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Distance Fibonacci Polynomials

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Abstract: In this paper, we introduce a new kind of generalized Fibonacci polynomials in the distance sense. We give a direct formula, a generating function and matrix generators for these polynomials. Moreover, we present a graph interpretation of these polynomials, their connections with Pascal's triangle and we prove some identities for them.

Keywords: generalized Fibonacci polynomials; Pascal's triangle; generating function; matrix generator; Cassini formula

MSC: 11B37; 11B39

1. Introduction

The well-known Fibonacci numbers F_n, with $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$, have many interesting interpretations, applications and generalizations. It is worth mentioning that the golden ratio, closely related to Fibonacci numbers, is still being discovered in many fields of modern science such as theoretical physics (hydrogen bonds, chaos, superconductivity), astrophysics (pulsating stars, black holes), chemistry (quasicrystals, protein AB models), biology (natural and artificial phyllotaxis, genetic code of DNA) and technology (resistors, quantum computing). In the literature one can find a lot of Fibonacci-like modifications of the classical Fibonacci numbers. Some of them, such as for example Lucas numbers L_n, with $L_0 = 2, L_1 = 1$ and the recursion $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$, are obtained by varying initial conditions. Others, like Pell numbers P_n defined by $P_0 = 0,\ P_1 = 1$ and $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$, are obtained by slight modifying the basic recursion. There are variants that simultaneously generalize more that one recursion. For instance, the k-Fibonacci numbers $F_{k,n}$ introduced by S. Falcon and A. Plaza [1] by the formula $F_{k,n} = kF_{k,n-1} + F_{k,n-2}$, where $n \geq 2, k \geq 1$ and $F_{k,0} = 0$, $F_{k,1} = 1$, generalize both Fibonacci and Pell numbers.

Another direction of modification of the classical Fibonacci sequence is changing distance between terms of a sequence. Narayana numbers N_n defined by the recursion $N_n = N_{n-1} + N_{n-3}$ for $n \geq 3$, with $N_0 = 0, N_1 = 1, N_2 = 1$, are one of such examples, as well as k-Narayana numbers introduced by J. Ramirez and V. Sirvent [2] by the recursion $N_{k,n} = kN_{k,n-1} + N_{k,n-3}$ for $n \geq 3, k \geq 1$, with $N_{0,k} = 0, N_{1,k} = 1, N_{2,k} = k$. Another one are generalized Fibonacci numbers $F(k,n)$ introduced by M. Kwaśnik and I. Włoch [3] given by the formula $F(k,n) = F(k,n-1) + F(k,n-k)$, for $n \geq k, k \geq 2$, with initial conditions $F(k,n) = n+1$ for $n = 0,1,\dots k-1$.

A natural way of generalization is a polynomial direction, which has over a century-long history. Fibonacci polynomials $f_n(x)$ defined by the recurrence relation

$$f_n(x) = xf_{n-1}(x) + f_{n-2}(x), \quad n \geq 2, \tag{1}$$

with initial conditions $f_0(x) = 0, f_1(x) = 1$, were introduced by the Belgian mathematician E. C. Catalan in 1883 and have been intensively studied by many authors since then. V. E. Hoggatt Jr. and M. Bicknell have revealed a relationship between these polynomials and Pascal's triangle, Zeckendorfs's theorem and have found roots of Fibonacci polynomials of degree n (see References [4–6]). W. A. Webb and A. E. Parberry [7] and recently L. Chen and X. Wang [8] have proved some divisibility properties of these polynomials, whereas P. Filipponi and A. F. Horadam [9] have investigated derivative sequences of Fibonacci polynomials. Binet's formula, a generating function and an extensive collection of identities for Fibonacci polynomials can be found in a book of T. Koshy [10]. Many interesting properties of these polynomials have also been proved by A. Lupas [11]. Obviously, $f_n(1) = F_n$ and $f_n(2) = P_n$. Similarly, if in the recursion of k-Fibonacci numbers k is a real variable, then k-Fibonacci numbers correspond to the Fibonacci polynomials defined by (1) (see References [12,13]).

It is necessary to mention that another concept of Fibonacci polynomials arises in the context of the graph theory. In 1984, G. Hopkins and W. Staton introduced Fibonacci polynomials of graphs as a number of all independent sets of the composition of two graphs (for details see Reference [14]). As the consequence, another type of Fibonacci polynomials $F_n(x)$ can be defined by the recurrence relation:

$$F_n(x) = F_{n-1}(x) + xF_{n-2}(x), \quad n \geq 2, \tag{2}$$

with initial conditions $F_0(x) = 1$ and $F_1(x) = x + 1$. The sequence $\{F_n(x)\}$ is different from $\{f_n(x)\}$, but also generalizes Fibonacci numbers, since $F_n(1) = F_{n+2}$. To compare, the first few Fibonacci polynomials $f_n(x)$ and $F_n(x)$ are given in Table 1.

Table 1. Fibonacci polynomials $f_n(x)$ and $F_n(x)$.

n	0	1	2	3	4	5
$f_n(x)$	0	1	x	x^2+1	x^3+2x	x^4+3x^2+1
$F_n(x)$	1	$x+1$	$2x+1$	x^2+3x+1	$3x^2+4x+1$	x^3+6x^2+5x+1

The interest in Fibonacci polynomials, both those given by the recursion (1) and by the recursion (2), has contributed to the emergence of many generalizations. Most of them, as in the case of Fibonacci numbers, are obtained by changing initial terms while preserving the recurrence relation (see References [15–17]). Some are obtained in the distance sense that is, by changing distance between terms of a sequence. Narayana polynomials defined by the formula $N_n(x) = xN_{n-1}(x) + N_{n-3}(x)$ for $n \geq 3$, with initial conditions $N_0(x) = 0, N_1(x) = 1, N_2(x) = x$, being a natural generalization of k-Narayana numbers, are a generalization in the distance sense. Such generalization are also the generalized Fibonacci polynomials $F_n(k, x)$ defined by I. Włoch. Based on the idea given in Reference [14] I. Włoch introduced generalized Fibonacci polynomials of graphs as a number of all distance independent sets in the generalized join of graphs (details can be found in Reference [18]). Consequently, for integers $n \geq 0, k \geq 2, x \geq 1$ generalized Fibonacci polynomials $F_n(k, x)$, being an extension of polynomials $F_n(x)$, are defined by $F_n(k, x) = F_{n-1}(k, x) + xF_{n-k}(k, x)$ for $n \geq k$ with $F_0(k, x) = 1$ and $F_n(k, x) = nx + 1$ for $n = 1, \ldots k - 1$. The sequence $\{F_n(k, x)\}$ generalizes numbers $F(k, n)$, because $F_n(k, 1) = F(k, n)$. It generalizes Narayana numbers too, because $F_n(3, 1) = N_{n+2}$.

In this paper we introduce distance Fibonacci polynomials being simultaneously a new generalization of Fibonacci polynomials $f_n(x)$ and Narayana polynomials. We give a graph interpretation of these polynomials which allows us to obtain the direct formula for distance Fibonacci polynomials. Special cases of this formula are direct formulas for Fibonacci polynomials $f_n(x)$, Narayana polynomials $N_n(x)$ as well as direct formulas for classical Fibonacci numbers F_n and Narayana numbers N_n. We reveal connections of distance Fibonacci polynomials with Pascal triangle and give combinatorial interpretations of coefficients of these polynomials. We prove some identities for the novel introduced distance Fibonacci polynomials and also derive the genering function and matrix generators for them.

2. Distance Fibonacci Polynomials and Their Interpretations

We begin this section with a definition. Let $k \geq 2$, $n \geq 0$ be integers. The distance Fibonacci polynomials $f_n(k, x)$ are given by the following recurrence relation

$$f_n(k, x) = x f_{n-1}(k, x) + f_{n-k}(k, x) \quad \text{for} \quad n \geq k, \tag{3}$$

with initial conditions $f_n(k, x) = x^n$ for $n = 0, 1, \ldots, k-1$.

In Table 2 we present some distance Fibonacci polynomials $f_n(k, x)$ for special values of k and n.

Table 2. Distance Fibonacci polynomials $f_n(k, x)$.

n	0	1	2	3	4	5	6
$f_n(2, x)$	1	x	x^2+1	x^3+2x	x^4+3x^2+1	x^5+4x^3+3x	$x^6+5x^4+6x^2+1$
$f_n(3, x)$	1	x	x^2	x^3+1	x^4+2x	x^5+3x^2	x^6+4x^3+1
$f_n(4, x)$	1	x	x^2	x^3	x^4+1	x^5+2x	x^6+3x^2
$f_n(5, x)$	1	x	x^2	x^3	x^4	x^5+1	x^6+2x

Note that for $k = 2$ we have $f_n(2, x) = f_{n+1}(x)$ and therefore $f_n(2, 1) = F_{n+1}$ and $f_n(2, 2) = P_{n+1}$. For $k = 3$ we have $f_n(3, x) = N_{n+1}(x)$ and consequently $f_n(3, 1) = N_{n+1}$. Moreover, $f_n(k, 1) = F(k, n-k+1)$.

By recursion (3) one can easy check that

$$f_{k+i}(k, x) = x^{k+i} + (i+1)x^i \quad \text{for} \quad i = 0, 1, \ldots, k-2. \tag{4}$$

Before giving a graph interpretation of the distance Fibonacci polynomials let us recall that a finite graph G consists of two finite sets $V(G)$ and $E(G)$. The elements of $V(G)$ are called vertices and the set $V(G)$ the vertex set, whereas the cardinality of $V(G)$ is called the order of a graph. The elements of $E(G)$, called edges, are unordered pairs of vertices. A sequence of distinct vertices $v_1, v_2, \ldots, v_n$ such that $\{v_i, v_{i+1}\} \in E(G)$ for $i = 1, 2, \ldots, n-1$ is called a path. It is worth noting that Fibonacci sequences and polynomials appear in graph theory. The first mention of the use of Fibonacci numbers in graphs occurred in 1982 in the paper of Prodinger and Tichy, see Reference [19]. They showed the relationship between independent sets (i.e., subsets of vertices being pairwise nonadjacent) and Fibonacci numbers. This interest has been multiplied due to the use of counting independent sets, and consequently Fibonacci numbers, in chemical combinatorics. Fibonacci numbers are used to describe the quantitative properties of molecular graphs. Two indices play a special role—Merrifield–Simmons index and Hosoya index, see Reference [20] and references therein. Independent sets have many applications in localization problems as well as in relation to their generalized distances. As a such example is the concept of secondary independent dominating sets introduced in Reference [21] and next studied in Reference [22]. In this context, the study of the distance Fibonacci numbers and polynomials seems to be essential and important. One of the research directions concerning the application of Fibonacci numbers and polynomials in graphs is finding graph interpretations for them. Using graph interpretations for Fibonacci numbers and polynomials, we can give new identities, direct formulas and other properties. Now we present a graph interpretation of the distance Fibonacci polynomials using a special kind of covering and coloring. For graph theory concepts not described here, please see Reference [23].

By $\mathbb{P}_n$, for $n \geq 1$, we mean a path of order n with the vertex set $V(\mathbb{P}_n)$. Let us consider a set of x colors, where $x \geq 1$. We cover the set $V(\mathbb{P}_n)$ by the subgraphs $\mathbb{P}_k$ and $\mathbb{P}_1$, with the vertex of a graph $\mathbb{P}_1$ additionally colored with one of x colors. This operation is called $(\mathbb{P}_k, \mathbb{P}_1)$-covering with $x\mathbb{P}_1$-coloring. By $\sigma(\mathbb{P}_n)$ we denote the number of all $(\mathbb{P}_k, \mathbb{P}_1)$-covering with $x\mathbb{P}_1$-coloring of the graph $\mathbb{P}_n$.

Theorem 1. *Let $k \geq 2, n \geq 1, x \geq 1$ be integers. Then $\sigma(\mathbb{P}_n) = f_n(k,x)$.*

Proof. We prove this theorem by induction on n. Let $k \geq 2, n \geq 1, x \geq 1$ be integers and $\mathbb{P}_n$ be a path of order n with the vertex set $V(\mathbb{P}_n) = \{t_1, t_2, \dots, t_n\}$.

If $n = 1, \dots, k-1$, then we cover the vertices only by subgraphs $\mathbb{P}_1$ with coloring by one of x colors. Hence $\sigma(\mathbb{P}_n) = x^n$ for $n = 1, \dots, k-1$. If $n = k$, then we can cover the vertices of a path $\mathbb{P}_k$ by k subgraphs $\mathbb{P}_1$ which are colored with one of x colors or we can cover such a graph by one path $\mathbb{P}_k$. Hence $\sigma(\mathbb{P}_k) = x^k + 1$.

Assume that $n \geq k+1$ and the theorem is valid for all integers less then n. We will prove that it is true for n. We have to consider two possibilities:

1. $t_n \in V(\mathbb{P}_1)$.
   Then a vertex t_n can be colored by one of x colors. Let $\sigma_1(\mathbb{P}_n)$ denote the number of all $(\mathbb{P}_k, \mathbb{P}_1)$-covering with $x\mathbb{P}_1$-coloring of a graph $\mathbb{P}_n$ with t_n belonging to $\mathbb{P}_1$. Thus, $\sigma_1(\mathbb{P}_n) = x\sigma(\mathbb{P}_{n-1})$.
2. $t_n \in V(\mathbb{P}_k)$.
   Therefore vertices $t_{n-1}, \dots, t_{n-k+1} \in V(\mathbb{P}_k)$. Let $\sigma_k(\mathbb{P}_n)$ denote the number of all $(\mathbb{P}_k, \mathbb{P}_1)$-covering with $x\mathbb{P}_1$-coloring of a graph $\mathbb{P}_n$ with t_n belonging to $\mathbb{P}_k$. Hence $\sigma_k(\mathbb{P}_n) = \sigma(\mathbb{P}_{n-k})$.

   Taking into account both cases and induction hypothesis we obtain

$$\sigma(\mathbb{P}_n) = \sigma_1(\mathbb{P}_n) + \sigma_k(\mathbb{P}_n) = xf_{n-1}(k,x) + f_{n-k}(k,x).$$

Thus the theorem is proved. □

Applying the above graph interpretation of $f_n(k,x)$ we can derive the direct formula for $f_n(k,x)$.

Theorem 2. *Let $k \geq 2, n \geq 0, x \geq 1$ be integers. Then*

$$f_n(k,x) = \sum_{j=0}^{\lfloor \frac{n}{k} \rfloor} \binom{n-(k-1)j}{j} x^{n-kj}. \tag{5}$$

Proof. If $n \leq k-1$, then $\lfloor \frac{n}{k} \rfloor = 0$ and $\sum_{j=0}^{\lfloor \frac{n}{k} \rfloor} \binom{n-(k-1)j}{j} x^{n-kj} = x^n$. Let us consider a path $\mathbb{P}_n, n \geq 1$ and a set of colors $x, x \geq 1$. Suppose that $n \geq k$. By Theorem 1 the number of all $(\mathbb{P}_k, \mathbb{P}_1)$-covering with $x\mathbb{P}_1$-coloring of the graph $\mathbb{P}_n$ is equal to $f_n(k,x)$. Each $(\mathbb{P}_k, \mathbb{P}_1)$-covering consists of j subgraphs $\mathbb{P}_k$ and $n-kj$ vertices $\mathbb{P}_1$, where $0 \leq j \leq \lfloor \frac{n}{k} \rfloor$. Moreover, for a fixed j, we have $\binom{n-(k-1)j}{j}$ possibilities of covering a path $\mathbb{P}_n$ by a subgraph $\mathbb{P}_k$. Each of the other $n-kj$ vertices $\mathbb{P}_1$ can be colored with one of the x colors. Hence

$$f_n(k,x) = \sum_{j=0}^{\lfloor \frac{n}{k} \rfloor} \binom{n-(k-1)j}{j} x^{n-kj}.$$

Thus the theorem is proved. □

For $k = 2$ by the Formula (5) and relation $f_n(2,x) = f_{n+1}(x)$ we get

$$f_{n+1}(x) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j}{j} x^{n-2j},$$

which is the well-known direct formula for the Fibonacci polynomials. Analogously, for $k = 3$, by the Formula (5) and relation $f_n(3,x) = N_{n+1}(x)$, we obtain

$$N_{n+1}(x) = \sum_{j=0}^{\lfloor \frac{n}{3} \rfloor} \binom{n-2j}{j} x^{n-3j},$$

which is the direct formula for Narayana polynomials.

It is known that Fibonacci polynomials $f_n(x)$ can be constructed using the binomial expansion of $(x+1)^n$. This observation is due to M. Bicknell who has noticed (see Table 3) that the sums of rising diagonals give various Fibonacci polynomials (for details see References [4,10]).

Table 3. Fibonacci polynomials $f_n(x)$ as rising diagonals.

n	Expansion of $(x+1)^n$						
0	1						
1	x	$+1$					
2	x^2	$+2x$	$+1$				
3	x^3	$+3x^2$	$+3x$	$+1$			
4	x^4	$+4x^3$	$+6x^2$	$+4x$	$+1$		
5	x^5	$+5x^4$	$+10x^3$	$+10x^2$	$+5x$	$+1$	
6	x^6	$+6x^5$	$+15x^4$	$+20x^3$	$+15x^2$	$+6x$	$+1$

It turns out that the distance Fibonacci polynomials $f_n(k,x)$ can also be constructed using binomials expansion of $(x+1)^n$. For this purpose for a fixed integer k we have to replace diagonals by steps of heights $k-1$. For $k=2$ instead of diagonals we obtain steps of hights 1. In Tables 4 and 5 we present a few distance Fibonacci polynomials obtained by steps method for $k=3$ and 4.

Table 4. Steps for $k=3$.

n	Expansion of $(x+1)^n$						
0	1						
1	x	$+1$					
2	x^2	$+2x$	$+1$				
3	x^3	$+3x^2$	$+3x$	$+1$			
4	x^4	$+4x^3$	$+6x^2$	$+4x$	$+1$		
5	x^5	$+5x^4$	$+10x^3$	$+10x^2$	$+5x$	$+1$	
6	x^6	$+6x^5$	$+15x^4$	$+20x^3$	$+15x^2$	$+6x$	$+1$

Table 5. Steps for $k=4$.

n	Expansion of $(x+1)^n$								
0	1								
1	x	$+1$							
2	x^2	$+2x$	$+1$						
3	x^3	$+3x^2$	$+3x$	$+1$					
4	x^4	$+4x^3$	$+6x^2$	$+4x$	$+1$				
5	x^5	$+5x^4$	$+10x^3$	$+10x^2$	$+5x$	$+1$			
6	x^6	$+6x^5$	$+15x^4$	$+20x^3$	$+15x^2$	$+6x$	$+1$		
7	x^7	$+7x^6$	$21x^5$	$+35x^4$	$+35x^3$	$+21x^2$	$+7x$	$+1$	
8	x^8	$+8x^7$	$+29x^6$	$+56x^5$	$70x^4$	$+50x^3$	$+29x^2$	$+8x$	$+1$

Because the sum of elements on steps beginning in the n^{th} row is $f_n(k, x)$, therefore we obtain in another way the direct Formula (5).

It is known (see Table 6) that if we arrange coefficients of the Fibonacci polynomials in increasing exponents then the elements on every rising diagonal on row $2n$ is zero and the alternate rising diagonals form Pascal's rows. Moreover if n is odd then the sum of the elements on the n^{th} rising diagonal is $2^{(n-1)/2}$ (for details see Reference [10]).

Table 6. Coefficients of $f_n(x)$ in ascending order.

n	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
1	1									
2	0	1								
3	1	0	1							
4	0	2	0	1						
5	1	0	3	0	1					
6	0	3	0	4	0	1				
7	1	0	6	0	5	0	1			
8	0	4	0	10	0	6	0	1		
9	1	0	10	0	15	0	7	0	1	

By steps method we can also compare coefficients of the distance Fibonacci polynomials with Pascal's triangle. Namely, if for a fixed k we arrange the coefficients of the distance Fibonacci polynomials $f_n(k, x)$ in ascending order starting with $n = 0$ then numbers on steps beginning in nk^{th} row form Pascal's row (as previously we build steps of height $k - 1$). In the Table 7 we present few steps for coefficients of $f_n(3, x)$.

Table 7. Coefficients of $f_n(3, x)$ in ascending order.

n	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
0	1									
1	0	1								
2	0	0	1							
3	1	0	0	1						
4	0	2	0	0	1					
5	0	0	3	0	0	1				
6	1	0	0	4	0	0	1			
7	0	3	0	0	5	0	0	1		
8	0	0	6	0	0	6	0	0	1	
9	1	0	0	10	0	0	7	0	0	1

The sum of elements on steps beginning in row nk is $2^{n/k}$. Moreover, the sum of the n^{th} row is equal to $f_n(k, 1)$. In particular, for $k = 2$ we obtain F_{n+1} and for $k = 3$ we get N_{n+1}.

Now we present combinatorial interpretations of the distance Fibonacci polynomials' coefficients. It turns out that the coefficients of a polynomial $f_n(k, x)$ are connected with ordered sums and tillings. Let us denote $f_n(k, x) = \sum_{i=0}^{n} a_i(k, n)x^i$. Then a coefficient $a_i(k, n)$ is the number of representations of n as an ordered sum of 1 and k in such a way that 1 is used exactly i times. For the polynomial $f_5(3, x)$ (see the above table) we have for example $a_2(3, 5) = 3$ that is the number of ways we can represent 5 as an ordered sum of 3 and 1 with 1 used exactly twice. Namely, $5 = 3 + 1 + 1, 5 = 1 + 3 + 1$ and $5 = 1 + 1 + 3$. Instead of ordered sums we can consider tilings of $1 \times n$ rectangle by 1×1 and $1 \times k$

shaped dominos. Then a coefficent $a_i(k,n)$ is the number of ways we can cover $1\times n$ rectangle by 1×1 and $1\times k$ rectangles in such a way that 1×1 domino is used i times.

3. Generating Function and Some Identities

In this section we present a generating function and some identities for distance Fibonacci polynomials $f_n(k,n)$. We also extend these polynomials to negative integers.

Let us recall that a generating function of a sequence $\{a_n\}$ is the function $g(x) = \sum\limits_{n=0}^{\infty} a_n x^n$. Generating functions are usefull tools for solving different kinds of counting problems, in discrete mathematics for solving recurrences.

Theorem 3. *Let $n \geq 0, k \geq 2$ be integers. The generating function of the distance Fibonacci polynomials sequence $\{f_n(k,x)\}$ has the following form* $g(t) = \dfrac{1}{1-xt-t^k}$.

Proof. Let $g(t) = \sum\limits_{n=0}^{\infty} f_n(k,x)t^n$. Then by recurrence relation (3) we have

$$g(t) = f_0(k,x) + f_1(k,x)t + \ldots + f_{k-1}(k,x)t^{k-1} + \sum_{n=k}^{\infty} f_n(k,x)t^n =$$

$$= 1 + xt + \ldots + x^{k-1}t^{k-1} + \sum_{n=k}^{\infty} [\,xf_{n-1}(k,x) + f_{n-k}(k,x)\,]t^n =$$

$$= 1 + xt + \ldots + x^{k-1}t^{k-1} + xt\sum_{n=k}^{\infty} f_{n-1}(k,x)t^{n-1} + t^k\sum_{n=k}^{\infty} f_{n-k}(k,x)t^{n-k} =$$

$$= 1 + xt + \ldots + x^{k-1}t^{k-1} + xt\sum_{n=k-1}^{\infty} f_n(k,x)t^n + t^k\sum_{n=0}^{\infty} f_n(k,x)t^n =$$

$$= 1 + xt + \ldots + x^{k-1}t^{k-1} + xt(\sum_{n=0}^{\infty} f_n(k,x)t^n - 1 - xt - \ldots - x^{k-2}t^{k-2}) + t^k\sum_{n=0}^{\infty} f_n(k,x)t^n =$$

$$= 1 + xt\sum_{n=0}^{\infty} f_n(k,x)t^n + t^k\sum_{n=0}^{\infty} f_n(k,x)t^n = 1 + xtg(t) + t^k g(t).$$

Thus,

$$g(t) = \frac{1}{1-xt-t^k},$$

which ends the proof. □

Note that from Theorem 3, the fact that $f_n(2,x) = f_{n+1}(x)$ and the right-shift rule for generating functions we obtain a function $g(t) = \dfrac{t}{1-xt-t^2}$ being a generating function for the classical Fibonacci polynomials $f_n(x)$.

Theorem 4. *Let $k \geq 2, n \geq 0$ be integers. Then*

(i) $x\sum\limits_{i=0}^{n} f_i(k,x) = \sum\limits_{i=n+2-k}^{n+1} f_i(k,x) - 1$ *for* $n \geq k-2$,

(ii) $x\sum\limits_{i=1}^{n} f_{ik-1}(k,x) = f_{nk}(k,x) - 1$,

(iii) $x\sum\limits_{i=0}^{n} f_{ik}(k,x) = f_{nk+1}(k,x)$,

(iv) $f_n(k,x) = \sum\limits_{i=0}^{k-1} x^i f_{n-k-i}(k,x) + x^k f_{n-k}(k,x)$,

(v) $xf_n(k,x) = x^2 f_{n-1}(k,x) + f_{n-k+1}(k,x) - f_{n-2k+1}(k,x)$ *for* $n \geq 2k-1$.

Proof. At the beginning we prove the identity (i) by induction on n. For $n = 0$ it is obvious. Assume that $n \geq 1$ and the equality (i) is true for an arbitrary n. We will prove that it holds for $n+1$.

By induction hypothesis and the recurrence relation (3) we have

$$x \sum_{i=0}^{n+1} f_i(k,x) = x \sum_{i=0}^{n} f_i(k,x) + xf_{n+1}(k,x) = \sum_{i=n+2-k}^{n+1} f_i(k,x) - 1 + xf_{n+1}(k,x) =$$
$$= \sum_{i=n+2-k}^{n+1} f_i(k,x) - 1 + f_{n+2}(k,x) - f_{n+2-k}(k,x) = f_{n+2-k}(k,x) + \sum_{i=n+3-k}^{n+1} f_i(k,x) + f_{n+2}(k,x) -$$
$$1 - f_{n+2-k}(k,x) = \sum_{i=n+3-k}^{n+2} f_i(k,x) - 1.$$

Thus the identity (i) is proved.

Analogously we can prove the identity (iii).

Now we prove the identity (ii). Using the recurrence relation (3) we have

$$xf_{n-1}(k,x) = f_n(k,x) - f_{n-k}(k,x), n \geq k.$$

Hence, for integers $k-1, 2k-1, \ldots, nk-1$, we obtain

$$xf_{k-1}(k,x) = f_k(k,x) - f_0(k,x)$$
$$xf_{2k-1}(k,x) = f_{2k}(k,x) - f_k(k,x)$$
$$xf_{3k-1}(k,x) = f_{3k}(k,x) - f_{2k}(k,x)$$
$$\vdots$$
$$xf_{nk-1}(k,x) = f_{nk}(k,x) - f_{(n-1)k}(k,x).$$

Adding these equalities we have

$$x \sum_{i=1}^{n} f_{ik-1}(k,x) = f_{nk}(k,x) - f_0(k,x) = f_{nk}(k,x) - 1.$$

Thus the identity (ii) is proved.

To prove the identity (iv) we use the definition of distance Fibonacci polynomials (3) by $k-1$ times. Then we obtain

$$f_n(k,x) = xf_{n-1}(k,x) + f_{n-k}(k,x) = x^2 f_{n-2}(k,x) + xf_{n-k-1}(k,x) + f_{n-k}(k,x) =$$
$$= x^2 f_{n-2}(k,x) + xf_{n-k-1}(k,x) + f_{n-k}(k,x) =$$
$$= x^3 f_{n-3}(k,x) + x^2 f_{n-k-2}(k,x) + xf_{n-k-1}(k,x) + f_{n-k}(k,x) = \vdots$$
$$= \sum_{i=0}^{k-1} x^i f_{n-k-i}(k,x) + x^k f_{n-k}(k,x).$$

Hence the identity (iv) holds.

Using the recurrence relation (3) once again we can prove the last identity (v). Let $n \geq 2k-1, k \geq 2$ be integers. Then

$$x^2 f_{n-1}(k,x) + f_{n-k+1}(k,x) - f_{n-2k+1}(k,x) =$$
$$x^2 f_{n-1}(k,x) + xf_{n-k}(k,x) + f_{n-2k+1}(k,x) - f_{n-2k+1}(k,x) =$$

$$x^2 f_{n-1}(k,x) + x f_{n-k}(k,x) = x f_n(k,x).$$

Thus the theorem is proved. □

Note that for $k = 2$ we obtain the well-known identities for the Fibonacci polynomials and for $x = 1$ we obtain well-known identities for Fibonacci numbers:

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1,$$

$$\sum_{i=1}^{n} F_{2i-1} = F_{2n},$$

$$\sum_{i=1}^{n} F_{2i} = F_{2n+1} - 1.$$

For $k = 3$ we obtain the identities for Narayana polynomials and if $x = 1$ we obtain identities for Narayana numbers:

$$\sum_{i=0}^{n} N_i = N_{n+3} - 1,$$

$$\sum_{i=0}^{n} N_{3i+1} = N_{3n+2},$$

$$\sum_{i=0}^{n} N_{3i} = N_{3n+1} - 1.$$

Moreover, for $x = 1$ and using the relation $f_n(k,1) = F(k, n-k+1)$ we obtain the identities for generalized Fibonacci numbers $F(k,n)$, see Reference [24].

The distance Fibonacci polynomials $f_n(k,x)$ can be extended to negative integers n. Let $k \geq 2$, $n \geq 0$ be integers. Then

$$f_{-n}(k,x) = f_{-n+k}(k,x) - x f_{-n+k-1}(k,x) \text{ for } n \geq 0, \quad (6)$$

with initial conditions $f_n(k,x) = x^n$, for $n = 0, 1, \ldots, k-1$.

The Table 8 includes the first few elements of $f_{-n}(k,x)$ polynomials for special k and negative n.

Table 8. Distance Fibonacci polynomials $f_{-n}(k,x)$.

n	-7	-6	-5	-4	-3	-2	-1	0	1
$f_n(2,x)$	$-x^5 - 4x^3 - 3x$	$x^4 + 3x^2 + 1$	$-x^3 - 2x$	$x^2 + 1$	$-x$	1	0	1	x
$f_n(3,x)$	x^2	1	$-x$	0	1	0	0	1	x
$f_n(4,x)$	$-x$	0	0	1	0	0	0	1	x
$f_n(5,x)$	0	0	1	0	0	0	0	1	x

Notice that setting $k = 2$ in (6), we get the well-known extension of Fibonacci polynomials for negative numbers

$$f_{-n}(2,x) = (-1)^n f_{n-2}(2,x).$$

For $k = 2$ and $k = 3$ and $x = 1$ we obtain the extension of classical Fibonacci numbers and Narayana numbers for negative numbers.

Moreover, if we set $n - k + 1$ in place of n and $x = 1$ in (6), then we obtain the extension of generalized Fibonacci numbers $F(k,n)$ for negative numbers. Hence we get the following relation

$$F(k,-n) = F(k,k-n) - F(k,k-n-1)$$

for $n \geq k+1, k \geq 2$ and $F(k,n) = n+1$ for $n = 0,1,\ldots,k-1$.

Proving analogously as Theorem 4, we get the following identities for the distance Fibonacci polynomials $f_{-n}(k,x)$ for negative integers.

Theorem 5. *Let $n \geq 1, k \geq 2$ be integers. Then*

(vi) $x\sum_{i=1}^{n} f_{-ik}(k,x) = -f_{-nk-k+1}(k,x)$,

(vii) $x\sum_{i=1}^{n} f_{-i}(k,x) = -\sum_{i=-n-k+1}^{-n} f_i(k,x) + 1$,

(viii) $x\sum_{i=1}^{n} f_{-2i}(k,x) = -f_{-2n}(k,x) + 1$,

(ix) $x\sum_{i=1}^{n} f_{-ik+1}(k,x) = -f_{-nk-k+2}(k,x)$ *for* $k \geq 3$.

4. Matrix Generators

In this section we introduce matrix generators for distance Fibonacci polynomials using special upper Hessenberg matrices. Matrix generators for Fibonacci numbers and polynomials have its long history and many applications (see References [10,25]).

Based on the recursion (3) let us define a matrix $Q_k(x) = [q_{ij}]_{k\times k}$, where $k \geq 2$ as follows. For $i = 1$ and $j = 1,2\ldots,k$ an element q_{ij} of the matrix $Q_k(x)$ is equal to the coefficient of $f_{n-j}(k,x)$ in the right-hand-side of the equation (3). For $2 \leq i \leq k$ and $j = 1,2\ldots k$ we put

$$q_{ij} = \begin{cases} 1 & \text{if } i = j+1, \\ 0 & \text{otherwise }. \end{cases}$$

By the above definition for $k = 2,3,\ldots$ we obtain the following matrices:

$$Q_2(x) = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix},\ Q_3(x) = \begin{bmatrix} x & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ \ldots,\ Q_k(x) = \begin{bmatrix} x & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

We will call the matrix $Q_k(x)$ as the distance Fibonacci matrix.

Note that $Q_2(x)$ is a well-known matrix generator for the classical Fibonacci polynomials. Moreover, putting $x = 1$ in $Q_2(x)$ we get the matrix $Q_2(1) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ which is a matrix generator for Fibonacci numbers, whereas for $x = 2$ we obtain the matrix $Q_2(2) = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ being a matrix generator for Pell numbers. The matrix $Q_3(x)$ is a matrix generator for Narayana polynomials and $Q_3(1) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ is a well-known matrix generator for Narayana numbers (see Reference [2]).

Now for a fixed integer $k \geq 2$ let us define a matrix $A_k(x)$ of order k being the matrix of initial conditions

$$A_k(x) = \begin{bmatrix} f_{2k-2}(k,x) & f_{2k-3}(k,x) & \cdots & f_{k-2}(k,x) & f_{k-1}(k,x) \\ f_{2k-3}(k,x) & f_{2k-4}(k,x) & \cdots & f_{k-3}(k,x) & f_{k-2}(k,x) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_k(k,x) & f_{k-1}(k,x) & \cdots & f_2(k,x) & f_1(k,x) \\ f_{k-1}(k,x) & f_{k-2}(k,x) & \cdots & f_1(k,x) & f_0(k,x) \end{bmatrix}.$$

Theorem 6. *Let $k \geq 2$, $n \geq 1$ be integers. Then*

$$Q_k^n(x)A_k(x) = \begin{bmatrix} f_{n+2k-2}(k,x) & f_{n+2k-3}(k,x) & \cdots & f_{n+k-2}(k,x) & f_{n+k-1}(k,x) \\ f_{n+2k-3}(k,x) & f_{n+2k-4}(k,x) & \cdots & f_{n+k-3}(k,x) & f_{n+k-2}(k,x) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{n+k}(k,x) & f_{n+k-1}(k,x) & \cdots & f_{n+2}(k,x) & f_{n+1}(k,x) \\ f_{n+k-1}(k,x) & f_{n+k-2}(k,x) & \cdots & f_{n+1}(k,x) & f_{n+0}(k,x) \end{bmatrix}. \tag{7}$$

Proof. (by induction on n). Let $k \geq 2$ be an integer. If $n = 1$, then by simple calculations and recursion (3) we get (7). Assume now that the statement is true for all integers $1, ..., n$. We will show that it is also true for an integer $n+1$.

Since $Q_k^{n+1}(x)A_k(x) = Q_k(x)Q_k^n(x)A_k(x)$, thus by our assumption and the recurrence relation (3) we obtain

$$Q_k^{n+1}(x)A_k(x) = \begin{bmatrix} x & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} f_{n+2k-2}(k,x) & \cdots & f_{n+k-1}(k,x) \\ f_{n+2k-3}(k,x) & \cdots & f_{n+k-2}(k,x) \\ \vdots & \ddots & \vdots \\ f_{n+k}(k,x) & \cdots & f_{n+1}(k,x) \\ f_{n+k-1}(k,x) & \cdots & f_{n+0}(k,x) \end{bmatrix} =$$

$$= \begin{bmatrix} xf_{n+2k-2}(k,x)+f_{n+k-1}(k,x) & \cdots & xf_{n+k-1}(k,x)+f_{n+0}(k,x) \\ f_{n+2k-2}(k,x) & \cdots & f_{n+k-1}(k,x) \\ \vdots & \ddots & \vdots \\ f_{n+k-1}(k,x) & \cdots & f_{n+2}(k,x) \\ f_{n+k}(k,x) & \cdots & f_{n+1}(k,x) \end{bmatrix} =$$

$$= \begin{bmatrix} f_{n+2k-1}(k,x) & f_{n+2k-2}(k,x) & \cdots & f_{n+k+1}(k,x) & f_{n+k}(k,x) \\ f_{n+2k-2}(k,x) & f_{n+2k-3}(k,x) & \cdots & f_{n+k}(k,x) & f_{n+k-1}(k,x) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{n+k+1}(k,x) & f_{n+k}(k,x) & \cdots & f_{n+3}(k,x) & f_{n+2}(k,x) \\ f_{n+k}(k,x) & f_{n+k-1}(k,x) & \cdots & f_{n+2}(k,x) & f_{n+1}(k,x) \end{bmatrix}.$$

□

Theorem 7. *Let $k \geq 2$, $n \geq 1$ be integers. Then*

$$det(\,Q_k^n(x)) = (-1)^{n(k+1)} \tag{8}$$

$$det\,(\,A_k(x)) = (-1)^{\lfloor \frac{k}{2} \rfloor}. \tag{9}$$

Proof. From a definition of $Q_k(x)$ and basic properties of determinants follows that $\det(Q_k(x)) = (-1)^{(k+1)}$. Hence by Cauchy's theorem for determinants we obtain equality (8). To prove equality (9) we will use relation (4).

$$det(A_k(x)) = \begin{vmatrix} x^{2k-2} + kx^{k-1} & x^{2k-3} + (k-1)x^{k-2} & \cdots & x^k + 1 & x^{k-1} \\ x^{2k-3} + (k-1)x^{k-2} & x^{2k-4} + (k-2)x^{k-3} & \cdots & x^{k-1} & x^{k-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x^k + 1 & x^{k-1} & \cdots & x^2 & x \\ x^{k-1} & x^{k-2} & \cdots & x & 1 \end{vmatrix}.$$

Multiplying the last column of the above determinant respectively by $x^{k-1}, x^{k-2}, \ldots, x$ and subtracting from the first, the second,..., the $(k-1)$th column respectively we obtain the following determinant

$$det(A_k(x)) = \begin{vmatrix} kx^{k-1} & (k-1)x^{k-2} & \cdots & 1 & x^{k-1} \\ (k-1)x^{k-2} & (k-2)x^{k-3} & \cdots & 0 & x^{k-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & x \\ 0 & 0 & \cdots & 0 & 1 \end{vmatrix}.$$

Expanding this determinant along the last row and then changing $\lfloor \frac{k}{2} \rfloor$ columns by places we get

$$det(A_k(x)) = (-1)^{\lfloor \frac{k}{2} \rfloor} \begin{vmatrix} 1 & 2x & \cdots & (k-1)x^{k-2} & kx^{k-1} \\ 0 & 1 & \cdots & (k-2)x^{k-3} & (k-1)x^{k-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{vmatrix}.$$

Hence $det\,(A_k(x)) = (-1)^{\lfloor \frac{k}{2} \rfloor}$. □

As an immediate consequence of Theorem 7 and Cauchy's theorem for determinants we obtain the Cassini formula for distance Fibonacci polynomials.

Corollary 1. *Let $k \geq 2$, $n \geq 1$ be integers. Then the generalized Cassini formula for the distance Fibonacci polynomials can be written as*

$$det\,(Q_k^n(x)\, A_k(x)) = (-1)^{n(k+1)\lfloor \frac{k}{2} \rfloor}. \tag{10}$$

Note that by Theorem 6 and the equallity $f_n(2,x) = f_{n+1}(x)$ we have

$$Q_2^{n-2}(x)\, A_2(x) = \begin{bmatrix} f_n(2,x) & f_{n-1}(2,x) \\ f_{n-1}(2,x) & f_{n-2}(2,x) \end{bmatrix} = \begin{bmatrix} f_{n+1}(x) & f_n(x) \\ f_n(x) & f_{n-1}(x) \end{bmatrix},$$

and then by Theorem 1 we obtain the well-known Cassini formula for the classical Fibonacci polynomials

$$f_{n+1}(x)f_{n-1}(x) - f_n^2(x) = (-1)^n.$$

5. Conclusions

Integers sequences defined by the homogenous linear recurrence relations were called as sequeneces of the Fibonacci type, see Reference [26], where the list of well known Fibonacci type sequences can be found. In that context introduced in this paper distance Fibonacci polynomials initiate studying the family of distance polynomials of the Fibonacci type. Distance Fibonacci type sequences which generalize Fibonacci, Lucas, Pell, Jacobsthal and others sequences can be extended to polynomials. Their properties will be interesting not only from the pure mathematical point of view but also from their applications.

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