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Multi-Granulation Picture Hesitant Fuzzy Rough Sets

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Abstract: We lay the theoretical foundations of a novel model, termed picture hesitant fuzzy rough sets, based on picture hesitant fuzzy relations. We also combine this notion with the ideas of multi-granulation rough sets. As a consequence, a new multi-granulation rough set model on two universes, termed a multi-granulation picture hesitant fuzzy rough set, is developed. When the universes coincide or play a symmetric role, the concept assumes the standard format. In this context, we put forward two new classes of multi-granulation picture hesitant fuzzy rough sets, namely, the optimistic and pessimistic multi-granulation picture hesitant fuzzy rough sets. Further, we also investigate the relationships among these two concepts and picture hesitant fuzzy rough sets.

Keywords: rough set; picture hesitant fuzzy rough relation; picture hesitant fuzzy rough set; multi-granulation picture hesitant fuzzy rough set; optimistic picture hesitant fuzzy rough set; pessimistic picture hesitant fuzzy rough set

1. Introduction

Vague and incomplete information poses many challenges to the researchers in science and engineering. Among the various techniques developed for tackling these burdensome constraints, fuzzy set (FS) theory and rough set (RS) theory [1,2] are prominent directions, which have produced multiple extensions and hybridizations. While fuzzy set theory enables one to model vagueness, rough set theory helps in modeling granular information. Even though RS theory was originally developed as the outcome of an indiscernibility relation, which can be identified mathematically as an equivalence relation, soon the shift was towards similarity or coverings as well as their fuzzy counterparts. Many hybrid models were proposed toward this aim [3–8]. With this exhaustive process, the distance between those ideas became shorter.

However, the partitions defined via indiscernibility relations in Pawlak's formulation of RS theory are too restrictive for many of its potential applications. In order to overcome that inadequacy, many alternatives have been suggested. To name but a few important cases: the dominance-based RS approach introduced by Greco et al. [9], the extension to similarity relations based on the work by Inuiguchi and Tanino [10], the covering-based models given by Bonikowski et al. [11], and the extension to general relations based on the proposal by Inuiguchi and Tanino [12]. Granular computing, introduced by Zadeh [13], is now a well developed computing paradigm of information processing. Here complex information will be divided into pieces or classes called granules [14–18] based on the needs and understanding of the problem being handled.

In most of the hybrid structures involving rough sets, researchers mainly focused on a single universe, as the original concept was designed. However, in reality, a variety of natural applications need to consider two or more universes. Zhang, Shu, and Liao [19] demonstrate this idea with the following situation. Patients in a hospital often show several simultaneous symptoms. A particular disease may have many clinical manifestations. It is thus sometimes difficult for a doctor to recognize whether a patient is suffering from a given disease. These situations are better handled by rough sets over two universes, one for the set of patients and the other for the set of clinical symptoms.

Pei and Xu [20] examine rough set models on two universes. A variable precision rough set model on two universes is developed by Shen and Wang [21]. In a different line, Qian et al. [22] also expanded rough set theory with the introduction of multi-granulation rough sets (MGRS). Their structure was clarified by She and He [23], who introduced the optimistic and pessimistic specifications of MGRS. In this way the opinions of various experts on the granulation of the information can be taken into consideration. Further, many extensions of MGRS were introduced.

For example, the multi-granulation fuzzy rough set model by Xu et al. [18], the multi-granulation rough sets based on an incomplete information system by Yang et al. [24,25], and the neighborhood-based multi-granulation rough set by Lin et al. [26] are important generalizations. Two fundamental approaches to MGRS theory are the optimistic and pessimistic multigranulation [27]. The pessimistic approach demands that the elements belong with certainty to a precise notion in every granular structure; however, the optimistic approach is characterized by the existence of at least one granular structure to which an element surely belongs.

Although fuzzy set theory is a useful form to reveal the uncertainty of evaluation information, it remains inefficient in solving some complex situations in real life. In practice, other types of uncertain and complex evaluations are often given by decision makers. Atanassov's intuitionistic fuzzy set theory expanded its scope with the addition of a degree of indeterminacy to fuzzy sets [28], and also in a temporal setting [29]. Real situations sometimes impose a neutral component in addition to indeterminacy. For modeling situations like this, picture fuzzy sets [30,31] were designed. They have positive, neutral, and negative or refusal membership functions. In a different line of thought, situations where the decision makers hesitate among various possible values appeal to hesitant fuzzy sets [32] and the hybrid models. For example, Wang and Li [33] have recently introduced picture hesitant fuzzy sets (PHFS).

The innovation that we pursue is the generalization of multi-granular rough sets that incorporate the nice features of PHFS. Thus in this paper we develop picture hesitant fuzzy rough sets defined from picture fuzzy relations, a novel concept that we introduce for this purpose. As single granulation has limitations, as already explained, we adopt a structure based on multi-granulation, which is more suited for multi-source information systems. We point out that multi-granulation rough sets have been used in various fields, including decision making [34], feature selection [35], etc.

We organize the contents of our paper into six sections. After this Introduction, in Section 2, we give basic definitions of hesitant fuzzy sets, picture fuzzy sets, and picture hesitant fuzzy sets. Then, Section 3 defines the notion of a picture hesitant fuzzy relation (PHFRS). Picture hesitant fuzzy rough sets are introduced using this relation, and an example is provided. Some properties of PHFRSs are also given in this section. Section 4 deals with multi-granulation picture hesitant roughs sets (MGPHFRS), where two types of MGPHFRSs are introduced, namely, optimistic MGPHFRS and pessimistic MGPHFRS. In Section 5, some relationships among these models are put forward. Section 6 gives a comparison of the new structure with the existing ones. Some conclusions and additional directions of research are provided in the concluding section of the paper.

2. Preliminaries

The fundamental concepts of hesitant fuzzy set, picture fuzzy set, and picture hesitant fuzzy set are given in the following definitions.

Definition 1. *[32] Let U denote a crisp set (of alternatives, options, ...). A hesitant fuzzy (HF) set $\mathbb{A}$ on the crisp set U consists of $h_{\mathbb{A}}$, a mapping that assigns a subset $h_{\mathbb{A}}(x)$ of $[0,1]$ with every element $x \in U$. Here $h_{\mathbb{A}}(x)$ means the collection of all acceptable membership degrees of $x \in U$ to $\mathbb{A}$.*

In more practical terms, this set is captured by the expression $\mathbb{A} = \{< x, h_{\mathbb{A}}(x) > | x \in U\}$ *[36].*

Following [36], each $h_{\mathbb{A}}(x) \subseteq [0,1]$ is a hesitant fuzzy element (HFE). We denote, by $l(h(x))$, the cardinality of $h(x)$ as a set (of possible membership degrees of $x \in U$ to $\mathbb{A}$). Then the score for $h(x)$ is $s(h) = \frac{1}{l(h)} \sum_{\gamma \in h} \gamma$, so that s is a score function of the HFEs [36]. Scores will be important tools in our analysis. They provide the following well known comparison law, as well as a novel methodology for set-theoretic composition of hesitant fuzzy sets (Definition 4 below):

Definition 2. *[36] When h_1 and h_2 are HF sets on X, then we denote $h_1 \preceq h_2$ if, and only if, $s(h_1(x)) \leq s(h_2(x)), \forall x \in X$.*

In our approach, we use score based unions and intersections instead of Torra's standard union and intersection, which has been already exploited by many authors. These elements arise from a reinterpretation of the definition above in the following terms:

Definition 3. *[3] Let h_1 and h_2 be HF sets on the crisp set X. Then, we declare that h_1 is a hesitant subset of h_2 (and we express it as $h_1 \preceq h_2$) when $s(h_1(x)) \leq s(h_2(x))$, for each $x \in X$.*

Now, we can define:

Definition 4. *[3] Let h_1 and h_2 be HF sets on X. Their score based intersection (denoted by $h_1 \tilde{\wedge} h_2$) is defined by:*

$$\left(h_1 \tilde{\wedge} h_2\right)(y) = \begin{cases} h_1(y) & \text{when } h_1(y) \prec h_2(y) \\ h_2(y) & \text{when } h_2(y) \prec h_1(y) \\ h_1(y) \cup h_2(y) & \text{when } h_1(y) \approx h_2(y) \end{cases} \quad \text{for every } y \in X.$$

Their score based union (denoted by $h_1 \tilde{\vee} h_2$) is given by:

$$\left(h_1 \tilde{\vee} h_2\right)(y) = \begin{cases} h_1(y) & \text{when } h_1(y) \succ h_2(y) \\ h_2(y) & \text{when } h_2(y) \succ h_1(y) \\ h_1(y) \cup h_2(y) & \text{when } h_1(y) \approx h_2(y) \end{cases} \quad \text{for every } y \in X.$$

Intuitionistic fuzzy sets extend fuzzy sets by incorporating a degree of indeterminacy to the idea of a fuzzy set. However, when real situations express an explicit neutral component in addition to generic indeterminacy, one needs the next definition that captures all these features:

Definition 5. *[30] A picture fuzzy set (abbreviated as PFS) on the non-empty set X is*

$$P = \{\langle x, \mu_P(x), \eta_P(x), v_P(x)\rangle \, | x \in X\}.$$

The figures $\mu_P(x)$, $\eta_P(x)$, and $v_P(x)$ denote the positive, the neutral, and the negative memberships of x, respectively. They are numbers that satisfy the restriction: for every $x \in X$, $0 \leq \mu_P(x) + \eta_P(x) + v_P(x) \leq 1$.

A recent contribution has merged the previous concept with the idea of hesitancy. The notion that arises is given in the next definition:

Definition 6. *[33] Let X be a non-empty set. A picture hesitant fuzzy set (abbreviated as PHFS) on the set X is*

$$P = \{\langle x, \mu_P(x), \eta_P(x), \nu_P(x)\rangle \, | x \in X\}.$$

The elements $\mu_P(x)$, $\eta_P(x)$, and $\nu_P(x)$ are subsets of $[0,1]$, which contain feasible sets of degrees of positive, neutral, and negative memberships of the element $x \in X$ in P, respectively. The degrees above satisfy the *restriction: For every $x \in X$, $0 \leq \sup(\mu_P(x)) + \sup(\eta_P(x)) + \sup(\nu_P(x)) \leq 1$. Moreover, $PHFS(X)$ denotes the collection of all PHFSs on X.*

We shall take advantage of this recent concept in order to define a novel model of granular knowledge in the next section.

3. Picture Hesitant Fuzzy Rough Sets

First, we present the new concept of picture hesitant fuzzy relations, which allows us to compare elements from two crisp sets, X and Y, with the advantages given by picture hesitant fuzzy evaluations. Let us denote the power set of $[0,1]$ by $\mathbb{P}\{[0,1]\}$. Then:

Definition 7. *Let X, Y be two non-empty and finite crisp sets. A picture hesitant fuzzy subset $\mathcal{R}$ of $X \times Y$ is called a picture hesitant fuzzy relation $\mathcal{R}$ from X to Y. In formal terms,*

$$\mathcal{R} = \{\langle (x,y), \mu_{\mathcal{R}}(x,y), \eta_{\mathcal{R}}(x,y), \nu_{\mathcal{R}}(x,y) \rangle | (x,y) \in X \times Y\}.$$

Here we use three mappings $\mu_{\mathcal{R}}, \eta_{\mathcal{R}}, \nu_{\mathcal{R}} : X \times Y \to \mathbb{P}\{[0,1]\}$.

Henceforth, $PHFR(X \times Y)$ shall represent the collection of all picture hesitant fuzzy relations from X to Y.

Example 1 below takes advantage of a natural tabular representation of picture hesitant fuzzy relations.

Next, we present some algebraic notions associated with the previous concept:

Definition 8. *Let $\mathcal{R} \in PHFR(X \times Y)$. The inverse relation $\mathcal{R}^{-1} : Y \to X$ is*

$$\mathcal{R}^{-1} = \{\langle (x,y), \mu_{\mathcal{R}}^{-1}(x,y), \eta_{\mathcal{R}}^{-1}(x,y), \nu_{\mathcal{R}}^{-1}(x,y) \rangle | (x,y) \in X \times Y\}$$

where $\mu_{\mathcal{R}}^{-1}(x,y) = \mu_{\mathcal{R}}(y,x)$, $\eta_{\mathcal{R}}^{-1}(x,y) = \eta_{\mathcal{R}}(y,x)$, and $\nu_{\mathcal{R}}^{-1}(x,y) = \nu_{\mathcal{R}}(y,x), \forall (x,y) \in X \times Y$.

As any picture hesitant fuzzy relation is also a picture hesitant fuzzy set, the properties of picture hesitant fuzzy relations coincide with the properties of picture hesitant fuzzy sets.

Now, we define some set-theoretic notions in the field of picture hesitant fuzzy sets. We do this in the following definition:

Definition 9. *For any $\mathcal{P}, \mathcal{R} \in PHFR(X \times Y)$, we denote:*

1. *Inclusion: $\mathcal{R} \subseteq \mathcal{P}$ if, and only if, $\mu_{\mathcal{R}}(x,y) \leq \mu_{\mathcal{P}}(x,y)$, $\eta_{\mathcal{R}}(x,y) \leq \eta_{\mathcal{P}}(x,y)$, $\nu_{\mathcal{R}}(x,y) \geq \nu_{\mathcal{P}}(x,y)$, for all $(x,y) \in X \times Y$;*
2. *Union of $\mathcal{P}$ and $\mathcal{R}$: $\mathcal{P} \cup \mathcal{R} = \{\langle (x,y),\ \mu_{\mathcal{P}}(x,y) \tilde{\vee} \mu_{\mathcal{R}}(x,y),\ \eta_{\mathcal{P}}(x,y) \tilde{\vee} \eta_{\mathcal{R}}(x,y),\ \nu_{\mathcal{P}}(x,y) \tilde{\wedge} \nu_{\mathcal{R}}(x,y) \rangle |\ x \in X,\ y \in Y\}$;*
3. *Intersection of $\mathcal{P}$ and $\mathcal{R}$: $\mathcal{P} \cap \mathcal{R} = \{\langle (x,y),\ \mu_{\mathcal{P}}(x,y) \tilde{\wedge} \mu_{\mathcal{R}}(x,y),\ \eta_{\mathcal{P}}(x,y) \tilde{\wedge} \eta_{\mathcal{R}}(x,y),\ \nu_{\mathcal{P}}(x,y) \tilde{\vee} \nu_{\mathcal{R}}(x,y) \rangle |\ x \in X,\ y \in Y\}$;*
4. *Complement of $\mathcal{P}$: $\mathcal{P}^c = \{\langle (x,y),\ \nu_{\mathcal{P}}(x,y),\ \eta_{\mathcal{P}}(x,y),\ \mu_{\mathcal{P}}(x,y) \rangle |\ x \in X,\ y \in Y\}$.*

The next properties follow from Definitions 8 and 9 immediately:

Proposition 1. *We fix $\mathcal{P}, \mathcal{Q}, \mathcal{R} \in PHFR(X \times Y)$. Then one has:*

1. *The inverse relation is involutive, i.e., $\mathcal{R} = (\mathcal{R}^{-1})^{-1}$;*
2. *The inverse relation is monotone with respect to inclusion, i.e., $\mathcal{R} \subseteq \mathcal{P}$ implies $\mathcal{R}^{-1} \subseteq \mathcal{P}^{-1}$;*
3. *The inverse relation preserves unions, i.e., $\mathcal{P}^{-1} \cup \mathcal{R}^{-1} = (\mathcal{P} \cup \mathcal{R})^{-1}$;*
4. *The inverse relation preserves intersections, i.e., $\mathcal{P}^{-1} \cap \mathcal{R}^{-1} = (\mathcal{P} \cap \mathcal{R})^{-1}$;*
5. *There is distributivity of unions with respect to intersections, i.e., $\mathcal{P} \cup (\mathcal{R} \cap \mathcal{Q}) = (\mathcal{P} \cup \mathcal{R}) \cap (\mathcal{P} \cup \mathcal{Q})$;*

6. *There is distributivity of intersections with respect to unions, i.e.,* $\mathcal{P} \cap (\mathcal{R} \cup \mathcal{Q}) = (\mathcal{P} \cap \mathcal{R}) \cup (\mathcal{P} \cap \mathcal{Q})$.

Definition 10. *We say that a picture hesitant fuzzy relation on X, namely* $\mathcal{P} = \{\langle (x,y), \mu_{\mathcal{P}}(x,y), \eta_P(x,y), \nu_{\mathcal{P}}(x,y)\rangle | (x,y) \in X \times Y\}$*, is reflexive when it is the case that for every* $x \in X$, $\mu_{\mathcal{P}}(x,x) = \{1\}, \eta_{\mathcal{P}}(x,x) = \{0\}$*, and* $\nu_{\mathcal{P}}(x,x) = \{0\}$*. We say that it is symmetric when it is the case that for every* $(x,y) \in X \times X$, $\mu_{\mathcal{P}}(x,y) = \mu_{\mathcal{P}}(y,x)$, $\eta_{\mathcal{P}}(x,y) = \eta_{\mathcal{P}}(y,x)$*, and* $\nu_{\mathcal{P}}(x,y) = \nu_{\mathcal{P}}(y,x)$.

Definition 11. *Let* U, V *be non-empty sets. When* $\mathcal{P}$ *is a picture hesitant fuzzy relation from U to V, we say that the triple* $(U, V, \mathcal{P})$ *is a picture hesitant fuzzy approximation space.*

The upper and lower approximations of $R \in PHFS(V)$ *with respect to* $(U, V, \mathcal{P})$ *are two picture hesitant fuzzy sets. They are respectively defined as:*

$$\mathcal{P}_*(R) = \{\langle x, \mu_{\mathcal{P}_*}(R)(x), \eta_{\mathcal{P}_*}(R)(x), \nu_{\mathcal{R}_*}(P)(x)\rangle | x \in U\}$$

$$\mathcal{P}^*(R) = \{\langle x, \mu_{\mathcal{P}^*}(R)(x), \eta_{\mathcal{P}_*}(R)(x), \nu_{\mathcal{P}^*}(R)(x)\rangle | x \in U\}$$

where

$$\mu_{\mathcal{P}_*}(R)(x) = \widetilde{\bigwedge_{y\in V}} [\nu_{\mathcal{P}}(x,y) \tilde{\vee} \mu_R(y)], \textit{for each } x \in U$$
$$\eta_{\mathcal{P}_*}(R)(x) = \widetilde{\bigwedge_{y\in V}} [\eta_{\mathcal{P}}(x,y) \tilde{\vee} \eta_R(y)], \textit{for each } x \in U$$
$$\nu_{\mathcal{P}_*}(R)(x) = \widetilde{\bigvee_{y\in V}} [\mu_{\mathcal{P}}(x,y) \tilde{\wedge} \nu_R(y)], \textit{for each } x \in U$$
$$\mu_{\mathcal{P}^*}(R)(x) = \widetilde{\bigvee_{y\in V}} [\mu_{\mathcal{P}}(x,y) \tilde{\wedge} \mu_R(y)], \textit{for each } x \in U$$
$$\eta_{\mathcal{P}^*}(R)(x) = \widetilde{\bigvee_{y\in V}} [\eta_{\mathcal{P}}(x,y) \tilde{\wedge} \eta_R(y)], \textit{for each } x \in U$$
$$\nu_{\mathcal{P}^*}(R)(x) = \widetilde{\bigwedge_{y\in V}} [\nu_{\mathcal{P}}(x,y) \tilde{\vee} \nu_R(y)], \textit{for each } x \in U.$$

Our next example illustrates the application of the previous concepts:

Example 1. *Let us fix two universes of discourse* $U = \{x_1, x_2\}$ *and* $V = \{y_1, y_2, y_3\}$*. Then we define* $\mathcal{R} \in PHFR(U \times V)$ *in the following tabular form:*

	y_1	y_2	y_3
x_1	{0.1,0.3,0.4},{0.2,0.3},{0.1,0.25}	{0.25,0.35},{0.1,0.15,0.2},{0.05,0.35}	{0.2,0.22},{0.11,0.29},{0.2,0.4}
x_2	{0.1,0.2},{0.13,2},{0.3,0.35}	{0.15,0.2,0.35},{0.2,0.25}, {0.15,0.3}	{0.1,0.4},{0.1,0.3}, {0.2,0.3}

$P = \{\langle y_1, \{0.2, 0.3\}, \{0.1, 0.15, 0.3\}\{0.2, 0.35, 0.4\}\rangle, \langle y_2, \{0.1, 0.2, 0.25\}, \{0.2, 0.3\}\{0.15, 0.3, 0.4\}\rangle, \langle y_3, \{0.2, 0.3\}, \{0.1, 0.35\}\{0.2, 0.3\}\rangle\}$.

From direct computations we have

$$\begin{aligned}\mu_{\mathcal{R}_*}(P)(x_1) &= [\nu_{\mathcal{R}}(x_1, y_1) \tilde{\vee} \mu_P(y_1)] \widetilde{\bigwedge} [\nu_{\mathcal{R}}(x_1, y_2) \tilde{\vee} \mu_P(y_2)] \widetilde{\bigwedge} [\nu_{\mathcal{R}}(x_1, y_3) \tilde{\vee} \mu_P(y_3)] \\ &= [\{0.1, 0.25\} \tilde{\vee} \{0.2, 0.3\}] \widetilde{\bigwedge} [\{0.05, 0.35\} \tilde{\vee} \{0.1, 0.2, 0.25\}] \\ &\quad \widetilde{\bigwedge} [\{0.2, 0.4\} \tilde{\vee} \{0.2, 0.3\}] \\ &= \{0.2, 0.3\} \widetilde{\bigwedge} \{0.05, 0.35\} \widetilde{\bigwedge} \{0.2, 0.4\} = \{0.05, 0.35\}\end{aligned}$$

$$\begin{aligned}\mu_{\mathcal{R}_*}(P)(x_2) &= [\nu_{\mathcal{R}}(x_2, y_1) \tilde{\vee} \mu_P(y_1)] \widetilde{\bigwedge} [\nu_{\mathcal{R}}(x_2, y_2) \tilde{\vee} \mu_P(y_2)] \widetilde{\bigwedge} [\nu_{\mathcal{R}}(x_2, y_3) \tilde{\vee} \mu_P(y_3)] \\ &= [\{0.3, 0.35\} \tilde{\vee} \{0.2, 0.3\}] \widetilde{\bigwedge} [\{0.15, 0.3\} \tilde{\vee} \{0.1, 0.2, 0.25\}] \\ &\quad \widetilde{\bigwedge} [\{0.2, 0.3\} \tilde{\vee} \{0.2, 0.3\}]\end{aligned}$$

$$
\begin{aligned}
&= \{0.2, 0.3\}\widetilde{\bigwedge}\{0.15, 0.3\}\widetilde{\bigwedge}\{0.2, 0.3\} = \{0.15, 0.3\} \\
\eta_{\mathcal{R}_*}(P)(x_1) &= [\eta_{\mathcal{R}}(x_1, y_1)\widetilde{\vee}\eta_P(y_1)]\widetilde{\bigwedge}[\eta_{\mathcal{R}}(x_1, y_2)\widetilde{\vee}\eta_P(y_2)]\widetilde{\bigwedge}[\eta_{\mathcal{R}}(x_1, y_3)\widetilde{\vee}\eta_P(y_3)] \\
&= [\{0.2, 0.3\}\widetilde{\vee}\{0.1, 0.15, 0.3\}]\widetilde{\bigwedge}[\{0.1, 0.15, 0.2\}\widetilde{\vee}\{0.2, 0.3\}] \\
&\qquad \widetilde{\bigwedge}[\{0.11, 0.29\}\widetilde{\vee}\{0.1, 0.35\}] \\
&= \{0.2, 0.3\}\widetilde{\bigwedge}\{0.2, 0.3\}\widetilde{\bigwedge}\{0.1, 0.35\} = \{0.1, 0.35\} \\
\eta_{\mathcal{R}_*}(P)(x_2) &= [\eta_{\mathcal{R}}(x_2, y_1)\widetilde{\vee}\eta_P(y_1)]\widetilde{\bigwedge}[\eta_{\mathcal{R}}(x_2, y_2)\widetilde{\vee}\eta_P(y_2)]\widetilde{\bigwedge}[\eta_{\mathcal{R}}(x_2, y_3)\widetilde{\vee}\eta_P(y_3)] \\
&= [\{0.13, 0.2\}\widetilde{\vee}\{0.1, 0.15, 0.3\}]\widetilde{\bigwedge}[\{0.2, 0.25\}\widetilde{\vee}\{0.2, 0.3\}] \\
&\qquad \widetilde{\bigwedge}[\{0.11, 0.29\}\widetilde{\vee}\{0.1, 0.35\}] \\
&= \{0.1, 0.15, 0.3\}\widetilde{\bigwedge}\{0.2, 0.3\}\widetilde{\bigwedge}\{0.1, 0.35\} = \{0.1, 0.15, 0.3\} \\
\nu_{\mathcal{R}_*}(P)(x_1) &= [\mu_{\mathcal{R}}(x_1, y_1)\widetilde{\wedge}\nu_P(y_1)]\widetilde{\bigvee}[\mu_{\mathcal{R}}(x_1, y_2)\widetilde{\wedge}\nu_P(y_2)]\widetilde{\bigvee}[\mu_{\mathcal{R}}(x_1, y_3)\widetilde{\wedge}\nu_P(y_3)] \\
&= [\{0.1, 0.3, 0.4\}\widetilde{\wedge}\{0.2, 0.35, 0.4\}]\widetilde{\bigvee}[\{0.25, 0.35\}\widetilde{\wedge}\{0.15, 0.3, 0.4\}] \\
&\qquad \widetilde{\bigvee}[\{0.2, 0.22\}\widetilde{\wedge}\{0.2, 0.3\}] \\
&= \{0.1, 0.3, 0.4\}\widetilde{\bigvee}\{0.15, 0.3, 0.4\}\widetilde{\bigvee}\{0.2, 0.22\} = \{0.15, 0.3, 0.4\} \\
\nu_{\mathcal{R}_*}(P)(x_2) &= [\mu_{\mathcal{R}}(x_2, y_1)\widetilde{\wedge}\nu_P(y_1)]\widetilde{\bigvee}[\mu_{\mathcal{R}}(x_2, y_2)\widetilde{\wedge}\nu_P(y_2)]\widetilde{\bigvee}[\mu_{\mathcal{R}}(x_2, y_3)\widetilde{\wedge}\nu_P(y_3)] \\
&= [\{0.1, 0.2\}\widetilde{\wedge}\{0.2, 0.35, 0.4\}]\widetilde{\bigvee}[\{0.15, 0.2, 0.35\}\widetilde{\wedge}\{0.15, 0.3, 0.4\}] \\
&\qquad \widetilde{\bigvee}[\{0.1, 0.4\}\widetilde{\wedge}\{0.2, 0.3\}] \\
&= \{0.1, 0.2\}\widetilde{\bigvee}\{0.15, 0.2, 0.35\}\widetilde{\bigvee}\{0.1, 0.4\} = \{0.1, 0.4\} \\
\mu_{\mathcal{R}^*}(P)(x_1) &= [\mu_{\mathcal{R}}(x_1, y_1)\widetilde{\wedge}\mu_P(y_1)]\widetilde{\bigvee}[\mu_{\mathcal{R}}(x_1, y_2)\widetilde{\wedge}\mu_P(y_2)]\widetilde{\bigvee}[\mu_{\mathcal{R}}(x_1, y_3)\widetilde{\wedge}\mu_P(y_3)] \\
&= [\{0.1, 0.3, 0.4\}\widetilde{\wedge}\{0.2, 0.3\}]\widetilde{\bigvee}[\{0.25, 0.35\}\widetilde{\wedge}\{0.1, 0.2, 0.25\}] \\
&\qquad \widetilde{\bigvee}[\{0.2, 0.22\}\widetilde{\wedge}\{0.2, 0.3\}] \\
&= \{0.2, 0.3\}\widetilde{\bigvee}\{0.1, 0.2, 0.25\}\widetilde{\bigvee}\{0.2, 0.22\} = \{0.2, 0.3\} \\
\mu_{\mathcal{R}^*}(P)(x_2) &= [\mu_{\mathcal{R}}(x_2, y_1)\widetilde{\wedge}\mu_P(y_1)]\widetilde{\bigvee}[\mu_{\mathcal{R}}(x_2, y_2)\widetilde{\wedge}\mu_P(y_2)]\widetilde{\bigvee}[\mu_{\mathcal{R}}(x_2, y_3)\widetilde{\wedge}\mu_P(y_3)] \\
&= [\{0.1, 0.2\}\widetilde{\wedge}\{0.2, 0.3\}]\widetilde{\bigvee}[\{0.15, 0.2, 0.35\}\widetilde{\wedge}\{0.1, 0.2, 0.25\}]\widetilde{\bigvee}[\{0.1, 0.4\}\widetilde{\wedge}\{0.2, 0.3\}] \\
&= \{0.1, 0.2\}\widetilde{\bigvee}\{0.1, 0.2, 0.25\}\widetilde{\bigvee}\{0.1, 0.4\} = \{0.1, 0.4\} \\
\eta_{\mathcal{R}^*}(P)(x_1) &= [\eta_{\mathcal{R}}(x_1, y_1)\widetilde{\wedge}\eta_P(y_1)]\widetilde{\bigvee}[\eta_{\mathcal{R}}(x_1, y_2)\widetilde{\wedge}\eta_P(y_2)]\widetilde{\bigvee}[\eta_{\mathcal{R}}(x_1, y_3)\widetilde{\wedge}\eta_P(y_3)] \\
&= [\{0.2, 0.3\}\widetilde{\wedge}\{0.1, 0.15, 0.3\}]\widetilde{\bigvee}[\{0.1, 0.15, 0.2\}\widetilde{\wedge}\{0.2, 0.3\}] \\
&\qquad \widetilde{\bigvee}[\{0.11, 0.29\}\widetilde{\wedge}\{0.1, 0.35\}] \\
&= \{0.1, 0.15, 0.3\}\widetilde{\bigvee}\{0.1, 0.15, 0.2\}\widetilde{\bigvee}\{0.11, 0.29\} = \{0.11, 0.29\} \\
\eta_{\mathcal{R}^*}(P)(x_2) &= [\eta_{\mathcal{R}}(x_2, y_1)\widetilde{\wedge}\eta_P(y_1)]\widetilde{\bigvee}[\eta_{\mathcal{R}}(x_2, y_2)\widetilde{\wedge}\eta_P(y_2)]\widetilde{\bigvee}[\eta_{\mathcal{R}}(x_2, y_3)\widetilde{\wedge}\eta_P(y_3)] \\
&= [\{0.13, 0.2\}\widetilde{\wedge}\{0.1, 0.15, 0.3\}]\widetilde{\bigvee}[\{0.2, 0.25\}\widetilde{\wedge}\{0.2, 0.3\}] \\
&\qquad \widetilde{\bigvee}[\{0.1, 0.3\}\widetilde{\wedge}\{0.1, 0.35\}] \\
&= \{0.13, 0.2\}\widetilde{\bigvee}\{0.2, 0.25\}\widetilde{\bigvee}\{0.1, 0.3\} = \{0.2, 0.25\} \\
\nu_{\mathcal{R}^*}(P)(x_1) &= [\nu_{\mathcal{R}}(x_1, y_1)\widetilde{\vee}\nu_P(y_1)]\widetilde{\bigwedge}[\nu_{\mathcal{R}}(x_1, y_2)\widetilde{\vee}\eta_P(y_2)]\widetilde{\bigwedge}[\nu_{\mathcal{R}}(x_1, y_3)\widetilde{\vee}\nu_P(y_3)] \\
&= [\{0.1, 0.25\}\widetilde{\vee}\{0.2, 0.35, 0.4\}]\widetilde{\bigwedge}[\{0.05, 0.35\}\widetilde{\vee}\{0.15, 0.3, 0.4\}] \\
&\qquad \widetilde{\bigwedge}[\{0.2, 0.4\}\widetilde{\vee}\{0.2, 0.3\}] \\
&= \{0.2, 0.35, 0.4\}\widetilde{\bigwedge}\{0.15, 0.3, 0.4\}\widetilde{\bigwedge}\{0.2, 0.4\} = \{0.15, 0.3, 0.4\} \\
\nu_{\mathcal{R}^*}(P)(x_2) &= [\nu_{\mathcal{R}}(x_2, y_1)\widetilde{\vee}\nu_P(y_1)]\widetilde{\bigwedge}[\nu_{\mathcal{R}}(x_2, y_2)\widetilde{\vee}\nu_P(y_2)]\widetilde{\bigwedge}[\nu_{\mathcal{R}}(x_2, y_3)\widetilde{\vee}\nu_P(y_3)] \\
&= [\{0.3, 0.35\}\widetilde{\vee}\{0.2, 0.35, 0.4\}]\widetilde{\bigwedge}[\{0.15, 0.3\}\widetilde{\vee}\{0.15, 0.3, 0.4\}] \\
&\qquad \widetilde{\bigwedge}[\{0.2, 0.3\}\widetilde{\wedge}\{0.2, 0.3\}] \\
&= \{0.3, 0.35\}\widetilde{\bigvee}\{0.15, 0.3, 0.4\}\widetilde{\bigvee}\{0.2, 0.3\} = \{0.2, 0.3\} \\
\therefore\ \mathcal{R}_*(P) &= \{\langle x_1, \{0.05, 0.35\}, \{0.1, 0.35\}, \{0.15, 0.3, 0.4\}\rangle, \\
&\qquad \langle x_2, \{0.15, 0.3\}, \{0.1, 0.15, 0.3\}, \{0.1, 0.4\}\rangle\} \\
\mathcal{R}^*(P) &= \{\langle x_1, \{0.2, 0.3\}, \{0.11, 0.29\}, \{0.15, 0.3, 0.4\}\rangle, \\
&\qquad \{\langle x_2, \{0.1, 0.4\}, \{0.2, 0.25\}, \{0.2, 0.3\}\rangle\}
\end{aligned}
$$

Now, we proceed to derive some properties of the new picture hesitant fuzzy approximation operators given by Definition 11:

Theorem 1. *Suppose $(U, V, \mathcal{R})$ is a PHF approximation space. Then the upper and lower PHF approximation operators satisfy that for every $P_1, P_2 \in PHFS(V)$:*

1. $P_1 \subseteq P_2$ *implies* $\mathcal{R}_*(P_1) \subseteq \mathcal{R}_*(P_2)$
   $P_1 \subseteq P_2$ *implies* $\mathcal{R}^*(P_1) \subseteq \mathcal{R}^*(P_2)$
2. $\mathcal{R}_*(P_1) \cap \mathcal{R}_*(P_2) = \mathcal{R}_*(P_1 \cap P_2)$
   $\mathcal{R}^*(P_1) \cap \mathcal{R}^*(P_2) = \mathcal{R}^*(P_1 \cap P_2)$
3. $\mathcal{R}_*(P_1 \cup P_2) = \mathcal{R}_*(P_1) \cup \mathcal{R}_*(P_2)$
   $\mathcal{R}^*(P_1 \cup P_2) = \mathcal{R}^*(P_1) \cup \mathcal{R}^*(P_2)$

Proof.

1.

$$\begin{aligned} P_1 \subseteq P_2 &\implies \mu_{P_1}(x) \leq \mu_{P_2}(x),\ \eta_{P_1}(x) \leq \eta_{P_2}(x),\ \nu_{P_1}(x) \geq \nu_{P_2}(x) \\ &\implies \mu_{\mathcal{R}_*}(P_1) \leq \mu_{\mathcal{R}_*}(P_2),\ \eta_{\mathcal{R}_*}(P_1) \leq \eta_{\mathcal{R}_*}(P_2),\ \nu_{\mathcal{R}_*}(P_1) \geq \nu_{\mathcal{R}_*}(P_2) \\ &\implies \mathcal{R}_*(P_1) \subseteq \mathcal{R}_*(P_2). \end{aligned}$$

We can prove that $\mathcal{R}^*(P_1) \subseteq \mathcal{R}^*(P_2)$ with a similar reasoning.

2.

$$\begin{aligned} \mu_{\mathcal{R}_*}(P_1 \cap P_2) &= \widetilde{\bigwedge_{y \in V}} [\nu_{\mathcal{R}}(x,y) \widetilde{\vee} \mu_{P_1 \cap P_2}(y)], x \in U \\ &= \widetilde{\bigwedge_{y \in V}} [\nu_{\mathcal{R}}(x,y) \widetilde{\vee} \{\mu_{P_1}(y) \widetilde{\wedge} \mu_{P_2}(y)\}, x \in U \\ &= \widetilde{\bigwedge_{y \in V}} [\{\nu_{\mathcal{R}}(x,y) \widetilde{\vee} \mu_{P_1}(y)\} \widetilde{\wedge} \{\nu_{\mathcal{R}}(x,y) \widetilde{\vee} \mu_{P_2}(y)\}], x \in U \\ &= \widetilde{\bigwedge_{y \in V}} [\nu_{\mathcal{R}}(x,y) \widetilde{\vee} \mu_{P_1}(y)] \widetilde{\wedge} \widetilde{\bigwedge_{y \in V}} [\nu_{\mathcal{R}}(x,y) \widetilde{\vee} \mu_{P_2}(y)], x \in U \\ &= \mu_{\mathcal{R}_*}(P_1) \wedge \mu_{\mathcal{R}_*}(P_2). \end{aligned}$$

Similarly, $\eta_{\mathcal{R}_*}(P_1 \cap P_2) = \eta_{\mathcal{R}_*}(P_1) \wedge \eta_{\mathcal{R}_*}(P_2)$ and $\nu_{\mathcal{R}_*}(P_1 \cap P_2) = \nu_{\mathcal{R}_*}(P_1) \vee \nu_{\mathcal{R}_*}(P_2)$.
Hence, $\mathcal{R}_*(P_1 \cap P_2) = \mathcal{R}_*(P_1) \cap \mathcal{R}_*(P_2)$.
In a similar way, one can prove $\mathcal{R}^*(P_1 \cap P_2) = \mathcal{R}^*(P_1) \cap \mathcal{R}^*(P_2)$.

3.

$$\begin{aligned} \mu_{\mathcal{R}_*}(P_1 \cup P_2) &= \widetilde{\bigvee_{y \in V}} [\mu_{\mathcal{R}}(x,y) \widetilde{\wedge} \mu_{P_1 \cup P_2}(y)], x \in U \\ &= \widetilde{\bigvee_{y \in V}} [\mu_{\mathcal{R}}(x,y) \widetilde{\wedge} \{\mu_{P_1}(y) \widetilde{\vee} \mu_{P_2}(y)\}, x \in U \\ &= \widetilde{\bigvee_{y \in V}} [\{\nu_{\mathcal{R}}(x,y) \widetilde{\wedge} \mu_{P_1}(y)\} \widetilde{\vee} \{\nu_{\mathcal{R}}(x,y) \widetilde{\wedge} \mu_{P_2}(y)\}], x \in U \\ &= \widetilde{\bigvee_{y \in V}} [\nu_{\mathcal{R}}(x,y) \widetilde{\wedge} \mu_{P_1}(y)] \widetilde{\vee} \widetilde{\bigvee_{y \in V}} [\nu_{\mathcal{R}}(x,y) \widetilde{\wedge} \mu_{P_2}(y)], x \in U \\ &= \mu_{\mathcal{R}_*}(P_1) \vee \mu_{\mathcal{R}_*}(P_2). \end{aligned}$$

Similarly, $\eta_{\mathcal{R}_*}(P_1 \cup P_2) = \eta_{\mathcal{R}_*}(P_1) \vee \eta_{\mathcal{R}_*}(P_2)$ and $\nu_{\mathcal{R}_*}(P_1 \cup P_2) = \nu_{\mathcal{R}_*}(P_1) \wedge \nu_{\mathcal{R}_*}(P_2)$.
Hence, $\mathcal{R}_*(P_1 \cup P_2) = \mathcal{R}_*(P_1) \cup \mathcal{R}_*(P_2)$.
In a similar way, one can prove $\mathcal{R}^*(P_1 \cup P_2) = \mathcal{R}^*(P_1) \cup \mathcal{R}^*(P_2)$.
□

Theorem 2. *Let U and V be arbitrary non-empty sets. Suppose that $\mathcal{R}_1, \mathcal{R}_2 \in PHFR(U \times V)$ are two PHF relations. If $\mathcal{R}_1 \subseteq \mathcal{R}_2$, for any $P \in PHFS(V)$, we have the following properties:*

1. $\mathcal{R}_{1_*}(P) \supseteq \mathcal{R}_{2_*}(P) \ \ \forall P \in PHF(V)$
2. $\mathcal{R}_1^*(P) \subseteq \mathcal{R}_2^*(P) \ \ \forall P \in PHF(V)$.

Proof. These properties can be derived directly from Definitions 9 and 11. □

4. Multi-Granulation Picture Hesitant Fuzzy Rough Sets

Now, we introduce two extended rough set models, namely, optimistic and pessimistic picture hesitant fuzzy rough sets. They are studied in separate subsections. Both models are multi-granular because they are defined in terms of various picture hesitant fuzzy binary relations, one for every expert providing assessments on the comparisons of the elements.

4.1. Optimistic Multi-Granulation Picture Hesitant Fuzzy Rough Sets

Definition 12. *Let U and V be arbitrary non-empty sets. We fix $\mathcal{R}_1, \ldots, \mathcal{R}_m \in PFR(U \times V)$, a list of PHF binary relations from U to V. When $P \in PHFS(V)$, its optimistic lower/upper approximations derived from $\{(U, V, \mathcal{R}_i)\}_{i=1}^m$ are:*

$$[\sum_{i=1}^{m} \mathcal{R}_i]_*^o(P) = \{\langle x, \mu_{\sum_{i=1}^m \mathcal{R}_{i_*}^o(P)}(x), \eta_{\sum_{i=1}^m \mathcal{R}_{i_*}^o(P)}(x), \nu_{\sum_{i=1}^m \mathcal{R}_{i_*}^o(P)}(x)\rangle | x \in U\},$$
$$[\sum_{i=1}^{m} \mathcal{R}_i]^{*o}(P) = \{\langle x, \mu_{\sum_{i=1}^m \mathcal{R}_i^{*o}(P)}(x), \eta_{\sum_{i=1}^m \mathcal{R}_i^{*o}(P)}(x), \nu_{\sum_{i=1}^m \mathcal{R}_i^{*o}(P)}(x)\rangle | x \in U\}.$$

In this expression we use

$$\mu_{\sum_{i=1}^m \mathcal{R}_{i_*}^o(P)}(x) = \widetilde{\bigvee}_{i=1}^m \{\widetilde{\bigwedge}_{y \in V}[\nu_{\mathcal{R}_i}(x,y) \tilde{\vee} \mu_P(y)]\}$$
$$\eta_{\sum_{i=1}^m \mathcal{R}_{i_*}^o(P)}(x) = \widetilde{\bigvee}_{i=1}^m \{\widetilde{\bigwedge}_{y \in V}[\eta_{\mathcal{R}_i}(x,y) \tilde{\vee} \eta_P(y)]\}$$
$$\nu_{\sum_{i=1}^m \mathcal{R}_{i_*}^o(P)}(x) = \widetilde{\bigwedge}_{i=1}^m \{\widetilde{\bigvee}_{y \in V}[\mu_{\mathcal{R}_i}(x,y) \tilde{\wedge} \nu_P(y)]\}$$
$$\mu_{\sum_{i=1}^m \mathcal{R}_i^{*o}(P)}(x) = \widetilde{\bigwedge}_{i=1}^m \{\widetilde{\bigvee}_{y \in V}[\mu_{\mathcal{R}_i}(x,y) \tilde{\wedge} \mu_P(y)]\}$$
$$\eta_{\sum_{i=1}^m \mathcal{R}_i^{*o}(P)}(x) = \widetilde{\bigwedge}_{i=1}^m \{\widetilde{\bigvee}_{y \in V}[\eta_{\mathcal{R}_i}(x,y) \tilde{\wedge} \eta_P(y)]\}$$
$$\nu_{\sum_{i=1}^m \mathcal{R}_i^{*o}(P)}(x) = \widetilde{\bigvee}_{i=1}^m \{\widetilde{\bigwedge}_{y \in V}[\nu_{\mathcal{R}_i}(x,y) \tilde{\vee} \nu_P(y)]\}.$$

We call the pair $([\sum_{i=1}^m \mathcal{R}_i]_*{}^o(P), [\sum_{i=1}^m \mathcal{R}_i]^{*o}(P))$ an optimistic PHFMGRS over two universes of P with respect to $\{(U, V, \mathcal{R}_i)\}_{i=1}^m$.

If $[\sum_{i=1}^m \mathcal{R}_i]_*{}^o(P) = [\sum_{i=1}^m \mathcal{R}_i]^{*o}(P)$, we say that P is definable in the optimistic multi-granulation picture hesitant fuzzy rough set model $\{(U, V, \mathcal{R}_i)\}_{i=1}^m$.

Theorem 3. *Let U and V be arbitrary non-empty sets. Let $\mathcal{R}_1, \ldots, \mathcal{R}_m \in PFR(U \times V)$ be PHF binary relations from U to V. For every $P \in PHFS(V)$, the optimistic PHFMGRS over two universes satisfies:*

1. $[\sum_{i=1}^{m}\mathcal{R}_i]_*^o(P)=\bigcup_{i=1}^{m}\mathcal{R}_{i_*}(P)$
2. $[\sum_{i=1}^{m}\mathcal{R}_i]^{*o}(P)=\bigcap_{i=1}^{m}\mathcal{R}_i{}^*(P)$.

Proof.

1.

$$\widetilde{\bigwedge}_{i=1}^{m}\mu_{\mathcal{R}_{i*}(P)}(x)=\widetilde{\bigvee}_{i=1}^{m}\{\widetilde{\bigwedge}_{y\in V}[\nu_{\mathcal{R}_i}(x,y)\tilde{\vee}\mu_P(y)]\}$$
$$=\mu_{\sum_{i=1}^{m}\mathcal{R}_{i_*}}{}^{o}(P).$$

Additionally,

$$\widetilde{\bigvee}_{i=1}^{m}\eta_{\mathcal{R}_{i*}(P)}(x)=\widetilde{\bigvee}_{i=1}^{m}\{\widetilde{\bigwedge}_{y\in V}[\eta_{\mathcal{R}_i}(x,y)\tilde{\vee}\eta_P(y)]\}$$
$$=\eta_{\sum_{i=1}^{m}\mathcal{R}_{i_*}}{}^{o}(P)$$

and

$$\widetilde{\bigwedge}_{i=1}^{m}\nu_{\mathcal{R}_{i*}(P)}(x)=\widetilde{\bigwedge}_{i=1}^{m}\{\widetilde{\bigvee}_{y\in V}[\mu_{\mathcal{R}_i}(x,y)\tilde{\wedge}\nu_P(y)]\}$$
$$=\nu_{\sum_{i=1}^{m}\mathcal{R}_{i_*}}{}^{o}(P)$$

$\therefore[\sum_{i=1}^{m}\mathcal{R}_i]_*^o(P)=\bigcup_{i=1}^{m}\mathcal{R}_{i_*}(P)$.

2.

$$\widetilde{\bigwedge}_{i=1}^{m}\mu_{\mathcal{R}_i{}^*(P)}(x)=\widetilde{\bigwedge}_{i=1}^{m}\{\widetilde{\bigvee}_{y\in V}[\mu_{\mathcal{R}_i}(x,y)\tilde{\wedge}\mu_P(y)]\}$$
$$=\mu_{\sum_{i=1}^{m}\mathcal{R}_i{}^*}{}^{o}(P).$$

Additionally,

$$\widetilde{\bigwedge}_{i=1}^{m}\eta_{\mathcal{R}_i{}^*(P)}(x)=\widetilde{\bigwedge}_{i=1}^{m}\{\widetilde{\bigvee}_{y\in V}[\eta_{\mathcal{R}_i}(x,y)\tilde{\wedge}\eta_P(y)]\}$$
$$=\eta_{\sum_{i=1}^{m}\mathcal{R}_i{}^*}{}^{o}(P)$$

and

$$\widetilde{\bigvee}_{i=1}^{m}\nu_{\mathcal{R}_i{}^*(P)}(x)=\widetilde{\bigvee}_{i=1}^{m}\{\widetilde{\bigwedge}_{y\in V}[\nu_{\mathcal{R}_i}(x,y)\tilde{\vee}\nu_P(y)]\}$$
$$=\nu_{\sum_{i=1}^{m}\mathcal{R}_i{}^*}{}^{o}(P)$$

$\therefore[\sum_{i=1}^{m}\mathcal{R}_i]^{*o}(P)=\bigcap_{i=1}^{m}\mathcal{R}_i{}^*(P)$.

□

Theorem 4. *Let U and V be arbitrary non-empty sets. Let $\mathcal{R}_1,\ldots,\mathcal{R}_m\in PFR(U\times V)$ be PHF binary relations from U to V. When $P,P_1,P_2\in PHFS(V)$, the optimistic PHFMGRS over two universes satisfies:*

1. $[\sum_{i=1}^{m}\mathcal{R}_i]_*^O(P)\subseteq\mathbb{P}$
   $[\sum_{i=1}^{m}\mathcal{R}_i]^{*O}(P)\supseteq\mathbb{P}$;
2. $P_1\subseteq P_2\implies[\sum_{i=1}^{m}\mathcal{R}_i]_*{}^o(P_1)\subseteq[\sum_{i=1}^{m}\mathcal{R}_i]_*{}^o(P_2)$
   $P_1\subseteq P_2\implies[\sum_{i=1}^{m}\mathcal{R}_i]^{*o}(P_1)\subseteq[\sum_{i=1}^{m}\mathcal{R}_i]^{*o}(P_2)$;
3. $[\sum_{i=1}^{m}\mathcal{R}_i]_*{}^o(P_1\cap P_2)=[\sum_{i=1}^{m}\mathcal{R}_i]_*{}^o(P_1)\cap[\sum_{i=1}^{m}\mathcal{R}_i]_*{}^o(P_1)$
   $[\sum_{i=1}^{m}\mathcal{R}_i]^{*o}(P_1\cap P_2)\subseteq[\sum_{i=1}^{m}\mathcal{R}_i]^{*o}(P_1)\cap[\sum_{i=1}^{m}\mathcal{R}_i]^{*o}(P_2)$;

4. $[\sum_{i=1}^{m} \mathcal{R}_i]_*{}^o(P_1 \cup P_2) = [\sum_{i=1}^{m} \mathcal{R}_i]_*{}^o(P_1) \cup [\sum_{i=1}^{m} \mathcal{R}_i]_*{}^o(P_1)$
$[\sum_{i=1}^{m} \mathcal{R}_i]^{*o}(P_1 \cup P_2) \supseteq [\sum_{i=1}^{m} \mathcal{R}_i]^{*o}(P_1) \cup [\sum_{i=1}^{m} \mathcal{R}_i]^{*o}(P_2)$.

Proof. Statements 1 and 2 follow directly from the definition. To prove claim 3,

$$\begin{aligned}
\mu_{\sum_{i=1}^{m} \mathcal{R}_{i_*}^{o}(P_1 \cap P_2)}(x) &= \widetilde{\bigvee}_{i=1}^{m}\{\widetilde{\bigwedge}_{y\in V}[\nu_{\mathcal{R}_i}(x,y)\widetilde{\vee}\mu_{P_1\cap P_2}(y)]\} \\
&= \widetilde{\bigvee}_{i=1}^{m}\{\widetilde{\bigwedge}_{y\in V}[\nu_{\mathcal{R}_i}(x,y)\widetilde{\vee}[\mu_{P_1}(y)\wedge\mu_{P_2}(y)]]\} \\
&= \widetilde{\bigvee}_{i=1}^{m}\{\widetilde{\bigwedge}_{y\in V}[\nu_{\mathcal{R}_i}(x,y)\widetilde{\vee}\mu_{P_1}(y)]\wedge[\nu_{\mathcal{R}_i}(x,y)\widetilde{\vee}\mu_{P_2}(y)]\} \\
&= \widetilde{\bigvee}_{i=1}^{m}\{\widetilde{\bigwedge}_{y\in V}[\nu_{\mathcal{R}_i}(x,y)\widetilde{\vee}\mu_{P_1}(y)]\}\wedge\widetilde{\bigvee}_{i=1}^{m}\{\widetilde{\bigwedge}_{y\in V}[\nu_{\mathcal{R}_i}(x,y)\widetilde{\vee}\mu_{P_2}(y)]\} \\
&= \mu_{\sum_{i=1}^{m} \mathcal{R}_{i_*}^{o}(P_1)}(x)\wedge\mu_{\sum_{i=1}^{m} \mathcal{R}_{i_*}^{o}(P_2)}(x).
\end{aligned}$$

$$\begin{aligned}
\text{Similarly, } &\eta_{\sum_{i=1}^{m} \mathcal{R}_{i_*}^{o}(P_1 \cap P_2)}(x) = \eta_{\sum_{i=1}^{m} \mathcal{R}_{i_*}^{o}(P_1)}(x)\wedge\eta_{\sum_{i=1}^{m} \mathcal{R}_{i_*}^{o}(P_2)}(x) \text{ and} \\
&\nu_{\sum_{i=1}^{m} \mathcal{R}_{i_*}^{o}(P_1 \cap P_2)}(x) = \nu_{\sum_{i=1}^{m} \mathcal{R}_{i_*}^{o}(P_1)}(x)\wedge\nu_{\sum_{i=1}^{m} \mathcal{R}_{i_*}^{o}(P_2)}(x) \\
&\therefore [\sum_{i=1}^{m}\mathcal{R}_i]_*^{\ o}(P_1\cap P_2) = [\sum_{i=1}^{m}\mathcal{R}_i]_*^{\ o}(P_1)\cap[\sum_{i=1}^{m}\mathcal{R}_i]_*^{\ o}(P_1).
\end{aligned}$$

$$\begin{aligned}
\mu_{\sum_{i=1}^{m} \mathcal{R}_i^{*o}(P_1 \cap P_2)}(x) &= \widetilde{\bigwedge}_{i=1}^{m}\{\widetilde{\bigvee}_{y\in V}[\mu_{\mathcal{R}_i}(x,y)\widetilde{\wedge}\mu_{P_1\cap P_2}(y)]\} \\
&= \widetilde{\bigwedge}_{i=1}^{m}\{\widetilde{\bigvee}_{y\in V}[\mu_{\mathcal{R}_i}(x,y)\widetilde{\wedge}[\mu_{P_1}(y)\wedge\mu_{P_2}(y)]]\} \\
&\le \widetilde{\bigwedge}_{i=1}^{m}\{\widetilde{\bigvee}_{y\in V}[\mu_{\mathcal{R}_i}(x,y)\widetilde{\wedge}\mu_{P_1}(y)]\wedge[\mu_{\mathcal{R}_i}(x,y)\widetilde{\wedge}\mu_{P_2}(y)]\} \\
&\le \widetilde{\bigwedge}_{i=1}^{m}\{\widetilde{\bigvee}_{y\in V}[\mu_{\mathcal{R}_i}(x,y)\widetilde{\wedge}\mu_{P_1}(y)]\}\wedge\widetilde{\bigwedge}_{i=1}^{m}\{\widetilde{\bigvee}_{y\in V}[\mu_{\mathcal{R}_i}(x,y)\widetilde{\wedge}\mu_{P_2}(y)]\} \\
&\le \mu_{\sum_{i=1}^{m} \mathcal{R}_i^{*o}(P_1)}(x)\wedge\mu_{\sum_{i=1}^{m} \mathcal{R}_i^{*o}(P_2)}(x).
\end{aligned}$$

$$\begin{aligned}
\text{Similarly, } &\eta_{\sum_{i=1}^{m} \mathcal{R}_i^{*o}(P_1 \cap P_2)}(x) \le \eta_{\sum_{i=1}^{m} \mathcal{R}_i^{*o}(P_1)}(x)\wedge\eta_{\sum_{i=1}^{m} \mathcal{R}_i^{*o}(P_2)}(x) \text{ and} \\
&\nu_{\sum_{i=1}^{m} \mathcal{R}_i^{*o}(P_1 \cap P_2)}(x) \ge \nu_{\sum_{i=1}^{m} \mathcal{R}_i^{*o}(P_1)}(x)\wedge\nu_{\sum_{i=1}^{m} \mathcal{R}_i^{*o}(P_2)}(x) \\
&\therefore [\sum_{i=1}^{m}\mathcal{R}_i]^{*\ o}(P_1\cap P_2) \subseteq [\sum_{i=1}^{m}\mathcal{R}_i]^{*\ o}(P_1)\cap[\sum_{i=1}^{m}\mathcal{R}_i]^{*\ o}(P_2).
\end{aligned}$$

The proof of claim 4 is similar to that of the result in claim 3. □

4.2. Pessimistic Multi-Granulation Picture Hesitant Fuzzy Rough Sets

Definition 13. *Let U and V denote two non-empty sets of alternatives. We fix $\mathcal{R}_1, \ldots, \mathcal{R}_m \in PFR(U \times V)$, a list of PHF binary relations from U to V. When $P \in PHFS(V)$, its pessimistic lower/upper approximations derived from $\{(U, V, \mathcal{R}_i)\}_{i=1}^{m}$ are:*

$$[\sum_{i=1}^{m}\mathcal{R}_i]_*^{p}(P) = \{\langle x, \mu_{\sum_{i=1}^{m}\mathcal{R}_{i_*}^{p}(P)}(x), \eta_{\sum_{i=1}^{m}\mathcal{R}_{i_*}^{p}(P)}(x), \nu_{\sum_{i=1}^{m}\mathcal{R}_{i_*}^{p}(P)}(x)\rangle | x \in U\}$$

$$[\sum_{i=1}^{m}\mathcal{R}_i]^{*p}(P) = \{\langle x, \mu_{\sum_{i=1}^{m}\mathcal{R}_i^{*P}(P)}(x), \eta_{\sum_{i=1}^{m}\mathcal{R}_i^{*p}(P)}(x), \nu_{\sum_{i=1}^{m}\mathcal{R}_i^{*p}(P)}(x)\rangle | x \in U\},$$

where

$$\mu_{\sum_{i=1}^m \mathcal{R}_{i_*}^p}(x) = \widetilde{\bigwedge}_{i=1}^m \{\widetilde{\bigwedge}_{y\in V}[\nu_{\mathcal{R}_i}(x,y)\widetilde{\vee}\mu_P(y)]\}$$
$$\eta_{\sum_{i=1}^m \mathcal{R}_{i_*}^p}(x) = \widetilde{\bigwedge}_{i=1}^m \{\widetilde{\bigwedge}_{y\in V}[\eta_{\mathcal{R}_i}(x,y)\widetilde{\wedge}\eta_P(y)]\}$$
$$\nu_{\sum_{i=1}^m \mathcal{R}_{i_*}^p}(x) = \widetilde{\bigvee}_{i=1}^m \{\widetilde{\bigvee}_{y\in V}[\mu_{\mathcal{R}_i}(x,y)\widetilde{\wedge}\nu_P(y)]\}$$
$$\mu_{\sum_{i=1}^m \mathcal{R}_i^{*p}}(x) = \widetilde{\bigvee}_{i=1}^m \{\widetilde{\bigvee}_{y\in V}[\mu_{\mathcal{R}_i}(x,y)\widetilde{\wedge}\mu_P(y)]\}$$
$$\eta_{\sum_{i=1}^m \mathcal{R}_i^{*p}}(x) = \widetilde{\bigvee}_{i=1}^m \{\widetilde{\bigvee}_{y\in V}[\nu_{\mathcal{R}_i}(x,y)\widetilde{\vee}\nu_P(y)]\}$$
$$\nu_{\sum_{i=1}^m \mathcal{R}_i^{*p}}(x) = \widetilde{\bigwedge}_{i=1}^m \{\widetilde{\bigwedge}_{y\in V}[\nu_{\mathcal{R}_i}(x,y)\widetilde{\vee}\nu_P(y)]\}.$$

We call the pair $([\sum_{i=1}^m \mathcal{R}_i]_*{}^p(P), [\sum_{i=1}^m \mathcal{R}_i]^{*p}(P))$ *a pessimistic* $PHFMGRS$ *over two universes of* P *with respect to* $(U, V, \mathcal{R}_i)$.

If $[\sum_{i=1}^m \mathcal{R}_i[_*{}^P(P) = [\sum_{i=1}^m \mathcal{R}_i]^{*P}(P)$*, then* P *is definable in the pessimistic multi-granulation picture hesitant fuzzy rough set model* $\{(U, V, \mathcal{R}_i)\}_{i=1}^m$.

Theorem 5. *Let* U *and* V *be arbitrary non-empty sets. Suppose that* $\mathcal{R}_1, \ldots, \mathcal{R}_m \in PFR(U \times V)$ *are two PHF binary relations from the set* U *to the set* V*. Then the pessimistic* $PHFMGRS$ *of* $P \in PHFS(V)$ *over two universes satisfies:*

1. $[\sum_{i=1}^m \mathcal{R}_i]_*{}^p(P) = \bigcap_{i=1}^m \mathcal{R}_{i_*}(P)$
2. $[\sum_{i=1}^m \mathcal{R}_i]^{*p}(P) = \bigcup_{i=1}^m \mathcal{R}_i{}^*(P)$.

Proof.

1.

$$\widetilde{\bigwedge}_{i=1}^m \mu_{\mathcal{R}_{i*}(P)}(x) = \widetilde{\bigwedge}_{i=1}^m \{\widetilde{\bigwedge}_{y\in V}[\nu_{\mathcal{R}_i}(x,y)\widetilde{\vee}\mu_P(y)]\}$$
$$= \mu_{\sum_{i=1}^m \mathcal{R}_{i_*}}{}^p(P).$$

Additionally,

$$\widetilde{\bigwedge}_{i=1}^m \eta_{\mathcal{R}_{i*}(P)}(x) = \widetilde{\bigwedge}_{i=1}^m \{\widetilde{\bigwedge}_{y\in V}[\eta_{\mathcal{R}_i}(x,y)\widetilde{\vee}\eta_P(y)]\}$$
$$= \eta_{\sum_{i=1}^m \mathcal{R}_{i_*}}{}^p(P)$$

and

$$\widetilde{\bigvee}_{i=1}^m \nu_{\mathcal{R}_{i*}(P)}(x) = \widetilde{\bigvee}_{i=1}^m \{\widetilde{\bigvee}_{y\in V}[\mu_{\mathcal{R}_i}(x,y)\widetilde{\wedge}\nu_P(y)]\}$$
$$= \nu_{\sum_{i=1}^m \mathcal{R}_{i_*}}{}^p(P)$$

$\therefore [\sum_{i=1}^m \mathcal{R}_i]_*{}^p(P) = \bigcap_{i=1}^m \mathcal{R}_{i_*}(P)$.

2.

$$\widetilde{\bigvee}_{i=1}^m \mu_{\mathcal{R}_i{}^*(P)}(x) = \widetilde{\bigvee}_{i=1}^m \{\widetilde{\bigvee}_{y\in V}[\mu_{\mathcal{R}_i}(x,y)\widetilde{\wedge}\mu_P(y)]\}$$
$$= \mu_{\sum_{i=1}^m \mathcal{R}_i{}^*}{}^p(P)$$

Additionally,

$$\widetilde{\bigvee}_{i=1}^{m}\eta_{\mathcal{R}_i{}^*(P)}(x) = \widetilde{\bigvee}_{i=1}^{m}\{\widetilde{\bigvee}_{y\in V}[\eta_{\mathcal{R}_i}(x,y)\widetilde{\wedge}\eta_P(y)]\}$$
$$= \eta_{\sum_{i=1}^{m}\mathcal{R}_i{}^{*p}}(P)$$

and

$$\widetilde{\bigwedge}_{i=1}^{m}\nu_{\mathcal{R}_i{}^*(P)}(x) = \widetilde{\bigwedge}_{i=1}^{m}\{\widetilde{\bigwedge}_{y\in V}[\nu_{\mathcal{R}_i}(x,y)\widetilde{\vee}\nu_P(y)]\}$$
$$= \nu_{\sum_{i=1}^{m}\mathcal{R}_i{}^{*p}}(P)$$

$\therefore [\sum_{i=1}^{m}\mathcal{R}_i]^{*p}(P) = \bigcap_{i=1}^{m}\mathcal{R}_i{}^*(P)$.
□

Theorem 6. *We fix U and V, two non-empty sets of alternatives. Suppose that $\mathcal{R}_1,\ldots,\mathcal{R}_m \in PFR(U\times V)$ are PHF binary relations from the set U to the set V. Then for each PHFS $P, P_{H_1}, P_{H_2} \in PHF(V)$, the pessimistic $PHFMGRS$ over two universes satisfies:*

1. $[\sum_{i=1}^{m}\mathcal{R}_i]_*^p(P) \subseteq \mathbb{P}$
   $[\sum_{i=1}^{m}\mathcal{R}_i]^{*p}(P) \supseteq \mathbb{P}$;
2. $P_1 \subseteq P_2 \implies [\sum_{i=1}^{m}\mathcal{R}_i]_*{}^p(P_1) \subseteq [\sum_{i=1}^{m}\mathcal{R}_i]_*{}^p(P_2)$
   $P_1 \subseteq P_2 \implies [\sum_{i=1}^{m}\mathcal{R}_i]^{*p}(P_1) \subseteq [\sum_{i=1}^{m}\mathcal{R}_i]^{*p}(P_2)$;
3. $[\sum_{i=1}^{m}\mathcal{R}_i]_*{}^p(P_1\cap P_2) = [\sum_{i=1}^{m}\mathcal{R}_i]_*{}^p(P_1) \cap [\sum_{i=1}^{m}\mathcal{R}_i]_*{}^p(P_2)$
   $[\sum_{i=1}^{m}\mathcal{R}_i]^{*p}(P_1\cap P_2) \subseteq [\sum_{i=1}^{m}\mathcal{R}_i]^{*p}(P_1) \cap [\sum_{i=1}^{m}\mathcal{R}_i]^{*p}(P_2)$;
4. $[\sum_{i=1}^{m}\mathcal{R}_i]_*{}^p(P_1\cup P_2) = [\sum_{i=1}^{m}\mathcal{R}_i]_*{}^p(P_1) \cup [\sum_{i=1}^{m}\mathcal{R}_i]_*{}^p(P_1)$
   $[\sum_{i=1}^{m}\mathcal{R}_i]^{*p}(P_1\cup P_2) \supseteq [\sum_{i=1}^{m}\mathcal{R}_i]^{*p}(P_1) \cup [\sum_{i=1}^{m}\mathcal{R}_i]^{*p}(P_2)$.

Proof. Claims 1 and 2 follow from the definition trivially.

3.

$$\mu_{\sum_{i=1}^{m}\mathcal{R}_{i_*}^{p}(P_1\cap P_2)}(x) = \widetilde{\bigwedge}_{i=1}^{m}\{\widetilde{\bigwedge}_{y\in V}[\nu_{\mathcal{R}_i}(x,y)\widetilde{\vee}\mu_{P_1\cap P_2}(y)]\}$$
$$= \widetilde{\bigwedge}_{i=1}^{m}\{\widetilde{\bigwedge}_{y\in V}[\nu_{\mathcal{R}_i}(x,y)\widetilde{\vee}[\mu_{P_1}(y)\wedge\mu_{P_2}(y)]]\}$$
$$= \widetilde{\bigwedge}_{i=1}^{m}\{\widetilde{\bigwedge}_{y\in V}[\nu_{\mathcal{R}_i}(x,y)\widetilde{\vee}\mu_{P_1}(y)]\wedge[\nu_{\mathcal{R}_i}(x,y)\widetilde{\vee}\mu_{P_2}(y)]\}$$

$$= \widetilde{\bigwedge}_{i=1}^{m}\{\widetilde{\bigwedge}_{y\in V}[\nu_{\mathcal{R}_i}(x,y)\widetilde{\vee}\mu_{P_1}(y)]\} \wedge \widetilde{\bigwedge}_{i=1}^{m}\{\widetilde{\bigwedge}_{y\in V}[\nu_{\mathcal{R}_i}(x,y)\widetilde{\vee}\mu_{P_2}(y)]\}$$
$$= \mu_{\sum_{i=1}^{m}\mathcal{R}_{i*}^{p}(P_1)}(x) \wedge \mu_{\sum_{i=1}^{m}\mathcal{R}_{i*}^{p}(P_2)}(x).$$

Similarly, $\eta_{\sum_{i=1}^{m}\mathcal{R}_{i*}^{p}(P_1\cap P_2)}(x) = \eta_{\sum_{i=1}^{m}\mathcal{R}_{i*}^{p}(P_1)}(x) \wedge \eta_{\sum_{i=1}^{m}\mathcal{R}_{i*}^{p}(P_2)}(x)$ and

$$\nu_{\sum_{i=1}^{m}\mathcal{R}_{i*}^{p}(P_1\cap P_2)}(x) = \nu_{\sum_{i=1}^{m}\mathcal{R}_{i*}^{p}(P_1)}(x) \wedge \nu_{\sum_{i=1}^{m}\mathcal{R}_{i*}^{p}(P_2)}(x)$$
$$\therefore [\sum_{i=1}^{m}\mathcal{R}_i]_*^{\ p}(P_1\cap P_2) = [\sum_{i=1}^{m}\mathcal{R}_i]_*^{\ p}(P_1) \cap [\sum_{i=1}^{m}\mathcal{R}_i]_*^{\ p}(P_2)$$
$$\mu_{\sum_{i=1}^{m}\mathcal{R}_i^{*p}(P_1\cap P_2)}(x) = \widetilde{\bigvee}_{i=1}^{m}\{\widetilde{\bigvee}_{y\in V}[\mu_{\mathcal{R}_i}(x,y)\widetilde{\wedge}\mu_{P_1\cap P_2}(y)]\}$$
$$= \widetilde{\bigvee}_{i=1}^{m}\{\widetilde{\bigvee}_{y\in V}[\mu_{\mathcal{R}_i}(x,y)\widetilde{\wedge}[\mu_{P_1}(y)\wedge\mu_{P_2}(y)]]\}$$
$$\leq \widetilde{\bigvee}_{i=1}^{m}\{\widetilde{\bigvee}_{y\in V}[\mu_{\mathcal{R}_i}(x,y)\widetilde{\wedge}\mu_{P_1}(y)] \wedge [\mu_{\mathcal{R}_i}(x,y)\widetilde{\wedge}\mu_{P_2}(y)]\}$$
$$\leq \widetilde{\bigvee}_{i=1}^{m}\{\widetilde{\bigvee}_{y\in V}[\mu_{\mathcal{R}_i}(x,y)\widetilde{\wedge}\mu_{P_1}(y)]\} \wedge \widetilde{\bigvee}_{i=1}^{m}\{\widetilde{\bigvee}_{y\in V}[\mu_{\mathcal{R}_i}(x,y)\widetilde{\wedge}\mu_{P_2}(y)]\}$$
$$\leq \mu_{\sum_{i=1}^{m}\mathcal{R}_i^{*p}(P_1)}(x) \wedge \mu_{\sum_{i=1}^{m}\mathcal{R}_i^{*p}(P_2)}(x).$$

Similarly, $\eta_{\sum_{i=1}^{m}\mathcal{R}_i^{*p}(P_1\cap P_2)}(x) \leq \eta_{\sum_{i=1}^{m}\mathcal{R}_i^{*p}(P_1)}(x) \wedge \eta_{\sum_{i=1}^{m}\mathcal{R}_i^{*p}(P_2)}(x)$ and

$$\nu_{\sum_{i=1}^{m}\mathcal{R}_i^{*p}(P_1\cap P_2)}(x) \geq \nu_{\sum_{i=1}^{m}\mathcal{R}_i^{*p}(P_1)}(x) \wedge \nu_{\sum_{i=1}^{m}\mathcal{R}_i^{*p}(P_2)}(x)$$
$$\therefore [\sum_{i=1}^{m}\mathcal{R}_i]^{*\ p}(P_1\cap P_2) \subseteq [\sum_{i=1}^{m}\mathcal{R}_i]^{*\ p}(P_1) \cap [\sum_{i=1}^{m}\mathcal{R}_i]^{*\ p}(P_2).$$

4. The argument for claim 3 can be adapted to prove claim 4.

□

5. The Relationship among PHFRS, the Optimistic MGPHFRS, and Pessimistic MGPHFRS

Theorem 7. *We fix U and V, two non-empty sets of alternatives. Suppose that $\mathcal{R}_1, \ldots, \mathcal{R}_m \in PFR(U \times V)$ are PHF binary relations from U to V. When $P \in PHFS(V)$:*

1. $[\sum_{i=1}^{m}\mathcal{R}_i]_*^{\ o}(P) \subseteq \left[\bigcap_{i=1}^{m}\mathcal{R}_i\right]_*(P)$;
2. $[\sum_{i=1}^{m}\mathcal{R}_i]^{*o}(P) \supseteq \left[\bigcap_{i=1}^{m}\mathcal{R}_i\right]^*(P)$;
3. $[\sum_{i=1}^{m}\mathcal{R}_i]_*^{\ p}(P) = \left[\bigcup_{i=1}^{m}\mathcal{R}_i\right]_*(P)$;
4. $[\sum_{i=1}^{m}\mathcal{R}_i]^{*p}(P) = \left[\bigcup_{i=1}^{m}\mathcal{R}_i\right]^*(P)$.

Proof.

1.

$$\begin{aligned}\mu_{[\bigcap_{i=1}^m \mathcal{R}_i]_*(P)}(x) &= \widetilde{\bigwedge}_{y\in V}[\nu_{[\bigcap_{i=1}^m \mathcal{R}_i]}(x,y)\tilde{\triangledown}\mu_P(y)]\\ &= \widetilde{\bigwedge}_{y\in V}[\widetilde{\bigvee}_{i=1}^m \nu_{\mathcal{R}_i}(x,y)\tilde{\triangledown}\mu_P(y)]\\ &\geq \widetilde{\bigvee}_{i=1}^m[\widetilde{\bigwedge}_{y\in V}\nu_{\mathcal{R}_i}(x,y)\tilde{\triangledown}\mu_P(y)]\\ &\geq \mu_{\sum_{i=1}^m \mathcal{R}_{i*}^o(P)}(x).\end{aligned}$$

$$\text{Similarly, } \eta_{[\bigcap_{i=1}^m \mathcal{R}_i]_*(P)}(x) \geq \eta_{\sum_{i=1}^m \mathcal{R}_{i*}^o(P)}(x) \text{ and}$$

$$\nu_{[\bigcap_{i=1}^m \mathcal{R}_i]_*(P)}(x) \geq \nu_{\sum_{i=1}^m \mathcal{R}_{i*}^o(P)}(x)$$

$$\therefore [\sum_{i=1}^m \mathcal{R}_i]_*^o(P) \subseteq [\bigcap_{i=1}^m \mathcal{R}_i]_*(P).$$

2. The proof is similar to the argument for claim 1.
3.

$$\begin{aligned}\mu_{[\bigcup_{i=1}^m \mathcal{R}_i]_*(P)}(x) &= \widetilde{\bigwedge}_{y\in V}[\nu_{[\bigcup_{i=1}^m \mathcal{R}_i]}(x,y)\tilde{\triangledown}\mu_P(y)]\\ &= \widetilde{\bigwedge}_{y\in V}[\widetilde{\bigwedge}_{i=1}^m \nu_{\mathcal{R}_i}(x,y)\tilde{\triangledown}\mu_P(y)]\\ &= \widetilde{\bigwedge}_{i=1}^m[\widetilde{\bigwedge}_{y\in V}\nu_{\mathcal{R}_i}(x,y)\tilde{\triangledown}\mu_P(y)]\\ &= \mu_{\sum_{i=1}^m \mathcal{R}_{i*}^p(P)}(x).\end{aligned}$$

$$\text{Similarly, } \eta_{[\bigcup_{i=1}^m \mathcal{R}_i]_*(P)}(x) = \eta_{\sum_{i=1}^m \mathcal{R}_{i*}^p(P)}(x) \text{ and}$$

$$\nu_{[\bigcup_{i=1}^m \mathcal{R}_i]_*(P)}(x) = \nu_{\sum_{i=1}^m \mathcal{R}_{i*}^p(P)}(x)$$

$$\therefore [\sum_{i=1}^m \mathcal{R}_i]_*^p(P) = [\bigcup_{i=1}^m \mathcal{R}_i]_*(P).$$

4. The argument for claim 3 can be adapted to prove claim 4.

□

Theorem 8. *Let U and V be non-empty universes of discourses that are finite. Suppose that $\mathcal{R}_1,\ldots,\mathcal{R}_m \in PFR(U\times V)$ are PHF binary relations from the set U to the set V. Then for each PHFS $P \in PHF(V)$, one has the following properties:*

1. $[\sum_{i=1}^m \mathcal{R}_i]_*{}^p(P) \subseteq \mathcal{R}_{i*}(P) \subseteq [\sum_{i=1}^m \mathcal{R}_i]_*{}^o(P)$;
2. $[\sum_{i=1}^m \mathcal{R}_i]_*{}^p(P) \supseteq \mathcal{R}_i{}^*(P) \supseteq [\sum_{i=1}^m \mathcal{R}_i]_*{}^o(P)$.

Proof. These properties can be directly obtained from Theorems 3 and 5. □

6. Comparison with Existing Structures

In comparison with existing structures, MGPHFRS have the following advantages. The fuzzy rough set models [37–40] use a crisp number for dealing with memberships in decision making problems. However in group decisions, this feature seems to be inadequate both because of the uncertainty of the information and the complexity of the problem. For example, if there are several experts assigning different membership values to an element in a set, a HF set can become a reasonable alternative and so MGHFRS, as the natural combination of the the MGRS and HF set theories, arise as a useful model [41].

Picture fuzzy sets have positive, neutral, and refusal or negative membership functions. Further, if the decision makers hesitate among various possible values they can appeal to hesitant fuzzy sets. All these beneficial features are present in their hybrid model MGPHFRS combining MGRS theory, HF set theory, and PFS theory as introduced in this paper. Their added advantages will result into a more efficient performance for decision making problems.

From a theoretical point of view, the introduced structure contains fuzzy, intuitionistic fuzzy, hesitant fuzzy, picture fuzzy, and single granulation structures as particular cases.

7. Conclusions and Future Work

Rough sets and picture hesitant fuzzy sets are two mathematical models that deal with uncertainty and imprecision in data analysis from different viewpoints. In this paper, the authors lay the theoretical foundations of a novel model, called picture hesitant fuzzy rough sets (PHFRS), defined from a picture hesitant fuzzy relation. In addition, a new multi-granulation rough set model on two universes, called a multi-granulation picture hesitant fuzzy rough set, is developed. Furthermore, this paper offers two types of multi-granulation picture hesitant fuzzy rough sets, namely, optimistic and pessimistic versions of multi-granulation picture hesitant fuzzy rough sets. In this way we have settled the grounds for further progress of the field of rough set theory and extensions. As this the paper concentrates on the theoretical developments, specific problems are not addressed here and neither are any benchmark problems. As is in the case of similar structures, PHFRS can also be applied in decision making and multi-criteria problems. This will be addressed in future works.

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Abbreviations

The following abbreviations are used in this manuscript:

RS	Rough Set
FS	Fuzzy Set
PHFRS	Picture Hesitant Fuzzy Rough Set
MGPHFRS	Multi-Granulation Picture Hesitant Fuzzy Rough Set
OMGPHFRS	Optimistic Multi-Granulation Picture Hesitant Fuzzy Rough Set
PMGPHFRS	Pessimistic Multi-Granulation Picture Hesitant Fuzzy Rough Set

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