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Transverse Kähler–Ricci Solitons of Five-Dimensional Sasaki–Einstein Spaces $Y^{p,q}$ and $T^{1,1}$

Mihai Visinescu

Department of Theoretical Physics, National Institute for Physics and Nuclear Engineering, Magurele P.O.Box M.G.-6, Romania; mvisin@theory.nipne.ro

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Abstract: We investigate the deformations of the Sasaki–Einstein structures of the five-dimensional spaces $T^{1,1}$ and $Y^{p,q}$ by exploiting the transverse structure of the Sasaki manifolds. We consider local deformations of the Sasaki structures preserving the Reeb vector fields but modify the contact forms. In this class of deformations, we analyze the transverse Kähler–Ricci flow equations. We produce some particular explicit solutions representing families of new Sasakian structures.

Keywords: contact geometry; Sasaki–Einstein spaces; Sasaki–Ricci flow

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1. Introduction

Sasakian geometry is often referred to as an odd-dimensional cousin of Kähler geometry and so is of independent interest. A Sasakian structure sits beween two Kähler structures, namely the one on its metric cone and the one on the normal bundle of its Reeb foliation. In physics, a prominent role is played by Sasaki–Einstein manifolds due to their applications in the so-called AdS/CFT correspondence.

During the last years, there is a lot of work done on the original AdS/CFT correspondence in maximally supersymmetric theories. Similar ideas have been applied to theories with less supersymmetry. Some of them are the gauge/gravity dualities that preserve $\mathcal{N} = 1$ supersymmetry. They are of the form $AdS_5 \times X^5$, where X^5 is a five-dimensional Sasaki–Einstein manifold [1,2].

The first nontrivial example in the AdS/CFT correspondence with the use of these Sasaki–Einstein manifolds was made in the case of homogeneous manifold $T^{1,1}$ [2]. Significant progress has been made in Sasaki–Einstein backgrounds and their dual field theories when an infinite family of inhomogeneous metrics on $Y^{p,q} \cong S^2 \times S^3$ was found [3]. All these manifolds have a Reeb vector field, which is a constant norm Killing vector field and, under the AdS/CFT correspondence, is isomorphic to the R-symmetry of dual field theory. The construction of Reference [3] was immediately generalized to higher dimensions [4]. For example, $AdS_4 \times X^7$ is a supersymmetric solution of eleven-dimensional supergravity that is expected to be dual to a three-dimensional superconformal field theory [5]. However, dimension five is the most interesting physically and the purpose of this work is to investigate deformations of Sasaki–Einstein structures in the frame of Sasaki–Ricci flow.

In the context of the relationship between the Sasakian structure and the two Kähler structures, we mention that, in the case of a Sasaki–Einstein manifold, the Riemannian metric cone is Ricci-flat and the transverse Kähler structure is Kähler–Einstein.

A well-known method for generating Einstein metrics on manifolds is the Ricci flow introduced by Hamilton in Reference [6]. Recently, the method was applied to Sasaki manifolds in Reference [7]. When one considers the problem of finding a Sasaki–Einstein metric, which is one of the main interests in physics, it is reduced to the problem of finding a transverse Kähler–Einstein metric. In Reference [8], the existence of transverse Kähler–Ricci solitons on compact toric Sasaki manifolds, of which the basic first Chern form of the normal bundle of the Reeb foliation is positive and the first Chern class of the contact bundle is trivial, is proven.

In this paper, we investigate the Sasaki–Ricci flow equations on five-dimensional Sasaki–Einstein spaces $T^{1,1}$ and $Y^{p,q}$. For this purpose, we perform deformations of Sasakian structures exploiting the transverse Kähler structure of Sasaki manifolds. These deformations of Sasaki–Einstein structures have important implications in holography and string theory [9]. We introduce local complex coordinates to parametrize the transverse holomorphic structure of the Sasaki–Kähler potential. In spite of the complexity of the Sasaki–Ricci flow equations, we are able to produce some explicit particular solutions. The perturbations that do not modify the transverse metric preserve the Sasaki–Einstein structures, whereas if the transverse metric is changed, the Sasaki structures are preserved but are not Einstein anymore. Preliminary results concerning the Sasaki–Ricci flow on these spaces have been reported in References [10–12]. Now, we give some explicit analytical solutions of the Sasaki–Ricci flow equations and discuss their relevance to the symmetries of the deformed Sasaki structures.

The paper is organized as follows. In the next section, we review fundamentals of Sasaki geometry, deformations of Sasaki structures, and Sasaki–Ricci flow. In Section 3, we investigate the Sasaki–Ricci flow equations on the Sasaki–Einstein spaces $T^{1,1}$ and $Y^{p,q}$. In the last section, we provide some closing remarks.

2. Background

In this section, we recall definitions and some basic facts of Sasakian manifolds and their deformations. We refer to the monograph of Reference [13] for details.

2.1. Sasakian Manifolds

Let (M, g) be a $(2n+1)$-dimensional Riemannian manifold, ∇ be the Levi–Civita connection of the Riemannian metric g, and Ric denote the Ricci tensor of ∇.

Definition 1. *A Riemannian manifold (M, g) is Sasakian if its metric cone $C(M, \bar{g}) = \mathbb{R}_+ \times M$ with metric $\bar{g} = dr^2 + r^2 g$ is Kähler with r as the coordinate on $\mathbb{R}_+ = (0, +\infty)$.*

M is a contact manifold with the contact 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. There is a canonical vector field ξ, called Reeb vector field, defined by

$$\eta(\xi) = 1 \quad , \quad d\eta(\xi, X) = 0 \tag{1}$$

for any vector field X on M.

η defines a $2n$-dimensional vector bundle $\mathcal{D} = \ker \eta$ over M, and the Sasakian metric g gives an orthogonal splitting of the tangential bundle TM:

$$TM = \mathcal{D} \oplus L_\xi \tag{2}$$

where L_ξ is the trivial bundle generated by the Reeb vector ξ. The restriction of the Sasaki metric g to $\mathcal{D}$ gives a well-defined Hermitian metric g^T, which is in fact Kähler. The metric g^T is related to the Sasakian metric g by

$$g = g^T + \eta \otimes \eta\,. \tag{3}$$

We define a tensor a tensor Φ of type $(1,1)$ satisfying

$$\Phi^2 = -I + \eta \otimes \xi \quad \text{and} \quad g(\Phi(X), \Phi(Y)) = g(X, Y) - \eta(X)\eta(Y) \tag{4}$$

for any vector fields X, Y on M.

Concerning the Einstein condition of a $(2n+1)$-dimensional Sasaki manifold (M, g), the following three conditions are equivalent:

1. g is Einstein with $Ric_g = 2ng$;
2. the metric cone $C(M, \bar{g})$ is a Ricci-flat Kähler manifold (i.e. Calabi–Yau manifold); and
3. the transverse Kähler metric g^T satisfies $Ric_{g^T} = (2n+2)g^T$.

Note that we often refer to $\frac{1}{2}d\eta$ as the Kähler form of the transverse Kähler metric g^T. The transverse Ricci form represents the first Chern class c_1^B, and let us denote by $c_1(\mathcal{D})$ the de Rham cohomology class of $\mathcal{D} = \ker\eta$.

We also recall the Sasakian structure and its transverse structure on local coordinates. Let U_α be an open covering of M and $\pi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{C}^n$ submersions such that

$$\pi_\alpha \circ \pi_\beta^{-1} : \pi_\beta(U_\alpha \cap U_\beta) \to \pi_\alpha(U_\alpha \cap U_\beta) \tag{5}$$

is biholomorphic when $U_\alpha \cap U_\beta$ is not empty. One can choose local coordinates charts $(z^1, \cdots, z^n)$ on V_α and local coordinates charts $(x, z^1, \cdots, z^n)$ on U_α such that $\xi = \partial_x$. We shall use the following notations:

$$\partial_x = \frac{\partial}{\partial x} \quad , \quad \partial_i = \frac{\partial}{\partial z^i} \quad , \quad \bar{\partial}_j = \partial_{\bar{j}} = \frac{\partial}{\partial \bar{z}^j} \,. \tag{6}$$

For the study of the deformations of the Sasaki structures, it is necessary to introduce the *basic forms*. A differential r-form α is said to be basic if

$$\iota_\xi \alpha = 0 \quad , \quad \iota_\xi d\alpha = 0 \tag{7}$$

where ι_ξ denotes the inner product. In the system of local coordinates $(x, z^1, \cdots, z^n)$, a basic r-form of type (p, q), $r = p + q$, has the form

$$\alpha = \alpha_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q} \,, \tag{8}$$

where $\alpha_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q}$ does not depend on x. In particular a function, φ is basic if and only if $\xi(\varphi) = 0$.

Note that, in the chart U_α, we may write

$$\xi = \frac{\partial}{\partial x} \,, \tag{9}$$

$$\eta = dx + i \sum_{j=1}^{n} (K_{,j} \mathrm{d}z^j) - i \sum_{\bar{j}=1}^{n} (K_{,\bar{j}} d\bar{z}^j) \,, \tag{10}$$

$$d\eta = -2i \sum_{j,\bar{k}=1}^{n} K_{,j\bar{k}} dz^j \wedge d\bar{z}^k \,, \tag{11}$$

$$g = \eta \otimes \eta + 2 \sum_{j,\bar{k}=1}^{n} K_{,j\bar{k}} dz^j \mathrm{d}\bar{z}^k \,, \tag{12}$$

$$\Phi = -i \sum_{j=1}^{n} [(\partial_j - iK_{,j}\partial_x) \otimes dz^j] + i \sum_{\bar{j}=1}^{n} (\partial_{\bar{j}} + iK_{,\bar{j}}\partial_x) \otimes d\bar{z}^j] \tag{13}$$

where $K : U \to \mathbb{R}$ is a local basic function called Sasaki potential [14] and $K_{,j} = \frac{\partial}{\partial z^j} K$, $K_{,j\bar{k}} = \frac{\partial^2}{\partial z^j \partial \bar{z}^k} K$.

Finally, we introduce the notion of toric Sasaki manifold.

Definition 2. *A Sasaki manifold (M, g) is said to be toric if the Kähler cone manifold $C(M)$ is toric, namely a $(n+1)$-dimensional torus G acts on $(C(M), \bar{g})$ effectively as holomorphic isometries.*

2.2. Deformations of Sasaki Structures

Let us assume that (η, ξ, Φ, g) defines a Sasaki structure on M. We deform the Sasakian structure keeping the Reeb vector field ξ fixed and perturbing the contact form η with a basic function φ:

$$\tilde{\eta} = \eta + 2d^c_B \varphi \,. \tag{14}$$

Here, we introduced the canonical basic Dolbeault operators

$$\partial_B = \sum_{j=1}^{n} dz^j \frac{\partial}{\partial z^j} \,, \quad \bar{\partial}_B = \sum_{j=1}^{n} d\bar{z}^j \frac{\partial}{\partial \bar{z}^j} \tag{15}$$

and $d^c_B = \frac{i}{2}(\bar{\partial}_B - \partial_B)$. Accordingly, the tensor Φ and the metric are modified as follows:

$$\tilde{\Phi} = \Phi - (\xi \otimes (d^c_B \varphi)) \circ \Phi \,, \tag{16}$$
$$\tilde{g} = d\tilde{\eta} \circ (I \otimes \tilde{\Phi}) + \tilde{\eta} \otimes \tilde{\eta} \,. \tag{17}$$

In Reference [7], the following is proven:

Lemma 1. *$(M, \xi, \tilde{\eta}, \tilde{\Phi}, \tilde{g})$ is also a Saskian structure.*

A simple proof can be done using the local frame and by observing that the Sasaki potential K is replaced by $K + \varphi$.

Definition 3. *A complex vector field X on a Sasaki manifold is called a Hamiltonian vector field if*

1. *$d\pi_\alpha(X)$ is a holomorphic vector field on V_α and*
2. *the complex vector field*

$$u_X = i\eta(X) \tag{18}$$

satisfies

$$\bar{\partial}_B u_X = -\frac{i}{2}\iota(X) d\eta \,. \tag{19}$$

Such a function u_X is called a Hamiltonian function [8].

In the foliation chart on U_α, X is written as

$$X = \eta(X)\frac{\partial}{\partial x} + \sum_{j=i}^{n} X^j \frac{\partial}{\partial z^j} - \eta\left(\sum_{j=i}^{n} X^j \frac{\partial}{\partial z^j}\right)\frac{\partial}{\partial x} \,. \tag{20}$$

Remark 1. *If the contact form η is modified according to Equation (14) with a basic function φ, the Hamilton function u_X is deformed to $u_X + X\varphi$.*

2.3. Sasaki–Ricci Flow

In what follows, we assume $c^B_1 > 0$ and $c_1 = 0$. We consider the flow $(\xi, \eta(t), \Phi(t), g(t))$ with an initial condition $(\xi, \eta(0), \Phi(0), g(0)) = (\xi, \eta, \Phi, g)$

$$\frac{d}{dt} g^T(t) = -Ric^T_{g(t)} + (2n+2)g^T \,. \tag{21}$$

This flow is called *Sasaki-Ricci flow*.

Let $\eta(t) = \eta + 2d_B^c\varphi(t)$ with $\varphi(t)$ as a family of basic functions similar to the deformations considered in Equation (14). The flow can be written as

$$\frac{d\varphi}{dt} = \log\det(g^T_{,i\bar{j}} + \varphi_{,i\bar{j}}) - \log\det(g^T_{,i\bar{j}}) + (2n+2)\varphi - h \tag{22}$$

where h is a basic function. It was proved in Reference [7] that this flow is well posed, preserving the Sasakian structure of M.

A Sasaki structure (M, ξ, η, Φ, g) with a Hamiltonian holomorphic vector field X is called a *transverse Kähler–Ricci soliton* or *Sasaki–Ricci soliton* if

$$Ric^T - (2n+2)g^T = \mathcal{L}_X g^T \tag{23}$$

where $\mathcal{L}_X$ stands for the Lie derivative by X. In Reference [8], it is proved that, on any toric Sasaki manifold, there exists a Sasaki–Ricci soliton.

Remark 2. *On a Sasaki–Einstein manifold, choosing the the vector field X proportional to the Reeb vector field, i.e., $X = c\xi$ with c a constant, the Hamilton function u_X is c.*

3. Sasaki–Ricci Flow on Spaces $\mathbf{T}^{1,1}$ and $\mathbf{Y}^{p,q}$

In what follows, we consider the Sasaki–Ricci flow on five-dimensional Sasaki–Einstein spaces $T^{1,1}$ and $Y^{p,q}$, looking for some explicit solutions of the flow equations.

3.1. Sasaki–Ricci flow on Sasaki–Einstein Space $T^{1,1}$

The Sasaki–Einstein space $T^{1,1}$ is one the most renowned examples of homogeneous Sasaki–Einstein space in five dimensions.

The standard metric on this manifold is as follows [15,16]:

$$ds^2(T^{1,1}) = \frac{1}{6}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2\theta_2 d\phi_2^2) + \frac{1}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 \tag{24}$$

where $\theta_i \in [0, \pi)$, $\phi_i \in [0, 2\pi)$, $i = 1, 2$, and $\psi \in [0, 4\pi)$. The contact 1-form η is

$$\eta = \frac{1}{3}(d\psi + \cos\theta_1\, d\phi_1 + \cos\theta_2\, d\phi_2) \tag{25}$$

and the Reeb vector field has the form

$$\xi = 3\frac{\partial}{\partial\psi}\,. \tag{26}$$

The isometries of the metric in Equation (24) form the group $SU(2)\times U(1)\times U(1)$, with the Reeb vector field in Equation (26) being one of the Killing vectors.

Writing the metric in Equation (24) as in Equation (3) with the contact form of Equation (25), we get for the transverse metric

$$g^T = \frac{1}{6}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2\theta_2 d\phi_2^2)\,. \tag{27}$$

As on $T^{1,1}$, the transverse structure is locally isomorphic to a product $S^2\times S^2$; for each S^2 sphere, the complex coordinate z^j is related to the spherical coordinates as

$$z^j = \tan\frac{\theta_j}{2}e^{i\phi_j}\,, \quad j = 1, 2\,. \tag{28}$$

The Sasaki potential of the transverse metric g^T is

$$K = \frac{1}{3}\sum_{j=1}^{2}\log(1+z^j\bar{z}^j) - \frac{1}{6}\sum_{j=1}^{2}\log(z^j\bar{z}^j)\,. \tag{29}$$

Assuming a deformation of the contact form with a basic function as in Equation (14), the Ricci flow equation has the following form [11]:

$$\begin{aligned}\frac{d\varphi}{dt} = &\log\left(\varphi_{1\bar{1}}\varphi_{2\bar{2}} - \varphi_{1\bar{2}}\varphi_{2\bar{1}} + \cos^4\frac{\theta_1}{2}\varphi_{2\bar{2}} + \cos^4\frac{\theta_2}{2}\varphi_{1\bar{1}} + \cos^4\frac{\theta_1}{2}\cos^4\frac{\theta_2}{2}\right)\\ &-\log\left(\cos^4\frac{\theta_1}{2}\cos^4\frac{\theta_2}{2}\right) + 6\varphi\,.\end{aligned} \tag{30}$$

This equation is quite involved, and it is not expected that the general solution can be found. Instead, we search for particular solutions imposing a factorization of the dependence on the variable t and angle coordinates as follows:

$$\varphi(t,\theta_1,\theta_2,\phi_1,\phi_2) = f(t)\cdot[g_1(\theta_1,\phi_1) + g_2(\theta_2,\phi_2)] \tag{31}$$

where the functions f, g_1, g_2 are to be determined. Note also that the dependence on the angles (θ_1,ϕ_1), (θ_2,ϕ_2) is separated.

At last, we look for solutions of the form in Equation (31), imposing the following additional constraints:

$$\frac{\partial^2\varphi}{\partial\theta_1^2} + \frac{1}{\sin^2\theta_1}\frac{\partial^2\varphi}{\partial\phi_1^2} + \frac{1}{\tan\theta_1}\frac{\partial\varphi}{\partial\theta_1} = c_1 f(t)\,, \tag{32}$$

$$\frac{\partial^2\varphi}{\partial\theta_2^2} + \frac{1}{\sin^2\theta_2}\frac{\partial^2\varphi}{\partial\phi_2^2} + \frac{1}{\tan\theta_2}\frac{\partial\varphi}{\partial\theta_2} = c_2 f(t)\,, \tag{33}$$

where c_j is some arbitrary real constants.

Owing to these assumptions, the Ricci flow equation in Equation (30) reduces to an ordinary differential equation for $f(t)$:

$$\begin{aligned}\frac{df(t)}{dt}\cdot[g_1(\theta_1,\phi_1) + g_2(\theta_2,\phi_2)] = &\log\left[f^2(t)(c_1c_2) + f(t)(c_1+c_2) + 1\right]\\ &+ 6f(t)\cdot[g_1(\theta_1,\phi_1) + g_2(\theta_2,\phi_2)]\,.\end{aligned} \tag{34}$$

Concerning the functions g_j, we get the following explicit expressions:

$$g_j(\theta_j,\phi_j) = \frac{1}{2}d_j\phi_j^2 + h_j(\theta_j) \tag{35}$$

with

$$h_j(\theta_1) = e_j\log u_j - \frac{d_j}{2}(\log u_j)^2 - c_j\log\sin\theta_j \tag{36}$$

where

$$u_j = \frac{\sin\theta_j}{1+\cos\theta_j}\quad,\quad j=1,2 \tag{37}$$

and d_j, e_j, $j = 1,2$ are other arbitrary real constants of integration.

As long as c_j is 0, we have the following:

Proposition 1. *Any metric of the form*

$$\tilde{g} = \tfrac{1}{9}\left(d\psi + \textstyle\sum_j(\cos\theta_j - \frac{3}{2}e_j + \frac{3}{2}d_j\log\tan\frac{\theta_j}{2})d\phi_j + \frac{3}{2}\sum_j d_j\frac{\phi_j}{\sin\theta_j}d\theta_j\right)^2 + \tfrac{1}{6}\textstyle\sum_j\left(d\theta_j^2 + \sin^2\theta_j d\phi_j^2\right) \tag{38}$$

with arbitrary real constants d_j, e_j, $j = 1, 2$, defined on the local chart considered above represents a deformation of the canonical metric on $T^{1,1}$. The deformed contact structure remains Sasaki–Einstein with the contact form

$$\tilde{\eta} = \eta - \frac{1}{2}\sum_j e_j\, d\phi_j + \frac{1}{2}\sum_j d_j \frac{\phi_j}{\sin\theta_j} d\theta_j + \frac{1}{2}\sum_j d_j \log\tan\frac{\theta_j}{2} d\phi_j\,. \tag{39}$$

Moreover, if the constants c_1, c_2 are 0 , Equation (34) becomes

$$\frac{df(t)}{dt} = 6f(t) \tag{40}$$

having the elementary solution with the initial condition $f(0) = 0$:

$$f(t) = e^{6t} - 1\,. \tag{41}$$

Remark 3. *The presence of the constants d_j in Equation (38) entails that the angles ϕ_j interfere in the deformed metric and that the Reeb vector field in Equation (26) remains the only Killing vector. Therefore, the initial toric symmetry of $T^{1,1}$ is broken in the deformed Sasaki–Einstein spaces.*

If the constants $c_j \neq 0$, the transverse metric is also modified but the contact structure remains Sasaki:

Proposition 2. *The deformed contact structure with the contact form*

$$\tilde{\eta} = \eta + \frac{1}{2}\sum_j c_j \cos\theta_j d\phi_j = \frac{1}{3}\Big(d\psi + (1 + \frac{3}{2}c_j)\cos\theta_j d\phi_j\Big) \tag{42}$$

remains Sasaki with the metric

$$\tilde{g} = \frac{1}{9}\big[d\psi + \sum_j (1 + \frac{3}{2}c_j)\cos\theta_j d\phi_j\big]^2 + \frac{1}{6}\sum_j (1 + \frac{3}{2}c_j)\big(d\theta_j^2 + \sin^2\theta_j d\phi_j^2\big)\,. \tag{43}$$

Regarding other tensors, they can be evaluated using Equations (11)–(13) with the Sasaki potential $K + \varphi$. Their explicit expressions are omitted here.

3.2. Sasaki–Ricci flow on Sasaki–Einstein Space $Y^{p,q}$

The Einstein–Sasaki geometries are the subject of much attention in connection with the supersymmetric backgrounds relevant to the AdS/CFT correspondence. An interesting class of inhomogeneous Sasaki–Einstein metrics is represented by the toric structures on $S^2 \times S^3$ denoted $Y^{p,q}$, where q, p are positive integers.

The metric of the Sasaki–Einstein space $Y^{p,q}$ is as follows [16]:

$$\begin{aligned} ds^2 \; &= \tfrac{1-y}{6}(d\theta^2 + \sin^2\theta\, d\phi^2) + \tfrac{1}{w(y)q(y)}dy^2 + \tfrac{w(y)q(y)}{36}(d\beta - \cos\theta\, d\phi)^2 \\ &+ \tfrac{1}{9}\left[d\psi + \cos\theta\, d\phi + y(d\beta - \cos\theta\, d\phi)\right]^2 \end{aligned} \tag{44}$$

where

$$\begin{aligned} w(y) &= \frac{2(a - y^2)}{1 - y}\,, \\ q(y) &= \frac{a - 3y^2 + 2y^3}{a - y^2}\,, \\ f(y) &= \frac{a - 2y + y^2}{6(a - y^2)}\,. \end{aligned} \tag{45}$$

The cubic equation

$$Q(y) = a - 3y^2 + 2y^3 = \frac{1-y}{2} w(y)\, q(y) = 0 \tag{46}$$

has three roots $y_1 \le y_2 \le y_3$. For any value of the constant $a \in (0,1)$, $y \in [y_1, y_2]$.

The range of the angular coordinates is $\theta \in [0,\pi]\,, \phi \in [0,2\pi]\,, \psi \in [0,2\pi]$. The variable β is connected with another variable α

$$\alpha = -\frac{1}{6}(\beta + \psi) \tag{47}$$

which has the range

$$0 \le \alpha \le 2\pi\ell \tag{48}$$

where

$$\ell = \frac{q}{3q^2 - 2p^2 + p(4p^2 - 3q^2)^{1/2}} \,. \tag{49}$$

The Sasaki–Einstein space $Y^{p,q}$ has the contact form

$$\eta = \frac{1}{3} d\psi + \frac{1}{3} y\, d\beta + \frac{1-y}{3} \cos\theta\, d\phi \tag{50}$$

and the Reeb vector field is

$$K_\eta = 3\frac{\partial}{\partial\psi} \,. \tag{51}$$

To describe the transverse structure of the $Y^{p,q}$ space, we introduce the following set of local complex coordinates [17]:

$$z^1 = \tan\frac{\theta}{2} e^{i\phi} \,, \tag{52}$$

$$z^2 = \frac{\sin\theta}{f_1(y)} e^{i\beta} \tag{53}$$

where

$$f_1(y) = \exp\left(\int \frac{1}{H^2(y)} dy \right) = \sqrt{(y-y_1)^{-\frac{1}{y_1}} (y_2 - y)^{-\frac{1}{y_2}} (y_3 - y)^{-\frac{1}{y_3}}} \tag{54}$$

with

$$H^2(y) = \frac{1}{6} w(y)\, q(y) = \frac{1}{3} \frac{Q(y)}{1-y} \,. \tag{55}$$

In terms of the above complex coordinates, the Sasaki–Kähler potential is

$$K = \frac{1}{3}\left[\left(1 + \frac{1}{z^1 \bar{z}^1} \right) f_2(y) \right] + \frac{1}{6} \ln(z^1 \bar{z}^1) \tag{56}$$

where

$$f_2(y) = \exp\left(\int \frac{y}{H^2(y)} dy \right) = \frac{1}{\sqrt{Q(y)}} \,. \tag{57}$$

In the case of $Y^{p,q}$ space, the Ricci flow equation is as follows [12]:

$$
\begin{aligned}
\frac{d\varphi}{dt} = \ln\Bigg\{ & \varphi_{,1\bar{1}}\varphi_{,2\bar{2}} - \varphi_{,1\bar{2}}\varphi_{,2\bar{1}} + \left[\frac{1}{3}(1-y)\cos^4\frac{\theta}{2} + \frac{w(y)q(y)}{72}\frac{\cos^2\theta}{\tan^2\frac{\theta}{2}}\right]\varphi_{,2\bar{2}} \\
& + \frac{w(y)q(y)}{72}\frac{f_1^2(y)}{\sin^2\theta}\varphi_{,1\bar{1}} + \frac{w(y)q(y)}{72}\cos\theta\cot\frac{\theta}{2}\frac{f_1(y)}{\sin\theta}e^{-i\phi+i\beta}\varphi_{,2\bar{1}} \\
& + \frac{w(y)q(y)}{72}\cos\theta\cot\frac{\theta}{2}\frac{f_1(y)}{\sin\theta}e^{i\phi-i\beta}\varphi_{,1\bar{2}} + \frac{w(y)q(y)}{216}f_1^2(y)(1-y)\cot^2\frac{\theta}{2}\Bigg\} \\
& - \ln\left[\frac{w(y)q(y)}{216}f_1^2(y)(1-y)\cot^2\frac{\theta}{2}\right] + 6\varphi
\end{aligned} \tag{58}
$$

which is more involved than in the case of the Sasaki space $T^{1,1}$.

Taking into account the complexity of this equation, we look for particular solutions. Using the same procedure as in the case of the $T^{1,1}$ space, we assume that the dependence of φ on t, z^1 and z^2 separates

$$\varphi = f(t)\left[g_1(z^1,\bar{z}^1) + g_2(z^2,\bar{z}^2)\right] \tag{59}$$

where the functions f, g_1, g_2 are to be determined.

Remark 4. *Compared to Equation (31), now, it is more convenient to write the separation of variables using the complex coordinates z^j in Equations (52) and (53). In fact the Sasaki–Ricci flow equation in Equation (22) is written in terms of complex coordinates of the transverse Kähler space. In the case of the $T^{1,1}$ space, the correspondence between the complex coordinates and angle coordinates is bijective according to Equation (28). A separation of variables written in spherical coordinates or complex coordinates is the same. Concerning the complex coordinates in Equations (52) and (53) describing the transverse structure of the $Y^{p,q}$ space, the angle variable θ intervenes in both complex coordinates z^j. Therefore, to analyze the Ricci flow equation in Equation (58), the separation assumption that is required is Equation (59).*

Some particular exact solutions of the Sasaki–Ricci flow equation can be obtained assuming

$$\varphi_{,1\bar{1}} = \cos^4\frac{\theta}{2}c_1 f(t)\,, \tag{60}$$

$$\varphi_{,2\bar{2}} = c_2 f(t)\,, \tag{61}$$

where c_1, c_2 are arbitrary constants.

With these assumptions, we get the following for g_1:

$$g_1(\theta,\phi) = \frac{d_1}{2}\phi^2 + e_1\ln\tan\frac{\theta}{2} - \frac{d_1}{2}\left(\ln\tan\frac{\theta}{2}\right)^2 - c_1\ln\sin\theta \tag{62}$$

involving the arbitrary constants c_1, d_1, e_1.

Similarly, for g_2, we obtain the following solution:

$$g_2(\rho,\beta) = \frac{d_2}{2}\beta^2 + c_2\rho^2 + e_2\ln\rho - \frac{d_2}{2}(\ln\rho)^2 \tag{63}$$

where c_2, d_2, e_2 are other arbitrary constants and

$$\rho = \frac{\sin\theta}{f_1(y)}\,. \tag{64}$$

As in the case of $T^{1,1}$ space, for $c_j \neq 0$, the contact structure remains Sasaki–Einstein and we can state the following:

Proposition 3. *The families of basic functions*

$$\varphi(t) = (e^{6t} - 1)\left[\frac{d_1}{2}\phi^2 + e_1 \ln\tan\frac{\theta}{2} - \frac{d_1}{2}\left(\ln\tan\frac{\theta}{2}\right)^2 + \frac{d_2}{2}\beta^2 + e_2\ln\rho - \frac{d_2}{2}(\ln\rho)^2\right], \tag{65}$$

with d_j, e_j arbitrary constants, stand as solutions of the transverse Kähler–Ricci flow equation on the manifold $Y^{p,q}$.

The corresponding deformed contact structures remain Sasaki–Einstein with the contact forms

$$\tilde{\eta} = \eta + \frac{e^{6t}-1}{2}\left[\frac{d_1\phi}{\sin\theta}d\theta + \left(-e_1 + d_1\ln\tan\frac{\theta}{2}\right)d\phi + \frac{d_2\beta}{\rho}\,d\rho + (-e_2 + d_2\ln\rho)d\beta\right]. \tag{66}$$

Remark 5. *We observe that the explicit presence of the coordinated ϕ, β in the functions g_1 of Equation (62) and g_2 of Equation (63) makes the Reeb vector field in Equation (51) the only Killing vector of the deformed metric. As it is noted in* Remark 3, *the inceptive toric symmetry is broken during the Ricci flow deformation.*

In the case $c_j \neq 0$, we have the following:

Proposition 4. *The deformed contact structures with the contact forms*

$$\tilde{\eta} = \eta + \frac{f(t)}{2}\left[c_1\cos\theta\,d\phi - c_2\rho^2 d\beta\right] \tag{67}$$

with c_j arbitrary constants remain Sasaki with the deformed metrics

$$\begin{aligned}\tilde{g} &= \tilde{\eta}\otimes\tilde{\eta} + g^T + \sum_{j=1}^{2}\phi_{j\bar{j}}dz^j d\bar{z}^j \\ &= \tilde{\eta}\otimes\tilde{\eta} + g^T + f(t)\left[\frac{c_1}{4}(d\theta^2 + \sin^2\theta\,d\phi^2) + c_2(d\rho^2 + \rho^2 d\beta^2)\right].\end{aligned} \tag{68}$$

In this case, the function $f(t)$ obeys a differential equation without a simple, explicit solution.

Again, making use of Equations (11)–(13), we can evaluate other tensors which describe the deformed contact structures.

4. Discussion

In this paper, we examine the Kähler structures of the Sasaki manifolds. We perform deformations of the contact structures fixing the Reeb vector field ξ but vary the contact form η by means of smooth basic functions.

The Sasaki–Ricci flow equations are quite involved, but eventually, we are able to produce some particular explicit analytical solutions representing deformations of the Sasaki–Einstein spaces $T^{1,1}$ and $Y^{p,q}$.

The deformations considered in this paper can be compared with the so-called β or TsT deformations of Sasaki–Einstein manifolds considered in Reference [9].

In Reference [18], the constants of geodesic motion in spaces $T^{1,1}$ and $Y^{p,q}$ were explicitly constructed. The angles ψ, ϕ_1, ϕ_2 and ψ, β, ϕ are cyclic variables for geodesic motion in spaces $T^{1,1}$ and $Y^{p,q}$, respectively. Consequently, the geodesic motion in these spaces are completely integrable. We note that, in the deformed Sasaki metrics considered in the present paper, some of the angles are no longer cyclic variables and the complete integrability is lost. It would be interesting to investigate the Hamiltonian holomorphic vector fields in connection with the integrability and action-angle variables on the perturbed Sasaki–Einstein spaces.

It is worth extending the study of deformations of contact structures and the Sasaki–Ricci flow on higher dimensional Sasaki–Einstein spaces.

Finally, we note the relevance of contact Hamiltonian systems in irreversible thermodynamics, statistical physics, systems with dissipation, etc. (see, e.g., Reference [19] for a recent review of applications of contact Hamiltonian dynamics in various fields).

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