

Symmetric Identities for Carlitz's Type Higher-Order (p, q) -Genocchi Polynomials

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Abstract: In this paper, we study Carlitz's type higher-order (p, q) -Genocchi polynomials. To be specific, we define Carlitz's type higher-order (p, q) -Genocchi polynomials and Carlitz's type higher-order (h, p, q) -Genocchi polynomials. This paper also explores properties including distribution relation and symmetric identities. In addition, we find alternating (p, q) -power sums. We identify symmetric identities using Carlitz's type higher-order (h, p, q) -Genocchi polynomials and alternating (p, q) -power sums.

Keywords: higher-order (p, q) -Genocchi polynomials; symmetric identities; alternating power sums

1. Introduction

The purpose of this paper is to discuss Carlitz's type higher-order (p, q) -Genocchi polynomials. To do so, we introduce notations and precedent researches related to the subject of this paper.

We utilize the following notations: $\mathbb{N} = \{1, 2, 3, \dots\}$ denotes the set of natural numbers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{Z}_0 = \mathbb{N} \cup \{0\}$ denotes the set of nonnegative integers, $\mathbb{C}$ denotes the set of complex numbers, and

$$\sum_{m_1, \dots, m_r=0}^{\infty} = \sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty}.$$

We would like to review several definitions related to q -number and (p, q) -number used in this paper. For any $m \in \mathbb{N}$, q -number can be defined as follows

$$[m]_q = \frac{1 - q^m}{1 - q} = \sum_{0 \leq i \leq m-1} q^i = 1 + q + q^2 + \cdots + q^{m-1}.$$

For $x \in \mathbb{C}$, the (p, q) -number is defined by

$$[x]_{p,q} = \frac{p^x - q^x}{p - q}, (p \neq q).$$

This shows that the (p, q) -number has a symmetric property. Let x be a natural number. If $p = 1$, then (p, q) -number is q -number and if $p = 1$ and $q \rightarrow 1$, then $[x]_{p,q} \rightarrow x$.

A number of researchers have been looking into Bernoulli polynomials, Euler polynomials, and Genocchi polynomials (see [1–15]). This paper reviews Bernoulli polynomials, Euler polynomials, and Genocchi polynomials. Especially, we would like to focus on Genocchi polynomials. The generating function of Genocchi polynomials is as follows

$$\frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}.$$

In 1955, Erdelyi et al. [4] introduced the definition, along with some special cases, of generalized Bernoulli polynomials. Afterwards, Srivastava and Pinter [14] not only discussed generalized Bernoulli polynomials, but also generalized Euler polynomials. They also studied the relations between generalized Bernoulli polynomials and classical Euler polynomials, as well as the connections between generalized Euler polynomials and classical Bernoulli polynomials. In [9], a new method to derive interesting properties related to higher-order Euler polynomials was discovered. Araci et al. [2] constructed higher-order Genocchi polynomials by using the method of [9]. Higher-order Genocchi polynomials are defined as the following generating function

$$\left(\frac{2t}{e^t + 1}\right)^k e^{xt} = \sum_{n=0}^{\infty} G_n^{(k)}(x) \frac{t^n}{n!}.$$

In the special case, $G_n^{(k)} = G_n^{(k)}(0)$ are called higher-order Genocchi numbers when $x = 0$.

Theorem 1. For $n \in \mathbb{N}$, higher-order Genocchi polynomials are expressed as follows.

$$G_n^{(k)}(x) = \sum_{l=0}^n \binom{n}{l} G_l^{(k)} x^{n-l}.$$

Choi and Kim [3] discussed a nonlinear ordinary differential equation and identified higher-order Bernoulli polynomials with ordinary differential equations. In [3], they came up with ordinary differential equations to determine the coefficients of a recurrence formula related to the generating function of Bernoulli numbers. Kim et al. [10] studied higher-order q -Euler polynomials, which are a special case that can be obtained from multiple q -Euler polynomials. Hwang and Ryoo [7] defined Carlitz's type higher-order (p, q) -Euler polynomials and multiple (p, q) -Hurwitz–Euler eta functions to discuss symmetric identities for the (p, q) -Hurwitz–Euler eta function in a complex field. In addition, Hwang and Ryoo [6] constructed generating functions of the Carlitz-type, higher-order twisted (p, q) -Euler polynomials. They gave some properties, which are related to Carlitz-type, higher-order twisted (p, q) -Euler polynomials, including symmetric identities. Ryoo [11] defined Carlitz's type (p, q) -Euler polynomials and gave some of their relevant identities, including distribution relation and the property of the complement. Kim [8] also defined Carlitz's type (p, q) -Genocchi polynomials as the following generating function

$$\sum_{n=0}^{\infty} G_{n,p,q}(x) \frac{t^n}{n!} = [2]_q t \sum_{m=0}^{\infty} (-1)^m q^m e^{[m+x]_{p,q} t}.$$

When $x = 0$, $G_{n,p,q} = G_{n,p,q}(0)$ are called Carlitz's type (p, q) -Genocchi numbers.

Theorem 2. For non-negative integer n , Carlitz's type (p, q) -Genocchi polynomials are

$$\frac{G_{n+1,p,q}(x)}{n+1} = \sum_{l=0}^n \binom{n}{l} [x]_{p,q}^{n-l} q^{xl} \frac{G_{l+1,p,q}^{(n-l)}}{l+1}.$$

Theorem 3. For non-negative integer n , the identity is

$$\frac{G_{n+1,p,q}^{(h)}(x)}{n+1} = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m p^{hm} [m+x]_{p,q}^n.$$

Theorem 4. (Distribution relation) For non-negative integer n and any positive odd integer m , the property is

$$G_{n+1,p,q}(x) = \frac{[2]_q}{[2]_{q^m}} [m]_{p,q}^n \sum_{a=0}^{m-1} (-1)^a q^a G_{n+1,p^m,q^m} \left(\frac{a+x}{m} \right).$$

The following diagram shows the variations of the different types of higher-order Genocchi polynomials, (p, q) -Genocchi polynomials, and (h, p, q) -Genocchi polynomials.

Those polynomials in the first row and the first column of the diagram are studied by Kim [2], Araci et al. [8], respectively. The studies of [2,8] have brought beneficial results in combinatorics and number theory. This paper has been inspired and reviewed with [2,8] in order to investigate some explicit identities for higher-order (p, q) -Genocchi polynomials and higher-order (h, p, q) -Genocchi polynomials in the second column of the diagram.

$$\begin{array}{ccc}
 \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} = \frac{2t}{e^t + 1} e^{xt} & \xrightarrow{\hspace{2cm}} & \sum_{n=0}^{\infty} G_n^{(r)}(x) \frac{t^n}{n!} = \left(\frac{2t}{e^t + 1} \right)^r e^{xt} \\
 \text{(Genocchi polynomials)} & & \text{(Higher-order Genocchi polynomials)} \\
 \downarrow & & \downarrow \\
 \sum_{n=0}^{\infty} G_{n,p,q}(x) \frac{t^n}{n!} & \xrightarrow{\hspace{2cm}} & \sum_{n=0}^{\infty} G_{n,p,q}^{(r)}(x) \frac{t^n}{n!} \\
 = [2]_q t \sum_{m=0}^{\infty} (-q)^m e^{[m+x]_{p,q}t} & & = [2]_q^r t^r \sum_{m_1, \dots, m_r=0}^{\infty} (-q)^{m_1+\dots+m_r} e^{[m_1+\dots+m_r+x]_{p,q}t} \\
 \text{((p, q)-Genocchi polynomials)} & & \text{(Higher-order (p, q)-Genocchi polynomials)} \\
 \downarrow & & \downarrow \\
 \sum_{n=0}^{\infty} G_{n,p,q}^{(h)}(x) \frac{t^n}{n!} & \xrightarrow{\hspace{2cm}} & \sum_{n=0}^{\infty} G_{n,p,q}^{(r,h)}(x) \frac{t^n}{n!} \\
 = [2]_q t \sum_{m=0}^{\infty} (-q)^m p^{hm} e^{[m+x]_{p,q}t} & & = [2]_q^r t^r \sum_{m_1, \dots, m_r=0}^{\infty} (-q)^{m_1+\dots+m_r} p^{h(m_1+\dots+m_r)} e^{[m_1+\dots+m_r+x]_{p,q}t} \\
 \text{((h, p, q)-Genocchi polynomials)} & & \text{(Higher-order (h, p, q)-Genocchi polynomials)}
 \end{array}$$

The aim of this paper is to research Carlitz's type higher-order (p, q) -Genocchi polynomials. In Section 2, we define Carlitz's type higher-order (p, q) -Genocchi polynomials and explore properties such as distribution relation. We also define Carlitz's type higher-order (h, p, q) -Genocchi polynomials to investigate the relations between Carlitz's type higher-order (p, q) -Genocchi polynomials and Carlitz's type higher-order (h, p, q) -Genocchi polynomials. In Section 3, we introduce multiple (p, q) -Hurwitz–Euler eta function in order to find symmetric properties. In addition, we find alternating (p, q) -power sums, and symmetric identities using Carlitz's type higher-order (h, p, q) -Genocchi polynomials and the alternating sums.

2. Carlitz's Type Higher-Order (p, q) -Genocchi Polynomials

In this section, we define Carlitz's type higher-order (p, q) -Genocchi polynomials and (h, p, q) -Genocchi polynomials, and explore their properties, including explicit formula and distribution relation.

Definition 1. For $r \in \mathbb{N}$ and $0 < q < p \leq 1$, Carlitz's type higher-order (p, q) -Genocchi polynomials are defined as the following generating function

$$F_{p,q}^{(r)}(x, t) = \sum_{n=0}^{\infty} G_{n,p,q}^{(r)}(x) \frac{t^n}{n!} = [2]_q^r t^r \sum_{m_1, \dots, m_r=0}^{\infty} (-1)^{m_1+\dots+m_r} q^{m_1+\dots+m_r} e^{[m_1+\dots+m_r+x]_{p,q}t}.$$

Here are a few examples:

$$\begin{aligned} G_{2,p,q}^{(1)}(x) &= -\frac{2(1+q)(-p^x - p^x q^2 + q^x + pq^{1+x})}{(p-q)(1+pq)(1+q^2)}, \\ G_{3,p,q}^{(2)}(x) &= -\frac{6(1+q)^2(-p^x - 2p^x q^2 - p^x q^4 + q^x + 2pq^{1+x} + p^2 q^{2+x})}{(p-q)(1+pq)^2(1+q^2)^2}, \\ G_{4,p,q}^{(3)}(x) &= -\frac{24(1+q)^3(-p^x - 3p^x q^2 - 3p^x q^4 - p^x q^6 + q^x + 3pq^{1+x} + 3p^2 q^{2+x} + p^3 q^{3+x})}{(p-q)(1+pq)^3(1+q^2)^3}. \end{aligned}$$

If we put $x = 0$ in Definition 1, then $G_{n,p,q}^{(r)} = G_{n,p,q}^{(r)}(0)$ are called Carlitz's type higher-order (p, q) -Genocchi numbers. Let us apply $r = 1$, $p = 1$, and $q \rightarrow 1$ in $G_{n,p,q}^{(r)}(x)$. Then, we get

$$\lim_{q \rightarrow 1} G_{n,1,q}^{(1)}(x) = G_n(x),$$

where $G_n(x)$ are Genocchi polynomials.

Theorem 5. Let $r \in \mathbb{N}$ and $n \in \mathbb{Z}_0$. Then, we have

$$\frac{G_{n+r,p,q}^{(r)}(x)}{\binom{n+r}{r} r!} = [2]_q^r \sum_{m_1, \dots, m_r=0}^{\infty} (-1)^{m_1+\dots+m_r} q^{m_1+\dots+m_r} [m_1 + \dots + m_r + x]_{p,q}^n$$

and $G_{0,p,q}^{(r)}(x) = \dots = G_{r-1,p,q}^{(r)}(x) = 0$.

Proof. From Definition 1, we derive

$$\begin{aligned} \sum_{n=0}^{\infty} G_{n,p,q}^{(r)}(x) \frac{t^n}{n!} &= [2]_q^r t^r \sum_{m_1, \dots, m_r=0}^{\infty} (-1)^{m_1+\dots+m_r} q^{m_1+\dots+m_r} e^{[m_1+\dots+m_r+x]_{p,q}t} \\ &= [2]_q^r \sum_{n=0}^{\infty} \sum_{m_1, \dots, m_r=0}^{\infty} (-1)^{m_1+\dots+m_r} q^{m_1+\dots+m_r} [m_1 + \dots + m_r + x]_{p,q}^n \frac{t^{n+r}}{n!}. \end{aligned} \quad (1)$$

Because of Equation (1), we obtain

$$\sum_{n=0}^{\infty} G_{n,p,q}^{(r)}(x) \frac{t^{n-r}}{n!} = [2]_q^r \sum_{n=0}^{\infty} \sum_{m_1, \dots, m_r=0}^{\infty} (-1)^{m_1+\dots+m_r} q^{m_1+\dots+m_r} [m_1 + \dots + m_r + x]_{p,q}^n \frac{t^n}{n!}. \quad (2)$$

Therefore, we can get $G_{0,p,q}^{(r)}(x) = 0, G_{1,p,q}^{(r)}(x) = 0, \dots, G_{r-1,p,q}^{(r)}(x) = 0$ by comparing the coefficient of $\frac{t^n}{n!}$ on both sides from the equation above (2), and express the equation as follows

$$\sum_{n=r}^{\infty} G_{n,p,q}^{(r)}(x) \frac{t^{n-r}}{n!} = [2]_q^r \sum_{n=0}^{\infty} \sum_{m_1, \dots, m_r=0}^{\infty} (-1)^{m_1+\dots+m_r} q^{m_1+\dots+m_r} [m_1 + \dots + m_r + x]_{p,q}^n \frac{t^n}{n!}. \quad (3)$$

If we calculate the left-hand side of the Equation (3), then

$$\begin{aligned} \sum_{n=r}^{\infty} G_{n,p,q}^{(r)}(x) \frac{t^{n-r}}{n!} &= \sum_{n=0}^{\infty} G_{n+r,p,q}^{(r)}(x) \frac{t^n}{(n+r)!} \\ &= \sum_{n=0}^{\infty} \frac{G_{n+r,p,q}^{(r)}(x)}{\binom{n+r}{r} r!} \frac{t^n}{n!}. \end{aligned} \quad (4)$$

Therefore, the proof is completed by comparing Equations (3) and (4). $\square$

Theorem 6. Let $r \in \mathbb{N}$ and $n \in \mathbb{Z}_0$. Then, we obtain

$$\frac{G_{n+r,p,q}^{(r)}(x)}{\binom{n+r}{r}r!} = \frac{[2]_q^r}{(p-q)^n} \sum_{k=0}^n \binom{n}{k} (-1)^k q^{kx} p^{(n-k)x} \left(\frac{1}{1+q^{1+k}p^{n-k}} \right)^r.$$

Proof. By using Theorem 5, we have

$$\begin{aligned} \frac{G_{n+r,p,q}^{(r)}(x)}{\binom{n+r}{r}r!} &= [2]_q^r \sum_{m_1, \dots, m_r=0}^{\infty} (-q)^{m_1+\dots+m_r} [m_1+\dots+m_r+x]_{p,q}^n \\ &= \frac{[2]_q^r}{(p-q)^n} \sum_{k=0}^n \binom{n}{k} (-1)^k q^{kx} p^{(n-k)x} \\ &\quad \times \sum_{m_1, \dots, m_r=0}^{\infty} (-1)^{m_1+\dots+m_r} q^{(1+k)(m_1+\dots+m_r)} p^{(n-k)(m_1+\dots+m_r)} \\ &= \frac{[2]_q^r}{(p-q)^n} \sum_{k=0}^n \binom{n}{k} (-1)^k q^{kx} p^{(n-k)x} \left(\frac{1}{1+q^{1+k}p^{n-k}} \right)^r. \end{aligned} \quad (5)$$

Hence, we finish the proof from the equation above (5). $\square$

Theorem 7. For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_0$, we obtain

$$\frac{G_{n+r,p,q}^{(r)}(x)}{\binom{n+r}{r}r!} = [2]_q^r \sum_{l=0}^{\infty} \binom{r+l-1}{l} (-1)^l q^l [l+x]_{p,q}^n.$$

Proof. Let us use Theorem 6. Then, because of the identity

$$\left(\frac{1}{1+q^{1+k}p^{n-k}} \right)^r = \sum_{l=0}^{\infty} \binom{r+l-1}{l} (-1)^l q^{l(1+k)} p^{l(n-k)},$$

we get

$$\begin{aligned} \frac{G_{n+r,p,q}^{(r)}(x)}{\binom{n+r}{r}r!} &= \frac{[2]_q^r}{(p-q)^n} \sum_{k=0}^n \binom{n}{k} (-1)^k q^{kx} p^{(n-k)x} \left(\frac{1}{1+q^{1+k}p^{n-k}} \right)^r \\ &= \frac{[2]_q^r}{(p-q)^n} \sum_{l=0}^{\infty} \sum_{k=0}^n \binom{r+l-1}{l} \binom{n}{k} (-1)^{k+l} q^l q^{(l+x)k} p^{(l+x)(n-k)} \\ &= \frac{[2]_q^r}{(p-q)^n} \sum_{l=0}^{\infty} \binom{r+l-1}{l} (-1)^l q^l (p^{l+x} - q^{l+x})^n \\ &= [2]_q^r \sum_{l=0}^{\infty} \binom{r+l-1}{l} (-1)^l q^l [l+x]_{p,q}^n. \end{aligned}$$

Therefore, we complete the proof of Theorem 7. $\square$

Definition 2. For $r \in \mathbb{N}$ and $0 < q < p \leq 1$, Carlitz's type higher-order (h, p, q) -Genocchi polynomials are defined as the following generating function

$$\sum_{n=0}^{\infty} G_{n,p,q}^{(r,h)}(x) \frac{t^n}{n!} = [2]_q^r t^r \sum_{m_1, \dots, m_r=0}^{\infty} (-q)^{m_1+\dots+m_r} p^{h(m_1+\dots+m_r)} e^{[m_1+\dots+m_r+x]_{p,q}t}.$$

When $x = 0$, $G_{n,p,q}^{(r,h)} = G_{n,p,q}^{(r,h)}(0)$ are called Carlitz's type higher-order (h, p, q) -Genocchi numbers. Especially, if we substitute $h = 0$ in Definition 2, Carlitz's type higher-order (h, p, q) -Genocchi

polynomials equal Carlitz's type higher-order (p, q) -Genocchi polynomials. That is $G_{n,p,q}^{(r,0)}(x) = G_{n,p,q}^{(r)}(x)$.

Theorem 8. Let $r \in \mathbb{N}$ and $n \in \mathbb{Z}_0$. Then, we get

$$\frac{G_{n+r,p,q}^{(r,h)}(x)}{\binom{n+r}{r} r!} = [2]_q^r \sum_{m_1, \dots, m_r=0}^{\infty} (-q)^{m_1+\dots+m_r} p^{h(m_1+\dots+m_r)} [m_1 + \dots + m_r + x]_{p,q}^n$$

and $G_{j,p,q}^{(r,h)}(x) = 0$, where j is a nonnegative integer with $0 \leq j < r$.

Theorem 9. Let $r \in \mathbb{N}$ and $n \in \mathbb{Z}_0$. Then, we obtain

$$\frac{G_{n+r,p,q}^{(r,h)}(x)}{\binom{n+r}{r} r!} = \frac{[2]_q^r}{(p-q)^n} \sum_{k=0}^n \binom{n}{k} (-1)^k q^{kx} p^{(n-k)x} \left(\frac{1}{1+q^{1+k} p^{h+n-k}} \right)^r.$$

Theorem 10. For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_0$, we have

$$\frac{G_{n+r,p,q}^{(r)}(x+y)}{\binom{n+r}{r}} = \sum_{k=0}^n \binom{n}{k} q^{ky} p^{(n-k)x} \frac{G_{k+r,p,q}^{(r,n-k)}(x)}{\binom{k+r}{r}} [y]_{p,q}^{n-k}.$$

Proof. By using Theorems 5 and 8, we obtain

$$\begin{aligned} \frac{G_{n+r,p,q}^{(r)}(x+y)}{\binom{n+r}{r} r!} &= [2]_q^r \sum_{m_1, \dots, m_r=0}^{\infty} (-1)^{m_1+\dots+m_r} q^{m_1+\dots+m_r} [m_1 + \dots + m_r + x + y]_{p,q}^n \\ &= [2]_q^r \sum_{m_1, \dots, m_r=0}^{\infty} (-1)^{m_1+\dots+m_r} q^{m_1+\dots+m_r} \\ &\quad \times \left(p^{m_1+\dots+m_r+x} [y]_{p,q} + q^y [m_1 + \dots + m_r + x]_{p,q} \right)^n \\ &= [2]_q^r \sum_{m_1, \dots, m_r=0}^{\infty} (-1)^{m_1+\dots+m_r} q^{m_1+\dots+m_r} \\ &\quad \times \sum_{k=0}^n \binom{n}{k} p^{(n-k)(m_1+\dots+m_r+x)} [y]_{p,q}^{n-k} q^{ky} [m_1 + \dots + m_r + x]_{p,q}^k \\ &= \sum_{k=0}^n \binom{n}{k} q^{ky} p^{(n-k)x} \frac{G_{k+r,p,q}^{(r,n-k)}(x)}{\binom{k+r}{r} r!} [y]_{p,q}^{n-k}. \end{aligned} \tag{6}$$

Therefore, we complete the proof of Theorem 10. $\square$

Let us put $x = 0$ and replace y with x in Theorem 10. Then, we get the following corollary.

Corollary 1. For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_0$, we obtain

$$\frac{G_{n+r,p,q}^{(r)}(x)}{\binom{n+r}{r}} = \sum_{k=0}^n \binom{n}{k} q^{kx} \frac{G_{k+r,p,q}^{(r,n-k)}(x)}{\binom{k+r}{r}} [x]_{p,q}^{n-k}.$$

Let $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$. Then, by Theorem 6, we can observe

$$\frac{G_{n+r,p,q}^{(r)}(x)}{\binom{n+r}{r} r!} = \frac{[2]_q^r}{(p-q)^n} \sum_{k=0}^n \binom{n}{k} (-1)^k q^{kx} p^{(n-k)x} \sum_{a_1, \dots, a_r=0}^{d-1} \sum_{m_1, \dots, m_r=0}^{\infty} (-1)^{a_1+\dots+a_r} (-1)^{m_1+\dots+m_r} \\ \times q^{(1+k)(a_1+dm_1+\dots+a_r+dm_r)} p^{(n-k)(a_1+dm_1+\dots+a_r+dm_r)}.$$

Theorem 11. Let $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$. Then, we have

$$\frac{[d]_{p,q}^n [2]_q^r}{[2]_{q^d}^r} \sum_{a_1, \dots, a_r=0}^{d-1} (-q)^{a_1+\dots+a_r} G_{n+r,p^d,q^d}^{(r)} \left(\frac{a_1+\dots+a_r+x}{d} \right) = G_{n+r,p,q}^{(r)}(x).$$

Proof. Let us replace x with $\frac{a_1+\dots+a_r+x}{d}$, p with p^d , and q with q^d in Theorem 6. Then, we get

$$G_{n+r,p^d,q^d}^{(r)} \left(\frac{a_1+\dots+a_r+x}{d} \right) \\ = \binom{n+r}{r} \frac{r! [2]_{q^d}^r}{(p^d-q^d)^n} \sum_{k=0}^n \binom{n}{k} (-1)^k q^{(a_1+\dots+a_r+k)x} p^{(n-k)(a_1+\dots+a_r+x)} \left(\frac{1}{1+q^{d(1+k)} p^{d(n-k)}} \right)^r. \quad (7)$$

If we multiply the equation above (7) by $\sum_{a_1, \dots, a_r=0}^{d-1} (-q)^{a_1+\dots+a_r}$, then

$$\sum_{a_1, \dots, a_r=0}^{d-1} (-q)^{a_1+\dots+a_r} G_{n+r,p^d,q^d}^{(r)} \left(\frac{a_1+\dots+a_r+x}{d} \right) \\ = \binom{n+r}{r} \frac{r! [2]_{q^d}^r}{(p^d-q^d)^n} \sum_{k=0}^n \binom{n}{k} (-1)^k q^{kx} p^{(n-k)x} \sum_{a_1, \dots, a_r=0}^{d-1} (-1)^{a_1+\dots+a_r} \\ \times q^{(1+k)(a_1+\dots+a_r)} p^{(n-k)(a_1+\dots+a_r)} \left(\frac{1}{1+q^{d(1+k)} p^{d(n-k)}} \right)^r \\ = \frac{[2]_{q^d}^r}{[d]_{p,q}^n [2]_q^r} \binom{n+r}{r} \frac{r! [2]_q^r}{(p-q)^n} \sum_{k=0}^n \binom{n}{k} (-1)^k q^{kx} p^{(n-k)x} \left(\frac{1}{1+q^{1+k} p^{n-k}} \right)^r \\ = \frac{[2]_{q^d}^r}{[d]_{p,q}^n [2]_q^r} G_{n+r,p,q}^{(r)}(x). \quad (8)$$

Hence, the proof is done. $\square$

In this section, we discuss two definitions of Carlitz's type higher-order (p, q) -Genocchi polynomials and Carlitz's type higher-order (h, p, q) -Genocchi polynomials. We identify the Carlitz's type higher-order (h, p, q) -Genocchi polynomial in this section, in order to find the symmetric properties using alternating (p, q) -power sums in Section 3. This section explores eighth theorems, including one corollary, and those theorems are such as addition formula, distribution relation, and the relation between Carlitz's type higher-order (p, q) -Genocchi polynomials and Carlitz's type higher-order (h, p, q) -Genocchi polynomials.

3. Symmetric Properties for Carlitz's Type Higher-Order (p, q) -Genocchi Polynomials and Its Alternating (p, q) -Power Sums

In this section, we examine symmetric property relations for Carlitz's type higher-order (p, q) -Genocchi polynomials. Furthermore, we find alternating (p, q) -power sums and symmetric identity for Carlitz's type higher-order (p, q) -Genocchi polynomials to use.

Hwang and Ryoo [7] defined the multiple (p, q) -Hurwitz–Euler eta function as follows

$$\eta_{p,q}^{(r)}(s, x) = [2]_q^r \sum_{m_1, \dots, m_r=0}^{\infty} \frac{(-1)^{m_1+\dots+m_r} q^{m_1+\dots+m_r}}{[m_1 + \dots + m_r + x]_{p,q}^s}, \quad (s, x \in \mathbb{C} \text{ with } \Re(x) > 0). \quad (9)$$

Theorem 12. For $s, x \in \mathbb{C}$ with $\Re(x) > 0$ and $r \in \mathbb{N}$, we have

$$\eta_{p,q}^{(r)}(s, x) = \frac{(-1)^r}{\Gamma(s)} \int_0^\infty F_{p,q}^{(r)}(x, -t) t^{s-r-1} dt,$$

where $\Gamma(s) = \int_0^\infty z^{s-1} e^{-z} dz$ is gamma function and $F_{p,q}^{(r)}(x, t)$ is generating function of the Carlitz's type higher-order (p, q) -Genocchi polynomials.

Proof. From Definition 1 and Equation (9), we get

$$\begin{aligned} \eta_{p,q}^{(r)}(s, x) &= [2]_q^r \sum_{m_1, \dots, m_r=0}^{\infty} \frac{(-1)^{m_1+\dots+m_r} q^{m_1+\dots+m_r}}{[m_1 + \dots + m_r + x]_{p,q}^s} \\ &= \frac{[2]_q^r}{\Gamma(s)} \sum_{m_1, \dots, m_r=0}^{\infty} \frac{(-1)^{m_1+\dots+m_r} q^{m_1+\dots+m_r}}{[m_1 + \dots + m_r + x]_{p,q}^s} \int_0^\infty z^{s-1} e^{-z} dz \\ &= \frac{(-1)^r [2]_q^r}{t^r \Gamma(s)} \sum_{m_1, \dots, m_r=0}^{\infty} (-1)^{m_1+\dots+m_r} q^{m_1+\dots+m_r} (-t)^r \int_0^\infty e^{-[m_1+\dots+m_r+x]_{p,q} t} t^{s-1} dt \\ &= \frac{(-1)^r}{\Gamma(s)} \int_0^\infty F_{p,q}^{(r)}(x, -t) t^{s-r-1} dt. \end{aligned}$$

Consequently, we complete the proof of Theorem 12. $\square$

Theorem 13. For $s, x \in \mathbb{C}$ with $\Re(x) > 0$ and $r, n \in \mathbb{N}$, we obtain

$$\eta_{p,q}^{(r)}(-n, x) = \frac{(-1)^r G_{n+r,p,q}^{(r)}(x)}{\binom{n+r}{r} r!}.$$

Proof. By using Definition 1 and Theorem 12, we have

$$\begin{aligned} \eta_{p,q}^{(r)}(s, x) &= \frac{(-1)^r}{\Gamma(s)} \int_0^\infty F_{p,q}^{(r)}(x, -t) t^{s-r-1} dt \\ &= \frac{(-1)^r}{\Gamma(s)} \int_0^\infty \sum_{m=0}^\infty G_{m,p,q}^{(r)}(x) \frac{(-t)^m}{m!} t^{s-r-1} dt. \end{aligned} \quad (10)$$

Let $n \in \mathbb{N}$. If we apply $s = -n$ in the equation above (10), then we get

$$\begin{aligned} \eta_{p,q}^{(r)}(-n, x) &= \lim_{s \rightarrow -n} \frac{1}{\Gamma(s)} \int_0^\infty \sum_{m=0}^\infty (-1)^m G_{m,p,q}^{(r)}(x) \frac{t^{m-n-r-1}}{m!} dt \\ &= \left(\lim_{s \rightarrow -n} \frac{1}{\Gamma(s)} \right) 2\pi i \frac{(-1)^{n+r} G_{n+r,p,q}^{(r)}(x)}{(n+r)!} \\ &= \frac{(-1)^r G_{n+r,p,q}^{(r)}(x)}{\binom{n+r}{r} r!}, \end{aligned}$$

because $\Gamma(-n) = 2\pi i \frac{(-1)^n}{n!}$. Thus, the proof is finished. $\square$

Theorem 14. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod{2}$ and $w_2 \equiv 1 \pmod{2}$. For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_0$, we get

$$\begin{aligned} & \frac{[w_2]_{p,q}^n}{[2]_{q^{w_2}}^r} \sum_{i_1, \dots, i_r=0}^{w_2-1} (-1)^{i_1+\dots+i_r} q^{w_1(i_1+\dots+i_r)} G_{n+r, p^{w_2}, q^{w_2}}^{(r)} \left(w_1 x + \frac{w_1}{w_2} (i_1 + \dots + i_r) \right) \\ &= \frac{[w_1]_{p,q}^n}{[2]_{q^{w_1}}^r} \sum_{i_1, \dots, i_r=0}^{w_1-1} (-1)^{i_1+\dots+i_r} q^{w_2(i_1+\dots+i_r)} G_{n+r, p^{w_1}, q^{w_1}}^{(r)} \left(w_2 x + \frac{w_2}{w_1} (i_1 + \dots + i_r) \right). \end{aligned}$$

Proof. Let us replace x with $w_1 x + \frac{w_1}{w_2} (i_1 + \dots + i_r)$, p with p^{w_2} , and q with q^{w_2} in Theorem 5. Then, we obtain

$$\begin{aligned} & \frac{G_{n+r, p^{w_2}, q^{w_2}}^{(r)} \left(w_1 x + \frac{w_1}{w_2} (i_1 + \dots + i_r) \right)}{\binom{n+r}{r} r!} \\ &= [2]_{q^{w_2}}^r \sum_{m_1, \dots, m_r=0}^{\infty} (-1)^{m_1+\dots+m_r} q^{w_2(m_1+\dots+m_r)} [m_1 + \dots + m_r + w_1 x + \frac{w_1}{w_2} (i_1 + \dots + i_r)]_{p^{w_2}, q^{w_2}}^n \\ &= [2]_{q^{w_2}}^r \sum_{m_1, \dots, m_r=0}^{\infty} (-1)^{m_1+\dots+m_r} q^{w_2(m_1+\dots+m_r)} \\ & \quad \times \frac{[w_2(m_1 + \dots + m_r) + w_1 w_2 x + w_1 (i_1 + \dots + i_r)]_{p,q}^n}{[w_2]_{p,q}^n} \\ &= \frac{[2]_{q^{w_2}}^r}{[w_2]_{p,q}^n} \sum_{m_1, \dots, m_r=0}^{\infty} \sum_{j_1, \dots, j_r=0}^{w_1-1} (-1)^{j_1+\dots+j_r} (-1)^{m_1+\dots+m_r} q^{w_2(j_1+w_1 m_1+\dots+j_r+w_1 m_r)} \\ & \quad \times [w_1 w_2 (m_1 + \dots + m_r + x) + w_1 (i_1 + \dots + i_r) + w_2 (j_1 + \dots + j_r)]_{p,q}^n. \end{aligned} \tag{11}$$

If we multiply the equation above (11) by $\sum_{i_1, \dots, i_r=0}^{w_2-1} (-1)^{i_1+\dots+i_r} q^{w_1(i_1+\dots+i_r)}$, then the identity is implemented as follows:

$$\begin{aligned} & \sum_{i_1, \dots, i_r=0}^{w_2-1} (-1)^{i_1+\dots+i_r} q^{w_1(i_1+\dots+i_r)} \frac{G_{n+r, p^{w_2}, q^{w_2}}^{(r)} \left(w_1 x + \frac{w_1}{w_2} (i_1 + \dots + i_r) \right)}{\binom{n+r}{r} r!} \\ &= \frac{[2]_{q^{w_2}}^r}{[w_2]_{p,q}^n} \sum_{m_1, \dots, m_r=0}^{\infty} \sum_{j_1, \dots, j_r=0}^{w_1-1} \sum_{i_1, \dots, i_r=0}^{w_2-1} (-1)^{\sum_{k=1}^r (i_k + j_k + m_k)} q^{w_1 \sum_{k=1}^r i_k} q^{w_2 \sum_{k=1}^r (j_k + w_1 m_k)} \\ & \quad \times [w_1 w_2 (m_1 + \dots + m_r + x) + w_1 (i_1 + \dots + i_r) + w_2 (j_1 + \dots + j_r)]_{p,q}^n. \end{aligned} \tag{12}$$

The following is calculated in the same way as Equations (11) and (12).

$$\begin{aligned} & \sum_{j_1, \dots, j_r=0}^{w_1-1} (-1)^{j_1+\dots+j_r} q^{w_2(j_1+\dots+j_r)} \frac{G_{n+r, p^{w_1}, q^{w_1}}^{(r)} \left(w_2 x + \frac{w_2}{w_1} (j_1 + \dots + j_r) \right)}{\binom{n+r}{r} r!} \\ &= \frac{[2]_{q^{w_1}}^r}{[w_1]_{p,q}^n} \sum_{m_1, \dots, m_r=0}^{\infty} \sum_{j_1, \dots, j_r=0}^{w_1-1} \sum_{i_1, \dots, i_r=0}^{w_2-1} (-1)^{\sum_{k=1}^r (i_k + j_k + m_k)} q^{w_2 \sum_{k=1}^r j_k} q^{w_1 \sum_{k=1}^r (i_k + w_2 m_k)} \\ & \quad \times [w_1 w_2 (m_1 + \dots + m_r + x) + w_1 (i_1 + \dots + i_r) + w_2 (j_1 + \dots + j_r)]_{p,q}^n. \end{aligned} \tag{13}$$

Therefore, we get the results of Theorem 14 by comparing Equations (12) and (13). $\square$

Let us take $w_2 = 1$ in Theorem 14. Then, we get the following corollary.

Corollary 2. Let $w_1 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod{2}$. For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_0$, we get

$$G_{n+r,p,q}^{(r)}(w_1 x) = \frac{[2]_q^r [w_1]_{p,q}^n}{[2]_{q^{w_1}}^r} \sum_{i_1, \dots, i_r=0}^{w_1-1} (-1)^{i_1+\dots+i_r} q^{i_1+\dots+i_r} G_{n+r,p^{w_1},q^{w_1}}^{(r)} \left(x + \frac{i_1+\dots+i_r}{w_1} \right).$$

Theorem 15. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod{2}$ and $w_2 \equiv 1 \pmod{2}$. For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_0$, we obtain

$$\begin{aligned} & \frac{[w_2]_{p,q}^n}{[2]_{q^{w_2}}^r} \sum_{i_1, \dots, i_r=0}^{w_2-1} (-1)^{i_1+\dots+i_r} q^{w_1(i_1+\dots+i_r)} \eta_{p^{w_2},q^{w_2}}^{(r)} \left(-n, w_1 x + \frac{w_1}{w_2} (i_1 + \dots + i_r) \right) \\ &= \frac{[w_1]_{p,q}^n}{[2]_{q^{w_1}}^r} \sum_{i_1, \dots, i_r=0}^{w_1-1} (-1)^{i_1+\dots+i_r} q^{w_2(i_1+\dots+i_r)} \eta_{p^{w_1},q^{w_1}}^{(r)} \left(-n, w_2 x + \frac{w_2}{w_1} (i_1 + \dots + i_r) \right). \end{aligned}$$

Proof. By comparing Theorems 13 and 14, we can easily get the results of Theorem 15. $\square$

If we put $w_2 = 1$ in Theorem 15, then we get the following corollary.

Corollary 3. Let $w_1 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod{2}$. For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_0$, we get

$$\eta_{p,q}^{(r)}(-n, w_1 x) = \frac{[2]_q^r [w_1]_{p,q}^n}{[2]_{q^{w_1}}^r} \sum_{i_1, \dots, i_r=0}^{w_1-1} (-1)^{i_1+\dots+i_r} q^{1(i_1+\dots+i_r)} \eta_{p,q}^{(r)} \left(-n, x + \frac{i_1+\dots+i_r}{w_1} \right).$$

Theorem 16. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod{2}$ and $w_2 \equiv 1 \pmod{2}$. For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_0$, we have

$$\begin{aligned} & \sum_{i_1, \dots, i_r=0}^{w_2-1} (-1)^{i_1+\dots+i_r} q^{w_1(i_1+\dots+i_r)} \frac{G_{n+r,p^{w_2},q^{w_2}}^{(r)} \left(w_1 x + \frac{w_1}{w_2} (i_1 + \dots + i_r) \right)}{\binom{n+r}{r}} \\ &= \sum_{k=0}^n \binom{n}{k} \frac{[w_1]_{p,q}^k}{[w_2]_{p,q}^k} p^{w_1 w_2 k x} \frac{G_{n-k+r,p^{w_2},q^{w_2}}^{(r,k)}(w_1 x)}{\binom{n-k+r}{r}} \\ &\quad \times \sum_{i_1, \dots, i_r=0}^{w_2-1} (-1)^{i_1+\dots+i_r} q^{w_1(n-k+1)(i_1+\dots+i_r)} [i_1 + \dots + i_r]_{p^{w_1},q^{w_1}}^k. \end{aligned}$$

Proof. From Theorem 10, we have

$$\begin{aligned} & \sum_{i_1, \dots, i_r=0}^{w_2-1} (-1)^{i_1+\dots+i_r} q^{w_1(i_1+\dots+i_r)} \frac{G_{n+r,p^{w_2},q^{w_2}}^{(r)} \left(w_1 x + \frac{w_1}{w_2} (i_1 + \dots + i_r) \right)}{\binom{n+r}{r}} \\ &= \sum_{i_1, \dots, i_r=0}^{w_2-1} (-1)^{i_1+\dots+i_r} q^{w_1(i_1+\dots+i_r)} \sum_{k=0}^n \binom{n}{k} q^{(n-k)w_1(i_1+\dots+i_r)} p^{w_1 w_2 k x} \\ &\quad \times \frac{G_{n-k+r,p^{w_2},q^{w_2}}^{(r,k)}(w_1 x)}{\binom{n-k+r}{r}} \left[\frac{w_1}{w_2} (i_1 + \dots + i_r) \right]_{p^{w_2},q^{w_2}}^k \\ &= \sum_{i_1, \dots, i_r=0}^{w_2-1} (-1)^{i_1+\dots+i_r} q^{w_1(i_1+\dots+i_r)} \sum_{k=0}^n \binom{n}{k} q^{(n-k)w_1(i_1+\dots+i_r)} p^{w_1 w_2 k x} \\ &\quad \times \frac{G_{n-k+r,p^{w_2},q^{w_2}}^{(r,k)}(w_1 x)}{\binom{n-k+r}{r}} [i_1 + \dots + i_r]_{p^{w_1},q^{w_1}}^k \left(\frac{[w_1]_{p,q}}{[w_2]_{p,q}} \right)^k. \end{aligned}$$

Hence, we finish the proof. $\square$

For $n \in \mathbb{Z}_0$, we define identity as follows:

$$\mathcal{A}_{n,k,p,q}^{(r)}(w) = \sum_{i_1, \dots, i_r=0}^{w-1} (-1)^{i_1+\dots+i_r} q^{(n-k+1)(i_1+\dots+i_r)} [i_1 + \dots + i_r]_{p,q}^k, \quad (14)$$

where $\mathcal{A}_{n,k,p,q}^{(r)}(w)$ is called alternating (p, q) -power sums.

Theorem 17. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod{2}$ and $w_2 \equiv 1 \pmod{2}$. For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_0$, we get

$$\begin{aligned} & [2]_{q^{w_1}}^r \sum_{k=0}^n \binom{n}{k} [w_1]_{p,q}^k [w_2]_{p,q}^{n-k} p^{w_1 w_2 k x} \frac{G_{n-k+r,p^{w_2},q^{w_2}}^{(r,k)}(w_1 x)}{\binom{n-k+r}{r}} \mathcal{A}_{n,k,p^{w_1},q^{w_1}}^{(r)}(w_2) \\ &= [2]_{q^{w_2}}^r \sum_{k=0}^n \binom{n}{k} [w_2]_{p,q}^k [w_1]_{p,q}^{n-k} p^{w_1 w_2 k x} \frac{G_{n-k+r,p^{w_1},q^{w_1}}^{(r,k)}(w_2 x)}{\binom{n-k+r}{r}} \mathcal{A}_{n,k,p^{w_2},q^{w_2}}^{(r)}(w_1). \end{aligned}$$

Proof. By using Theorems 14 and 16, we have

$$\begin{aligned} & [2]_{q^{w_1}}^r [w_2]_{p,q}^n \sum_{i_1, \dots, i_r=0}^{w_2-1} (-1)^{i_1+\dots+i_r} q^{w_1(i_1+\dots+i_r)} \frac{G_{n+r,p^{w_2},q^{w_2}}^{(r)}\left(w_1 x + \frac{w_1}{w_2}(i_1 + \dots + i_r)\right)}{\binom{n+r}{r}} \\ &= [2]_{q^{w_1}}^r \sum_{k=0}^n \binom{n}{k} [w_1]_{p,q}^k [w_2]_{p,q}^{n-k} p^{w_1 w_2 k x} \frac{G_{n-k+r,p^{w_2},q^{w_2}}^{(r,k)}(w_1 x)}{\binom{n-k+r}{r}} \mathcal{A}_{n,k,p^{w_1},q^{w_1}}^{(r)}(w_2) \end{aligned}$$

and

$$\begin{aligned} & [2]_{q^{w_2}}^r [w_1]_{p,q}^n \sum_{i_1, \dots, i_r=0}^{w_1-1} (-1)^{i_1+\dots+i_r} q^{w_2(i_1+\dots+i_r)} \frac{G_{n+r,p^{w_1},q^{w_1}}^{(r)}\left(w_2 x + \frac{w_2}{w_1}(i_1 + \dots + i_r)\right)}{\binom{n+r}{r}} \\ &= [2]_{q^{w_2}}^r \sum_{k=0}^n \binom{n}{k} [w_2]_{p,q}^k [w_1]_{p,q}^{n-k} p^{w_1 w_2 k x} \frac{G_{n-k+r,p^{w_1},q^{w_1}}^{(r,k)}(w_2 x)}{\binom{n-k+r}{r}} \mathcal{A}_{n,k,p^{w_2},q^{w_2}}^{(r)}(w_1). \end{aligned}$$

Consequently, we finish the proof from the result above. $\square$

If $p = 1$ and $q \rightarrow 1$ in Theorem 17, then we get the following corollary.

Corollary 4. Let $w_1, w_2 \in \mathbb{N}$, with $w_1 \equiv 1 \pmod{2}$ and $w_2 \equiv 1 \pmod{2}$. For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_0$, we have

$$\sum_{k=0}^n \binom{n}{k} w_1^k w_2^{n-k} \frac{G_{n-k+r}^{(r)}(w_1 x)}{\binom{n-k+r}{r}} \mathcal{A}_{n,k}^{(r)}(w_2) = \sum_{k=0}^n \binom{n}{k} w_2^k w_1^{n-k} \frac{G_{n-k+r}^{(r)}(w_2 x)}{\binom{n-k+r}{r}} \mathcal{A}_{n,k}^{(r)}(w_1),$$

where $\mathcal{A}_{n,k}^{(r)}(w) = \sum_{i_1, \dots, i_r=0}^{w-1} (-1)^{i_1+\dots+i_r} (i_1 + \dots + i_r)^k$ and $G_n^{(k)}(x)$ are higher-order Genocchi polynomials.

In Section 3, we investigate the multiple (p, q) -Hurwitz–Euler eta function that is defined by Hwang and Ryoo [7] using gamma function and generating functions of the Carlitz’s type higher-order (p, q) -Genocchi polynomials. We also derive symmetric properties for Carlitz’s type higher-order (p, q) -Genocchi polynomials and the multiple (p, q) -Hurwitz–Euler eta function. Furthermore, we find the alternating (p, q) -power sums. We look for the symmetric property related to the alternating (p, q) -power sums and Carlitz’s type higher-order (h, p, q) -Genocchi polynomials.

4. Roots of the Carlitz's Type Higher-Order (p, q) -Genocchi Polynomials

In this section, we would like to show some patterns for the roots of the higher-order (p, q) -Genocchi polynomials $G_{n,p,q}^{(r)}(x) = 0$ for a given n and r , using numerical experiments. The higher-order (p, q) -Genocchi polynomials $G_{n,p,q}^{(r)}(x)$ can be found explicitly by using a mathematical program on computer. With the computer program, we examine the roots of the higher-order (p, q) -Genocchi polynomials $G_{n,p,q}^{(r)}(x) = 0$ for a given r . The roots of the higher-order (p, q) -Genocchi polynomials $G_{n,p,q}^{(r)}(x) = 0$ for $n = 20, p = 1/2, q = 1/10, r = 3, 33, 333, 3333$ and $x \in \mathbb{C}$ are displayed in Figure 1.

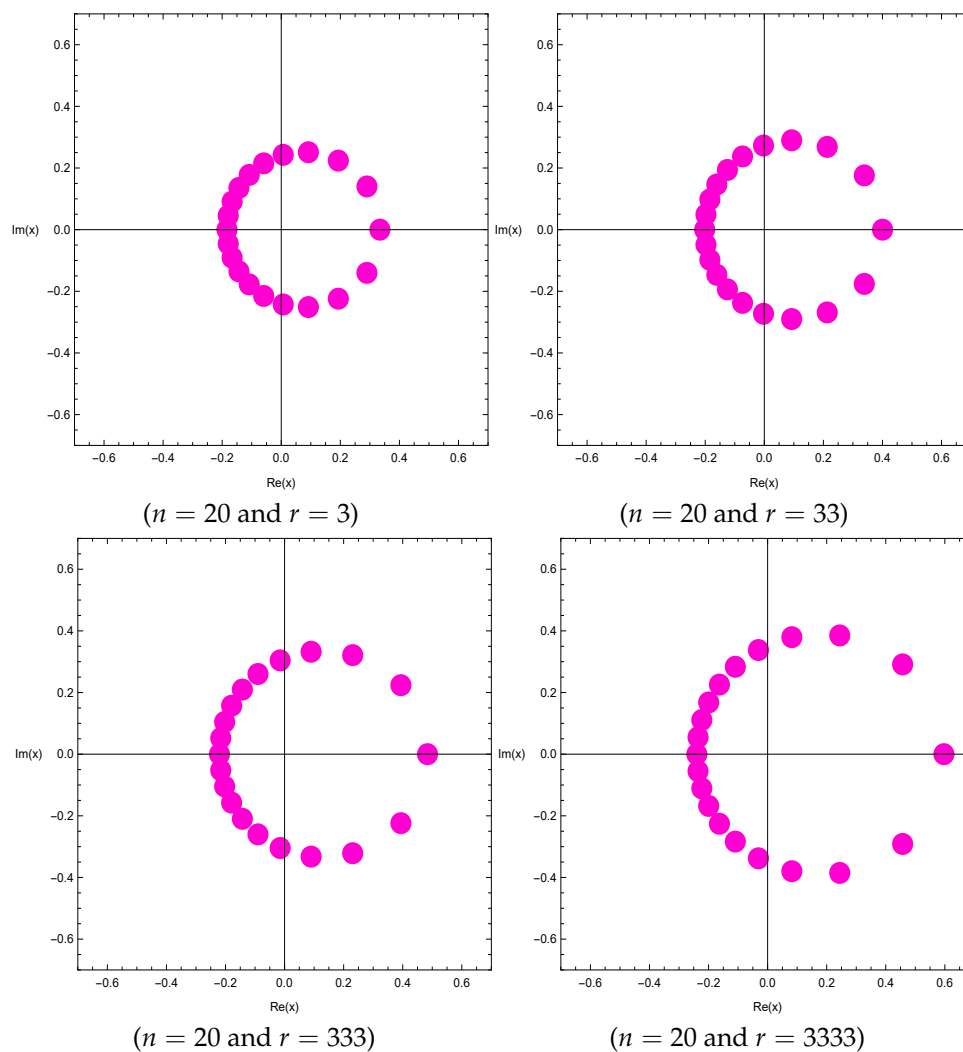


Figure 1. Roots of $G_{n,p,q}^{(r)}(x) = 0$.

For the top left picture in Figure 1, we select $n = 20$ and $r = 3$. For the top right picture in Figure 1, we select $n = 20$ and $r = 33$. For the bottom left picture in Figure 1, we select $n = 20$ and $r = 333$. For the bottom right picture in Figure 1, we select $n = 20$ and $r = 3333$.

We can see that the root structure of the higher-order (p, q) -Genocchi polynomials $G_{n,p,q}^{(r)}(x) = 0$ is very general. Therefore, the theoretical prediction of the regular structure of the zeros of the higher-order (p, q) -Genocchi polynomials $G_{n,p,q}^{(r)}(x) = 0$ will have been a popular research problem (Figure 1). Prove or disprove that $G_{n,p,q}^{(r)}(x), x \in \mathbb{C}$, has $\text{Im}(x) = 0$ reflection symmetry analytic complex functions. Although many more values of n have been checked, it still remains unknown if the open problem holds or fails for any value n (see Figure 1).

We show our numerical experiments for approximate solutions of real roots of the higher-order (p, q) -Genocchi polynomials $G_{n,p,q}^{(r)}(x) = 0$ (Tables 1 and 2).

Table 1. Numbers of real and complex roots of $G_{n,p,q}^{(r)}(x) = 0$.

Degree n	$r = 1, p = 1/2, q = 1/10$		$r = 3, p = 1/2, q = 1/10$	
	Real Roots	Complex Roots	Real Roots	Complex Roots
2	1	0	*	*
3	2	0	*	*
4	1	2	1	0
5	1	4	*	*
6	2	4	1	2
7	1	6	*	*
8	*	*	1	4
9	1	8	2	4
10	2	8	1	6

The * mark in Table 1 shows no solution of $G_{n,p,q}^{(r)}(x) = 0$. We can also find out that there is a regular pattern in the complex roots of the higher-order (p, q) -Genocchi polynomials $G_{n,p,q}^{(r)}(x) = 0$. Furthermore, we would like to verify the same kind of regular structure of the complex roots of the higher-order (p, q) -Genocchi polynomials $G_{n,p,q}^{(r)}(x) = 0$ (Table 1).

Next, we show the approximate roots satisfying $G_{n,p,q}^{(r)}(x) = 0, x \in \mathbb{C}$ for given $n, r = 3, p = 1/2, q = 1/10$ in Table 2.

Table 2. Approximate roots of $G_{n,p,q}^{(r)}(x) = 0, r = 3, p = 1/2, q = 1/10, x \in \mathbb{C}$.

Degree n	x			
4	0.0723976			
5	*			
6	$-0.0919934 - 0.114792\text{ i},$	$-0.0919934 + 0.114792\text{ i},$	0.206956	
7	*			
8	$-0.141585 - 0.0805248\text{ i},$	$-0.141585 + 0.0805248\text{ i},$	$0.0152158 - 0.201828\text{ i},$	$0.0152158 + 0.201828\text{ i},$
9	$-0.163912,$	$-0.114908 - 0.13525\text{ i},$	$-0.114908 + 0.13525\text{ i},$	$0.0615869 - 0.214241\text{ i},$
	$0.0615869 - 0.214241\text{ i},$	$0.0615869 + 0.214241\text{ i},$	0.273465	

The * mark in Table 2 means that there is no solution of $G_{n,p,q}^{(r)}(x) = 0$.

5. Conclusions

In previous papers [2,8], these studies show some identities of symmetry on the higher-order Genocchi polynomials and Carlitz's type (p, q) -Genocchi polynomials. This paper focuses on some explicit identities for Carlitz's type higher-order (p, q) -Genocchi polynomials $G_{n,p,q}^{(r)}(x)$ in the second column of the diagram at page 3. Thus, we define the Carlitz's type higher-order (p, q) -Genocchi polynomials $G_{n,p,q}^{(r)}(x)$ in Definition 1 and obtain the formulas, including explicit formula (Theorem 6) and distribution relation (Theorem 11 and Corollary 2). In Theorem 14 and Theorem 17, we give some symmetric identity related to the Carlitz's type higher-order (p, q) -Genocchi polynomials $G_{n,p,q}^{(r)}(x)$, the Carlitz's type higher-order (h, p, q) -Genocchi polynomials $G_{n,p,q}^{(r,h)}(x)$, and the alternating (p, q) -power sums. In addition, if we put $r = 1$ in this paper, we will have same special case of [8].

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