Consequences of $f(\mathcal{T})$ Cosmology in Thermal Leptogenesis and Gravitino Late Abundance

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Abstract: Thermal Leptogenesis and the gravitino problem are reviewed in the framework of non-standard cosmologies. We consider in particular the $f(\mathcal{T})$ cosmology, where $\mathcal{T}$ is the torsion field. We constrain the parameters space of these cosmological models consistently with thermal Leptogenesis scenario (with degenerate mass spectrum of light neutrinos), and we show that they allow to solve the gravitino problem as well. Owing to the similar characteristics to $f(\mathcal{T})$ cosmology, we shortly discuss also the case of the shear dominated Universe.

1. Introduction

Modified theories of gravity have been proposed in the last years as a possible solution of the shortcomings related to the Cosmological Standard Model, based on General Relativity (GR). This is the case, for example, of higher order curvature invariants than the simple Ricci scalar R that allow to get inflationary behaviors, removing the primordial singularity, and to explain the flatness and horizon problems [1] (see also [2–18]). This approach and, of course, all those related to it, are fundamentally motivated by the fact that, at high curvature regimes, further curvature invariants have to be considered for constructing self-consistent effective actions in curved spacetime [19–21]. In this respect, modified gravity could provide the unification of the early-time (inflation) and the later-time acceleration of the Universe [22,23], providing at the same time, an unified description of Dark Energy (DE) and Dark Matter (DM).

One of the consequences to deal with the non standard cosmology is that the thermal history of particles gets modified. In fact, the expansion rate of the Universe can be typically written in the form [24],

$$H(T) = A(T)H_{GR}(T), \tag{1}$$

where H_{GR} is the expansion rate of the Universe in GR, and $A(T)$ accounts for deviations from GR. The latter is defined such that the successful predictions of the abundance of primordial light elements is preserved, i.e., $A(T) \neq 1$ at early time, or equivalently, at temperatures $T > T_{BBN}$, where T_{BBN} is the big bang nucleosynthesis (BBN) temperature, and $A(T) \to 1$ before or when the BBN begin. Therefore, it is common to refer to the so-called pre-BBN epoch, since this era is not directly constrained by cosmological observations. Typically this factor is parameterized as [24]

$$A(T) = \eta_* \left(\frac{T}{T_*} \right)^\nu, \tag{2}$$

where T_* is a parameter with dimensions of the temperature (interpreted as the transition temperature from the modified cosmology to the standard cosmology), and $\{\eta_*, \nu\}$ are free parameters. All these parameters do depend on the cosmological model under consideration. Investigations along these lines have been performed in different cosmological scenarios [24,25]: (*i*). $\nu = 2$ in Randall-Sundrum type II brane cosmology [26]; (*ii*). $\nu = 1$ in kination models [27]; (*iii*). $\nu = 0$ in cosmologies with an overall boost of the Hubble expansion rate [24,28–34]; (*iv*). Gauss–Bonnet scenario $\nu = -2/3$ [35]; (*v*). $\nu = -0.8$ in scalar-tensor cosmology [24,36]; (*vi*). $\nu = 2/n - 2$ in $f(x)$ cosmology, with $f(x) = x + \alpha x^n$ and $x = R, \mathcal{T}$, where R [2,4] and $\mathcal{T}$ [37–39] stand for the scalar curvature and the scalar torsion, respectively.

The aim of this paper is twice. First we study the origin of the matter-antimatter asymmetry in the Universe in the case in which the evolution of the Universe is described by non-standard cosmologies. The matter-antimatter asymmetry represents one of the open issue of the modern cosmology and particle physics. The successful predictions of BBN combined with the observations of the CMB anisotropies show that the parameter that characterizes such a asymmetry is $\eta = \frac{n_B - n_{\bar{B}}}{s} < (9.2 \pm 0.5) \times 10^{-11}$, where n_B ($n_{\bar{B}}$) is the baryon (antibaryon) number and $s = \frac{2\pi^2 g_* T^3}{45}$ the entropy density ($g_* = 106.7$ counts the total degrees of freedom for the relativistic particles). Among the models proposed in literature for explaining the origin of baryon/antibaryon asymmetry, the thermal Leptogenesis is certainly one of the favorite candidates. The asymmetry is generated in the lepton sector, and through non–perturbative electroweak processes involving sphalerons physics, is totally or partially converted to an asymmetry in the baryon sector. Leptogenesis is the simplest extension of the Standard Model (SM) of particle physics that includes right handed neutrinos N_R. They implement the see-saw mechanism, providing from a side a possible explanation of light neutrinos mass of the SM, from the other side to generate the lepton asymmetry from the decays of right handed neutrinos. Moreover, Leptogenesis is very attractive since allows a possible lepton number violation in the neutrino sector [40,41] (for a recent review see Ref. [42–44]), as well as the Sakharov conditions for explaining the matter-antimatter asymmetry are easily realized.

In thermal Leptogenesis, the out-of-equilibrium condition occurs when $\Gamma_{N_1} < H_{GR}$ at $T = M_1$, where $\Gamma_{N_i} = \frac{(\lambda^\dagger \lambda) M_1}{8\pi}$ is the decay rate of heavy Majorana right-handed neutrinos, $H_{GR} = \sqrt{\frac{8\pi^3 g_*}{45}} \frac{T^2}{M_{Pl}}$, M_1 is the heavy right-handed neutrino mass, and finally λ_{ik} are the complex Yukawa coupling constants (such a study for the brane cosmology has been performed in [45]). In terms of the scaled quantities

$$\tilde{m}_1 \equiv \frac{8\pi v^2}{M_1^2} \Gamma_{N_1} = \frac{(\lambda^\dagger \lambda)_{11} v^2}{M_1}, \tag{3}$$

$$m_* \equiv \frac{8\pi v^2}{M_1^2} H_{GR}(T = M_1) \simeq 1.1 \times 10^{-3} \text{eV}, \tag{4}$$

where $(\lambda^\dagger \lambda)_{11} = \sum_j (\lambda_{1j}^\dagger \lambda_{1j})$, the out-of equilibrium condition reads

$$\tilde{m}_1 < m_* \simeq 10^{-3} \text{eV}. \tag{5}$$

This condition provides an upper bound on the lightest neutrino mass m_1 owing to the inequality $m_1 \lesssim \tilde{m}_1$ [46–49]. Such a bound normally implies that thermal Leptogenesis cannot generate sufficient baryon asymmetry for degenerate masses of light neutrinos, as immediately follows from the expression of the CP parameter $\epsilon \simeq 10^{-6} \sin \delta \frac{m_3}{5 \times 10^{-2} \text{eV}} \frac{M_1}{10^{10}} \text{GeV}$ entering the lepton asymmetry $\eta \sim \epsilon$, see Appendix A. As we will show, in the case in which the cosmological background evolves according to (1), hence for modified cosmologies, the bound (5) turns out to be $\tilde{m}_1 < m_* A(T)$. As a

consequence, the parameters characterizing non–standard cosmologies can be constrained by thermal Leptogenesis in the case of degenerate light neutrino mass spectrum.

Moreover, we also explore the implications of modified cosmologies in relation to the gravitino problem (see for example [50]). Gravitino couples to ordinary matter only through the gravitational interaction, with a (quite) long lifetime $\tau_{3/2} \sim \dfrac{M_{Pl}^2}{m_{3/2}^3} \simeq 10^5 \left(\dfrac{1\text{TeV}}{m_{3/2}}\right)^3$ sec, where $m_{3/2}$ is the mass of the particle $\sim 10^2$ GeV. The decay of these particles with such a long lifetime is problematic in cosmology since their decay products would destroy light elements, destroying the successful predictions of BBN. This problem can be avoided by putting an upper bound on the reheating temperature, that for the standard cosmological model is given by $T_R \lesssim (10^6 - 10^7)\,\text{GeV}$ for $m_{3/2} \sim \mathcal{O}(10^2\,\text{GeV})$. It is worth to mention another mechanism to generate gravitinos by means of primordial black holes (PBHs) evaporation [51]. As shown in [51], as source of gravitions is also provided by the evaporation of light PBHs, with initial $M_i \lesssim 10^9$ gr. The subsequent decay of evaporated gravitinos into cascades of non-equilibrium particles, as occurs in minimal supergravity, leads to the formation of elements whose abundance is constrained by observations, and one has to require that their density does not overclose the Universe. A consequence of this mechanism is that cosmological models with substantial inhomogeneities on small scales are excluded.

This bound opens a serious question for the inflationary models (and for some models of Leptogenesis) since these models predict a larger reheating temperature [50]. Such a difficulty can be avoided, as we will see, in the framework of non-standard cosmologies.

The paper is organized as follows. In next Section we discuss the thermal Leptogenesis for degenerate mass spectrum of the light neutrinos in the context of non-standard cosmologies. The gravitino problem is also discussed. Here we provide a general discussion considering non-standatd cosmologies characterized by a Friedamn equation of the form $H^2 \sim \rho^n$. In Section 3 we analyzed specific examples of non-standard cosmologies, focusing in particular on Teleparallel Equivalent of General Relativity and shear dominated Universe. Conclusions are discussed in Section 4.

2. Thermal Leptogenesis in Modified Cosmologies

We consider the case in which the lightest heavy neutrino N_1 decays during the phase of the Universe expansion described by the modified cosmology. The expansion rate of the Universe is therefore given by (1) with $A(T)$ defined in (2). In such a case, the out-of-equilibrium condition reads

$$\Gamma(T = M_1) < H(T = M_1) = H_{GR}(M_1)\,A(M_1)\,, \tag{6}$$

or equivalently, in terms of the scaled masses, Equation (5) assumes the form

$$\tilde{m}_1 < m_* A(T) \simeq 10^{-3} A(T)\text{eV}\,. \tag{7}$$

The upper bound turns out to be hence modulated by the factor $A(T)$. It then follows that for low temperatures, i.e., $T_* < T \lesssim M_1$, the non-standard cosmologies are consistent with thermal Leptogenesis with a degenerate light neutrino mass spectrum. In fact one gets

$$K_1 \equiv \frac{\Gamma_{N_1}}{H(T=M_1)} = \frac{1}{A(M_1)} \frac{\Gamma_{N_1}}{H_{GR}(T=M_1)} = \frac{1}{A(M_1)} \frac{\tilde{m}_1}{m_*} = \frac{1}{A(M_1)} \frac{\tilde{m}_1}{10^{-3}\text{eV}} < 1 \quad \text{for} \quad A(M_1) \gg 1\,, \tag{8}$$

so that, the effects of modified gravity is to avoid the suppression of Leptogenesis even in the case of degenerate light neutrino masses, determining a sufficient baryon asymmetry.

The above discussion relies on the assumption that the neutrinos N_1 are in thermal equilibrium at temperature T greater than the heavy Majorana neutrino mass M_1, $T > M_1$ (this assumption underlies the derivation of (7)). We shall now verify that such an assumption can be realized in modified cosmologies. Owing to the fact that N_1 are singlets under the Standard Model gauge group, the only

interaction that allows to N_1 to be in thermal equilibrium is the Yukawa coupling (see Equation (A1) in Appendix A). The thermal average (pair) annihilation rate is [45]

$$\Gamma_{ann}(T) = n_{N_1}\langle \sigma v \rangle \simeq \frac{3\xi(3)T^3}{2\pi^2}\frac{N_{ann}\lambda^4}{a_{Kin}T^2}, \tag{9}$$

where N_{ann} is the number of annihilation channels, λ denotes the dominant Yukawa's coupling, $a_{Kin} \sim 10^2$ is the kinematical phase factor, and $N_{ann}\lambda^4 \simeq \mathcal{O}(10)$. The freeze-out temperature T_f, determined as usual by $\Gamma_{ann}(T_f) = H(T_f) = H_{GR}(T_f)A(T_f)$, is given by

$$
\begin{aligned}
T_f &= \left[\frac{3\xi(3)}{2\pi^2\eta_*}\frac{N_{ann}\lambda^4}{a_{Kin}}\sqrt{\frac{45}{8\pi^3 g_*}}\right]^{\frac{1}{\nu+1}}\left(\frac{T_*}{M_{Pl}}\right)^{\frac{\nu}{\nu+1}}M_{Pl} \\
&\simeq \left(\frac{7.5\times 10^{-3}}{\eta_*}\frac{M_{Pl}}{T_*}\right)^{\frac{1}{\nu+1}}\frac{T_*}{\text{GeV}}\,\text{GeV}.
\end{aligned}
\tag{10}
$$

In order that thermal Leptogenesis can be realized in modified cosmologies, one has to require that the mass of Majorana neutrinos is such that

$$T_* < M_1 \lesssim T_f. \tag{11}$$

In what follows we shall assume $M_1 \simeq (10^8 - 10^9)\text{GeV}$ (see Appendix A).

2.1. Kinetic Equations

Here we shortly discuss the kinetics equations in the framework of non standard cosmology. The kinetic equation for the leptogenesis are given by (see for example [52])

$$\frac{dN_1}{dz} = (D+S)(N_1 - N_1^{eq}), \quad \frac{dN_{B-L}}{dz} = -\epsilon D(N_1 - N_1^{eq}) - WN_{B-L}, \tag{12}$$

with $z = M_1/T$, $(D, S, W) = (\Gamma_D, \Gamma_S, \Gamma_W)/(Hz)$, where $\Gamma_{D,S,W}$ are the scattering rates defined as $\Gamma_D = \frac{M_1}{8\pi}(\lambda^\dagger\lambda)_{11}\frac{K_1(z)}{K_2(z)}$ ($K_{1,2}(z)$ are the Bessel functions), $\Gamma_S = 2\Gamma_{N_1\rightarrow\phi,t} + 4\Gamma_{N_1\rightarrow\phi,s}$ ($\Gamma_{N_1\rightarrow\phi,\{t,s\}}$ are the N_1 scattering rates in which the Higgs ϕ processes in t- and s-channels are involved), and Γ_W is the washout-rate. The quantities $\Omega = D, S, W$ are proportional to $\lambda^\dagger\lambda$, so that the reduced rates entering (12) are such that [52] $\Omega - \Delta W \propto \frac{M_{Pl}\tilde{m}_1^D}{v^2}$, where $\Delta W \propto \frac{M_{Pl}\bar{m}^2 M_1}{v^4}$, with $\bar{m}^2 \simeq 3m_i^2$ for quasi degenerate neutrinos, and $\tilde{m}_1^D = \frac{(m_D^\dagger m_D)_{11}}{M_1}$, with m_D the Dirac neutrino mass. The latter equation shows that the dependence of the generated lepton asymmetry on M_1 is relaxed as long as ΔW is negligible. The Boltzamann equations of the number of heavy Majorana neutrinos N_1 and of $B-L$ number N_{B-L}, Equation (12), are dependent on $A(T)$. However, in this paper we are mainly interested to fulfill the conditions allowing the out-of-equilibrium, and the consequent constraints on the parameter spaces of the non-standard cosmology, therefore the numerical analysis of the N_1 and N_{B-L} quantities are beyond the scope of the present paper, and will be faced elsewhere (in the regime to which we are interested in this paper, i.e., $A(T) \sim T$ ($\nu = 1$) and $A(T \simeq M_1) \sim \mathcal{O}(10 - 10^2)$, see next Section, one gets $\frac{H(z)}{H(z=1)} = \frac{1}{z}\frac{H_{GR}(z)}{H_{GR}(z=1)}$, $\frac{\Gamma_{D,S,W}}{H(z)} = \frac{z}{A(M_1)}\frac{\Gamma_{D,S,W}}{H_{GR}(z)}$, while the abundance of N_1 and N_{B-L} in non-standard cosmologies may remain unaffected, with respect results obtained in the standard cosmological model [52], for appropriate initial conditions).

2.2. The Gravitino Problem

The gravitino abundance is described, in the standard cosmological model, by the Boltzmann equation (here we follow Ref. [45], where the gravitino problem has been discussed in the framework of Brane Cosmology)

$$\frac{dY_{3/2}}{dT} = -\Theta_{GR}, \qquad \Theta_{GR} \equiv \frac{s\langle\sigma v\rangle Y_{rad}^2}{H_{GR}T},$$ (13)

where $Y_{3/2} = n_{3/2}/s$ the gravitino abundance, $Y_{rad} = n_{rad}/s$ the radiation abundance, and s the entropy density, it follows $Y_{3/2} \simeq 10^{-11}\frac{T_R}{10^{10}\text{GeV}}$, where T_R is the reheating temperature. The adiabatic evolution of the Universe and the conservation of the entropy is assumed. Note that the quantity Θ_{GR} is independent on T since $\langle\sigma v\rangle \sim M_{Pl}^{-2}$, $H_{GR} \sim T^2/M_{Pl}$, and $s \sim T^3$. To solve the gravitino problem one has to impose that the reheating temperature is constrained to $T_R \lesssim (10^6 - 10^7)$ GeV (for gravitino masses of the order of 100 GeV) [50]. This bound is problematic for thermal Leptogenesis in supersymmetric models after inflation since in these frameworks the reheating temperature is many order of magnitudes greater than $(10^6 - 10^7)$ GeV [50,53–55].

However, if the Universe evolution is described by a non standard cosmology, then the expansion rate is replaced by (1). This implies that the factor Θ_{GR} in (13) is replaced by $\Theta_{GR} \to \Theta = \Theta_{GR}/A(T)$ [45,56,57]. In such a circumstance, the reheating temperature T_R entering the above expression for $Y_{3/2}$ is replaced by the transition temperature T_*. In fact, referring to cosmological models studied in the next Sections, $\nu = 1$ and $\nu > 1$, the integration of (13) from T_* to T_R (assuming $T_R \gg T_*$ and using (2)), gives

$$Y_{3/2} \sim \begin{cases} \frac{T_*}{\eta}\log\frac{T_R}{T_*} & \nu = 1 \\[2mm] \frac{T_*^{2\nu-1}}{\eta(\nu-1)} & \nu > 1 \end{cases}$$

Therefore, in the case in which the transition temperature is low enough, i.e., $T_* < (10^6–10^7)$ GeV, the gravitino problem is avoided even if the reheating temperature is much higher.

Let us check that the condition $T_R \gg T_*$ occurs also in non-standard cosmology. By comparing the expansion rate of the Universe (1) with the decay rate of the inflaton Γ_I one obtains the reheating temperature

$$T_R = T_*\left(\frac{M_{Pl}\Gamma_I}{T_*^2}\right)^{\frac{1}{\nu+2}}.$$ (14)

On the other hand, in the standard cosmology, the relation $H_{GR}(T_R^{GR}) = \Gamma_I$ gives $T_R^{GR} \simeq (M_{Pl}\Gamma_I)^{1/2}$. Combining this relation with the reheating temperature (14), one infers

$$T_R = T_*\left(\frac{T_R^{GR}}{T_*}\right)^{\frac{2}{\nu+2}}.$$ (15)

Thus $T_R \gg T_*$ provided $T_R^{GR} \gg T_*$, as in the standard cosmology (such a relation, however, is such that the values $\nu \gg 1$ are excluded).

3. Examples of Non Standard Cosmologies

As pointed out, results of previous Section are general, and apply to cosmological models in which the expansion rate can be cast in the form of Equations (1) and (2). In this Section we shall consider some specific models of modified cosmologies. We first focus on $f(\mathcal{T})$ model, deriving a bound on the parameter of the model, then we shortly discuss the shear dominated Universe ($\nu = 1$).

3.1. $f(\mathcal{T})$ Cosmology

An interesting cosmological model recently proposed in literature is the Teleparallel Equivalent of General Relativity (TEGR) [37]. This theory of gravity is based on the Weitzenböck connection (instead of the usual Levi-Civita connection), and the gravitational field is described by the torsion tensor (instead of the curvature tensor)

$$\mathcal{T}^{\lambda}_{\mu\nu} = \hat{\Gamma}^{\lambda}_{\nu\mu} - \hat{\Gamma}^{\lambda}_{\mu\nu} = e^{\lambda}_i (\partial_{\mu} e^i_{\nu} - \partial_{\nu} e^i_{\mu}), \tag{16}$$

where $e^i_{\mu}(x)$ are the vierbein fields defined as $g_{\mu\nu}(x) = \eta_{ij} e^i_{\mu}(x) e^j_{\nu}(x)$. The action is given by

$$S_I = \frac{1}{16\pi G} \int d^4 x e \left[\mathcal{T} + f(\mathcal{T}) \right], \tag{17}$$

where $\mathcal{T} = S_{\rho}{}^{\mu\nu} \mathcal{T}^{\rho}_{\mu\nu}$ is the torsion scalar, $e = det(e^i_{\mu}) = \sqrt{-g}$, $f(\mathcal{T})$ is a generic function of the torsion, and

$$S_{\rho}{}^{\mu\nu} = \frac{1}{2} \left[\frac{1}{4} (\mathcal{T}^{\mu\nu}{}_{\rho} - \mathcal{T}^{\nu\mu}{}_{\rho} - \mathcal{T}_{\rho}{}^{\mu\nu}) + \delta^{\mu}_{\rho} \mathcal{T}^{\theta\nu}{}_{\theta} - \delta^{\nu}_{\rho} \mathcal{T}^{\theta\mu}{}_{\theta} \right]. \tag{18}$$

TEGR represents an alternative to inflation models, as well as to effective DE models, in which the Universe acceleration is driven by the torsion terms [37,58] (see the review [59,60]). For a homogeneous and isotropic Universe (FRW Universe) one finds that $e^A_{\mu} = \text{diag}(1, a, a, a)$ and $\mathcal{T} = -6H^2$. The cosmological field equations are [60]

$$12H^2[1 + f_{\mathcal{T}}] + [\mathcal{T} + f] = 16\pi G \rho, \tag{19}$$

$$48H^2 f_{\mathcal{T}\mathcal{T}} \dot{H} - (1 + f_{\mathcal{T}})[12H^2 + 4\dot{H}] - (\mathcal{T} - f) = 16\pi G p,$$

where $f_{\mathcal{T}} = df/d\mathcal{T}$. As an explicit example, we consider the power-law $f(T)$ model $f(\mathcal{T}) = \beta_{\mathcal{T}} |\mathcal{T}|^{n_{\mathcal{T}}}$ [61,62]. By rewriting (19) in the form $H^2_{GR} + H^2_{\mathcal{T}} = \frac{8\pi}{3M^2_{Pl}} \rho$, where $H^2_{\mathcal{T}} \equiv \frac{f}{6} - \frac{T f_{\mathcal{T}}}{3} = 6^{n-1} \beta_{\mathcal{T}} (2n_{\mathcal{T}} + 1) H^{2n}$, and assuming $H_{\mathcal{T}} \gg H_{GR}$, one gets $H_{\mathcal{T}} = A(T) H_{GR}$, where $A(T)$ is of the form (2) with

$$\eta = 1, \quad \nu = \frac{2}{n_{\mathcal{T}}} - 2, \tag{20}$$

$$T_* \equiv \left(\frac{24\pi^3 g_*}{45} \right)^{\frac{1}{4}} (2n_{\mathcal{T}} + 1)^{\frac{1}{4(1-n_{\mathcal{T}})}} \left(\frac{\beta_{\mathcal{T}}}{\text{GeV}^{2(1-n_{\mathcal{T}})}} \right)^{\frac{1}{4(1-n_{\mathcal{T}})}} \left(\frac{M_{Pl}}{\text{GeV}} \right)^{\frac{1}{2}} \text{GeV}.$$

From (20) it follows

$$A(T = M_1) = \left(\frac{1}{(2n_{\mathcal{T}} + 1)} \frac{\text{GeV}^{2(1-n_{\mathcal{T}})}}{\beta_{\mathcal{T}}} \right)^{\frac{1}{2n_{\mathcal{T}}}} \left[\left(\frac{45}{24\pi^3 g_*} \right)^{\frac{1}{2}} \left(\frac{M_1}{\text{GeV}} \right)^2 \frac{\text{GeV}}{M_{Pl}} \right]^{\frac{1}{n_{\mathcal{T}}} - 1}. \tag{21}$$

According to recent results in literature, one has that $\nu \geqslant 1$, i.e., $n_{\mathcal{T}} \leqslant 2/3$, in order that the gravitino problem can be solved in non-standard cosmologies, and that the BBN temperature is $T_{BBN} \simeq (10 - 50)$ GeV, so that the enhancement to the annihilation cross section is able to reproduce the observed Dark Matter density [63]. We discuss the case $n_{\mathcal{T}} = 2/3$, i.e., $\nu = 1$, and $n_{\mathcal{T}} < 2/3$, i.e., $\nu > 1$. In Table 1 are reported the values of $\{n_{\mathcal{T}}, \beta_{\mathcal{T}}, M_1\}$ such that the relation $T_* < M_1 \lesssim T_f$, see Equation (11), is fulfilled. We have taken $M_1 = (10^8, 10^9)$ GeV (values $M_1 \gtrsim 10^{10}$ GeV are difficult to accommodate in this scenario). We see, for example, that the parameter $n_{\mathcal{T}}$ may vary in the range $0.65 \lesssim n_{\mathcal{T}} \lesssim 2/3$ as $M_1 = 10^9$ GeV, for a temperature $T_* \gtrsim \mathcal{O}(10 \text{ MeV})$. Notice that a large value of $A(T)$ also occurs in scalar tensor theories [24]. Although very tiny, there is a range of parameters $\{n_{\mathcal{T}}, \beta_{\mathcal{T}}\}$ for which the

constraints from thermal leptogenesis (for massive degenerate neutrinos) are consistent with the ones needed to solve the gravitino problem (see Section 2.1).

Table 1. Values of the parameters $\{n_{\mathcal{T}}, \beta_{\mathcal{T}}, M_1\}$ such that $T_* < M_1 \lesssim T_f$, according to Equation (11). Here we take $M_1 = (10^8, 10^9)$GeV and $\tilde{\beta}_{\mathcal{T}} \equiv \beta_{\mathcal{T}}/\mathrm{GeV}^{2(1-n_{\mathcal{T}})}$.

$\{n_{\mathcal{T}}, \tilde{\beta}_{\mathcal{T}}, M_1(\mathbf{GeV})\}$	$T_*(\mathbf{GeV})$	$T_f(\mathbf{GeV})$	$A(T = M_1)$
$\{2/3, 10^{-16}, 10^9\}$	0.038	1.39×10^9	2.5×10^{10}
$\{0.67, 10^{-16}, 10^9\}$	0.029	1.87×10^9	2.3×10^{10}
$\{0.66, 10^{-17}, 10^9\}$	0.012	1.80×10^9	1.7×10^{11}
$\{0.65, 10^{-18}, 10^9\}$	0.005	1.68×10^9	1.4×10^{12}
$\{2/3, 10^{-16}, 10^8\}$	0.038	1.39×10^9	2.5×10^9
$\{0.62, 10^{-18}, 10^8\}$	0.050	1.4×10^8	2.5×10^{11}
$\{0.615, 10^{-18}, 10^8\}$	0.070	1.01×10^8	2.8×10^{11}

3.2. Shear Dominated Universe

The shear dominated Universe, or Bianchi's type I Universe, is a Universe homogeneous but anisotropic, characterized by three time dependent scale factors $a_i(t)$, $i = 1, 2, 3$, one for each spatial directions. For $a_1 = a_2 = a_3$ one recovers the standard FRW cosmology. The Friedman equation for this Universe reads [64–66].

$$H^2 = \frac{8\pi}{3M_{Pl}^2}(\rho + \rho_s),\tag{22}$$

where $\rho = \frac{\pi^4 g_* T^4}{30}$, and ρ_s is the shear energy density

$$\rho_s = \frac{M_{Pl2}}{48\pi}\left[(H_1 - H_2)^2 + (H_1 - H_3)^2 + (H_2 - H_3)^2\right],\tag{23}$$

where $H_i = \frac{\dot{a}_i}{a_i}$, $i = 1, 2, 3$, are the expansion rate of the three scale factors. Expressing results in terms of the temperature T, one finds that the expansion rate of the Universe can be written in the form

$$H(T) = H_{GR}\sqrt{1 + \frac{g_*}{g_*^e}\frac{T^2}{T_e^2}},\tag{24}$$

where T_e is the temperature such that $\rho = \rho_s$ (it is related to the transition temperature T_* as we will see in a moment), and g_*^e is the number of degree of freedom evaluated at T_e. The Universe is therefore shear dominated when $T \gg T_e$ (before the BBN), while for $T \ll T_e$, the Universe is radiation dominated and the background evolves according to the standard cosmological model. BBN constraints require $T_e \gtrsim 2.5$ MeV. During the shear dominated phase $T \gg T_e$ the factor $A(T)$ reads

$$A(T) = \sqrt{1 + \frac{g_*}{g_*^e}\left(\frac{T}{T_e}\right)^2} \simeq \frac{g_*}{g_*^e}\frac{T}{T_e},\tag{25}$$

From (25) it follows $\eta_* = 1$, $\nu = 1$ and $T_* = \frac{g_*^e}{g_*}T_e$. Requiring that $A(T = M_1) > 1$ and $g_*^e = g_*$, one gets $T_* \lesssim 10^{-2}M_1$, and $T_f \lesssim 2.5 \times 10^7\sqrt{M_1\mathrm{GeV}}$. Taking, for example, $A(M_1) \simeq 10^2$ and the lower bound on Majorana heavy neutrino $M_1 = 10^9$ GeV, it follows $T_f \simeq 0.8 \times 10^{12}$ GeV and $T_* \simeq 10^7$ GeV (according to (11)). The latter value for the transition temperature $T_* \sim 10^7$ GeV allows in turn to solve the gravitino problem, as discussed in Section 2.1.

4. Conclusions

The cosmological evolution of the Universe is, at the moment, observationally tested at temperature $T \sim 1$ MeV, and for $T > 1$ MeV it is not excluded that new physics (non-standard

cosmology, Dark Matter, Dark Energy, and so on) might be occurred opening a serious of questions which are still under a deep scrutiny.

In this paper we focused on some consequences of non-standard cosmologies in the framework of thermal Leptogenesis and gravitino late abundance. Since the out-of-equilibrium condition gets modified whether or not the cosmic evolution of the Universe is described by non-standard cosmology, then the consistence with thermal Leptogenesis, in the case of degenerate mass spectrum of light neutrinos, constrains the parameters space of these cosmological models. We have studied also the possibility that these constraints can solve the gravitino problem. As specific examples, we have considered the $f(\mathcal{T})$ cosmology and the shear dominated Universe.

Results derived in this paper show once more the intriguing interplay between particle physics and the non standard cosmological models needed for a consistent description of the evolution of the very early Universe.

Appendix A. Thermal Leptogenesis Scenario

In this Appendix, we shortly recall the main features of thermal Leptogenesis. The Lagrangian density is [67,68]

$$\mathcal{L} = h^*_\beta (\bar{L}_\beta\, \phi^{c*}) E_\beta - \lambda^*_{\alpha k}(\bar{L}_\alpha\, \phi^*) N_k - \frac{1}{2}\bar{N}_j M_j N^c_j + h.c.\,, \tag{A1}$$

where $L = (e\, \nu)^T_L$ is the Standard Model left-handed doublet, $E = e_R$ is the right-handed singlet, ϕ is the scalar field, N_j are the singlet fermions (Majorana neutrinos), α, β are the flavor indices of the Standard Model ($\alpha, \beta = e, \mu, \tau$), M is the mass matrix, h are coupling constants, and finally λ are Yakawa matrices. Equation (A1) is written in a basis such that h and M are real and diagonal, and λ is complex. Integrating out the heavy fermions N_i, one infers the effective light neutrino masses (see-saw mechanism): $m_{\nu\,\alpha\beta} = \lambda_{\alpha k} M^{-1}_k \lambda_{\beta k}$.

Consider, for simplicity, the lightest Majorana singlet N_1 (with $M_1 \ll M_{2,3}$). The allowed channel decays are $N_1 \to L_\alpha + \phi$ and $N_1 \to \bar{L}_\alpha + \phi^\dagger$. The net lepton asymmetry is given by [67,68]

$$\eta \simeq \frac{\epsilon}{g_*}\, \eta_{eff}\,. \tag{A2}$$

Here η_{eff} is the efficiency factor. It may assume values in the range $0 < \eta_{eff} < 1$. In fact, define the freeze-out temperature T_F as the temperature for which $\Gamma(\phi + L \to N_1)\big|_{T_F} < H(T_F)$, where $\Gamma(\phi + L \to N_1) \simeq \frac{1}{2}\Gamma_{N_1} e^{-M_1/T}$, with $\Gamma_{N_1} = \frac{(\lambda^\dagger \lambda)_{11} M_1}{8\pi}$ and, in the standard cosmological model, $H = \left(\frac{8\pi}{3M_P^2}\rho\right)^{1/2} = 1.66 g_*^{1/2} \frac{T^2}{M_P}$. Below T_F the density of the fermion N_1 is Boltzmann suppressed ($N_1 \sim e^{-M_1/T}$), so that the decay of N_1 at $T < T_F$ contributes to the lepton asymmetry, with

$\eta_{eff} \simeq \frac{n_{N_1}(T_F)}{n_{N_1}(T \gg M_1)} \simeq e^{-M_1/T_F} < 1$. The efficiency factor depends on how much the N_1 decays are out-of-equilibrium and ϵ is the *CP* parameter related to the asymmetry in the N_1 decays

$$\epsilon \equiv \frac{\Gamma(N_1 \to \phi + L) - \Gamma(N_1 \to \phi^\dagger + \bar{L})}{\Gamma(N_1 \to \phi + L) + \Gamma(N_1 \to \phi^\dagger + \bar{L})} \tag{A3}$$

$$\simeq 10^{-6} \sin\delta \frac{m_3}{5 \times 10^{-2}\text{eV}} \frac{M_1}{10^{10}\text{GeV}}, \tag{A4}$$

where δ is the phase coming from the complex Yukawa couplings, m_3 the heaviest light neutrino mass normalized to the atmospheric neutrino oscillation data $\sqrt{\Delta m_\oplus} \simeq 0.05$ eV [69], and $M_1 \ll M_j, j = 2, 3$. This quantity is generated from the interference of the tree-level and one-loop amplitudes, with complex Yakawa couplings. From Equation (A2) it follows that the observed baryon asymmetry ($\eta \sim 10^{-10}$) provides a lower bound on the mass of the Majorana neutrino N_1 given by $M_1 \gtrsim (10^8 - 10^9)$ GeV [47]. In terms of the scaled quantities (3) and (4), the condition of out-of-equilibrium decay ($\Gamma(T) < H(T)$) can be cast in the form $\tilde{m}_1 < m_*$. This equation provides an upper bound on the lightest neutrino mass m^ν being $m^\nu \lesssim \tilde{m}_1$ [46,47].

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