Magnetized Flow of Cu + Al₂O₃ + H₂O Hybrid Nanofluid in Porous Medium: Analysis of Duality and Stability

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Abstract: In this analysis, we aim to examine the heat transfer and flow characteristics of a copper-aluminum/water hybrid nanofluid in the presence of viscous dissipation, magnetohydrodynamic (MHD), and porous medium effect over the shrinking sheet. The governing equations of the fluid model have been acquired by employment of the model of Tiwari and Das, with additional properties of the hybrid nanofluid. The system of partial differential equations (PDEs) has been converted into ordinary differential equations (ODEs) by adopting the exponential similarity transformation. Similarity transformation is an essential class of phenomenon where the symmetry of the scale helps to reduce the number of independent variables. Note that ODE solutions demonstrate the PDEs symmetrical behavior for the velocity and temperature profiles. With BVP4C solver in the MATLAB program, the system of resulting equations has been solved. We have compared the present results with the published results and found in excellent agreements. The findings of the analysis are also displayed and discussed in depth graphically and numerically. It is discovered that two solutions occur in definite ranges of suction and magnetic parameters. Dual (no) similarity solutions can be found in the range of \( S_c \leq S \) and \( M_c \leq M \) \( (S_c > S \) and \( M_c > M \)). By performing stability analysis, the smallest values of eigenvalue are obtained, suggesting that a stable solution is the first one. Furthermore, the graph of the smallest eigenvalue shows symmetrical behavior. By enhancing the Eckert number values the temperature of the fluid is raised.

Keywords: hybrid nanofluid; porous medium; dual solutions; viscous dissipation; stability analysis

1. Introduction

The concept of nanofluid was first proposed in 1995, by Choi and Eastman [1]. In their spearheading research, they found that the enhancement of the heat transfer rate in nanofluid is higher...
as compared to any kind of simple viscous fluids. It is made by mixing solid nanoparticles in the base fluids [2]. Until now, various kinds of base fluids with different kinds of solid nanoparticles have been mixed and examined in the literature. Nanoparticles can be polymeric nanoparticles, magnetic nanoparticles, dendrimers, liposomes, metallic nanoparticles, quantum dots, and numerous others. The ethylene glycol, water, oil, sodium alginate and, etc. can be used as the base fluids. Further, carbon nanotubes and the well-known graphite are categorized as the no-metallic nanoparticles, while nitrides, metal oxides, copper, carbides, and, alumina are the metallic nanoparticles. Ghazvini et al. [3] used an experimental approach to examine the water-based nanofluid which contained the CuFe2O4 nanoparticles. Tassaddiq et al. [4] investigated the sodium alginate nanofluid through a numerical approach where they used the model of Brinkman. Molybdenum disulphide (MoS2) nanoparticles were considered as solid particles. Further, single-wall carbon nanotubes (SWCNT) and multi-wall carbon nanotubes (MWCNT) carbon nanotubes in the water base fluid were considered by Nadeem et al. [5]. Mitra et al. [6] tried to find the maximum heat transfer rate of TiO2 in the examination by considering the various sizes and phases of the nanoparticles. Sahoo and Kumar [7] investigated the various mixtures of nanoparticles in order to find the maximum heat transfer rate. In this regard, three nanoparticles Al2O3, CuO, and TiO2 were considered, and also Al2O3–CuO–TiO2 ternary hybrid nanofluid were examined. Shafiq et al. [8] numerically examined the single-and multi-wall carbon nanotubes. Further, Gireesha et al. [9] used a two-phase nanofluid model to investigate Jeffrey nanofluid in a three-dimensional framework. Lund et al. [10] used the same model of Gireesha et al. [9] for the examination of the micropolar nanofluid. It is worth mentioning here that various mixes of nanoparticles in the different base fluids have been investigated. Nevertheless, nobody concluded which nanoparticle mixture and base fluid may offer superior heat transfer rate improvement (refer to the work of [11–18]).

The new development in the innovation of technologies required more heat transfer rates. The most recent investigation of the nanofluids revealed that the better heat transfer rate can be achieved with a new kind of nanofluid which is a hybrid nanofluid. It is composed by dispersing two different types of nanoparticles in the base fluid. This new kind of nanofluid can support the technologies of the industries and engineering with the minimum cost as it works effectively in vehicle thermal management/engine cooling, generator cooling, heating and cooling in buildings, biomedical, electronic cooling, nuclear system cooling, etc. By suspending the suitable combination of nanoparticles, even for the small volume fraction of nanoparticles, the required effects of heat transfer rate can be achieved [19]. However, not many investigations were accounted for on the synthesis and preparation of the hybrid nanofluid as it is a new kind of fluid [20]. It tends to be closed from the overview of the published literature that the exceptionally limited studies have been focused on the heat transfer and the fluid flow of the hybrid nanofluids numerically. Lund et al. [21] considered the shrinking surface to investigate the hybrid nanofluid by considering copper and alumina as solid particles and found double solutions. We have looked at Devi and Devi’s thermophysical model [22]. Devi and Devi [22] compared their numerical findings with the experimental outcomes of Suresh et al. [23] and found a good contrast between them. Therefore, it can be expected that our present results would be beneficial for those who are working in this area, as the model of Devi and Devi [23] has good validation with the experimental work. Hayat and Nadeem [24] investigated the hybrid nanofluid and found that hybrid nanofluid has more capacity to transfer heat than simple nanofluid. Jamshed et al. [25] studied hybrid nanofluid using engine oil as the regular fluid and observed that higher heat transfer is possible only for minimum value of shape feature parameter. Dual solutions with stability analysis of stagnation point flow of hybrid nanofluid were examined by Rostami et al. [26]. The magnetohydrodynamic (MHD) flow from hybrid Nanofluid based on water over shrinking/stretching sheets has been studied by Aly and Pop [27] and double solutions have been found. In comparison with the hybrid nanofluid TiO2–Cu/H2O, Khan et al. [28] revealed that the lower Nusselt is the Cu-Water nanofluid. Olatundun and Makinde [29] modified the model of Blasius for the hybrid nanofluid in which convective condition had also been considered. Chamkha et al. [30] examined the hybrid nanofluid in the rotating system where they found that “Nusselt number acts as an ascending function of injection and
radiation parameters, as well as volume fraction of nanofluid”]. Maskeen et al. [31] investigated the hydromagnetic alumina–copper/water hybrid nanofluid. An interesting development of the hybrid nanofluid can be seen in these papers [32–38].

The exponentially shrinking/stretching surface is commonly utilized with the fluid flow and the heat transfer in daily life and industrial problems. It seems that the Magyari and Keller [39] paper is the first paper on the exponentially stretching sheet to look at the fluid boundary layer flow. Mushtaq et al. [40] utilized the exponential similarity variables to transform the governing PDEs into ODEs. Further, Reddy et al. [41] considered the mixed convection flow of nanofluid over the exponential surface where they found that for the highest value of viscous ratio parameter then concentration, momentum, thermal boundary layers thicknesses enhance. Rahman et al. [42] developed the MHD flow model for the stagnation point where exponential similarity variables were used to get ODEs. Some studies of fluid flow over the exponential surface can be found in these articles [43–47]. Bachok et al. [48] used a single-phase model of nanofluid over the exponential surface and concluded that two solutions existed in an exponentially shrinking sheet. Moreover, unsteady stagnation point flow of nanofluid was considered by Dzulkifli et al. [49]. Anuar et al. [50] examined the hybrid nanofluid flow over an exponential sheet and noticed double solutions. Meanwhile, the stability of the solutions was also evaluated because solutions were not uniquely present. Waini et al. [51] have recently considered and obtained two branches for a hybrid nanofluid MHD flow in the impact of the heat radiation effect. They considered alumina and copper as the nanoparticles with water and concluded that heat transfer was reduced in both solutions when the thermal radiation parameter was enhanced.

In this study, we have extended the research of the Waini et al. [51] to examine the MHD flow of the Al₂O₃–Cu/water hybrid on the exponentially shrinking sheet. Moreover, the effects of the porous medium and viscous dissipation have been taken into account. By applying exponential similarity transformation variables, momentum, and energy conservations are converted to the system of ODEs. The numerical solutions of the resulting equations have been determined by employing the shooting technique. Further, analysis of the stability of solutions has also been performed to specify a stable solution with a bvp4c solver. The effects of the different application parameters are shown graphically on the heat transfer rate and skin friction coefficient. Lastly, this work can be extended in the following directions: (i) considering the vertical exponential surface with thermal radiation effect; (ii) considering the first and second-order slip conditions, and (iii) considering the entire model for the three-dimensional flow.

2. Mathematical Modeling

Let us take the two-dimensional, steady, MHD, and incompressible flow of a hybrid nanofluid in the presence of viscous dissipation and porosity on the exponentially shrinking surface (allude to Figure 1). The governing momentum, mass, and energy conservation are expressed in the following terms by considering all assumptions (see [51–53]):

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]  
\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\mu_{nf}}{\rho_{nf}} \frac{\partial^2 u}{\partial y^2} - \frac{\mu_{nf}}{\rho_{nf} k} u - \frac{\alpha_{nf}}{\rho_{nf}} \beta^2 u \]  
\[ u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{k_{nf}}{(\rho c_p)_{nf}} \frac{\partial^2 T}{\partial y^2} + \frac{\mu_{nf}}{\rho c_p} \left( \frac{\partial u}{\partial y} \right)^2 \]
The boundary conditions are

\[
\begin{align*}
v &= v_w(x), \quad u = u_w, \quad T = T_w \text{ as } y = 0 \\
u &\rightarrow 0, \quad T \rightarrow T_\infty \text{ as } y \rightarrow \infty
\end{align*}
\]  
(4)

denotes velocities along the y-axis and x-axis are \(v\) and \(u\), respectively, \(T\) and \(T_w(x)\) are the temperature of fluid and surface, respectively, where \(T_w(x) = T_\infty + T_0 e^{\tau x}\), \(T_\infty\) is free stream temperature, \(B = B_0 e^{\tau x}\) is the field of magnetic where \(B_0\) is constant magnetic strength, \(K = 2K_0 e^{\tau x}\) is considered as the permeability of porous medium where \(K_0\) is the reference permeability, \((\rho C_p)_{lmf}, \rho_{lmf}, \sigma_{lmf}, k_{lmf}\), and \(\mu_{lmf}\), are corresponding effective heat capacity, density, electrical conductivity, thermal conductivity, and viscosity of hybrid nanofluid. Further, \(u_w = -U_w e^{\tau x}\) is the surface velocity, and \(v_w = \sqrt{\frac{2 U_w}{\tau^2}} e^{\tau x} S\) where \(S\) is blowing/suction parameter.

**Figure 1.** Physical model and coordinate system.

In the current work, we use the thermophysical features of nanomaterials, base fluid, and thermophysical properties of hybrid nanofluid [51]. In these regards, Tables 1 and 2 are presented.

**Table 1.** Properties of thermophysical of hybrid nanofluid.

<table>
<thead>
<tr>
<th>Names</th>
<th>Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>Viscosity of Dynamic Density</td>
<td>(\mu_{lmf} = \frac{\mu}{(1-\phi_{Cu})(1-\phi_{Al2O3})}^{\frac{1}{3}})</td>
</tr>
<tr>
<td>Density</td>
<td>(\rho_{lmf} = (1-\phi_{Cu}) \left( 1 - \phi_{Al2O3} \right) \left( 1 - \phi_{Cu} \phi_{Al2O3} \right) + \phi_{Cu} \rho_{Cu})</td>
</tr>
<tr>
<td>Thermal conductivity</td>
<td>(k_{lmf} = \frac{k_{Cu} + 2k_{Al2O3}(k_{Cu}-k_{Al2O3})}{k_{Cu} + 2k_{Al2O3}(k_{Cu}+k_{Al2O3})} \times (k_{f}))</td>
</tr>
<tr>
<td>where (k_{f} = \frac{k_{Cu} + 2k_{Al2O3}(k_{Cu}-k_{Al2O3})}{k_{Cu} + 2k_{Al2O3}(k_{Cu}+k_{Al2O3})} \times (k_{f}))</td>
<td></td>
</tr>
<tr>
<td>Heat capacity</td>
<td>(\left(\rho C_p\right)<em>{lmf} = (1-\phi</em>{Cu}) \left( 1 - \phi_{Al2O3} \right) \left( 1 - \phi_{Cu} \phi_{Al2O3} \right) \left( \rho C_p \right)<em>{Cu} + \phi</em>{Cu} \left( \rho C_p \right)_{Al2O3})</td>
</tr>
<tr>
<td>Electrical conductivity</td>
<td>(\sigma_{lmf} = \frac{\sigma_{Cu} + 2\sigma_{Al2O3}(\sigma_{Cu}-\sigma_{Al2O3})}{\sigma_{Cu} + 2\sigma_{Al2O3}(\sigma_{Cu}+\sigma_{Al2O3})} \times (\sigma_{f}))</td>
</tr>
<tr>
<td>where (\sigma_{f} = \frac{\sigma_{Cu} + 2\sigma_{Al2O3}(\sigma_{Cu}-\sigma_{Al2O3})}{\sigma_{Cu} + 2\sigma_{Al2O3}(\sigma_{Cu}+\sigma_{Al2O3})} \times (\sigma_{f}))</td>
<td></td>
</tr>
</tbody>
</table>
To reduce the system into ODEs, we have the following variables of similarity transformation

$$\psi = \sqrt{2\frac{\partial f}{\partial \eta}} \frac{U_w e^{\frac{\xi}{2}}} {\lambda} f(\eta); \quad \theta(\eta) = \frac{T - T_{\infty}} {T_w - T_{\infty}}; \quad \eta = y \sqrt{\frac{U_w}{2\frac{\partial f}{\partial \eta}}}$$

(5)

where stream function is $\psi$ and velocities are explained as $u = \frac{\partial \psi}{\partial \eta}$ and $v = \frac{\partial \psi}{\partial \xi}$. By substituting Equation (5) in the Equations (2) and (3), which implies that

$$f'''' + \xi_1 \left(f'' - 2(f')^2\right) - \left[\gamma + \frac{\alpha_{lnf}}{\alpha_f} \xi_2 M\right] f' = 0$$

(6)

$$\frac{k_{lnf}}{k_f} \frac{\eta_1}{\eta_3} \theta'' + \theta f - \theta f' + \frac{Ec} {\xi_2 \xi_3} (f''')^2 = 0$$

(7)

$$\left\{ \begin{align*}
\xi_1 &= \xi_2 \left(1 - \phi_{Cu}\right) \left[1 - \phi_{Al_2O_3} + \phi_{Al_2O_3} \left(\frac{\rho_{Al_2O_3}} {\rho_f}\right)\right] + \phi_{Cu} \left(\frac{\rho_{Cu}} {\rho_f}\right) \\
\xi_2 &= \xi_3 \left(1 - \phi_{Cu}\right) ^{2.5} \left(1 - \phi_{Al_2O_3}\right) ^{2.5} \\
\xi_3 &= \left(1 - \phi_{Cu}\right) \left[1 - \phi_{Al_2O_3} + \phi_{Al_2O_3} \left(\frac{\rho_{Al_2O_3}} {\rho_f}\right)\right] + \phi_{Cu} \left(\frac{\rho_{Cu}} {\rho_f}\right)
\end{align*} \right.$$ (8)

Along with boundary conditions

$$\left\{ \begin{align*}
f(0) &= S, f'(0) = -1, \quad \theta(0) = 1 \\
f'(\eta) &\rightarrow 0; \quad \theta(\eta) &\rightarrow 0 \text{ as } \eta \rightarrow \infty
\end{align*} \right.$$ (9)

where $M = \frac{2\alpha_f (\beta_{0f})^2}{\nu_f U_w}$ is the magnetic number $\gamma = \frac{\lambda_f}{\lambda_w K}$ is the permeability parameter, $Pr = \frac{\alpha_f}{\alpha_w}$ is Prandtl number, and $Ec = \frac{\nu_f}{(\rho_w) T_0}$ is Eckert number.

The significant physical factors are skin friction coefficient $C_f$ and local Nusselt number $Nu_x$ and explained as

$$C_f = \frac{\mu_{lnf}} {\rho_f \mu_0} \left(\frac{\partial u}{\partial y}\right)|_{y=0}, \quad Nu_x = -\frac{x k_{lnf}} {k_f (T_w - T_{\infty})} \left(\frac{\partial T}{\partial y}\right)|_{y=0}$$ (10)

By using Equation (5), these are obtained

$$\sqrt{Re} C_f = \frac{1}{\xi_2} f''(0); \quad \sqrt{\frac{1}{Re}} Nu_x = -\frac{k_{lnf}} {k_f} \theta'(0)$$ (11)

3. Stability Analysis

To perform a stability analysis of the solution, the governing Equations (2) and (3) are needed to be reduced to unsteady flow problems. Thus, we get

$$\frac{\partial u} {\partial t} + u \frac{\partial u} {\partial x} + v \frac{\partial u} {\partial y} = \frac{\mu_{lnf}} {\rho_n f} \frac{\partial^2 u} {\partial x^2} - \frac{\mu_{lnf}} {\rho_{lnf} K} u - \frac{\sigma_{lnf}} {\rho_{lnf}} B^2 u$$ (12)
we have to solve the following linear eigenvalues problems.

\[
\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha_{nf} \frac{\partial^2 T}{\partial y^2} + \left( \rho c_p \right)_{inf} \left( \frac{\partial \psi}{\partial y} \right)^2
\]

By introducing a time dependent variable \( \tau \), we have the following new non-dimensionless similarity transformation variables as mentioned in the paper of Lund et al. [54]:

\[
\psi = \sqrt{2 \delta \left( H \right)} f(\eta, \tau); \quad \eta = y \sqrt{\frac{U_w}{2 \theta}}; \quad \tau = \frac{U_w}{2 \theta} \cdot t; \quad \theta(\eta, \tau) = \frac{(T - T_\infty)}{(T_w - T_\infty)}
\]

By substituting Equation (14) into Equations (12) and (13), we have

\[
\frac{\partial^3 f(\eta, \tau)}{\partial \eta^3} + \xi_1 \left\{ \frac{\partial^2 f(\eta, \tau)}{\partial \eta^2} f(\eta, \tau) - 2 \left[ \frac{\partial f(\eta, \tau)}{\partial \eta} \right]^2 - \frac{\partial^2 f(\eta, \tau)}{\partial \eta \partial \eta} \right\} - \left[ \gamma + \frac{\sigma_{inf}}{\sigma_f} \xi_2 M \right] \frac{\partial f(\eta, \tau)}{\partial \eta} = 0
\]

\[
\frac{k_{inf} / k_f}{Pr \xi_3} \frac{\partial^2 \theta(\eta, \tau)}{\partial \eta^2} + f(\eta, \tau) \frac{\partial \theta(\eta, \tau)}{\partial \eta} - \frac{\partial f(\eta, \tau)}{\partial \eta} \theta(\eta, \tau) + \frac{Ec}{\xi_2 \xi_3} \left( \frac{\partial^2 f(\eta, \tau)}{\partial \eta^2} \right)^2 \frac{\partial \theta(\eta, \tau)}{\partial \tau} = 0
\]

Along with boundary conditions

\[
\begin{align*}
 f(0, \tau) &= S \frac{\delta f}{\delta \eta}(0, \tau) = -1, \theta(0, \tau) = 1 \\
 f'(\eta, \tau) &\rightarrow 0, \theta'(\eta, \tau) \rightarrow 0 \text{ as } \eta \rightarrow \infty
\end{align*}
\]

To test the stability analysis of solutions of the steady-state flow, we have \( f(\eta) = f_0(\eta) \) and \( \theta(\eta) = \theta_0(\eta) \) which must satisfy the boundary value problems (BVPs) (6)–(9), we have

\[
\theta(\eta, \tau) = \theta_0(\eta) + e^{-\xi} G(\eta, \tau); \quad \theta(\eta, \tau) = f_0(\eta) + e^{-\xi} F(\eta, \tau)
\]

Here, \( G(\eta, \tau) \) and \( F(\eta, \tau) \) are the small concerned to \( \theta_0(\eta) \). Additionally, \( f_0(\eta) \) and \( \xi \) is the unknown eigenvalue and the solutions of eigenvalues problems (16)–(18) provide an unlimited set of the eigenvalues \( \xi_1 < \xi_2 < \xi_3 \ldots \). Substituting Equation (18) into Equations (15)–(17). The solutions \( f(\eta) = f_0(\eta) \) and \( \theta(\eta) = \theta_0(\eta) \) of steady state Equations (8) and (9) are found by setting \( \tau = 0 \). Thus, we have to solve the following linear eigenvalues problems.

\[
F'' + \xi_1 \left[ f_0F'' + F_0f_0' - 4f_0'F_0 + \epsilon F_0' \right] - \left[ \gamma + \frac{\sigma_{inf}}{\sigma_f} \xi_2 M \right] F_0 = 0
\]

\[
\frac{k_{inf} / k_f}{Pr \xi_3} \epsilon'' + f_0G'0 + F_0\theta_0' - f_0'G_0 - F_0'\theta_0 + \frac{2Ec}{\xi_2 \xi_3} \left( f_0'' \right) + \xi G_0 = 0
\]

Along with reduced boundary conditions

\[
\begin{align*}
 F_0(0) &= 0, \quad F_0'(0) = 0, \quad G_0(0) = 0 \\
 F_0'(\eta) &\rightarrow 0, \quad G_0(\eta) \rightarrow 0 \text{ as } \eta \rightarrow \infty
\end{align*}
\]

Harris et al. [55] proposed that in order to obtain the \( \xi_1 \), one boundary condition should be relaxed. Therefore, in this problem \( F_0'(\eta) \rightarrow 0 \) as \( \eta \rightarrow \infty \) is relaxed to a new initial condition such that \( F_0''(0) = 1 \).

4. Results and Discussion

In this section, we discuss the results of the considered flow problem. Before going on to discuss the results, our numerical method is needed to validate and check the accuracy of the used method. For the validation, the values of \( \sqrt{RcC_f} \) has been compared with the outcomes of Waini et al. [51] in Figure 2 graphically. The results show the same features as those noticed in the published paper (refer to Figure 2 of Waini et al. [51]). The results found excellent agreements as the critical values of
$S_c$ of the current study are the same up to three decimal points those gotten by the Waini et al. [51]. Therefore, the three-stage Lobatto IIIa formula can be used confidently in this problem. The detailed description of this method can be seen in Lund et al.’s paper [54] and Rehman et al.’s paper [56]. Moreover, the present values of $f''(0)$ and $-\theta'(0)$ are correspondingly given in Tables 3 and 4.

Figure 2. Variation of $\sqrt{ReC_f}$ for the comparison with Waini et al. [51].

### Table 3. Outcomes of $f''(0)$ surface where $Pr = 6.2$, $\phi_{Al_2O_3} = 0.1$, and $Ec = 0.3$.

<table>
<thead>
<tr>
<th>$\phi_{Cu}$</th>
<th>$M$</th>
<th>$\gamma$</th>
<th>$S$</th>
<th>$f''(0)$</th>
<th>First Solution</th>
<th>Second Solution</th>
</tr>
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<tbody>
<tr>
<td>0.01</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>2.4863</td>
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<td>-1.1077</td>
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<tr>
<td>0.05</td>
<td>0.1</td>
<td>0.3</td>
<td>0.5</td>
<td>3.0749</td>
<td></td>
<td>-2.0807</td>
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<tr>
<td>0.1</td>
<td>0</td>
<td>0.1</td>
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<td>3.1146</td>
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<tr>
<td></td>
<td>0.3</td>
<td>3.1908</td>
<td>2</td>
<td>3.2633</td>
<td></td>
<td>-2.8077</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>3.2967</td>
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<td>-3.1999</td>
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<tr>
<td></td>
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<td>2.5</td>
<td>3.0944</td>
<td></td>
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<td>2.5</td>
<td>3.0944</td>
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<td>-1.0062</td>
</tr>
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</table>

### Table 4. The results of $-\theta'(0)$ where $\gamma = 0$, $\phi_{Al_2O_3} = 0.1$, and $S = 3$.

<table>
<thead>
<tr>
<th>$\phi_{Cu}$</th>
<th>$Pr$</th>
<th>$M$</th>
<th>$Ec$</th>
<th>$-\theta'(0)$</th>
<th>First Solution</th>
<th>Second Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>6.2</td>
<td>0</td>
<td>0</td>
<td>12.7302</td>
<td>12.5387</td>
<td></td>
</tr>
<tr>
<td>0.05</td>
<td>0.1</td>
<td>3</td>
<td>5</td>
<td>9.6302</td>
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<td></td>
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<td>3</td>
<td>8.7893</td>
<td>7.2426</td>
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</tr>
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<td>0.1</td>
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<td>9.2613</td>
<td></td>
</tr>
<tr>
<td></td>
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<td>0.1</td>
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<td>9.6386</td>
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<td>0.3</td>
<td>0.1</td>
<td>0.1</td>
<td>6.4827</td>
<td>0.6929</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>0.5</td>
<td>0.3</td>
<td>4.3787</td>
<td>7.2883</td>
<td></td>
</tr>
</tbody>
</table>
From Figures 3–8, it is noticed that the similarity solutions of a system of reduced ODEs have a non-uniqueness of solutions in certain ranges of \( M \) and \( S \). It should be noted that there does not exist a similarity solution when \( S_c > S \) and \( M_c > M \). Further, \( S_c \) and \( M_c \) are respective critical values of \( S \) and \( M \) where solutions exist. Variation of \( f''(0) \) for numerous permeability parameter \( \gamma \) values is displayed in Figure 3 where \( \phi_{Al_2O_3} = 0.1, \phi_{Cu} = 0.04, M = 0.5, Pr = 6.2, \) and \( Ec = 0.3 \) are kept constant. It is noticed that for confident ranges of the \( S \) there are two solutions. Further, \( S_{c1} = 1.7758 \), \( S_{c2} = 1.6169 \), and \( S_{c3} = 1.4463 \) are the critical values for the respective \( \gamma = 0, \gamma = 0.2, \) and \( \gamma = 0.4 \). It is also examined that an increase in \( \gamma \) creates the enhancement in the coefficient of skin friction for the stable solution, while the reverse tendency is noted for the unstable solution. Physically, the reduction in skin friction is caused by the suction force that facilitates the separation of the boundary layer in the second solution. This nature of flow happens because of the pressure gradient and the antagonistic roles of transpiration on the fluid flow.

Figures 4 and 5 demonstrate the effect of \( \phi_{Cu} \) on \( f''(0) \) and \( -\theta'(0) \) for several values of \( M \) by keeping \( \phi_{Al_2O_3} = \gamma = 0.1, S = 1.75, Pr = 6.2, \) and \( Ec = 0.3 \) constant. It is found that the higher \( \phi_{Cu} \) values postpone separation of the layer since the \( \phi_{Cu} \)'s critical values shift to the left. The coefficient of skin friction increases when Hartmann number \( M \) and copper volume fraction \( \phi_{Cu} \) are increased in the first solution, but the contradictory trend of \( f''(0) \) is found in the second solution. Furthermore, heat transfer enhances in a stable solution for the intensity of the magnetic effect, while it reduces in the second solution. Physically, the explanation of these natures can be explained as “the Lorenz force suppressed the vorticity produced by the shrinking of the sheet inside the boundary layer” [57]. The critical values of \( \phi_{Cu} = 0.001, \phi_{Cu} = 0.01, \) and \( \phi_{Cu} = 0.1 \) are \( M_{c1} = 0.4885, M_{c2} = 0.4768, \) and \( M_{c3} = 0.3718 \), respectively.

Figures 6 and 7 were drawn to demonstrate the effect of \( \phi_{Cu} \) and \( S \) on \( f''(0) \) and \( -\theta'(0) \), respectively. Usually, flow over the shrinking surface generates the vorticity, therefore solutions do not occur subsequently vorticity has not been restricted inside the boundary layer. It is noticed from figures, so enough suction efficiency is required to sustain the fluid flow on a shrinking surface. These findings and behavior of fluid flow are supported by the statements of the Miklavčič and Wang [58] and Fang [59]. For high \( S \) values in the first solution, the rate of heat transfer and skin friction coefficient increase monotonically, while skin friction in the second solution decreases, but the rate of heat transfer upsurges initially for some instances and then starts to decrease. Further, \( S_{c1} = 1.7401, S_{c2} = 1.6973, \) and \( S_{c3} = 1.6747 \) are the critical values for the respective \( \phi_{Cu} = 0.0, \phi_{Cu} = 0.02, \) and \( \phi_{Cu} = 0.04 \).
displayed in Figure 3 where $\phi_{Al_{2}O_{3}} = 0.1$, $\phi_{Cu} = 0.04$, $M = 0.5$, $Pr = 6.2$, and $Ec = 0.3$ are kept constant. It is noticed that for confident ranges of the $S$ there are two solutions. Further, $S_{05} = 1.7758$, $S_{06} = 1.6169$, and $S_{07} = 1.4463$ are the critical values for the respective $\gamma = 0$, $\gamma = 0.2$, and $\gamma = 0.4$.

It is also examined that an increase in $\gamma$ creates the enhancement in the coefficient of skin friction for the stable solution, while the reverse tendency is noted for the unstable solution. Physically, the reduction in skin friction is caused by the suction force that facilitates the separation of the boundary layer in the second solution. This nature of flow happens because of the pressure gradient and the antagonistic roles of transpiration on the fluid flow.

Figure 3. Variation of $f''(0)$ with $\gamma$ for various values of $S$.

Figure 4. Variation of $f''(0)$ with $\phi_{Cu}$ for various values of $M$.

Figure 5. Variation of $-\theta'(0)$ with $\phi_{Cu}$ for various values of $M$.

Figure 6. Variation of $f''(0)$ with $\phi_{Cu}$ for various values of $S$. 

Figure 7. Variation of $-\theta'(0)$ with $\phi_{Cu}$ for various values of $S$. 

We also found that the fluid temperature rises with the rising $Ec$ when the porosity enhances. Physically, it shows that resistance exists due to the direct effect on the viscosity of the fluid. The effect of permeability and temperature profiles fulfill infinite boundary conditions asymptotically. It is examined that the increment of $\gamma$ contributes to the rise of $f'(\eta)$ and $\theta(\eta)$ in the second solution, while no large variation is perceived in the first solution. It is examined that the velocity of the hybrid nanofluid reduces in the first solution when the porosity enhances. Physically, it shows that resistance exists due to the direct effect on the viscosity of the fluid. The effect of $Ec$ on the profile of temperature $\theta(\eta)$ was drawn in Figures 9 and 10. It should be noted that for $f'(\eta)$ and $\theta(\eta)$ profiles, dual solutions exist and these profiles fulfill infinite boundary conditions asymptotically. It is examined that the increment of $\gamma$ contributes to the rise of $f'(\eta)$ and $\theta(\eta)$ in the second solution, while no large variation is perceived in the first solution. It is examined that the velocity of the hybrid nanofluid reduces in the first solution when the porosity enhances. Physically, it shows that resistance exists due to the direct effect on the viscosity of the fluid. The effect of $Ec$ on the profile of temperature $\theta(\eta)$ is revealed in Figure 11. We also found that the fluid temperature rises with the rising $Ec$ values for both branches. Physically, the increase in Eckert number can be clarified in order to minimize enthalpy influence.

![Figure 7](image_url)

**Figure 7.** Variation of $-\theta'(0)$ with $\phi_{Cu}$ for various values of $S$.

![Figure 8](image_url)

**Figure 8.** Variation of $-\theta'(0)$ with $Ec$ for various values of $S$. 

The effect of Eckert number on $-\theta'(0)$ is shown in Figure 8 where $\phi_{Al_2O_3} = \gamma = 0.1$, $\phi_{Cu} = 0.04$, $M = 0.5$, and $Pr = 6.2$ are kept as the constant. First solution demonstrations that the rate of heat transfer increases, while it reduces (enhances) for $Ec = 0.3, 0.6$ ($Ec = 0.1$) for the unstable solution. The effect of permeability $\gamma$ on profiles of velocity $f'(\eta)$ and temperature $\theta(\eta)$ was drawn in Figures 9 and 10. It should be noted that for $f'(\eta)$ and $\theta(\eta)$ profiles, dual solutions exist and these profiles fulfill infinite boundary conditions asymptotically. It is examined that the increment of $\gamma$ contributes to the rise of $f'(\eta)$ and $\theta(\eta)$ in the second solution, while no large variation is perceived in the first solution. It is examined that the velocity of the hybrid nanofluid reduces in the first solution when the porosity enhances. Physically, it shows that resistance exists due to the direct effect on the viscosity of the fluid. The effect of $Ec$ on the profile of temperature $\theta(\eta)$ is revealed in Figure 11. We also found that the fluid temperature rises with the rising $Ec$ values for both branches. Physically, the increase in Eckert number can be clarified in order to minimize enthalpy influence.
Figure 9. Variation of $f'(\eta)$ with $\eta$ for various values of $\gamma$.

Figure 10. Variation of $\theta(\eta)$ with $\eta$ for various values of $\gamma$. 
Figure 11. Variation of $\theta(\eta)$ with $\eta$ for various values of $E_c$.

The graph of the values of the smallest eigenvalue $\varepsilon_1$ against suction $S$ is depicted in Figure 12. According to Hamid et al. [60], positive (negative) values of $\gamma$ show initial growth of decay (disturbance), and the solution of flow can be stable (unstable). From Figure 12, the first solution is clearly stable and the second one is unstable. Moreover, the graph of the smallest eigenvalue shows symmetrical behavior.

Figure 12. Smallest eigenvalues $\varepsilon_1$ for various values of $S$. 

The effect of MHD flow of hybrid nanofluid on the exponentially permeable shrinking surface with the porous medium in the presence of viscous dissipation is examined. The effect of numerous
5. Conclusions

The effect of MHD flow of hybrid nanofluid on the exponentially permeable shrinking surface with the porous medium in the presence of viscous dissipation is examined. The effect of numerous emerging physical parameters on the heat transfer of hybrid nanofluid flow has been examined from the figures, and can be summarized as follows:

1. The present results show good agreements with the previously published results.
2. Dual solutions exist when $S_c \leq S$ and $M_c \leq M$, while no solution exists $S_c > S$ and $M_c > M$.
3. Shear stress rises in the first solution then declines in the second solution for the rising values of $\phi_{Cu}, M, S$, and $\gamma$.
4. For the first solution, the heat transfer rate rises as $S$ and $M$ parameters are enhanced, while this is lower when $\phi_{Cu}$ is up.
5. Enhancement in the volume fraction of the nanoparticles pushes forward the boundary layer separation. Therefore, ranges of solutions increase.
6. Compared with nanofluid and viscous fluid, hybrid nanofluid seems to be more efficient in cooling processes.
7. The first is stable, and the second is unstable.
8. The Eckert number and temperature profiles are directly proportional.
9. The highest value of Eckert number does not affect the boundary layer separation against suction.
10. This model does not function outside the critical points, so there is no solution.

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