ &\quad - \int_{t_1}^t H(t, s) \delta(s) \frac{\alpha_1 \theta M \sigma'(s) \sigma^{n-2}(s)}{(\rho(s) r(s))^{\mu-1}} \omega^\mu(s) ds. \\ &\leq - \int_{t_1}^t \left[- \frac{\partial}{\partial s} (H(t, s) \delta(s)) - H(t, s) \delta(s) \frac{\rho'(t)}{\rho(t)} \right] \omega(s) ds \\ &\quad - H(t, s) \delta(s) \omega(s) \Big|_{t_1}^t \\ &\quad - \int_{t_1}^t H(t, s) \delta(s) \frac{\alpha_1 \theta M \sigma'(s) \sigma^{n-2}(s)}{(\rho(s) r(s))^{\mu-1}} \omega^\mu(s) ds. \end{aligned}$$

This implies that

$$\begin{aligned} \int_{t_1}^t H(t, s) \delta(s) \rho(s) A(s) ds &\leq - \int_{t_1}^t |h(t, s)| \omega(s) ds + H(t, t_1) \delta(t_1) \omega(t_1) \\ &\quad - \int_{t_1}^t H(t, s) \delta(s) \frac{\alpha_1 \theta M \sigma'(s) \sigma^{n-2}(s)}{(\rho(s) r(s))^{\mu-1}} \omega^\mu(s) ds. \end{aligned} \quad (15)$$

Using Lemma 1 with

$$\begin{aligned} C &= \frac{(\alpha_1 \theta M \sigma'(s) \sigma^{n-2}(s) H(t, s) \delta(s))^{1/\mu}}{(\rho(s) r(s))^{1/(\alpha_1+1)}} \omega(s); \\ D &= \left(\frac{|h(t, s)|^{\alpha_1} \rho(s) r(s)}{\mu^{\alpha_1} (\alpha_1 \theta M \sigma'(s) \sigma^{n-2}(s) H(t, s) \delta(s))^{\alpha_1}} \right)^{1/\mu}, \end{aligned}$$

we have

$$\int_{t_1}^t (H(t, s) \delta(s) \rho(s) A(s) - \Theta(s)) ds \leq H(t, t_1) \delta(t_1) \omega(t_1),$$

which contradicts (14). Theorem 2 is proved. \square

Corollary 2. If the condition (14) in Theorem 2 is replaced by the following conditions:

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_1)} \int_{t_1}^t H(t, s) \delta(s) \rho(s) A(s) ds = \infty \quad (16)$$

and

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_1)} \int_{t_1}^t \Theta(s) ds < \infty, \quad (17)$$

then every solution of (3) is oscillatory.

Example 1. Consider the differential equation

$$\left(t \left(y(t) + \frac{1}{2} y \left(\frac{t}{3} \right) \right) \right)' + y \left(\frac{t}{2} \right) + y^2(t) + y^3 \left(\frac{t}{4} \right) = 0, \quad (18)$$

where $t \geq 1$. Note that $r(t) = t$, $n = 2$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 3$, $a = 1/2$, $\tau(t) = t/3$, $\sigma_1(t) = t/2$, $\sigma_2(t) = t$, $\sigma_3(t) = t/4$, and $q_i(t) = 1$. Hence, we have $m_1 = m_2 = 3$, $A_k(t) = 2^{-k}$

$$A(s) = \frac{1}{4} \sqrt[3]{18} + \frac{1}{2}$$

and

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\alpha_1}(t)} dt = \int_{t_0}^{\infty} \frac{1}{t} dt = \infty.$$

If we set $\rho(t) = 1$, then condition (4) is satisfied. Therefore, from Theorem 1, every solution of Equation (18) is oscillatory.

4. Conclusions

In this work, by using the generalized Riccati transformation technique and the integral averaging technique, we establish a new oscillation criteria for (3) under (1). Further, in future work, we can attempt to find some oscillation criteria of Equation (3), if $z(t) = y(t) - a(t)y(\tau(t))$.

Author Contributions: The authors have contributed equally and significantly in this paper. All authors have read and agreed to the published version of the manuscript.

Funding: The authors received no direct funding for this work.

Acknowledgments: The authors thank the reviewers for their useful comments, which led to the improvement of the content of the paper.

Conflicts of Interest: There are no competing interests between the authors.

References

- Hale, J.K. *Theory of Functional Differential Equations*; Springer: New York, NY, USA, 1977.
- Bazighifan, O.; Cesarano, C. Some New Oscillation Criteria for Second-Order Neutral Differential Equations with Delayed Arguments. *Mathematics* **2019**, *7*, 619. [[CrossRef](#)]
- Bazighifan, O.; Elabbasy, E.M.; Moaaz, O. Oscillation of higher-order differential equations with distributed delay. *J. Inequal. Appl.* **2019**, *55*, 55. [[CrossRef](#)]
- Cesarano, C.; Pinelas, S.; Al-Showaiikh, F.; Bazighifan, O. Asymptotic Properties of Solutions of Fourth-Order Delay Differential Equations. *Symmetry* **2019**, *11*, 628. [[CrossRef](#)]
- Cesarano, C.; Bazighifan, O. Qualitative behavior of solutions of second order differential equations. *Symmetry* **2019**, *11*, 777. [[CrossRef](#)]
- El-Nabulsi, R.A.; Bazighifan, O.; Moaaz, O. New Results for Oscillatory Behavior of Fourth-Order Differential Equations. *Symmetry* **2020**, *12*, 136. [[CrossRef](#)]
- Elabbasy, E.M.; Cesarano, C.; Bazighifan, O.; Moaaz, O. Asymptotic and oscillatory behavior of solutions of a class of higher order differential equation. *Symmetry* **2019**, *11*, 1434. [[CrossRef](#)]
- Moaaz, O. New criteria for oscillation of nonlinear neutral differential equations. *Adv. Differ. Equ.* **2019**, *2019*, 484. [[CrossRef](#)]
- Moaaz, O.; Elabbasy, E.M.; Bazighifan, O. On the asymptotic behavior of fourth-order functional differential equations. *Adv. Differ. Equ.* **2017**, *2017*, 261. [[CrossRef](#)]
- Moaaz, O.; Elabbasy, E.M.; Muhib, A. Some new oscillation results for fourth-order neutral differential equations. *Adv. Differ. Equ.* **2019**, *2019*, 297. [[CrossRef](#)]
- Moaaz, O.; Elabbasy, E.M.; Shaaban, E. Oscillation criteria for a class of third order damped differential equations. *Arab J. Math. Sci.* **2018**, *24*, 16–30. [[CrossRef](#)]
- Trench, W.F. Canonical forms and principal systems for general disconjugate equations. *Trans. Amer. Math. Soc.* **1973**, *189*, 319–327. [[CrossRef](#)]
- Baculiková, B.; Džurina, J. Oscillation theorems for higher order neutral differential equations. *Appl. Math. Comput.* **2012**, *219*, 3769–3778. [[CrossRef](#)]
- Baculikova, B.; Dzurina, J.; Li, T. Oscillation results for evenorder quasilinear neutral functional differential equations. *Electron. J. Differ. Equ.* **2011**, *143*, 1–9.

15. Chatzarakis, G.E.; Elabbasy, E.M.; Bazighifan, O. An oscillation criterion in 4th-order neutral differential equations with a continuously distributed delay. *Adv. Differ. Equ.* **2019**, *336*, 1–9.
16. Xing, G.; Li, T.; Zhang, C. Oscillation of higher-order quasilinear neutral differential equations. *Adv. Differ. Equ.* **2011**, *2011*, 45. [[CrossRef](#)]
17. Li, T.; Rogovchenko, Y.V. Asymptotic behavior of higher-order quasilinear neutral differential equations. In *Abstract and Applied Analysis*; Hindawi: London, UK, 2014; Volume 2014, p. 395368.
18. Liu, S.; Zhang, Q.; Yu, Y. Oscillation of even-order half-linear functional differential equations with damping. *Compu. Math. Appl.* **2011**, *61*, 2191–2196. [[CrossRef](#)]
19. Agarwal, R.P.; Grace, S.R.; O'Regan, D. *Oscillation Theory for Difference and Functional Differential Equations*; Kluwer Acad. Publ.: Dordrecht, The Netherlands, 2000.
20. Philos, C. On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delay. *Arch. Math.* **1981**, *36*, 168–178. [[CrossRef](#)]



© 2020 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<http://creativecommons.org/licenses/by/4.0/>).