

Article

Direct Comparison between Two Third Convergence Order Schemes for Solving Equations

Samundra Regmi ¹, Ioannis K. Argyros ^{1,*} and Santhosh George ²

¹ Department of Mathematical Sciences, Cameron University, Lawton, OK 73505, USA; samundra.regmi@cameron.edu

² Department of Mathematical and Computational Sciences, National Institute of Technology Karnataka, Karnataka 575 025, India; sgeorge@nitk.edu.in

* Correspondence: iargyros@cameron.edu

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Abstract: We provide a comparison between two schemes for solving equations on Banach space. A comparison between same convergence order schemes has been given using numerical examples which can go in favor of either scheme. However, we do not know in advance and under the same set of conditions which scheme has the largest ball of convergence, tighter error bounds or best information on the location of the solution. We present a technique that allows us to achieve this objective. Numerical examples are also given to further justify the theoretical results. Our technique can be used to compare other schemes of the same convergence order.

Keywords: banach space; third convergence order schemes; ball convergence; Chebyshev-type scheme

1. Introduction

In this study we compare two third convergence order schemes for solving nonlinear equation

$$G(x) = 0, \quad (1)$$

where $G : D \subset B_1 \rightarrow B_2$ be a continuously differentiable nonlinear operator and D stands for an open non empty subset of B_1 . Here B_1 and B_2 denote Banach spaces. It is desirable to obtain a unique solution p of (1). However, this can rarely be achieved. So researchers develop iterative schemes which converge to p . Some popular schemes are

Chebyshev-Type Scheme:

$$\begin{aligned} y_n &= x_n + \alpha G'(x_n)^{-1} G(x_n) \\ x_{n+1} &= x_n - \frac{1}{2\alpha} G'(x_n)^{-1} A_n G'(x_n)^{-1} G(x_n), \end{aligned} \quad (2)$$

Simplified Chebyshev-Type Scheme:

$$\begin{aligned} y_n &= x_n - G'(x_n)^{-1} G(x_n) \\ x_{n+1} &= y_n + \frac{1}{2} G'(x_n)^{-1} (G'(y_n) - G'(x_n)) G'(x_n)^{-1} G(x_n), \end{aligned} \quad (3)$$

Two-Step-Newton-Type Scheme:

$$\begin{aligned} y_n &= x_n - G'(x_n)^{-1} G(x_n) \\ x_{n+1} &= y_n - G'(x_n)^{-1} G(y_n), \end{aligned} \quad (4)$$

where $A_n = (2\alpha - 1)G'(x_n) + G'(y_n)$, $\alpha \in \mathbb{R} - \{0\}$ or $\alpha \in \mathbb{C} - \{0\}$. Notice that (2) specializes to (3) for $\alpha = -1$.

The analysis of these schemes uses assumptions on the fourth order derivatives of G which are not on these schemes. The assumptions on fourth order derivatives reduce the applicability of these schemes. For example: Let $B_1 = B_2 = \mathbb{R}$, $D = [-\frac{1}{2}, \frac{3}{2}]$. Define G on D by

$$G(t) = \begin{cases} t^3 \log t^2 + t^5 - t^4, & t \neq 0 \\ 0, & t = 0. \end{cases}$$

Then, we get

$$\begin{aligned} G'(t) &= 3t^2 \log t^2 + 5t^4 - 4t^3 + 2t^2, \\ G''(t) &= 6t \log t^2 + 20t^3 - 12t^2 + 10t, \\ G'''(t) &= 6 \log t^2 + 60t^2 - 24t + 22, \end{aligned}$$

where the solution $p = 1$. Obviously $G'''(t)$ is not bounded on D . Hence, the convergence of the above schemes are not guaranteed by the earlier studies. In this study we use only assumptions on the first derivative to prove our results. The advantages of our approach include: larger radius needed on scheme of convergence (i.e., more initial points), tighter upper bounds on $\|x_k - p\|$ (i.e., fewer iterates to achieve a desired error tolerance). It is worth noting that these advantages are obtained without any additional conditions [1–35].

So far a comparison is given between iterative schemes of the same order using numerical examples [1–35]. However, not a direct comparison is given theoretically, so we know in advance under the same set of convergence conditions which scheme has the largest radius, tighter error bounds and better results on the uniqueness of the solution. The novelty of our paper is that, we introduce a technique under we can answer that scheme (4) is the best when compared to scheme (3). The same technique can be used to draw conclusions on other same order schemes.

Notice also that scheme (3) requires two derivative evaluations, one inverse and one operator evaluation. However, scheme (4) is less expensive requiring two function evaluations and one inverse. Both schemes have been studied in the literature under assumptions reaching the fourth derivative which does not appear in these schemes. However, we use only conditions on the first derivative that does appear on the schemes.

Throughout this paper $U(x, r)$ stand for open ball with center at x and radius $r > 0$ and $\bar{U}(x, r)$ denote the closure of $U(x, r)$.

Rest of the paper is structured as follows. The convergence analysis of schemes are given in Section 2 and examples is given in Section 3.

2. Ball Convergence

We present the ball convergence scheme (2), scheme (3) and scheme (4), respectively in this section.

To achieve this introduce certain functions and parameters. Suppose that there exists a continuous and increasing function defined on the interval $[0, \infty)$ with values in itself such that equation

$$\omega_0(t) - 1 = 0, \tag{5}$$

has a real positive zero denoted as R_0 . Suppose that there exists functions ω and ω_1 defined on $[0, R_0)$ with values in $[0, \infty)$. Define functions g_α^1 and h_α^1 on $[0, R_0)$ as

$$g_\alpha^1(s) = \frac{\int_0^1 \omega((1-\tau)s) d\tau + |1+\alpha| \int_0^1 \omega_1(\tau s) d\tau}{1 - \omega_0(s)}$$

and

$$h_\alpha^1(s) = g_\alpha^1(s) - 1.$$

Suppose that equation

$$h_\alpha^1(s) = 0 \tag{6}$$

has a least zero denoted by R_α^1 in $(0, R_0)$. Moreover, define functions g_α^2, h_α^2 on $[0, R_0)$ as

$$g_\alpha^2(s) = g_\alpha^1(s) + \frac{(\omega_0(s) + \omega_0(g_\alpha^1(s)s)) \int_0^1 \omega_1(\tau s) d\tau}{2|\alpha|(1 - \omega_0(s))^2}$$

and

$$h_\alpha^2(s) = g_\alpha^2(s) - 1.$$

Suppose equation

$$h_\alpha^2(s) = 0 \tag{7}$$

has a least zero denoted by R_α^2 in $(0, R_0)$. Define a radius of convergence R_α as

$$R_\alpha = \min\{R_\alpha^1, R_\alpha^2\}. \tag{8}$$

It follows that

$$0 \leq \omega_0(s) < 1, \tag{9}$$

$$0 \leq g_\alpha^i(s) < 1, i = 1, 2 \tag{10}$$

for all $s \in [0, R_\alpha)$. Set $e_n = \|x_n - p\|$. We introduce a set of conditions (A) under which the ball convergence for all schemes will be obtained.

- (A1) $G : D \rightarrow B_2$ is differentiable; there exists a simple zero p of equation $G(x) = 0$.
- (A2) There exists a continuous and increasing function ω_0 defined on $[0, \infty)$ with values in $[0, \infty)$ such that for all $x \in D$

$$\|G'(p)^{-1}(G'(x) - G'(p))\| \leq \omega_0(\|x - p\|),$$

provided R_0 exists and is defined in (5). Set $D_0 = D \cap U(p, R_0)$.

- (A3) There exist continuous and increasing functions ω and ω_1 on the interval $[0, R_0)$ with values in $[0, \infty)$ such that for all $x, y \in D_0$

$$\|G'(p)^{-1}(G'(y) - G'(x))\| \leq \omega(\|y - x\|)$$

and

$$\|G'(p)^{-1}G'(x)\| \leq \omega_1(\|x - p\|).$$

- (A4) $\bar{U}(p, R_\alpha) \subset D$, and R_α^1, R_α^2 exist and are given by (6) and (7), respectively, where R_α is defined by (8).
- (A5) There exists $S_\alpha \geq R_\alpha$ such that

$$\int_0^1 \omega_0(\tau S_\alpha) d\tau < 1.$$

Set $D_1 = D \cap U(p, S_\alpha)$.

Next, the main ball convergence result for scheme (2) is displayed.

Theorem 1. Under the conditions (A) choose $x_0 \in U(p, R_\alpha) - \{p\}$. Then, the following assertions hold true

$$\{x_n\} \subset U(p, R_\alpha), \lim_{n \rightarrow \infty} x_n = p, \tag{11}$$

$$\|y_n - p\| \leq g_\alpha^1(e_n)e_n \leq e_n < R_\alpha, \quad (12)$$

and

$$\|x_{n+1} - p\| \leq g_\alpha^2(e_n)e_n \leq e_n, \quad (13)$$

where functions g_α^1, g_α^2 were introduced earlier and R_α is defined in (8). The vector p is the only zero of Equation (1) in the set D_1 introduced in condition (A5).

Proof. It is based on induction which assists us to show (12) and (13). If $z \in U(p, R_\alpha) - \{p\}$, we can use (A1), (A2), (8) and (9) to see that

$$\|G'(p)^{-1}(G'(z) - G'(p))\| \leq \omega_0(\|z - p\|) < \omega_0(R_\alpha) \leq 1,$$

so by the perturbation Banach result for invertible operators [30] $G'(z)^{-1} \in L(B_2, B_1)$ with

$$\|G'(z)^{-1}G'(p)\| \leq \frac{1}{1 - \omega_0(\|z - p\|)}, \quad (14)$$

so y_0 and x_1 exist by scheme (2) for $n = 0$. We use the identity

$$G(z) - G(p) = \int_0^1 G'(p + \tau(z - p))d\tau(z - p)$$

and the second condition in (A3) to obtain

$$\|G'(p)^{-1}G'(z)\| \leq \int_0^1 \omega_1(\tau\|z - p\|)d\tau\|z - p\|. \quad (15)$$

Next, in view of (8), (10), (10) (for $i = 1$), (A1), (A3), (14) (for $z = x_0$), scheme (2) and (16) the following are obtained in sequence

$$\begin{aligned} \|y_0 - p\| &= \|x_0 - p - G'(x_0)^{-1}G(x_0) + (1 + \alpha)G'(x_0)^{-1}G(x_0)\| \\ &= \|(G'(x_0)^{-1}G'(p))(G'(p)^{-1} \int_0^1 (G'(p + \tau(x_0 - p)) - G'(x_0))(x_0 - p))d\tau \\ &\quad + (1 + \alpha)G'(x_0)^{-1}G(x_0)\| \\ &\leq \frac{\int_0^1 \omega((1-\tau)e_0)d\tau e_0 + [1+\alpha] \int_0^1 \omega_1(\tau e_0)d\tau e_0}{1 - \omega_0(e_0)} \\ &= g_\alpha^1(e_0)e_0 \leq e_0 < R_\alpha, \end{aligned} \quad (16)$$

leading to $y_0 \in U(p, R_\alpha)$ and the verification of (12) for $n = 0$. Moreover, as in (16) the following are obtained in sequence

$$\begin{aligned} \|x_1 - p\| &\leq \|(x_0 - p - G'(x_0)^{-1}G(x_0)) \\ &\quad + (I - \frac{1}{2\alpha}G'(x_0)^{-1}A_0)G'(x_0)^{-1}G(x_0)\| \\ &\leq \|(y_0 - p + \frac{1}{2\alpha}(G'(x_0)^{-1}G'(p))(G'(p)^{-1}(G'(x_0) - G'(y_0))) \\ &\quad \times (G'(x_0)^{-1}G'(p))(G'(p)^{-1}G(x_0))\| \\ &\leq [g_\alpha^1(e_0) + \frac{(\omega_0(e_0) + \omega_0(\|y_0 - p\|)) \int_0^1 \omega_1(\tau e_0)d\tau}{2|\alpha|(1 - \omega_0(e_0))^2}]e_0 \\ &\leq g_\alpha^2(e_0)e_0 \leq e_0, \end{aligned} \quad (17)$$

leading to $x_1 \in U(p, R_\alpha)$ and the verification of (13). Thus, estimates (12) and (13) hold true for $n = 0$. Suppose they hold true for all $j \leq n - 1$. Then, by exchanging x_0, y_0, x_1 by x_j, y_j, x_{j+1} in the preceding calculations, we terminate the induction for (12) and (13). Next, by the estimate

$$\|x_{n+1} - p\| \leq be_0 < R_\alpha, \quad (18)$$

where $b = g_\alpha^2(e_0) \in [0, 1)$, we deduce $\lim_{n \rightarrow \infty} x_n = p$ and $x_{n+1} \in U(p, R_\alpha)$. Further, let $v \in D_1$ with $G(v) = 0$ and set $E = \int_0^1 G'(v + \tau(p - v))d\tau$. By (A1) and (A5) we get

$$\|G'(p)^{-1}(E - G'(p))\| \leq \int_0^1 \omega_0((1 - \tau)\|p - v\|)d\tau \leq \int_0^1 \omega_0(\tau S_\alpha)d\tau < 1,$$

so the invertibility is implied leading together with the estimate $0 = G(p) - G(v) = E(p - v)$ to the conclusion that $p = v$. □

Remark 1.

1. In view of (A3) and the estimate

$$\begin{aligned} \|G'(p)^{-1}G'(x)\| &= \|G'(p)^{-1}(G'(x) - G'(p)) + I\| \\ &\leq 1 + \|G'(p)^{-1}(G'(x) - G'(p))\| \leq 1 + \omega_0(\|x - p\|) \end{aligned}$$

condition (C5) can be dropped and ω_1 can be replaced by

$$\omega_1(t) = 1 + \omega_0(t) \text{ or } \omega_1(t) = 1 + \omega_0(R_0).$$

2. The results obtained here can be used for operators F satisfying autonomous differential equations [3] of the form

$$G'(x) = P(G(x))$$

where P is a continuous operator. Then, since $G'(p) = P(G(p)) = P(0)$, we can apply the results without actually knowing p . For example, let $G(x) = e^x - 1$. Then, we can choose: $P(x) = x + 1$.

3. If ω_0 and ω_1 are constant functions, say $\omega_0(t) = L_0t, \omega(t) = Lt$, for some $L_0 > 0$ and $L > 0$, then the radius $r_1 = \frac{2}{2L_0+L}$ was shown by us to be the convergence radius of Newton’s method [5,6]

$$x_{n+1} = x_n - G'(x_n)^{-1}G(x_n) \text{ for each } n = 0, 1, 2, \dots \tag{19}$$

under the conditions (C1)–(C4). It follows from the definition of r that the convergence radius R_α of the method (2) cannot be larger than the convergence radius r_1 of the second order Newton’s method (19). As already noted in [5,6] r_1 is at least as large as the convergence ball given by Rheinboldt [14]

$$r_R = \frac{2}{3L_1}, \tag{20}$$

where L_1 is the Lipschitz constant on D . In particular, for $L_0 < L_1$ we have that

$$r_R < r$$

and

$$\frac{r_R}{r_1} \rightarrow \frac{1}{3} \text{ as } \frac{L_0}{L_1} \rightarrow 0.$$

That is our convergence ball r_1 is at most three times larger than Rheinboldt’s. The same value for r_R was given by Traub [15].

4. It is worth noticing that method (2) is not changing when we use simpler methods the conditions of Theorem 1 instead of the stronger conditions used in [10]. Moreover, we can compute the computational order of convergence (COC) defined by

$$\zeta = \ln \left(\frac{\|x_{n+1} - p\|}{\|x_n - p\|} \right) / \ln \left(\frac{\|x_n - p\|}{\|x_{n-1} - p\|} \right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln \left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|} \right) / \ln \left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|} \right).$$

This way we obtain in practice the order of convergence in a way that avoids the bounds involving estimates using estimates higher than the first Fréchet derivative of operator F .

5. Method (2) can be generalized as

$$y_n = G(x_n) \quad (21)$$

$$x_{n+1} = H(y_n), \quad (22)$$

where $G : D \rightarrow X$, $H : X \rightarrow X$ are continuous operators

(i) For all $x \in U(x_*, R)$, we have $y = G(x) \in U(x_*, R)$ for $R > 0$, $x_* \in D$ such that $U(x_*, R) \subset D$ and $G(x_*) = 0$;

(ii) $\|H(y) - x_*\| \leq d\|x - x_*\|$ for some $d \in [0, 1)$.

Then, it follows $\lim_{n \rightarrow \infty} x_n = x_*$ provided that $x_0 \in U(x_*, R)$. Indeed, we have

$$\|x_{n+1} - x_*\| = \|H(y_n) - x_*\| \leq d\|x_n - x_*\| \leq d^{n+1}\|x_0 - x_*\| < R$$

so $x_{n+1} \in U(x_*, R)$ and $\lim_{n \rightarrow \infty} x_n = x_*$. Notice that if $G(x) = x - A(x)^{-1}G(x)$, $H(y) = y - B(x)^{-1}G(y)$, then (21) specializes to scheme (2).

6. The ball convergence result for scheme (3) clearly is obtained from Theorem 1 for $\alpha = -1$.

7. In our earlier works with other schemes we assumed $\omega_0(0) = \omega_1(0) = 0$ to show the existence of R_α^1 and R_α^2 using the intermediate value theorem. But these initial conditions on ω_0 and ω_1 . This is not necessary with our approach. This way we further expand the applicability of scheme (2) and (3). The same is true for scheme (4) whose ball convergence follows.

Theorem 2. Under the conditions of Theorem 1, the conclusions of it hold for scheme (3) for $\alpha = -1$.

To deal with scheme (4) our real functions and parameters are $g_{-1}^1, h_{-1}^1, R_{-1}^1$. Moreover, we suppose that equation

$$\omega_0(g_{-1}^1(s)s) - 1 = 0$$

has a least solution in $(0, R_0)$ denoted by R_0^1 and set $\bar{R}_0 = \min\{R_0, R_0^1\}$. Define functions \bar{g}_2 and \bar{h}_2 on $(0, \bar{R}_0)$ as

$$\bar{g}_2(s) = \left[1 + \frac{\omega_0(s) + \omega_0(g_{-1}^1(s)s) \int_0^1 \omega_1(\tau g_{-1}^1(s)s) d\tau}{(1 - \omega_0(s))(1 - \omega_0(g_{-1}^1(s)s))} \right] g_{-1}^1(s)$$

and

$$\bar{h}_2(s) = \bar{g}_2(s) - 1.$$

Suppose equation $\bar{h}_2(s) = 0$ has a least positive zero in $(0, \bar{R}_0)$ denoted by \bar{R}_2 . Define a radius of convergence \bar{R} by

$$\bar{R} = \min\{R_{-1}^1, \bar{R}_2\}. \quad (23)$$

We also use

$$\begin{aligned} x_{n+1} - p &= y_n - p + (F'(y_n)^{-1} - F'(x_n)^{-1})F(y_n) \\ &= y_n - p + F'(y_n)^{-1}(F'(x_n) - F'(y_n))F'(x_n)^{-1}F(y_n) \end{aligned}$$

instead of (17) to obtain

$$\begin{aligned} \|x_{n+1} - p\| &\leq \left[1 + \frac{\omega_0(e_n) + \omega_0(\|y_n - p\|) \int_0^1 \omega_1(\tau \|y_n - p\|) d\tau}{(1 - \omega_0(e_n))(1 - \omega_0(\|y_n - p\|))}\right] \|y_n - p\| \\ &\leq \bar{g}_2(e_n) e_n \leq e_n < \bar{R}. \end{aligned} \tag{24}$$

Hence, we arrive at:

Theorem 3. Under the conditions (A) but with $\bar{R}, R_*, \bar{b} = \bar{g}_2(e_0)$, replacing R, S_α, b , respectively, the following hold for method (4), $\{x_n\} \in U(p, \bar{R})$, $\lim_{n \rightarrow \infty} x_n = p$,

$$\|y_n - p\| \leq g_{-1}^1(e_n) e_n \leq e_n < \bar{R} \tag{25}$$

and

$$\|x_{n+1} - p\| \leq \bar{g}_2(e_n) e_n \leq e_n. \tag{26}$$

Moreover, p is the unique solution of Equation (1) in the set D_1 .

Remark 2. We have

$$\bar{g}(s) \leq g_{-1}^1(s) \tag{27}$$

and

$$\bar{h}_2(s) \leq h_{-1}(s) \tag{28}$$

for all $s \in (0, R_0)$, so

$$R \leq \bar{R} \tag{29}$$

and

$$\bar{b} \leq b. \tag{30}$$

Hence, the radius \bar{R} of scheme (4) is at least as large as that of scheme (3) whereas the ratio of convergence of scheme (4) is at least as small as that of scheme (3) (see also the numerical examples).

3. Numerical Examples

We compute the radii provided that $\alpha = -1$.

Example 1. Let us consider a system of differential equations governing the motion of an object and given by

$$F_1'(x) = e^x, F_2'(y) = (e - 1)y + 1, F_3'(z) = 1$$

with initial conditions $F_1(0) = F_2(0) = F_3(0) = 0$. Let $H = (F_1, F_2, F_3)$. Let $B_1 = B_2 = \mathbb{R}^3, D = \bar{U}(0, 1), p = (0, 0, 0)^T$. Define function F on D for $w = (x, y, z)^T$ by

$$F(w) = (e^x - 1, \frac{e - 1}{2} y^2 + y, z)^T.$$

We need the Fréchet-derivative defined by

$$F'(w) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e - 1)y + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

to compute function ω_0 (see (A2)) and functions ω, ω_1 (see (A3)). Notice that using the ((A)) conditions, we get $\omega_0(t) = (e - 1)t, \omega(t) = e^{\frac{1}{e-1}t}, \omega_1(t) = e^{\frac{1}{e-1}}$. The radii are given in Table 1.

Table 1. Comparison Table of Chebyshev (2) and Simplified Chebyshev (3) method with Two-Step-Newton method (4).

$\alpha = -1$	(2) & (3)	(4)
	$R_\alpha^1 = 0.38269191223238574472986783803208$ $R_\alpha^2 = 0.21292963191331051864274570561975$	$R_{-1}^1 = 0.38269191223238574472986783803208$ $\bar{R}_2 = 1.6195946293252685421748537919484$
radius	$R_\alpha = R_\alpha^2$	$\bar{R} = R_{-1}^1$

Example 2. Let $B_1 = B_2 = C[0, 1]$, the space of continuous functions defined on $[0, 1]$ be equipped with the max norm. Let $D = \bar{U}(0, 1)$. Define function F on D by

$$F(\varphi)(x) = \varphi(x) - 5 \int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{31}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15 \int_0^1 x\theta\varphi(\theta)^2 \xi(\theta) d\theta, \text{ for each } \xi \in D.$$

Then, we get that $x^* = 0$, so $\omega_0(t) = 7.5t, \omega(t) = 15t$ and $\omega_1(t) = 2$. Then they are given in Table 2.

Table 2. Comparison Table of Chebyshev (2) and Simplified Chebyshev (3) method with Two-Step-Newton method (4).

$\alpha = -1$	(2) & (3)	(4)
	$R_\alpha^1 = 0.06666666666666666666666666666667$ $R_\alpha^2 = 0.040057008172481922692043099232251$	$R_{-1}^1 = 0.06666666666666666666666666666667$ $\bar{R}_2 = 0.014407266709891463490889051968225$
radius	$R_\alpha = R_\alpha^2$	$\bar{R} = \bar{R}_2$

Example 3. The example at the introduction of this study, gives $\omega_0(t) = \omega(t) = 96.6629073t$ and $\omega_1(t) = 2$. The parameters are given in Table 3.

Table 3. Comparison Table of Chebyshev (2) and Simplified Chebyshev (3) with Two-Step-Newton method (4)

$\alpha = -1$	(2) & (3)	(4)
	$R_\alpha^1 = 0.0068968199414654552878434223828208$ $R_\alpha^2 = 0.00377575830521247697221798311773$	$R_{-1}^1 = 0.0068968199414654552878434223828208$ $\bar{R}_2 = 0.000205625336098942375698261919581$
radius	$R_\alpha = R_\alpha^2$	$\bar{R} = \bar{R}_2$

4. Conclusions

A new technique is introduced allowing to compare schemes of the same convergence order under the same set of conditions. Hence, we know how to choose in advance among all third convergence order schemes the one providing larger choice of initial points the least number of iterates for a predetermined error tolerance and the best location on the solution. This technique can be used on other schemes along the same lines. In particular, we have shown that scheme (4) is better to use than scheme (3) under the condition (A).

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