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An Approach for Studying Asymptotic Properties of Solutions of Neutral Differential Equations

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Received: 11 February 2020; Accepted: 16 March 2020; Published: 4 April 2020

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Abstract: The paper is devoted to the study of oscillation of even-order neutral differential equations. New Kamenev-type oscillation criteria are established, and they essentially improve and complement some the well-known results reported in the literature. Ideas of symmetry help us determine the correct ways to study these topics and show us the correct direction, because they are often invisible. To illustrate the main results, some examples are mentioned.

Keywords: Oscillation; even order; neutral differential equation

1. Introduction

The objective of this paper is to investigate the oscillation of solutions to the following equation:

$$\left(a\left(v\right)u^{(n-1)}\left(v\right)\right)' + q\left(v\right)\phi\left(y\left(\delta\left(v\right)\right)\right) = 0,\ v \geq v_0, \tag{1}$$

where n is an even natural number, $n \geq 2$, $\beta \geq 1$ is a constant and

$$u\left(v\right) := y^{\beta}\left(v\right) + \vartheta\left(v\right)y\left(\varsigma\left(v\right)\right). \tag{2}$$

We assume throughout that the following conditions are satisfied:

(P_1) $a, \vartheta, q \in C\left(\left[v_0, \infty\right), \left[0, \infty\right)\right)$, $a\left(v\right) > 0$, $a'\left(v\right) \geq 0$, $0 \leq \vartheta\left(v\right) < 1$ and

$$\int_{v_0}^{\infty} \frac{1}{a\left(s\right)} \mathrm{d}s = \infty; \tag{3}$$

(P_2) $\phi \in C\left(\mathbb{R}, \mathbb{R}\right)$, $\phi\left(y\right) \geq y^{\beta}$ for $y \neq 0$;
(P_3) $\varsigma \in C\left(\left[v_0, \infty\right), \left(0, \infty\right)\right)$, $\varsigma\left(v\right) \leq v$ and $\lim_{v\to\infty} \varsigma\left(v\right) = \infty$; $\delta \in C\left(\left[v_0, \infty\right), \mathbb{R}\right)$, $\delta\left(v\right) \leq v$, $\delta'\left(v\right) > 0$ and $\lim_{v\to\infty} \delta\left(v\right) = \infty$.

We consider only those solutions x of Equation (1) which satisfy $\sup\{|y\left(v\right)| : v \geq L\} > 0$, for all $L > L_y$. We consider only those solutions y of (1) which satisfy $\sup\{|y\left(v\right)| : v \geq L\} > 0$, for all $L > L_y$. We assume that (1) possesses such a solution. Differential equations have many applications in this life, it is related to biology, physics, dynamica, and so on. In particular, the oscillatory behavior of ordinary differential equations plays a crucial role in this applications, so there was an interest of many authors in studying the qualitative behavior of differential equations see [1–28].

For instance, Zhang et al. [25] examined the oscillation of even-order neutral differential equations

$$u^{(n)}\left(v\right) + q\left(v\right)f\left(y\left(\delta\left(v\right)\right)\right) = 0$$

and established the criteria for the solution to be oscillatory when $0 \le p(v) < 1$.

Xing et al. [21] proved that the equation

$$\left(r(v)\left(u^{(n-1)}(v)\right)^{\alpha}\right)' + q(v)\,y^{\alpha}(\delta(v)) = 0,$$

is oscillatory if

$$\left(\delta^{-1}(v)\right)' \ge \delta_0 > 0,\ \varsigma'(v) \ge \varsigma_0 > 0,\ \varsigma^{-1}(\delta(v)) < v$$

and

$$\liminf_{v\to\infty} \int_{\varsigma^{-1}(\delta(v))}^{v} \frac{\widehat{q}(s)}{r(s)}\left(s^{n-1}\right)^{\alpha} \mathrm{d}s > \left(\frac{1}{\delta_0} + \frac{p_0^{\alpha}}{\delta_0\varsigma_0}\right)\frac{((n-1)!)^{\alpha}}{\mathrm{e}},$$

where α is a quotient of odd positive integers and $\widehat{q}(v) := \min\left\{q\left(\delta^{-1}(v)\right), q\left(\delta^{-1}(\varsigma(v))\right)\right\}$.

In this article, using the technique of Riccati and comparison with first-order differential equations, we establish new Kamenev-type oscillation criteria of an even-order neutral differential equation. To illustrate the main results, some examples are mentioned.

Notation 1. *For convenience, we use the following notations:*

$$\varphi(v) := q(v)\left(1 - \vartheta(\delta(v))\right)$$

and

$$\tilde{\varphi}(v) := \frac{\mu\delta^{n-1}(v)}{(n-1)!a(\delta(v))} Q(v).$$

2. Some Auxiliary Lemmas

We shall employ the following lemmas:

Lemma 1 ([9])**.** *Let γ be a ratio of two odd numbers, $V > 0$ and U are constants. Then*

$$Uy - Vy^{(\gamma+1)/\gamma} \le \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{U^{\gamma+1}}{V^{\gamma}}.$$

Lemma 2 ([17])**.** *Let $u \in C^n([v_0,\infty),(0,\infty))$. If $u^{(n)}(v)$ is eventually of one sign for all large v, then there exist a $v_y > v_1$ for some $v_1 > v_0$ and an integer m, $0 \le m \le n$ with $n+m$ even for $u^{(n)}(v) \ge 0$ or $n+m$ odd for $u^{(n)}(v) \le 0$ such that $m > 0$ implies that $u^{(k)}(v) > 0$ for $v > v_y, k = 0,1,...,m-1$ and $m \le n-1$ implies that $(-1)^{m+k}u^{(k)}(v) > 0$ for $v > v_y, k = m, m+1, ..., n-1$.*

Lemma 3 ([18])**.** *Let $u \in C^n([v_0,\infty),(0,\infty))$. If $u^{(n-1)}(v)\,u^{(n)}(v) \le 0$ for $v \ge v_0$, then for every $\lambda \in (0,1)$ there exists a constant $k > 0$ such that*

$$|y(\lambda v)| \ge k v^{n-1}\left|y^{(n-1)}(v)\right|,$$

for all v large enough.

Lemma 4 ([19])**.** *Let $u \in C^n([v_0,\infty),(0,\infty))$. Assume that $u^{(n)}(v)$ is of a fixed sign, on $[v_0,\infty)$, $u^{(n)}(v)$ not identically zero and that there exists a $v_1 \ge v_0$ such that, for all $v \ge v_1$,*

$$u^{(n-1)}(v)\,u^{(n)}(v) \le 0.$$

If we have $\lim_{v\to\infty} u(v) \neq 0$*, then there exists* $v_\lambda \geq v_0$ *such that*

$$u(v) \geq \frac{\lambda}{(n-1)!} v^{n-1} \left| u^{(n-1)}(v) \right|,$$

for every $\lambda \in (0,1)$ *and* $v \geq v_\lambda$*.*

We define the generalized Riccati substitutions

$$\varphi(v) := \pi(v) \frac{a(v) u^{(n-1)}(v)}{u(\lambda\delta(v))}. \tag{4}$$

Lemma 5. *Assume that* $y(v)$ *is an eventually positive solution of Equation (1). Then*

$$u(v) > 0,\ u'(v) > 0,\ u^{(n-1)}(v) \geq 0 \ \text{ and } \ u^{(n)}(v) \leq 0, \tag{5}$$

for $v \geq v_2$*.*

Proof. Suppose $y(v)$ is an eventually positive solution of (1). Then, we can assume that $y(v) > 0$, $y(\varsigma(v)) > 0$ and $y(\delta(v)) > 0$ for $v \geq v_1$. Hence, we deduce $u(v) > 0$ for $v \geq v_1$ and

$$\left(a(v) u^{(n-1)}(v)\right)' = -q(v) \phi(y(\delta(v))) \leq 0. \tag{6}$$

This means that $a(v) u^{(n-1)}(v)$ is decreasing and $u^{(n-1)}(v)$ is eventually of one sign. We claim that $u^{(n-1)}(v) \geq 0$. Otherwise, if there exists a $v_2 \geq v_1$ such that $u^{(n-1)}(v) < 0$ for $v \geq v_2$, and

$$\left(a(v) u^{(n-1)}(v)\right) \leq \left(a(v_2) u^{(n-1)}(v_2)\right)(v_2) = -L,$$

where $L > 0$. Integrating the above inequality from v_2 to v we get

$$u^{(n-2)}(v) \leq u^{(n-2)}(v_2) - L \int_{v_2}^{v} \frac{1}{a(s)} \mathrm{d}s.$$

Letting $v \to \infty$, we have $\lim_{v\to\infty} u^{(n-2)}(v) = -\infty$, which contradicts the fact that $u(v)$ is a positive solution by Lemma 2. Hence, we have that $u^{(n-1)}(v) \geq 0$ for $v \geq v_1$. Furthermore, from Equation (1) and (P_1), we have

$$\left(a(v) u^{(n)}(v)\right) = -\left(a'(v) u^{(n-1)}(v)\right)(v) - q(v) \phi(y(\delta(v))) \leq 0,$$

this implies that $u^{(n)}(v) \leq 0,\ v \geq v_1$. From Lemma 2, we obtain that (5) are satisfied. This completes the proof of the lemma. □

3. Oscillation Criterion

In this section, we study the results of oscillation for (1) by using the technique of comparison with first order delay equations.

Theorem 1. *If for some constant* $\mu \in (0,1)$*, the differential equation*

$$y'(v) + \tilde{\varphi}(v) y(\delta(v)) = 0 \tag{7}$$

is oscillatory, then every solution of (1) is oscillatory.

Proof. Suppose that Equation (1) has a nonoscillatory solution in $[v_0, \infty)$. Without loss of generality, in our proof we only need to be concerned with positive solutions of Equation (1). Using Lemma 5, we get that (5) holds. From definition (2), we get

$$\begin{aligned} y^{\beta}(v) &= u(v) - \vartheta(v) y(\varsigma(v)) \geq u(v) - \vartheta(v) u(\varsigma(v)) \geq u(v) - \vartheta(v) u(v) \\ &\geq (1 - \vartheta(v)) u(v) \end{aligned}$$

and so

$$y^{\beta}(\delta(v)) \geq u(\delta(v))(1 - \vartheta(\delta(v))). \tag{8}$$

From (P_2) and (8), we find

$$\phi(y(\delta(v))) \geq u(\delta(v))(1 - \vartheta(\delta(v))). \tag{9}$$

Combining (1) and (9), we obtain

$$\begin{aligned} \left(a(v) u^{(n-1)}(v)\right)' &\leq -q(v) u(\delta(v))(1 - \vartheta(\delta(v))) \\ &\leq -u(\delta(v)) q(v)(1 - \vartheta(\delta(v))) \\ &= -\varphi(v) u(\delta(v)). \end{aligned} \tag{10}$$

In view of Lemma 4, we find

$$u(v) \geq \frac{\mu}{(n-1)!} v^{n-1} u^{(n-1)}(v),$$

for all $v \geq v_2 \geq \max\{v_1, v_\mu\}$. Thus, by using (10), we obtain

$$\left(a(v) u^{(n-1)}(v)\right)' + \frac{\mu \delta^{n-1}(v) \varphi(v)}{(n-1)! a(\delta(v))} \left(a(\delta(v)) u^{(n-1)}(\delta(v))\right) \leq 0.$$

Therefore, we see that $y(v) := a(v) u^{(n-1)}(v)$ is a positive solution of the differential inequality

$$y'(v) + \tilde{\varphi}(v) y(\delta(v)) \leq 0.$$

From ([19], Corollary 1), we have that the associated differential Equation (7) also has a positive solution, which yields a contradiction. This completes the proof. □

By using Theorem 2.1.1 in [20], we get the following corollary.

Corollary 1. *If*

$$\liminf_{v \to \infty} \int_{\delta(v)}^{v} \frac{\delta^{n-1}(s)}{a(\delta(s))} \varphi(s) \, ds > \frac{(n-1)!}{\mu e},$$

for some constant $\mu \in (0,1)$, then every solution of (1) is oscillatory.

Lemma 6. *Assume that y be an eventually positive solution of (1) and (5) holds. If we have the function $\varsigma \in C^1[v, \infty)$ defined as (4), where $\pi \in C^1([v_0, \infty), (0, \infty))$ and constants $\lambda \in (0,1)$, $k > 0$, then*

$$\varsigma'(v) \leq \frac{\pi'(v)}{\pi(v)} \varsigma(v) - \pi(v) \varphi(v) - \frac{\lambda}{\eta(v)} \varsigma^2(v), \tag{11}$$

for all $v > v_1$, where v_1 large enough.

Proof. Let y is an eventually positive solution of (1) and (5) holds. As in the proof of Theorem 1, we arrive at (10).

Now, by using Lemma 3 with $y = u'$, there exists $k > 0$ such that

$$\begin{aligned} u'(\lambda\delta(v)) &\geq k(\delta(v))^{n-2} u^{(n-1)}(\delta(v)) \\ &\geq k(\delta(v))^{n-2} u^{(n-1)}(v). \end{aligned} \tag{12}$$

From (4), we see that $\varsigma(v) > 0$ for $v \geq v_1$, and

$$\varsigma'(v) = \frac{\pi'(v)}{\pi(v)}\varsigma(v) + \pi(v)\frac{\left(a(v)u^{(n-1)}(v)\right)'}{u(\lambda\delta(v))} - \lambda\pi(v)\frac{a(v)u^{(n-1)}(v)u'(\lambda\delta(v))\delta'(v)}{(u(\lambda\delta(v)))^2}.$$

From (10), we obtain

$$\varsigma'(v) \leq \frac{\pi'(v)}{\pi(v)}\varsigma(v) - \pi(v)\varphi(v) - \lambda\frac{u'(\delta(v))\delta'(v)}{u(\lambda\delta(v))}\varsigma(v).$$

By using (12), we have

$$\varsigma'(v) \leq \frac{\pi'(v)}{\pi(v)}\varsigma(v) - \pi(v)\varphi(v) - \lambda\frac{k(\delta(v))^{n-2}u^{(n-1)}(v)\delta'(v)}{u(\lambda\delta(v))}\varsigma(v),$$

which yields

$$\varsigma'(v) \leq \frac{\pi'(v)}{\pi(v)}\varsigma(v) - \pi(v)\varphi(v) - \frac{\lambda}{\eta(v)}\varsigma^2(v).$$

The proof is complete. □

In this theorem, we establish new Kamenev-type oscillation criteria for (1).

Theorem 2. *If there exist a function $\pi \in C^1([v_0,\infty), \mathbb{R}^+)$ and constants $\lambda \in (0,1)$, $k > 0$, $m \in \mathbb{N}$ such that*

$$\limsup_{v\to\infty}\frac{1}{v^m}\int_{v_0}^{v}(v-s)^m\left(\pi(s)\varphi(s) - \frac{1}{4\lambda}\left(\frac{\pi'(s)}{\pi(s)}\right)^2\frac{a(s)\pi(s)}{k(\delta(s))^{n-2}\delta'(s)}\right)\mathrm{d}s = \infty, \tag{13}$$

then every solution of (1) is oscillatory.

Proof. Suppose that Equation (1) has a nonoscillatory solution in $[v_0,\infty)$. Without loss of generality, in our proof we only need to be concerned with positive solutions of Equation (1). From Lemma 1, we set $U = \pi'/\pi$, $V = \lambda k\delta^{n-2}(v)\delta'(v)/(a(v)\pi(v))$ and $y = \varsigma(v)$, thus, we have

$$\varsigma'(v) \leq -\pi(v)\varphi(v) + \frac{1}{4\lambda}\left(\frac{\pi'(v)}{\pi(v)}\right)^2\frac{a(v)\pi(v)}{k(\delta(v))^{n-2}\delta'(v)}.$$

Thus, we have

$$-\int_{v_0}^{v}(v-s)^m\varsigma'(s)(s)\,\mathrm{d}s \geq \int_{v_0}^{v}(v-s)^m\left(\pi(s)\varphi(s) - \frac{1}{4\lambda}\left(\frac{\pi'(s)}{\pi(s)}\right)^2\frac{a(s)\pi(s)}{k(\delta(s))^{n-2}\delta'(s)}\right)\mathrm{d}s.$$

Since

$$\int_{v_0}^{v}(v-s)^m\varsigma'(s)\,\mathrm{d}s = m\int_{v_0}^{v}(v-s)^{m-1}\varsigma(s)\,\mathrm{d}s - (v-v_0)^m\varsigma(v_0). \tag{14}$$

Thus, we get

$$\begin{aligned}&\left(\frac{v-v_0}{v}\right)^m \varsigma(v_0) - \frac{m}{v^m}\int_{v_0}^{v}(v-s)^{m-1}\varsigma(s)\,\mathrm{ds}\\ \geq\ & \frac{1}{v^m}\int_{v_0}^{v}(v-s)^m\left(\pi(s)\varphi(s) - \frac{1}{4\lambda}\left(\frac{\pi'(s)}{\pi(s)}\right)^2 \frac{a(s)\pi(s)}{k(\delta(s))^{n-2}\delta'(s)}\right)\mathrm{ds}.\end{aligned}$$

Hence,

$$\frac{1}{v^m}\int_{v_0}^{v}(v-s)^m\left(\pi(s)\varphi(s) - \frac{1}{4\lambda}\left(\frac{\pi'(s)}{\pi(s)}\right)^2 \frac{a(s)\pi(s)}{k(\delta(s))^{n-2}\delta'(s)}\right)\mathrm{ds} \leq \left(\frac{v-v_0}{v}\right)^m \varsigma(v_0),$$

and so

$$\limsup_{v\to\infty}\frac{1}{v^m}\int_{v_0}^{v}(v-s)^m\left(\pi(s)\varphi(s) - \frac{1}{4\lambda}\left(\frac{\pi'(s)}{\pi(s)}\right)^2 \frac{a(s)\pi(s)}{k(\delta(s))^{n-2}\delta'(s)}\right)\mathrm{ds} \to \varsigma(v_0),$$

which contradicts (13) and this completes the proof. □

Example 1. *For $v \geq 1$, consider the equation*

$$\left(v\left(y(v) + \frac{1}{2}y\left(\frac{v}{3}\right)\right)\right)'' + \frac{q_0}{v}y\left(\frac{v}{2}\right) = 0, \tag{15}$$

where $q_0 > 0$ is a constant. Note that $\beta = 1$, $n = m = 2$, $a(v) = v$, $\vartheta(v) = 1/2$, $q(v) = q_0/v$, $\delta(v) = v/2$ and $\varsigma(v) = v/3$. If we set $\pi(v) = v$, $k = 1$, then

$$\int_{v_0}^{\infty}\frac{1}{a(s)}ds = \int_{v_0}^{\infty}\frac{1}{s}ds = \infty$$

and

$$\varphi(v) := q(v)(1-\vartheta(\delta(v))) = \frac{q_0}{2v}.$$

Thus, we get

$$\begin{aligned}&\limsup_{v\to\infty}\frac{1}{v^m}\int_{v_0}^{v}(v-s)^m\left(\pi(s)\varphi(s) - \frac{1}{4\lambda}\left(\frac{\pi'(s)}{\pi(s)}\right)^2 \frac{a(s)\pi(s)}{k(\delta(s))^{n-2}\delta'(s)}\right)\mathrm{ds}\\ &= \limsup_{v\to\infty}\frac{1}{v^2}\int_{v_0}^{v}(v-s)^2(q_0-1)\,\mathrm{ds} = \infty.\end{aligned}$$

Therefore, by Theorem 2, all solution of (15) is oscillatory if $q_0 > 1$.

Example 2. *For $v \geq 1$, consider the equation*

$$\left(v\left(y(v) + \frac{1}{3}y\left(\frac{v}{2}\right)\right)\right)'' + \frac{c}{v}y\left(\frac{v}{3}\right) = 0, \tag{16}$$

where $c > 0$ is a constant. Note that $\beta = 1$, $n = m = 2$, $a(v) = v$, $\vartheta(v) = 1/3$, $q(v) = q_0/v$, $\delta(v) = v/3$ and $\varsigma(v) = v/2$. If we set $\pi(v) = v$, $k = 1$, then

$$\int_{v_0}^{\infty}\frac{1}{a(s)}ds = \int_{v_0}^{\infty}\frac{1}{s}ds = \infty$$

and

$$\varphi(v) := q(v)(1 - \vartheta(\delta(v))) = \frac{2c}{3v}.$$

By using Corollary 1, we find

$$\liminf_{v\to\infty} \int_{\delta(v)}^{v} \frac{\delta^{n-1}(s)}{a(\delta(s))} \varphi(s)\, ds$$
$$\liminf_{v\to\infty} \frac{2q_0}{3} \int_{\delta(v)}^{v} \frac{1}{s} ds.$$

Thus, all solution of (16) is oscillatory if $c > 0.5$.

4. Conclusions

In this paper, a class of even-order neutral differential equations is studied. We establish a new Kamenev-type oscillation criterion using the Riccati transformation and theory of comparison. Furthermore, in future work, we can to get some Hille and Nehari types and Philos type oscillation criteria of (1).

Funding: The author received no direct funding for this work.

Acknowledgments: The author thanks the reviewers for their useful comments, which led to the improvement of the content of the paper.

Conflicts of Interest: The author declares no conflicts of interest.

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