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Some New $q-$Integral Inequalities Using Generalized Quantum Montgomery Identity via Preinvex Functions

Miguel Vivas-Cortez [1,*,†], Artion Kashuri [2,†], Rozana Liko [2,†] and Jorge E. Hernández Hernández [3,†]

1 Escuela de Ciencias Físicas y Matemáticas, Facultad de Ciencias Exactas y Naturales, Pontificia Universidad Católica del Ecuador, Av. 12 de Octubre 1076. Apartado, Quito 17-01-2184, Ecuador
2 Department of Mathematics, Faculty of Technical Science, University Ismail Qemali, L. Pavaresia, Vlora 1001, Vlore, Albania; artionkashuri@gmail.com (A.K.); rozanaliko86@gmail.com (R.L.)
3 Departamento de Técnicas Cuantitativas, Decanato de Ciencias Económicas y Empresariales, Universidad Centroccidental Lisandro Alvarado, Av. 20. esq. Av. Moran, Edf. Los Militares, Piso 2, Ofc.2, Barquisimeto 3001, Venezuela; jorgehernandez@ucla.edu.ve
* Correspondence: mjvivas@puce.edu.ec
† These authors contributed equally to this work.

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Abstract: In this work the authors establish a new generalized version of Montgomery's identity in the setting of quantum calculus. From this result, some new estimates of Ostrowski type inequalities are given using preinvex functions. Given the generality of preinvex functions, particular $q-$integral inequalities are established with appropriate choice of the parametric bifunction. Some new special cases from the main results are obtained and some known results are recaptured as well. At the end, a briefly conclusion is given.

Keywords: Quantum Montgomery identity; ϕ-convex functions; integral inequalities

MSC: 26D10; 52A01; 26D07

1. Introduction

Quantum calculus, or $q-$calculus, has had an important development in recent decades, both in pure mathematics and its applicability, for example in Physics [1]. The convexity of a function has played an important role as a tool in the development of inequalities. Some fields of Mathematics have used this property: harmonic analysis, interpolation theory, and control theory, as can be seen in the works of C.P. Niculescu [2], C. Bennett and R. Sharpley [3], ¸S. Mititelu and S. Trenţă [4], S. Trenţă [5,6].

Furthermore, it is important to note that in recent decades, the evolution of the concept of convexity has been extended and its evolution has been subject of many studies as is shown in the works of Ben-Israel A. and Mond B. [7], Hernández Hernández, J. E. [8,9], Niculescu C.P. [2], Mitrinović D.S. et al. [10], Noor M. et.al. [11,12], Sarikaya M.Z. et. al. [13], Vivas-Cortez M.J. et al. [14], Weir, T.; Mond, B. [15] and others.

Recently, Tariboon et al. in [16], defined q-derivative and q-integral as follows:

Definition 1. *Let* $\Upsilon : [a_1, a_2] \to \mathbb{R}$ *be a continuous function and let* $x \in [a_1, a_2]$ *and* $0 < \mathrm{q} < 1$ *be a constant. Then the* q*-derivative on* $[a_1, a_2]$ *of function* Υ *at* x *is defined as*

$$ {}_{a_1}D_{\mathrm{q}}\Upsilon(x) = \frac{\Upsilon(x) - \Upsilon(qx + (1-\mathrm{q})a_1)}{(1-\mathrm{q})(x-a_1)}, \quad x \neq a_1, \tag{1}$$

We say that Υ is q-differentiable on $[a_1, a_2]$ provided ${}_{a_1}D_q\Upsilon(x)$ exists for all $x \in [a_1, a_2]$.

Definition 2. *Let $\Upsilon : [a_1, a_2] \subset \mathbb{R} \to \mathbb{R}$ be a continuous function. Then q-integral on $[a_1, a_2]$ is defined as*

$$\int_{a_1}^{x} \Upsilon(\nu)\, {}_{a_1}d_q\nu = (1-q)(x-a_1)\sum_{n=0}^{\infty} q^n \Upsilon(q^n x + (1-q^n)a_1), \tag{2}$$

for $x \in [a_1, a_2]$.

Some properties of interest regarding these definitions are the following.

Theorem 1. *([17]) Let $f : J \to \mathbb{R}$ be a $q-$differentiable functions. Then we have*

1. *The sum $(f+g)$ is $q-$differentiable on J with*

$$ {}_aD_q(f(t)+g(t)) = {}_aD_qf(t) + {}_aD_qg(t)$$

2. *For any constant $\alpha \in \mathbb{R}$, the function $(\alpha f$ is $q-$differentiable and*

$$ {}_aD_q(\alpha f(t)) = \alpha\, {}_aD_qf(t)$$

3. *The function (fg) is $q-$differentiable with*

$$\begin{aligned} {}_aD_q(fg)(t) &= f(t)\,{}_aD_qg(t) + g(qt+(1-q)a)\,{}_aD_qf(t) \\ &= g(t)\,{}_aD_qf(t) + f(qt+(1-q)a)\,{}_aD_qg(t) \end{aligned}$$

Lemma 1. *[17]. Let $\alpha \in \mathbb{R}$, then we have*

$$ {}_aD_q(x-a)^{\alpha} = \left(\frac{1-q^{\alpha}}{1-q}\right)(x-a)^{\alpha-1}$$

Theorem 2. *[17] Let $f : J \to \mathbb{R}$ be a continuous function. Then we have*

1. ${}_aD_q \int_a^x f(t)\,{}_ad_qt = f(x)$
2. $\int_c^x {}_aD_qf(t)\,{}_ad_qt = f(x) - f(c)$ *for $c \in (a, x)$*

Theorem 3. *[17] Let $f, g : J \to \mathbb{R}$ be a continuous functions and $\alpha \in \mathbb{R}$. Then, for $x \in J$ we have*

1. $\int_a^x (f(t)+g(t))\,{}_ad_qt = \int_a^x f(t)\,{}_ad_qt + \int_a^x g(t)\,{}_ad_qt$
2. $\int_a^x (\alpha f(t))\,{}_ad_qt = \alpha \int_a^x f(t)\,{}_ad_qt$
3. $\int_c^x f(t)\,{}_aD_qg(t)\,{}_ad_qt = f(t)g(t)|_c^x - \int_c^x g(qt+(1-q)a)\,{}_aD_qf(t)\,{}_ad_qt$

For more details on q-calculus and certain q-analogues of classical inequalities, see [16,18–29].

The following famous identity in [10], is called Montgomery identity:

$$f(x) = \frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(\nu)\,d\nu + \frac{1}{a_2-a_1}\int_{a_1}^{x}(\nu-a_1)f'(\nu)\,d\nu + \frac{1}{a_2-a_1}\int_{x}^{a_2}(\nu-a_2)f'(\nu)\,d\nu, \tag{3}$$

where $f(x)$ is continuous function on $[a_1, a_2]$ with a continuous first derivative in (a_1, a_2). By changing the variable, the Montgomery identity (3) could be expressed as follows:

$$f(x) - \frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(\nu)\,d\nu = (a_2-a_1)\int_0^1 K(\nu)\, f'((1-\nu)a_1 + \nu a_2)\,d\nu, \tag{4}$$

where

$$K(\nu) = \begin{cases} \nu & , \quad \nu \in \left[0, \frac{x-a_1}{a_2-a_1}\right], \\ \nu - 1 & , \quad \nu \in \left(\frac{x-a_1}{a_2-a_1}, 1\right]. \end{cases}$$

This identity has been used in various works to establish bounds for quadrature rules via specialized algorithms [30].

We recall now some basic definitions for our study as follows:

Let $\mathbf{E} \subset \mathbb{R}^n$ be a non-empty set, $\Upsilon : \mathbf{E} \to \mathbb{R}$ be a continuous functions and $\theta : \mathbf{E} \times \mathbf{E} \to \mathbb{R}^n$ be a continuous bifunction.

Definition 3. *[7] A set $\mathbf{E} \subset \mathbb{R}^n$ is said to be invex with respect to bifunction $\theta(.,.)$, if*

$$a_1 + \nu\theta(a_2, a_1) \in \mathbf{E}, \quad \forall\, a_1, a_2 \in \mathbf{E},\ \nu \in [0,1].$$

Definition 4. *[15] A function $\Upsilon : \mathbf{E} \to \mathbb{R}$ is said to be preinvex with respect to bifunction $\theta(.,.)$, if*

$$\Upsilon(a_1 + \nu\theta(a_2, a_1)) \leq (1-\nu)\Upsilon(a_1) + \nu\Upsilon(a_2), \quad \forall\, a_1, a_2 \in \mathbf{E},\ \nu \in [0,1].$$

Some properties of this class of functions can be found in [31].

Motivated by the above literatures, the main objective of this article is to obtain a generalization of the Montgomery identity given in (4) using the concepts of q-calculus. From this identity, several new and known q-analogues of integral inequalities involving preinvex functions will be obtain. We also discuss some new special cases of the main results. At the end, a briefly conclusion is provided as well.

2. Main Results

In this section, before we derive our main results, for brevity we define the following notations:

$$P = [a_1, a_1 + \theta(a_2, a_1)], \quad \wp_\theta(x) = \frac{x - a_1}{\theta(a_2, a_1)}, \text{ where } \theta(a_2, a_1) > 0.$$

Lemma 2. *(Generalized quantum Montgomery identity) If $\Upsilon : P \to \mathbb{R}$ is a q-differentiable function such that ${}_eD_q\Upsilon$ is quantum integrable on P° (the interior of P), then the following identity holds:*

$$\Upsilon(x) - \frac{1}{\theta(a_2, a_1)} \int_{a_1}^{a_1+\theta(a_2,a_1)} \Upsilon(\nu)\, {}_{a_1}d_q\nu = \theta(a_2, a_1) \int_0^1 T_q(\nu)\, {}_{a_1}D_q\Upsilon(a_1 + \nu\theta(a_2, a_1))\, {}_0d_q\nu, \tag{5}$$

where

$$T_q(\nu) = \begin{cases} q\nu, & \text{if } \nu \in [0, \wp_\theta(x)]; \\ q\nu - 1, & \text{if } \nu \in (\wp_\theta(x), 1]. \end{cases}$$

Proof. By using Definitions 1 and 2, we have

$$
\begin{aligned}
&\theta(a_2,a_1)\int_0^1 T_q(\nu)\,{}_{a_1}D_q\Upsilon(a_1+\nu\theta(a_2,a_1))\,{}_0d_q\nu\\
&=\theta(a_2,a_1)\left[\int_0^{\wp_\theta(x)} q\nu\,{}_{a_1}D_q\Upsilon(a_1+\nu\theta(a_2,a_1))\,{}_0d_q\nu+\int_{\wp_\theta(x)}^1 (q\nu-1)\,{}_{a_1}D_q\Upsilon(a_1+\nu\theta(a_2,a_1))\,{}_0d_q\nu\right]\\
&=\theta(a_2,a_1)\left[\int_0^{\wp_\theta(x)} q\nu\,{}_{a_1}D_q\Upsilon(a_1+\nu\theta(a_2,a_1))\,{}_0d_q\nu+\int_0^1 (q\nu-1)\,{}_{a_1}D_q\Upsilon(a_1+\nu\theta(a_2,a_1))\,{}_0d_q\nu\right.\\
&\qquad\left.-\int_0^{\wp_\theta(x)} (q\nu-1)\,{}_{a_1}D_q\Upsilon(a_1+\nu\theta(a_2,a_1))\,{}_0d_q\nu\right]\\
&=\theta(a_2,a_1)\left[\int_0^1 (q\nu-1)\,{}_{a_1}D_q\Upsilon(a_1+\nu\theta(a_2,a_1))\,{}_0d_q\nu+\int_0^{\wp_\theta(x)} {}_{a_1}D_q\Upsilon(a_1+\nu\theta(a_2,a_1))\,{}_0d_q\nu\right]\\
&=\theta(a_2,a_1)\left[\int_0^1 q\nu\,{}_{a_1}D_q\Upsilon(a_1+\nu\theta(a_2,a_1))\,{}_0d_q\nu-\int_0^1 {}_{a_1}D_q\Upsilon(a_1+\nu\theta(a_2,a_1))\,{}_0d_q\nu\right.\\
&\qquad\left.+\int_0^{\wp_\theta(x)} {}_{a_1}D_q\Upsilon(a_1+\nu\theta(a_2,a_1))\,{}_0d_q\nu\right]\\
&=\frac{1}{1-q}\left[q\left[\int_0^1 \Upsilon(a_1+\nu\theta(a_2,a_1))\,{}_0d_q\nu-\int_0^1 \Upsilon(a_1+q\nu\theta(a_2,a_1))\,{}_0d_q\nu\right]\right.\\
&\qquad-\left[\int_0^1 \frac{\Upsilon(a_1+\nu\theta(a_2,a_1))}{\nu}d_q\nu-\int_0^1 \frac{\Upsilon(a_1+q\nu\theta(a_2,a_1))}{\nu}d_q\nu\right]\\
&\qquad\left.+\left[\int_0^{\wp_\theta(x)} \frac{\Upsilon(a_1+\nu\theta(a_2,a_1))}{\nu}d_q\nu-\int_0^{\wp_\theta(x)} \frac{\Upsilon(a_1+q\nu\theta(a_2,a_1))}{\nu}d_q\nu\right]\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{1-q}\Bigg[q(1-q)\left(\sum_{n=0}^{\infty} q^n \Upsilon(a_1+q^n\theta(a_2,a_1)) - \sum_{n=0}^{\infty} q^n \Upsilon(a_1+q^{n+1}\theta(a_2,a_1))\right) \\
&\qquad -(1-q)\left(\sum_{n=0}^{\infty} q^n \frac{\Upsilon(a_1+q^n\theta(a_2,a_1))}{q^n} - \sum_{n=0}^{\infty} q^n \frac{\Upsilon(a_1+q^{n+1}\theta(a_2,a_1))}{q^n}\right) \\
&\qquad +(1-q)\wp_\theta(x)\left(\sum_{n=0}^{\infty} q^n \frac{\Upsilon\left(a_1+q^n\wp_\theta(x)\theta(a_2,a_1)\right)}{q^n\wp_\theta(x)} - \sum_{n=0}^{\infty} q^n \frac{\Upsilon\left(a_1+q^{n+1}\wp_\theta(x)\theta(a_2,a_1)\right)}{q^n\wp_\theta(x)}\right)\Bigg] \\
&= q\left(\sum_{n=0}^{\infty} q^n \Upsilon(a_1+q^n\theta(a_2,a_1)) - \sum_{n=0}^{\infty} q^n \Upsilon(a_1+q^{n+1}\theta(a_2,a_1))\right) \\
&\qquad -\left(\sum_{n=0}^{\infty} \Upsilon(a_1+q^n\theta(a_2,a_1)) - \sum_{n=0}^{\infty} \Upsilon(a_1+q^{n+1}\theta(a_2,a_1))\right) \\
&\qquad\qquad +\left(\sum_{n=0}^{\infty} \Upsilon\left(a_1+q^n\wp_\theta(x)\theta(a_2,a_1)\right) - \sum_{n=0}^{\infty} \Upsilon\left(a_1+q^{n+1}\wp_\theta(x)\theta(a_2,a_1)\right)\right) \\
&= q\left(\sum_{n=0}^{\infty} q^n \Upsilon(a_1+q^n\theta(a_2,a_1)) - \frac{1}{q}\sum_{n=1}^{\infty} q^n \Upsilon(a_1+q^n\theta(a_2,a_1))\right) \\
&\qquad -\left(\sum_{n=0}^{\infty} \Upsilon(a_1+q^n\theta(a_2,a_1)) - \sum_{n=1}^{\infty} \Upsilon(a_1+q^n\theta(a_2,a_1))\right) \\
&\qquad\qquad +\left(\sum_{n=0}^{\infty} \Upsilon\left(a_1+q^n\wp_\theta(x)\theta(a_2,a_1)\right) - \sum_{n=1}^{\infty} \Upsilon\left(a_1+q^n\wp_\theta(x)\theta(a_2,a_1)\right)\right) \\
&= q\left[(1-\frac{1}{q})\sum_{n=0}^{\infty} q^n \Upsilon(a_1+q^n\theta(a_2,a_1)) + \frac{\Upsilon(a_1+\theta(a_2,a_1))}{q}\right] \\
&\qquad\qquad - \Upsilon(a_1+\theta(a_2,a_1)) + \Upsilon\left(a_1+\wp_\theta(x)\theta(a_2,a_1)\right) \\
&= \Upsilon(x) - (1-q)\sum_{n=0}^{\infty} q^n \Upsilon(a_1+q^n\theta(a_2,a_1)) \\
&= \Upsilon(x) - \frac{1}{\theta(a_2,a_1)}\int_{a_1}^{a_1+\theta(a_2,a_1)} \Upsilon(\nu)\,{}_{a_1}d_q\nu.
\end{aligned}
$$

The proof is complete. □

Remark 1. *Taking* $q \to 1^-$ *in Lemma 2, we have*

$$
\begin{aligned}
&\Upsilon(x) - \frac{1}{\theta(a_2,a_1)}\int_{a_1}^{a_1+\theta(a_2,a_1)} \Upsilon(\nu)d\nu \\
&= \theta(a_2,a_1)\left[\int_0^{\wp_\theta(x)} \nu\Upsilon'(a_1+\nu\theta(a_2,a_1))d\nu + \int_{\wp_\theta(x)}^{1} (\nu-1)\Upsilon'(a_1+\nu\theta(a_2,a_1))d\nu\right].
\end{aligned}
$$

Remark 2. *Taking* $\theta(a_2,a_1) = a_2 - a_1$ *in Lemma 2, we get ([20], Lemma 3).*

Remark 3. *Taking* $q \to 1^-$ *and* $\theta(a_2,a_1) = a_2 - a_1$ *in Lemma 2, we obtain Montgomery identity given in (4).*

Remark 4. *Taking $x = \frac{2a_1+\theta(a_2,a_1)}{2}$ in Remark 1, we get ([11], Lemma 3.10).*

$$\Upsilon\left(\frac{2a_1+\theta(a_2,a_1)}{2}\right) - \frac{1}{\theta(a_2,a_1)} \int_{a_1}^{a_1+\nu\theta(a_2,a_1)} \Upsilon(\nu)d\nu$$
$$= \theta(a_2,a_1)\left[\int_0^{\frac{1}{2}} \nu\Upsilon'(a_1+\nu\theta(a_2,a_1))d\nu + \int_{\frac{1}{2}}^{1} (\nu-1)\Upsilon'(a_1+\nu\theta(a_2,a_1))d\nu\right].$$

Remark 5. *Taking $x = \frac{qa_1+a_2}{1+q}$ and $\theta(a_2,a_1) = a_2 - a_1$ in Lemma 2, we obtain equality (4.1) of [18].*

$$\Upsilon\left(\frac{qa_1+a_2}{1+q}\right) - \frac{1}{a_2-a_1}\int_{a_1}^{a_2} \Upsilon(\nu)\,{}_{a_1}d_q\nu$$
$$= (a_2-a_1)\left[\int_0^{\frac{1}{1+q}} q\nu\,{}_{a_1}D_q\Upsilon((1-\nu)a_1+\nu a_2)\,{}_0d_q\nu\right.$$
$$\left.+\int_{\frac{1}{1+q}}^{1} (q\nu-1)\,{}_{a_1}D_q\Upsilon((1-\nu)a_1+\nu a_2)\,{}_0d_q\nu\right].$$

Remark 6. *Taking $x = \frac{a_1+q(a_1+\theta(a_2,a_1))}{1+q}$ in Lemma 2, we have*

$$\Upsilon\left(\frac{a_1+q(a_1+\theta(a_2,a_1))}{1+q}\right) - \frac{1}{\theta(a_2,a_1)}\int_{a_1}^{a_1+\theta(a_2,a_1)} \Upsilon(\nu)\,{}_{a_1}d_q\nu$$
$$= \theta(a_2,a_1)\left[\int_0^{\frac{q}{1+q}} q\nu\,{}_{a_1}D_q\Upsilon(a_1+\nu\theta(a_2,a_1))\,{}_0d_q\nu\right.$$
$$\left.+\int_{\frac{q}{1+q}}^{1} (q\nu-1)\,{}_{a_1}D_q\Upsilon(a_1+\nu\theta(a_2,a_1))\,{}_0d_q\nu\right].$$

Now using Lemma 2, we are in position to derive our main results for the class of preinvex functions.

Theorem 4. *Let $\Upsilon : P \to \mathbb{R}$ be a function such that ${}_{a_1}D_q\Upsilon$ is q-integrable on P° (the interior of P). If $|{}_{a_1}D_q\Upsilon|^r$ is preinvex function on P, then for $r > 1$ and $p^{-1} + r^{-1} = 1$, the following inequality holds:*

$$\left|\Upsilon(x) - \frac{1}{\theta(a_2,a_1)}\int_{a_1}^{a_1+\theta(a_2,a_1)} \Upsilon(\nu)\,{}_{a_1}d_q\nu\right|$$
$$\leq q\theta(a_2,a_1)\left[(L_1(q,a_1,a_2,x))^{\frac{1}{p}}\left(|{}_{a_1}D_q\Upsilon(a_1)|^r L_2(q,a_1,a_2,x) + |{}_{a_1}D_q\Upsilon(a_2)|^r L_3(q,a_1,a_2,x)\right)^{\frac{1}{r}}\right.$$
$$\left.+(L_4(q,a_1,a_2,x))^{\frac{1}{p}}\left(|{}_{a_1}D_q\Upsilon(a_1)|^r L_5(q,a_1,a_2,x) + |{}_{a_1}D_q\Upsilon(a_2)|^r L_6(q,a_1,a_2,x)\right)^{\frac{1}{r}}\right],$$

where

$$L_1(\mathrm{q}, a_1, a_2, x) = [\wp_\theta(x)]^{p+1} \frac{1-\mathrm{q}}{1-\mathrm{q}^{p+1}},$$
$$L_2(\mathrm{q}, a_1, a_2, x) = \wp_\theta(x) - \frac{1}{1+\mathrm{q}}[\wp_\theta(x)]^2,$$
$$L_3(\mathrm{q}, a_1, a_2, x) = \frac{1}{1+\mathrm{q}}[\wp_\theta(x)]^2,$$
$$L_4(\mathrm{q}, a_1, a_2, x) = (1-\mathrm{q}) \left[\sum_{n=0}^{\infty} \mathrm{q}^n \left(\mathrm{q}^n - \frac{1}{\mathrm{q}} \right)^p - \wp_\theta(x) \sum_{n=0}^{\infty} \mathrm{q}^n \left(\mathrm{q}^n \wp_\theta(x) - \frac{1}{\mathrm{q}} \right)^p \right],$$
$$L_5(\mathrm{q}, a_1, a_2, x) = \frac{\mathrm{q}}{1+\mathrm{q}} - \wp_\theta(x) + \frac{1}{1+\mathrm{q}}[\wp_\theta(x)]^2,$$
$$L_6(\mathrm{q}, a_1, a_2, x) = \frac{1}{1+\mathrm{q}} \left(1 - [\wp_\theta(x)]^2 \right).$$

Proof. Using Lemma 2, preinvexity of $|{}_{a_1}D_\mathrm{q}\Upsilon|^r$ and Hölder's inequality, we get

$$\begin{aligned}
&\left| \Upsilon(x) - \frac{1}{\theta(a_2, a_1)} \int_{a_1}^{a_1+\theta(a_2,a_1)} \Upsilon(\nu)\, {}_{a_1}d_\mathrm{q}\nu \right| \\
&\leq \theta(a_2, a_1) \left[\int_0^{\wp_\theta(x)} \mathrm{q}\nu |{}_{a_1}D_\mathrm{q}\Upsilon(a_1 + \nu\theta(a_2, a_1))|\, {}_0d_\mathrm{q}\nu \right. \\
&\qquad \left. + \int_{\wp_\theta(x)}^1 (\mathrm{q}\nu - 1)|{}_{a_1}D_\mathrm{q}\Upsilon(a_1 + \nu\theta(a_2, a_1))|\, {}_0d_\mathrm{q}\nu \right] \\
&\leq \theta(a_2, a_1) \left[\left(\int_0^{\wp_\theta(x)} (\mathrm{q}\nu)^p\, {}_0d_\mathrm{q}\nu \right)^{\frac{1}{p}} \left(\int_0^{\wp_\theta(x)} |{}_{a_1}D_\mathrm{q}\Upsilon(a_1 + \nu\theta(a_2, a_1))|^r\, {}_0d_\mathrm{q}\nu \right)^{\frac{1}{r}} \right. \\
&\qquad \left. + \left(\int_{\wp_\theta(x)}^1 (\mathrm{q}\nu - 1)^p\, {}_0d_\mathrm{q}\nu \right)^{\frac{1}{p}} \left(\int_{\wp_\theta(x)}^1 |{}_{a_1}D_\mathrm{q}\Upsilon(a_1 + \nu\theta(a_2, a_1))|^r\, {}_0d_\mathrm{q}\nu \right)^{\frac{1}{r}} \right] \\
&\leq \mathrm{q}\theta(a_2, a_1) \left[\left(\int_0^{\wp_\theta(x)} \nu^p\, {}_0d_\mathrm{q}\nu \right)^{\frac{1}{p}} \left(|{}_{a_1}D_\mathrm{q}\Upsilon(a_1)|^r \int_0^{\wp_\theta(x)} (1-\nu)\, {}_0d_\mathrm{q}\nu + |{}_{a_1}D_\mathrm{q}\Upsilon(a_2)|^r \int_0^{\wp_\theta(x)} \nu\, {}_0d_\mathrm{q}\nu \right)^{\frac{1}{r}} \right. \\
&\left. + \left(\int_{\wp_\theta(x)}^1 \left(\nu - \frac{1}{\mathrm{q}} \right)^p {}_0d_\mathrm{q}\nu \right)^{\frac{1}{p}} \left(|{}_{a_1}D_\mathrm{q}\Upsilon(a_1)|^r \int_{\wp_\theta(x)}^1 (1-\nu)\, {}_0d_\mathrm{q}\nu + |{}_{a_1}D_\mathrm{q}\Upsilon(a_2)|^r \int_{\wp_\theta(x)}^1 \nu\, {}_0d_\mathrm{q}\nu \right)^{\frac{1}{r}} \right].
\end{aligned}$$

Letting

$$L_1(\mathrm{q}, a_1, a_2, x) = \int_0^{\wp_\theta(x)} \nu^p \, {}_0d_\mathrm{q}\nu = [\wp_\theta(x)]^{p+1} \frac{1-\mathrm{q}}{1-\mathrm{q}^{p+1}},$$

$$L_2(\mathrm{q}, a_1, a_2, x) = \int_0^{\wp_\theta(x)} (1-\nu) \, {}_0d_\mathrm{q}\nu = \wp_\theta(x) - \frac{1}{1+\mathrm{q}}[\wp_\theta(x)]^2,$$

$$L_3(\mathrm{q}, a_1, a_2, x) = \int_0^{\wp_\theta(x)} \nu \, {}_0d_\mathrm{q}\nu = \frac{1}{1+\mathrm{q}}[\wp_\theta(x)]^2,$$

$$L_4(\mathrm{q}, a_1, a_2, x) = \int_{\wp_\theta(x)}^{1} \left(\nu - \frac{1}{\mathrm{q}}\right)^p {}_0d_\mathrm{q}\nu = (1-\mathrm{q}) \left[\sum_{n=0}^{\infty} \mathrm{q}^n \left(\mathrm{q}^n - \frac{1}{\mathrm{q}}\right)^p - \wp_\theta(x) \sum_{n=0}^{\infty} \mathrm{q}^n \left(\mathrm{q}^n \wp_\theta(x) - \frac{1}{\mathrm{q}}\right)^p\right],$$

$$L_5(\mathrm{q}, a_1, a_2, x) = \int_{\wp_\theta(x)}^{1} (1-\nu) \, {}_0d_\mathrm{q}\nu = \frac{\mathrm{q}}{1+\mathrm{q}} - \wp_\theta(x) + \frac{1}{1+\mathrm{q}}[\wp_\theta(x)]^2,$$

$$L_6(\mathrm{q}, a_1, a_2, x) = \int_{\wp_\theta(x)}^{1} \nu \, {}_0d_\mathrm{q}\nu = \frac{1}{1+\mathrm{q}} \left(1 - [\wp_\theta(x)]^2\right),$$

we have the desired result. The proof is complete. □

We point out some special cases of Theorem 4.

Corollary 1. **I.** *Taking* $\mathrm{q} \to 1^-$ *in Theorem 4, we have*

$$\begin{aligned} &\left| \Upsilon(x) - \frac{1}{\theta(a_2, a_1)} \int_{a_1}^{a_1+\theta(a_2,a_1)} \Upsilon(\nu) d\nu \right| \\ &\leq \theta(a_2, a_1) \left[[L_7(a_1, a_2, x)]^{\frac{1}{p}} \left[|\Upsilon'(a_1)|^r L_8(a_1, a_2, x) + |\Upsilon'(a_2)|^r L_9(a_1, a_2, x) \right]^{\frac{1}{r}} \right. \\ &\left. + [L_{10}(a_1, a_2, x)]^{\frac{1}{p}} \left[|\Upsilon'(a_1)|^r L_{11}(a_1, a_2, x) + |\Upsilon'(a_2)|^r L_{12}(a_1, a_2, x) \right]^{\frac{1}{r}} \right], \end{aligned}$$

where

$$L_7(a_1,a_2,x)=\int_0^{\wp_\theta(x)} \nu^p d\nu=\frac{1}{p+1}[\wp_\theta(x)]^{p+1},$$

$$L_8(a_1,a_2,x)=\int_0^{\wp_\theta(x)} (1-\nu) d\nu=\wp_\theta(x)-\frac{1}{2}[\wp_\theta(x)]^{2},$$

$$L_9(a_1,a_2,x)=\int_0^{\wp_\theta(x)} \nu d\nu=\frac{1}{2}[\wp_\theta(x)]^{2},$$

$$L_{10}(a_1,a_2,x)=\int_{\wp_\theta(x)}^{1} (1-\nu)^p d\nu=\frac{1}{p+1}\left(\frac{a_1+\theta(a_2,a_1)-x}{\theta(a_2,a_1)}\right)^{p+1},$$

$$L_{11}(a_1,a_2,x)=\int_{\wp_\theta(x)}^{1} (1-\nu) d\nu=\frac{a_1+\theta(a_2,a_1)-x}{\theta(a_2,a_1)}-\frac{1}{2}\left(1-[\wp_\theta(x)]^2\right),$$

$$L_{12}(a_1,a_2,x)=\int_{\wp_\theta(x)}^{1} \nu d\nu=\frac{1}{2}\left(1-[\wp_\theta(x)]^2\right).$$

II. *Taking* $\mathrm{q}\to 1^-$ *and* $x=\frac{2a_1+\theta(a_2,a_1)}{2}$ *in Theorem 4, we get ([13], Theorem 6).*

$$\left|\Upsilon\left(\frac{2a_1+\theta(a_2,a_1)}{2}\right)-\frac{1}{\theta(a_2,a_1)}\int_{a_1}^{a_1+\theta(a_2,a_1)}\Upsilon(\nu)d\nu\right|$$
$$\leq \frac{\theta(a_2,a_1)}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[\left(3|\Upsilon'(a_1)|^r+|\Upsilon'(a_2)|^r\right)^{\frac{1}{r}}+\left(|\Upsilon'(a_1)|^r+3|\Upsilon'(a_2)|^r\right)^{\frac{1}{r}}\right].$$

III. *Taking* $x=\frac{qa_1+a_2}{1+\mathrm{q}}$ *and* $\theta(a_2,a_1)=a_2-a_1$ *in Theorem 4, we obtain ([18], Theorem 18).*

$$\left|\Upsilon\left(\frac{qa_1+a_2}{1+\mathrm{q}}\right)-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}\Upsilon(\nu)\,{}_{a_1}d_{\mathrm{q}}\nu\right|$$
$$\leq \mathrm{q}(a_2-a_1)\left[\left(\frac{1}{(1+\mathrm{q})^{p+1}}\frac{(1-\mathrm{q})}{1-\mathrm{q}^{p+1}}\right)\left[\frac{(\mathrm{q}^2+2\mathrm{q})|{}_{a_1}D_{\mathrm{q}}\Upsilon(a_1)|^r}{(1+\mathrm{q})^3}+\frac{\mathrm{q}^2|{}_{a_1}D_{\mathrm{q}}\Upsilon(a_2)|^r}{(1+\mathrm{q})^3}\right]^{\frac{1}{r}}\right.$$
$$\left.+\left(\int_{\frac{1}{1+\mathrm{q}}}^{1}\left(\nu-\frac{1}{\mathrm{q}}\right)^p\,{}_{a_1}d_{\mathrm{q}}\nu\right)^{\frac{1}{p}}\left[\frac{(\mathrm{q}^3+\mathrm{q}^2-\mathrm{q})|{}_{a_1}D_{\mathrm{q}}\Upsilon(a_1)|^r}{(1+\mathrm{q})^3}+\frac{(\mathrm{q}^2+2\mathrm{q})|{}_{a_1}D_{\mathrm{q}}\Upsilon(a_2)|^r}{(1+\mathrm{q})^3}\right]^{\frac{1}{r}}\right],$$

where

$$\int_{\frac{1}{1+\mathrm{q}}}^{1}\left(\nu-\frac{1}{\mathrm{q}}\right)^p\,{}_0d_{\mathrm{q}}\nu=(1-\mathrm{q})\left[\sum_{n=0}^{\infty}\mathrm{q}^n\left(\mathrm{q}^n-\frac{1}{\mathrm{q}}\right)^p-\frac{1}{1+\mathrm{q}}\sum_{n=0}^{\infty}\mathrm{q}^n\left(\mathrm{q}^n\left(\frac{1}{1+\mathrm{q}}\right)-\frac{1}{\mathrm{q}}\right)^p\right].$$

IV. *Taking* $x = \frac{a_1+\mathrm{q}(a_1+\theta(a_2,a_1))}{1+\mathrm{q}}$ *in Theorem 4, we get*

$$\left| \Upsilon\left(\frac{a_1+\mathrm{q}(a_1+\theta(a_2,a_1))}{1+\mathrm{q}}\right) - \frac{1}{\theta(a_2,a_1)} \int_{a_1}^{a_1+\theta(a_2,a_1)} \Upsilon(\nu)\, {}_{a_1}d_{\mathrm{q}}\nu \right|$$
$$\leq \mathrm{q}\theta(a_2,a_1) \left[\left(\frac{\mathrm{q}^p}{(1+\mathrm{q})^{p+1}} \frac{(1-\mathrm{q})}{1-\mathrm{q}^{p+1}} \right) \left[\frac{(\mathrm{q}^3+\mathrm{q}^2+\mathrm{q})|{}_{a_1}D_{\mathrm{q}}\Upsilon(a_1)|^r}{(1+\mathrm{q})^3} + \frac{\mathrm{q}^2|{}_{a_1}D_{\mathrm{q}}\Upsilon(a_2)|^r}{(1+\mathrm{q})^3} \right]^{\frac{1}{r}} \right.$$
$$\left. + \left(\int_{\frac{\mathrm{q}}{1+\mathrm{q}}}^{1} \left(\nu - \frac{1}{\mathrm{q}}\right)^p {}_0d_{\mathrm{q}}\nu \right)^{\frac{1}{p}} \left[\frac{\mathrm{q}^2|{}_{a_1}D_{\mathrm{q}}\Upsilon(a_1)|^r}{(1+\mathrm{q})^3} + \frac{(1+2\mathrm{q})|{}_{a_1}D_{\mathrm{q}}\Upsilon(a_2)|^r}{(1+\mathrm{q})^3} \right]^{\frac{1}{r}} \right],$$

where

$$\int_{\frac{\mathrm{q}}{1+\mathrm{q}}}^{1} \left(\nu - \frac{1}{\mathrm{q}}\right)^p {}_0d_{\mathrm{q}}\nu = (1-\mathrm{q}) \left[\sum_{n=0}^{\infty} \mathrm{q}^n \left(\mathrm{q}^n - \frac{1}{\mathrm{q}}\right)^p - \frac{\mathrm{q}}{1+\mathrm{q}} \sum_{n=0}^{\infty} \mathrm{q}^n \left(\mathrm{q}^n \left(\frac{\mathrm{q}}{1+\mathrm{q}}\right) - \frac{1}{\mathrm{q}} \right)^p \right].$$

Theorem 5. *Let* $\Upsilon : P \to \mathbb{R}$ *be a function such that* ${}_{a_1}D_{\mathrm{q}}\Upsilon$ *is* q*-integrable on* P° *(the interior of* P*). If* $|{}_{a_1}D_{\mathrm{q}}\Upsilon|^r$ *is preinvex function on* P*, then for* $r \geq 1$*, the following inequality holds:*

$$\left| \Upsilon(x) - \frac{1}{\theta(a_2,a_1)} \int_{a_1}^{a_1+\theta(a_2,a_1)} \Upsilon(\nu)\, {}_{a_1}d_{\mathrm{q}}\nu \right| \leq \theta(a_2,a_1) \times$$
$$\left[[J_1(\mathrm{q},a_1,a_2,x)]^{1-\frac{1}{r}} \left[|{}_{a_1}D_{\mathrm{q}}\Upsilon(a_1)|^r J_2(\mathrm{q},a_1,a_2,x) + |{}_{a_1}D_{\mathrm{q}}\Upsilon(a_2)|^r J_3(\mathrm{q},a_1,a_2,x) \right]^{\frac{1}{r}} \right.$$
$$\left. + [J_4(\mathrm{q},a_1,a_2,x)]^{1-\frac{1}{r}} \left[|{}_{a_1}D_{\mathrm{q}}\Upsilon(a_1)|^r J_5(\mathrm{q},a_1,a_2,x) + |{}_{a_1}D_{\mathrm{q}}\Upsilon(a_2)|^r J_6(\mathrm{q},a_1,a_2,x) \right]^{\frac{1}{r}} \right],$$

where

$$J_1(\mathrm{q},a_1,a_2,x) = \int_0^{\wp_\theta(x)} \mathrm{q}\nu\, {}_0d_{\mathrm{q}}\nu = \frac{\mathrm{q}}{1+\mathrm{q}}[\wp_\theta(x)]^2,$$
$$J_2(\mathrm{q},a_1,a_2,x) = \int_0^{\wp_\theta(x)} (\mathrm{q}\nu - \mathrm{q}\nu^2)\, {}_0d_{\mathrm{q}}\nu = J_1(\mathrm{q},a_1,a_2,x) - J_3(\mathrm{q},a_1,a_2,x),$$
$$J_3(\mathrm{q},a_1,a_2,x) = \int_0^{\wp_\theta(x)} \mathrm{q}\nu^2\, {}_0d_{\mathrm{q}}\nu = \frac{\mathrm{q}}{1+\mathrm{q}+\mathrm{q}^2}[\wp_\theta(x)]^3,$$
$$J_4(\mathrm{q},a_1,a_2,x) = \int_{\wp_\theta(x)}^{1} (\mathrm{q}\nu - 1)\, {}_0d_{\mathrm{q}}\nu = \frac{\mathrm{q}}{1+\mathrm{q}} \left(\frac{a_1+\theta(a_2,a_1)-x}{\theta(a_2,a_1)} \right)^2,$$
$$J_5(\mathrm{q},a_1,a_2,x) = \int_{\wp_\theta(x)}^{1} (1 - \mathrm{q}\nu - \nu + \mathrm{q}\nu^2)\, {}_0d_{\mathrm{q}}\nu = J_4(\mathrm{q},a_1,a_2,x) - J_6(\mathrm{q},a_1,a_2,x),$$
$$J_6(\mathrm{q},a_1,a_2,x) = \int_{\wp_\theta(x)}^{1} (\mathrm{q}\nu^2 - \nu)\, {}_0d_{\mathrm{q}}\nu = \frac{1}{(1+\mathrm{q})(1+\mathrm{q}+\mathrm{q}^2)} - \frac{1}{1+\mathrm{q}}[\wp_\theta(x)]^2 + \frac{\mathrm{q}}{1+\mathrm{q}+\mathrm{q}^2}[\wp_\theta(x)]^3.$$

Proof. Using Lemma 2, preinvexity of $|{}_{a_1}D_qY|^r$ and the well-known power mean inequality, we have

$$\begin{aligned}
&\left|Y(x) - \frac{1}{\theta(a_2,a_1)}\int_{a_1}^{a_1+\theta(a_2,a_1)} Y(\nu)\,{}_{a_1}d_q\nu\right| \\
&\leq \theta(a_2,a_1)\left[\int_0^{\wp_\theta(x)} q\nu|{}_{a_1}D_qY(a_1+\nu\theta(a_2,a_1))|\,{}_0d_q\nu + \int_{\wp_\theta(x)}^1 (q\nu-1)|{}_{a_1}D_qY(a_1+\nu\theta(a_2,a_1))|\,{}_0d_q\nu\right] \\
&\leq \theta(a_2,a_1)\left[\left(\int_0^{\wp_\theta(x)} q\nu\,{}_0d_q\nu\right)^{1-\frac{1}{r}}\left(\int_0^{\wp_\theta(x)} q\nu|{}_{a_1}D_qY(a_1+\nu\theta(a_2,a_1))|^r\,{}_0d_q\nu\right)^{\frac{1}{r}}\right. \\
&\left.+\left(\int_{\wp_\theta(x)}^1 (q\nu-1)\,{}_0d_q\nu\right)^{1-\frac{1}{r}}\left(\int_{\wp_\theta(x)}^1 (q\nu-1)|{}_{a_1}D_qY(a_1+\nu\theta(a_2,a_1))|^r\,{}_0d_q\nu\right)^{\frac{1}{r}}\right]
\end{aligned}$$

$$\begin{aligned}
&\leq \theta(a_2,a_1)\left[\left(\int_0^{\wp_\theta(x)} q\nu\,{}_0d_q\nu\right)^{1-\frac{1}{r}}\left(|{}_{a_1}D_qY(a_1)|^r\int_0^{\wp_\theta(x)}(q\nu-q\nu^2)\,{}_0d_q\nu + |{}_{a_1}D_qY(a_2)|^r\int_0^{\wp_\theta(x)} q\nu^2\,{}_0d_q\nu\right)^{\frac{1}{r}}\right. \\
&\left.+\left(\int_{\wp_\theta(x)}^1 (q\nu-1)\,{}_0d_q\nu\right)^{1-\frac{1}{r}}\left(|{}_{a_1}D_qY(a_1)|^r\int_{\wp_\theta(x)}^1(1-q\nu-\nu+q\nu^2)\,{}_0d_q\nu + |{}_{a_1}D_qY(a_2)|^r\int_{\wp_\theta(x)}^1(q\nu^2-\nu)\,{}_0d_q\nu\right)^{\frac{1}{r}}\right].
\end{aligned}$$

The proof of Theorem 5 is completed. □

We point out some special cases of Theorem 5.

Corollary 2. I. *Taking $r=1$ in Theorem 5, we have*

$$\begin{aligned}
\left|Y(x) - \frac{1}{\theta(a_2,a_1)}\int_{a_1}^{a_1+\theta(a_2,a_1)} Y(\nu)\,{}_{a_1}d_q\nu\right| &\leq \theta(a_2,a_1)\,\big[|{}_{a_1}D_qY(a_1)|[J_2(q,a_1,a_2,x)+J_5(q,a_1,a_2,x)] \\
&+|{}_{a_1}D_qY(a_2)|[J_3(q,a_1,a_2,x)+J_6(q,a_1,a_2,x)]\big].
\end{aligned}$$

II. *Taking $r=1$ and $q\to 1^-$ in Theorem 5, we get*

$$\begin{aligned}
\left|Y(x) - \frac{1}{\theta(a_2,a_1)}\int_{a_1}^{a_1+\theta(a_2,a_1)} Y(\nu)d\nu\right| &\leq \theta(a_2,a_1)\,\big[|Y'(a_1)|[J_7(a_1,a_2,x)+J_9(a_1,a_2,x)] \\
&+|Y'(a_2)|[J_8(a_1,a_2,x)+J_{10}(a_1,a_2,x)]\big],
\end{aligned}$$

where

$$J_7(a_1,a_2,x) = \int_0^{\wp_\theta(x)} \nu(1-\nu)d\nu = \frac{1}{2}[\wp_\theta(x)]^2 - \frac{1}{3}[\wp_\theta(x)]^3,$$

$$J_8(a_1,a_2,x) = \int_0^{\wp_\theta(x)} \nu^2 d\nu = \frac{1}{3}[\wp_\theta(x)]^3,$$

$$J_9(a_1,a_2,x) = \int_{\wp_\theta(x)}^1 (1-2\nu+\nu^2)d\nu = \frac{1}{3} - \wp_\theta(x) + [\wp_\theta(x)]^2 - \frac{1}{3}[\wp_\theta(x)]^3,$$

$$J_{10}(a_1,a_2,x) = \int_{\wp_\theta(x)}^1 (\nu-\nu^2)d\nu = \frac{1}{6} - \frac{1}{2}[\wp_\theta(x)]^2 + \frac{1}{3}[\wp_\theta(x)]^3.$$

III. *Taking* $\mathrm{q} \to 1^-$ *and* $x = \frac{2a_1+\theta(a_2,a_1)}{2}$ *in Theorem 5, we obtain ([13], Theorem 8).*

$$\left| \Upsilon\left(\frac{2a_1+\theta(a_2,a_1)}{\theta(a_2,a_1)}\right) - \frac{1}{\theta(a_2,a_1)} \int_{a_1}^{a_1+\theta(a_2,a_1)} \Upsilon(\nu)d\nu \right|$$
$$\leq \frac{\theta(a_2,a_1)}{8}\left[\left(\frac{2|\Upsilon'(a_1)|^r + |\Upsilon'(a_2)|^r}{3}\right)^{\frac{1}{r}} + \left(\frac{|\Upsilon'(a_1)|^r + 2|\Upsilon'(a_2)|^r}{3}\right)^{\frac{1}{r}}\right].$$

IV. *Taking* $r = 1$, $\mathrm{q} \to 1^-$ *and* $x = \frac{2a_1+\theta(a_2,a_1)}{2}$ *in Theorem 5, we get ([13], Theorem 5).*

$$\left| \Upsilon\left(\frac{2a_1+\theta(a_2,a_1)}{2}\right) - \frac{1}{\theta(a_2,a_1)} \int_{a_1}^{a_1+\theta(a_2,a_1)} \Upsilon(\nu)d\nu \right| \leq \frac{\theta(a_2,a_1)}{8}\left[|\Upsilon'(a_1)| + |\Upsilon'(a_2)|\right].$$

IV. *Taking* $x = \frac{qa_1+a_2}{1+\mathrm{q}}$ *and* $\theta(a_2,a_1) = a_2 - a_1$ *in Theorem 5, we have the following inequalities, for more details, see [20].*

$$\left| \Upsilon\left(\frac{qa_1+a_2}{1+\mathrm{q}}\right) - \frac{1}{a_2-a_1}\int_{a_1}^{a_2} \Upsilon(\nu)\, {}_{a_1}d_{\mathrm{q}}\nu \right|$$
$$\leq (a_2-a_1)\left[\frac{1}{(1+\mathrm{q})^{3-\frac{3}{r}}}\left[|{}_{a_1}D_{\mathrm{q}}\Upsilon(a_1)|^r \frac{\mathrm{q}^2(1+\mathrm{q})}{(1+\mathrm{q}+\mathrm{q}^2)(1+\mathrm{q})^3} + |{}_{a_1}D_{\mathrm{q}}\Upsilon(a_2)|^r \frac{\mathrm{q}}{(1+\mathrm{q}+\mathrm{q}^2)(1+\mathrm{q})^3}\right]^{\frac{1}{r}}\right.$$
$$\left. + \left(\frac{\mathrm{q}}{1+\mathrm{q}}\right)^{3-\frac{3}{r}}\left[|{}_{a_1}D_{\mathrm{q}}\Upsilon(a_1)|^r \frac{(\mathrm{q}^5+\mathrm{q}^4+\mathrm{q}^3-2\mathrm{q})}{(1+\mathrm{q}+\mathrm{q}^2)(1+\mathrm{q})^3} + |{}_{a_1}D_{\mathrm{q}}\Upsilon(a_2)|^r \frac{2\mathrm{q}}{(1+\mathrm{q}+\mathrm{q}^2)(1+\mathrm{q})^3}\right]^{\frac{1}{r}}\right].$$

V. *Taking* $r = 1$, $x = \frac{qa_1+a_2}{1+\mathrm{q}}$ *and* $\theta(a_2,a_1) = a_2 - a_1$ *in Theorem 5, we obtain ([18], Theorem 13).*

$$\left| \Upsilon\left(\frac{qa_1+a_2}{1+\mathrm{q}}\right) - \frac{1}{a_2-a_1}\int_{a_1}^{a_2} \Upsilon(\nu)\, {}_{a_1}d_{\mathrm{q}}\nu \right|$$
$$\leq (a_2-a_1)\left[|{}_{a_1}D_{\mathrm{q}}\Upsilon(a_1)| \frac{(\mathrm{q}^5+\mathrm{q}^4+2\mathrm{q}^3+\mathrm{q}^2-2\mathrm{q})}{(1+\mathrm{q}+\mathrm{q}^2)(1+\mathrm{q})^3} + |{}_{a_1}D_{\mathrm{q}}\Upsilon(a_2)| \frac{3\mathrm{q}}{(1+\mathrm{q}+\mathrm{q}^2)(1+\mathrm{q})^3}\right].$$

VI. *Taking* $x = \frac{a_1+\mathrm{q}(a_1+\theta(a_2,a_1))}{1+\mathrm{q}}$ *in Theorem 5, we get*

$$\left| \Upsilon\left(\frac{a_1+\mathrm{q}(a_1+\theta(a_2,a_1))}{1+\mathrm{q}}\right) - \frac{1}{\theta(a_2,a_1)} \int_{a_1}^{a_1+\theta(a_2,a_1)} \Upsilon(\nu)\, {}_{a_1}d_{\mathrm{q}}\nu \right| \leq \theta(a_2,a_1)\times$$

$$\left[\left(\frac{\mathrm{q}}{1+\mathrm{q}}\right)^{3-\frac{3}{r}} \left[|{}_{a_1}D_{\mathrm{q}}\Upsilon(a_1)|^r \frac{\mathrm{q}^3(1+\mathrm{q}^2)}{(1+\mathrm{q}+\mathrm{q}^2)(1+\mathrm{q})^3} + |{}_{a_1}D_{\mathrm{q}}\Upsilon(a_2)|^r \frac{\mathrm{q}^4}{(1+\mathrm{q}+\mathrm{q}^2)(1+\mathrm{q})^3}\right]^{\frac{1}{r}}\right.$$

$$\left. + \left(\frac{\mathrm{q}}{(1+\mathrm{q})^3}\right)^{1-\frac{1}{r}} \left[|{}_{a_1}D_{\mathrm{q}}\Upsilon(a_1)|^r \frac{(2\mathrm{q}^3+\mathrm{q}^2-\mathrm{q}-1)}{(1+\mathrm{q}+\mathrm{q}^2)(1+\mathrm{q})^3} + |{}_{a_1}D_{\mathrm{q}}\Upsilon(a_2)|^r \frac{(1+2\mathrm{q}-\mathrm{q}^3)}{(1+\mathrm{q}+\mathrm{q}^2)(1+\mathrm{q})^3}\right]^{\frac{1}{r}}\right].$$

3. Applications to Special Means

Recalling the following means for arbitrary real numbers a and b with $a \neq b$:

$$A(a_1,a_2) = \frac{a_1+a_2}{2} \qquad \text{Arithmetic mean}$$

$$L_p(a_1,a_2) = \left(\frac{a_2^{p+1}-a_1^{p+1}}{(p+1)(a_2-a_1)}\right)^{1/p}, p \in \mathbb{R}\setminus\{-1,0\} \qquad \text{p- Logarithmic mean}$$

it is possible to relate them through the previous results.

Proposition 1. *Let* $p \in \mathbb{R}\setminus\{-1,0\}$ *and* a_1, a_2 *real numbers such that* $a_2 \geq a_2$*, then*

$$\left|A^p(a_1,a_2) - \frac{1}{a_2-a_1}L_p^p(a_1,a_2)\right| \leq \frac{|p|}{8}(a_2-a_1)\left(\left|a_1^{p-1}\right| + \left|a_2^{p-1}\right|\right)$$

Proof. Let $f(x) = x^p$ for some arbitrary $p \in \mathbb{R}\setminus\{-1,0\}$, $x = (qa_1+a_2)/(1+q)$ and $r = 1$. Then

$$f\left(\frac{qa_1+a_2}{1+q}\right) = \left(\frac{qa_1+a_2}{1+q}\right)^p,$$

$$\int_{a_1}^{a_2} t^p\, {}_{a_1}d_q t = \left[\frac{1-q}{1-q^{p+1}}\right]\left(\frac{a_2^{p+1}-a_1^{p+1}}{a_2-a_1}\right)$$

and

$${}_{a_1}D_{\mathrm{q}}\Upsilon(a_1) = \left[\frac{1-q^p}{1-q}\right]a_1^{p-1} \text{ and } {}_{a_1}D_{\mathrm{q}}\Upsilon(a_2) = \left[\frac{1-q^p}{1-q}\right]a_2^{p-1}$$

So, using the Corollary 2 part **IV**, we have

$$\left|\left(\frac{qa_1+a_2}{1+q}\right)^p - \frac{1}{a_2-a_1}\left[\frac{1-q}{1-q^{p+1}}\right]\left(\frac{a_2^{p+1}-a_1^{p+1}}{a_2-a_1}\right)\right|$$

$$\leq (a_2-a_1)\left[\frac{1}{(1+\mathrm{q})^{3-\frac{3}{r}}}\left[\left|\left[\frac{1-q^p}{1-q}\right]a_1^{p-1}\right|\frac{\mathrm{q}^2(1+\mathrm{q})}{(1+\mathrm{q}+\mathrm{q}^2)(1+\mathrm{q})^3} + \left|\left[\frac{1-q^p}{1-q}\right]a_2^{p-1}\right|\frac{\mathrm{q}}{(1+\mathrm{q}+\mathrm{q}^2)(1+\mathrm{q})^3}\right]^{\frac{1}{r}}\right.$$

$$\left. + \left(\frac{\mathrm{q}}{1+\mathrm{q}}\right)^{3-\frac{3}{r}}\left[\left|\left[\frac{1-q^p}{1-q}\right]a_1^{p-1}\right|\frac{(\mathrm{q}^5+\mathrm{q}^4+\mathrm{q}^3-2\mathrm{q})}{(1+\mathrm{q}+\mathrm{q}^2)(1+\mathrm{q})^3} + \left|\left[\frac{1-q^p}{1-q}\right]a_2^{p-1}\right|\frac{2\mathrm{q}}{(1+\mathrm{q}+\mathrm{q}^2)(1+\mathrm{q})^3}\right]^{\frac{1}{r}}\right].$$

Taking limit when $q \to 1$

$$\left| \left(\frac{a_1 + a_2}{2} \right)^p - \frac{1}{a_2 - a_1} \left(\frac{a_2^{p+1} - a_1^{p+1}}{(p+1)(a_2 - a_1)} \right) \right|$$
$$\leq (a_2 - a_1) \left[\left| p a_1^{p-1} \right| \frac{1}{12} + \left| p a_2^{p-1} \right| \frac{1}{24} + \left| p a_1^{p-1} \right| \frac{1}{24} + \left| p a_2^{p-1} \right| \frac{1}{12} \right]$$
$$\leq \frac{|p|}{8} (a_2 - a_1) \left(\left| a_1^{p-1} \right| + \left| a_2^{p-1} \right| \right).$$

The proof is complete. □

4. Conclusions

It is expected that from the results obtained, and following the methodology applied, additional special functions may also be evaluated. Future works can be developed in the area of numerical analysis and even contributions using quantum algorithms, using the theorems and corollaries presented. Finally, our results can be applied to derive some inequalities using special means. The authors hope that the ideas and techniques of this paper will inspire interested readers working in this fascinating field.

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