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Subordination and Superordination Properties for Certain Family of Analytic Functions Associated with Mittag–Leffler Function

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Abstract: We obtain new outcomes of analytic functions linked with operator $\mathcal{H}^{\eta,k}_{\alpha,\beta}(f)$ defined by Mittag–Leffler function. Moreover, new theorems of differential sandwich-type are obtained.

Keywords: differential subordinations; differential superordinations; subordinant; Mittag–Leffler function

1. Basic Definitions and Preliminaries

Let $\mathbb{A}$ define the class of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{H}[a, n]$ be the subclass of $\mathbb{A}$, which is

$$f(z) = a + a_n z^n + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + a_{n+3}z^{n+3} + \dots \qquad (a \in \mathbb{C}),$$

Furthermore, let $\mathcal{H}$ be the subclass of $\mathbb{A}$ of all the functions $f(z) \in \mathcal{H}$ normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Attiya [1] introduced and investigated the operator $\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z)) : \mathcal{H} \to \mathcal{H}$, which $\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))$ is defined by

$$\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z)) = Q^{\eta,k}_{\alpha,\beta}(z) * f(z), \quad (z \in \mathbb{U}),$$

for $f(z) \in \mathcal{H}$ given by (1), the symbol $*$ denotes the Hadamard product, and

$$Q^{\eta,k}_{\alpha,\beta}(z) = \frac{\Gamma(\alpha+\beta)}{(\eta)_k}\left(E^{\eta,k}_{\alpha,\beta}(z) - \frac{1}{\Gamma(\beta)}\right), \quad (z \in \mathbb{U}).$$

Moreover, the function $E^{\eta,k}_{\alpha,\beta}(z)$ is called the general Mittag–Leffler function defined by

$$E^{\eta,k}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\eta)_{nk} z^n}{\Gamma(\alpha n+\beta) n!}, \quad (\alpha, \beta, \eta \in \mathbb{C}; Re(\alpha) > max\{0, Re(k) - 1\}; Re(k) > 0),$$

where

$$(\eta)_n = \frac{\Gamma(\eta+n)}{\Gamma(\eta)} = \begin{cases} 1, & n = 0, \\ \eta(\eta+1)(\eta+2)\dots(\eta+n-1), & n \in \mathbb{N}. \end{cases}$$

The function $E^{\eta,k}_{\alpha,\beta}(z)$ was investigated by Srivastava and Tomovski [2]. Many authors studied and investigated Mittag–Leffler function; for more details on Mittag–Leffler function and general Mittag–Leffler function see, e.g., [1,3–19].

Moreover, Attiya [1] deduced that $\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))$ can be put in

$$\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z)) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\eta+nk)\Gamma(\alpha+\beta)}{\Gamma(\eta+k)\Gamma(n\alpha+\beta)} a_n z^n \quad (z \in \mathbb{U}). \tag{2}$$

It follows from (2) that (see [1])

$$kz\big(\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))\big)' = (\eta+k)\big(\mathcal{H}^{\eta+1,k}_{\alpha,\beta}(f(z))\big) - \eta\big(\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))\big), \tag{3}$$

and

$$\alpha z\big(\mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z))\big)' = (\alpha+\beta)\big(\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))\big) - \beta\big(\mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z))\big). \tag{4}$$

It should be remarked that the operator $\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))$ for some special cases of α, β, η, and k provides many special functions, e.g.,

$$\begin{aligned} \mathcal{H}^{1,1}_{0,\beta}(f)(z) &= f(z). \\ \mathcal{H}^{2,1}_{0,\beta}(f)(z) &= \tfrac{1}{2}\big(f(z) + zf'(z)\big). \\ \mathcal{H}^{0,1}_{0,\beta}(f)(z) &= \int_0^z \tfrac{1}{t} f(t)dt. \\ \mathcal{H}^{1,1}_{1,0}\big(\tfrac{z}{1-z}\big) &= ze^z. \\ \mathcal{H}^{1,1}_{1,1}\big(\tfrac{z}{1-z}\big) &= e^z - 1. \\ \mathcal{H}^{1,1}_{2,1}\big(\tfrac{z}{1-z}\big) &= -2 + \cosh(\sqrt{z}). \end{aligned}$$

Definition 1. *Let functions $f(z)$ and $g(z)$ be analytic in the open unit disk $\mathbb{U}$. Then $f(z)$ is subordinate to $g(z)$ if there exists a Schwarz function $\omega(z)$, analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1, (z \in \mathbb{U})$, such that $f(z) = g(\omega(z)), (z \in \mathbb{U})$, we denote this subordination by $f(z) \prec g(z)$. In particular, if $g(z)$ is univalent in $\mathbb{U}$, then subordination is equivalent to $f(z) \prec g(z) \Leftrightarrow f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.*

Definition 2. *If $\mathbb{Q}$ the set of all functions $q(z)$ that are analytic and univalent on $\overline{\mathbb{U}} \setminus E(q)$, where*

$$E(q) = \{\xi \in \partial\mathbb{U} : \lim_{z \to \xi} q(z) = \infty\},$$

and $min|q'(\xi)| = \rho > 0$ for $\xi \in \partial\mathbb{U} \setminus E(q)$. Further, let $\mathbb{Q}(a) = \{q(z) \in \mathbb{U} : q(0) = a\}$ and $\mathbb{Q}_1 = \mathbb{Q}(1)$.

Definition 3. *If $\psi : \mathbb{C}^4 \times \mathbb{U} \longrightarrow \mathbb{C}$ and $h(z)$ be univalent in $\mathbb{U}$. If $p(z)$ is analytic in $\mathbb{U}$, and satisfies the third-order differential subordination*

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \prec h(z), \tag{5}$$

then $p(z)$ is called a solution of the differential subordination and $q(z)$ is called a dominant of the solutions of the differential subordination as well as a dominant if $p(z) \prec q(z)$ for all $p(z)$ satisfying (5). $\widetilde{q}(z)$ that satisfies $\widetilde{q}(z) \prec q(z)$ for all dominants of (5) is called the best dominant of (5).

Definition 4. *Let $\Omega \subseteq \mathbb{C}$, $q(z) \in \mathbb{Q}$ and $n \in \mathbb{N} \setminus \{1\}$. The class of admissible functions $\Psi_n[\Omega, q(z)]$ consists of those functions $\psi : \mathbb{C}^4 \times \mathbb{U} \longrightarrow \mathbb{C}$ that satisfy the admissibility condition:*

$$\psi(r, s, t, u; z) \notin \Omega,$$

whenever

$$r = q(\zeta), \quad s = \ell\zeta q'(\zeta), \quad Re\Big(\frac{t}{s} + 1\Big) \geq \ell \quad Re\Big(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\Big),$$

and

$$Re\Big(\frac{u}{s}\Big) \geq \ell^2 \quad Re\Big(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\Big),$$

where $z \in \mathbb{U}$; $\zeta \in \partial\mathbb{U} \setminus E(q)$ and $\ell \geq n$.

Analogous to the second order differential super-ordinations introduced by Miller and Mocanu [20], Tang et al. [21] defined the differential super-ordinations as follows:

Definition 5. *Let $\psi : \mathbb{C}^4 \times \mathbb{U} \longrightarrow \mathbb{C}$ and the function $h(z)$ be analytic in $\mathbb{U}$. If functions $p(z)$ and $\psi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z))$ are univalent in $\mathbb{U}$, and satisfy the following third-order differential super-ordination*

$$h(z) \prec \psi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z)), \tag{6}$$

then $p(z)$ is called a solution of the differential superordination and $q(z)$ is called a subordinant of the solutions of the differential super-ordinations as well as a subordinant if $p(z) \prec q(z)$ for all $p(z)$ satisfying Equation (6). A univalent subordinant $\widetilde{q}(z)$ that satisfies $\widetilde{q}(z) \prec q(z)$ for all super-ordinations of (6) is the best superordinant.

Definition 6. *Let $\Omega \subseteq \mathbb{C}$, $q(z) \in \mathcal{H}[a, n]$ with $n \in \mathbb{N} \setminus \{1\}$ and $q'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Omega, q(z)]$ consists of those functions $\psi : \mathbb{C}^4 \times \mathbb{U} \longrightarrow \mathbb{C}$ that satisfy the admissibility condition:*

$$\psi(r, s, t, u; \zeta) \in \Omega,$$

whenever

$$r = q(z), \quad s = \frac{zq'(z)}{m}, \quad Re\Big(\frac{t}{s} + 1\Big) \leq \frac{1}{m} \quad Re\Big(\frac{zq''(z)}{q'(z)} + 1\Big),$$

and

$$Re\Big(\frac{u}{s}\Big) \leq \frac{1}{m^2} \quad Re\Big(\frac{z^2 q'''(z)}{q'(z)}\Big),$$

where $z \in \mathbb{U}$; $\zeta \in \partial\mathbb{U}$ and $m \geq n \geq 2$.

Here, we use the following theorems given by Antonino and Miller [22]:

Theorem 1 ([22])**.** *Let $p(z) \in \mathcal{H}[a, n]$ with $n \in \mathbb{N} \setminus \{1\}$. Also, let $q(z) \in \mathbb{Q}(a)$ and satisfy the following conditions:*

$$Re\Big(\frac{\zeta q''(\zeta)}{q'(\zeta)}\Big) > 0, \qquad \Big|\frac{zp'(z)}{q'(\zeta)}\Big| \leq \ell, \quad (z \in \mathbb{U}; \zeta \in \partial\mathbb{U} \setminus E(q); \ell \geq n),$$
$$\psi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z) \in \Omega,$$

and, if $\Omega \subseteq \mathbb{C}$, $\psi \in \Psi_n[\Omega, q(z)]$, then $p(z) \prec q(z)$.

Theorem 2. *Let $q(z) \in \mathcal{H}[a,n]$ and $\psi \in \Psi'_n[\Omega, q(z)]$. If $p(z) \in \mathbb{Q}(a)$ and $\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z)$ is univalent in $\mathbb{U}$ and*

$$Re\Big(\frac{zq''(z)}{q'(z)}\Big) \geq 0, \quad \Big|\frac{\zeta p'(\zeta)}{q'(z)}\Big| \leq m, \quad (z \in \mathbb{U}; \zeta \in \partial\mathbb{U}; m \geq n \geq 2)$$
$$\Omega \subset \Big\{\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) : z \in \mathbb{U}\Big\},$$

then $q(z) \prec p(z)$.

Here, we study a certain family of admissible functions by using the third-order differential subordination and superordination given by Antonino and Miller [22] and Tang et al. [21]—see also Attiya et al. [23]—we obtain new results of subordination and superordination properties of analytic functions linked with the operator $\mathcal{H}^{\eta,k}_{\alpha,\beta}(f)$.

2. Main Results

Definition 7. *Let $\Omega \subseteq \mathbb{C}$ and $q(z) \in \mathbb{Q}$. The class of admissible functions $\Psi_\Gamma[\Omega, q(z)]$ consists of those functions $\phi : \mathbb{C}^4 \times \mathbb{U} \longrightarrow \mathbb{C}$ that satisfy the admissibility condition*

$$\phi(a_1, a_2, a_3, a_4; z) \notin \Omega,$$

whenever

$$a_1 = q(\zeta), \quad a_2 = \frac{\ell k \zeta q'(\zeta) + bq(\zeta)}{b},$$
$$Re\Big(\frac{(b+1)(a_3 - a_1)}{k(a_2 - a_1)} - \frac{2b+1}{k}\Big) \geq \ell \, Re\Big(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\Big),$$

$$Re\Big(\frac{(b+1)(b+2)(a_4 - a_1) - 3(b+1)(b+k+1)(a_3 - a_1)}{k^2(a_2 - a_1)} + \frac{3b(b+1)+1}{k^2} + \frac{6b+3}{k} + 2\Big) \geq \ell^2 Re\Big(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\Big),$$

where $z \in \mathbb{U}$; $\zeta \in \partial\mathbb{U} \setminus E(q)$, $\ell \in \mathbb{N} \setminus \{1\}$ and $b = \eta + k$.

Theorem 3. *Let $\phi \in \Psi_\Gamma[\Omega, q(z)]$. If $f(z) \in \mathcal{H}$ and $q(z) \in \mathbb{Q}_1$ satisfy:*

$$Re\Big(\frac{\zeta q''(\zeta)}{q'(\zeta)}\Big) \geq 0, \qquad \Big|\frac{\mathcal{H}^{\eta+1,k}_{\alpha,\beta}(f(z)) - \mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{zq'(\zeta)}\Big| \leq |\frac{k}{b}|\ell, \tag{7}$$

$$\Big\{\phi\Big(\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+1,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+2,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+3,k}_{\alpha,\beta}(f(z))}{z}; z\Big) : z \in \mathbb{U}\Big\} \subset \Omega, \tag{8}$$

then

$$\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z)) \prec q(z).$$

Proof. Let

$$\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z)) = zp(z) \qquad z \in \mathbb{U}, \tag{9}$$

From (3), we have

$$\frac{\mathcal{H}^{\eta+1,k}_{\alpha,\beta}(f(z))}{z} = (\frac{k}{b})(zp'(z) + \frac{b}{k}p(z)), \tag{10}$$

which implies

$$\frac{\mathcal{H}^{\eta+2,k}_{\alpha,\beta}(f(z))}{z} = \frac{k^2}{b(b+1)}\big(z^2p''(z) + (\frac{2b+1}{k}+1)zp'(z) + \frac{b(b+1)}{k^2}p(z)\big). \tag{11}$$

Furthermore, we have

$$\begin{aligned}\frac{\mathcal{H}^{\eta+3,k}_{\alpha,\beta}(f(z))}{z} &= \frac{k^3}{b(b+1)(b+2)}\Big(z^3p'''(z) + 3(\frac{b+1}{k}+1)z^2p''(z) \\ &+ (\frac{3b^2+6b+2}{k^2} + \frac{3(b+1)}{k}+1)zp'(z) + \frac{b(b+1)(b+2)}{k^3}p(z)\Big).\end{aligned} \tag{12}$$

Now, we define the parameters a_1, a_2, a_3, and a_4 as

$$a_1 = r, \quad a_2 = (\frac{k}{b})(s+\frac{b}{k}r), \quad a_3 = \frac{k^2}{b(b+1)}\big(t + (\frac{2b+1}{k}+1)s + \frac{b(b+1)}{k^2}r\big),$$

and

$$\begin{aligned}a_4 = \frac{k^3}{b(b+1)(b+2)}\Big(u + 3(\frac{b+1}{k}+1)t + (\frac{3b^2+6b+2}{k^2} + \frac{3(b+1)}{k}+1)s \\ + \frac{b(b+1)(b+2)}{k^3}r\Big).\end{aligned}$$

Then, transformation $\psi : \mathbb{C}^4 \times \mathbb{U} \longrightarrow \mathbb{C}$ as

$$\psi(r,s,t,u;z) = \phi(a_1,a_2,a_3,a_4;z), \tag{13}$$

by using the relations from (9) to (12), we have

$$\begin{aligned}&\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) = \\ &\phi\Big(\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+1,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+2,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+3,k}_{\alpha,\beta}(f(z))}{z}; z\Big),\end{aligned} \tag{14}$$

therefore, we recompute (8) as

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \in \Omega,$$

then, the proof is completed by showing that the admissibility condition for $\phi \in \Psi_\Gamma[\Omega, q(z)]$ is equivalent to the admissibility condition for ψ as given in Definition 3, since

$$\frac{t}{s}+1 = \frac{(b+1)(a_3-a_1)}{k(a_2-a_1)} - \frac{2b+1}{k},$$

and

$$\begin{aligned}\frac{u}{s} = \frac{(b+1)(b+2)(a_4-a_1) - 3(b+1)(b+k+1)(a_3-a_1)}{k^2(a_2-a_1)} \\ + \frac{3b(b+1)+1}{k^2} + \frac{6b+3}{k} + 2,\end{aligned}$$

we also note that

$$\Big|\frac{zp'(z)}{q'(\zeta)}\Big| = \Big|\frac{(\frac{b}{zk})\big(\mathcal{H}^{\eta+1,k}_{\alpha,\beta}(f(z)) - \mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))\big)}{q'(\zeta)}\Big| \le \ell,$$

therefore, $\psi \in \Psi_\Gamma[\Omega, q(z)]$ and by Theorem 1, $p(z) \prec q(z)$. $\square$

In a similar way, we define the parameters a_1, a_2, a_3, and a_4 as follows:

Definition 8. *Let $\Omega \subseteq \mathbb{C}$ and $q(z) \in \mathbb{Q}$. The class of admissible functions $\Psi_\Gamma[\Omega, q(z)]$ consists of those functions $\phi : \mathbb{C}^4 \times \mathbb{U} \longrightarrow \mathbb{C}$ that satisfy the admissibility condition*

$$\phi(a_1, a_2, a_3, a_4; z) \notin \Omega,$$

whenever

$$a_1 = q(\zeta), \qquad a_2 = \frac{\ell\alpha\zeta q'(\zeta) + cq(\zeta)}{c},$$

$$Re\Big(\frac{(c-1)(a_3 - a_1)}{\alpha(a_2 - a_1)} - \frac{2c-1}{\alpha}\Big) \geq \ell \quad Re\Big(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\Big),$$

$$Re\Big(\frac{(c-1)(c-2)(a_4 - a_1) - 3(c-1)(c+\alpha-1)(a_3 - a_1)}{\alpha^2(a_2 - a_1)} + \frac{3c(c-1)+1}{\alpha^2} + \frac{6c-3}{\alpha} + 2\Big) \geq \ell^2 Re\Big(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\Big),$$

where $z \in \mathbb{U}$; $\zeta \in \partial\mathbb{U} \setminus E(q)$, $\ell \in \mathbb{N} \setminus \{1\}$ and $c = \alpha + \beta$.

Theorem 4. *Let $\phi \in \Psi_\Gamma[\Omega, q(z)]$. If $f(z) \in \mathcal{H}$ and $q(z) \in \mathbb{Q}_1$ satisfy the following conditions:*

$$Re\Big(\frac{\zeta q''(\zeta)}{q'(\zeta)}\Big) \geq 0, \qquad \Big|\frac{\frac{1}{z}\big[\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z)) - \mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z))\big]}{q'(\zeta)}\Big| \leq |\frac{\alpha}{c}|\ell, \tag{15}$$

$$\Big\{\phi\Big(\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta-1}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta-2}(f(z))}{z}; z\Big) : z \in \mathbb{U}\Big\} \subset \Omega, \tag{16}$$

then

$$\mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z)) \prec q(z).$$

Proof. Let

$$\mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z)) = zp(z) \qquad z \in \mathbb{U}, \tag{17}$$

From (4), we have

$$\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z} = (\frac{\alpha}{c})(zp'(z) + \frac{c}{\alpha}p(z)), \tag{18}$$

which implies

$$\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta-1}(f(z))}{z} = \frac{\alpha^2}{c(c-1)}(z^2p''(z) + (\frac{2c-1}{\alpha} + 1)zp'(z) + \frac{c(c-1)}{\alpha^2}p(z)). \tag{19}$$

Moreover, we have

$$\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta-2}(f(z))}{z} = \frac{\alpha^3}{c(c-1)(c-2)}\Big(z^3p'''(z) + 3(\frac{c-1}{\alpha} + 1)z^2p''(z) + (\frac{3c^2-6c+2}{\alpha^2} + \frac{3(c-1)}{\alpha} + 1)zp'(z) + \frac{c(c-1)(c-2)}{\alpha^3}p(z)\Big). \tag{20}$$

Parameters a_1, a_2, a_3 and, a_4 as

$$a_1 = r, \quad a_2 = (\frac{\alpha}{c})(s + \frac{c}{\alpha}r), \quad a_3 = \frac{\alpha^2}{c(c-1)}\big(t + (\frac{2c-1}{\alpha}+1)s + \frac{c(c-1)}{\alpha^2}r\big),$$

and

$$a_4 = \frac{\alpha^3}{c(c-1)(c-2)}\Big(u + 3(\frac{c-1}{\alpha}+1)t + (\frac{3c^2-6c+2}{\alpha^2} + \frac{3(c-1)}{\alpha}+1)s + \frac{c(c-1)(c-2)}{\alpha^3}r\Big).$$

The transformation $\psi : \mathbb{C}^4 \times \mathbb{U} \longrightarrow \mathbb{C}$

$$\psi(r, s, t, u; z) = \phi(a_1, a_2, a_3, a_4; z), \tag{21}$$

by using the relations from (17) to (20), we have

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) = \phi\Big(\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+2,k}_{\alpha,\beta-1}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta-2}(f(z))}{z}; z\Big),$$

we recompute (16) as

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \in \Omega,$$

This completes the proof by showing that the admissibility condition for $\phi \in \Psi_\Gamma[\Omega, q(z)]$ is equivalent to the admissibility condition for ψ as given in Definition 3, since

$$\frac{t}{s} + 1 = \frac{(c-1)(a_3-a_1)}{\alpha(a_2-a_1)} - \frac{2c-1}{\alpha},$$

and

$$\frac{u}{s} = \frac{(c-1)(c-2)(a_4-a_1) - 3(c-1)(c+\alpha-1)(a_3-a_1)}{\alpha^2(a_2-a_1)} + \frac{3c(c-1)+1}{\alpha^2} + \frac{6c-3}{\alpha} + 2,$$

we also note that

$$\Big|\frac{zp'(z)}{q'(\zeta)}\Big| = \Big|\frac{(\frac{c}{z\alpha})\big(\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z)) - \mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z))\big)}{q'(\zeta)}\Big| \le \ell,$$

therefore, $\psi \in \Psi_\Gamma[\Omega, q(z)]$ and hence by Theorem 1, $p(z) \prec q(z)$. □

If $\Omega \ne \mathbb{C}$ is simply connected to the domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω. In this case, the class $\Psi_\Gamma[h(\mathbb{U}), q(z)]$ is written as $\Psi_\Gamma[h, q]$; the following theorem is a direct consequence of Theorems 3 and 4.

Theorem 5. *Let $\phi \in \Psi_\Gamma[h, q]$. If $f(z) \in \mathcal{H}$ and $q(z) \in \mathbb{Q}_1$ satisfy the following conditions:*

$$(i) Re\Big(\frac{\zeta q''(\zeta)}{q'(\zeta)}\Big) \ge 0, \qquad \Big|\frac{\frac{1}{z}\big[\mathcal{H}^{\eta+1,k}_{\alpha,\beta}(f(z)) - \mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))\big]}{q'(\zeta)}\Big| \le |\frac{k}{b}|\ell, \tag{22}$$

$$(ii)\phi\Big(\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+1,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+2,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+3,k}_{\alpha,\beta}(f(z))}{z}; z\Big) \prec h(z),$$

then

$$\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z)) \prec q(z).$$

Theorem 6. *Let $\phi \in \Psi_\Gamma[h,q]$. If $f(z) \in \mathcal{H}$ and $q(z) \in \mathbb{Q}_1$ satisfy the following conditions:*

$$Re\Big(\frac{\zeta q''(\zeta)}{q'(\zeta)}\Big) \geq 0, \qquad \Big|\frac{\frac{1}{z}\big[\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z)) - \mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z))\big]}{q'(\zeta)}\Big| \leq |\frac{\alpha}{c}|\ell, \tag{23}$$

$$\phi\Big(\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta-1}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta-2}(f(z))}{z}; z\Big) \prec h(z),$$

then

$$\mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z)) \prec q(z).$$

The next corollaries extend Theorems 3 and 4, when the behavior of $q(z)$ on $\partial\mathbb{U}$ is not known.

Corollary 1. *Let $\Omega \subset \mathbb{C}$ and let $q(z)$ be univalent in $\mathbb{U}$; $q(0) = 1$. Let $\phi \in \Psi_\Gamma[\Omega, q_\sigma]$ for some $\sigma \in (0,1)$ where $q_\sigma(z) = q(\sigma z)$. If $f(z) \in \mathcal{H}$ satisfies the following conditions:*

$$Re\Big(\frac{\zeta q''_\sigma(\zeta)}{q'_\sigma(\zeta)}\Big) \geq 0, \qquad \Big|\frac{\frac{1}{z}\big[\mathcal{H}^{\eta+1,k}_{\alpha,\beta}(f(z)) - \mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))\big]}{q'_\sigma(\zeta)}\Big| \leq |\frac{k}{b}|\ell,$$

$$\phi\Big(\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+1,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+2,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+3,k}_{\alpha,\beta}(f(z))}{z}; z\Big) \in \Omega,$$

then

$$\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z)) \prec q(z),$$

where $z \in \mathbb{U}$ and $\zeta \in \partial\mathbb{U} \setminus E(q_\sigma)$.

Proof. by using Theorem 3, we have $\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z)) \prec q_\sigma(z)$. Then we obtain the result from $q_\sigma(z) \prec q(z)$. □

Corollary 2. *Let $\Omega \subset \mathbb{C}$ and let $q(z)$ be univalent in $\mathbb{U}$; $q(0) = 1$. Let $\phi \in \Psi_\Gamma[\Omega, q_\sigma]$ for some $\sigma \in (0,1)$, where $q_\sigma(z) = q(\sigma z)$. If $f(z) \in \mathcal{H}$ satisfy the following conditions:*

$$Re\Big(\frac{\zeta q''_\sigma(\zeta)}{q'_\sigma(\zeta)}\Big) \geq 0, \qquad \Big|\frac{\frac{1}{z}\big[\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z)) - \mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z))\big]}{q'_\sigma(\zeta)}\Big| \leq |\frac{\alpha}{c}|\ell,$$

$$\phi\Big(\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta-1}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta-2}(f(z))}{z}; z\Big) \in \Omega,$$

then

$$\mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z)) \prec q(z),$$

where $z \in \mathbb{U}$ and $\zeta \in \partial\mathbb{U} \setminus E(q_\sigma)$.

Proof. By using Theorem 4, we have $\mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z)) \prec q_\sigma(z)$. Then we obtain the result from $q_\sigma(z) \prec q(z)$. □

Corollary 3. *Let $\Omega \subset \mathbb{C}$ and let $q(z)$ be univalent in $\mathbb{U}$; $q(0) = 1$. Let $\phi \in \Psi_\Gamma[\Omega, q_\sigma]$ for some $\sigma \in (0,1)$, where $q_\sigma(z) = q(\sigma z)$. If $f(z) \in \mathcal{H}$ satisfy the following conditions:*

$$Re\Big(\frac{\zeta q''_\sigma(\zeta)}{q'_\sigma(\zeta)}\Big) \geq 0, \quad \Big|\frac{\frac{1}{z}\left[\mathcal{H}^{\eta+1,k}_{\alpha,\beta}(f(z)) - \mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))\right]}{q'_\sigma(\zeta)}\Big| \leq |\frac{k}{b}|\ell,$$

$$\phi\Big(\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+1,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+2,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+3,k}_{\alpha,\beta}(f(z))}{z}; z\Big) \prec h(z), \tag{24}$$

then

$$\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z)) \prec q(z),$$

where $z \in \mathbb{U}$ and $\zeta \in \partial\mathbb{U} \setminus E(q_\sigma)$.

Corollary 4. *Let $\Omega \subset \mathbb{C}$ and let $q(z)$ be univalent in $\mathbb{U}$; $q(0) = 1$. Let $\phi \in \Psi_\Gamma[\Omega, q_\sigma]$ for some $\sigma \in (0,1)$, where $q_\sigma(z) = q(\sigma z)$. If $f(z) \in \mathcal{H}$ satisfy the following conditions:*

$$Re\Big(\frac{\zeta q''_\sigma(\zeta)}{q'_\sigma(\zeta)}\Big) \geq 0, \quad \Big|\frac{\frac{1}{z}\left[\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z)) - \mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z))\right]}{q'_\sigma(\zeta)}\Big| \leq |\frac{\alpha}{c}|\ell,$$

$$\phi\Big(\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta-1}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta-2}(f(z))}{z}; z\Big) \prec h(z), \tag{25}$$

then

$$\mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z)) \prec q(z),$$

where $z \in \mathbb{U}$ and $\zeta \in \partial\mathbb{U} \setminus E(q_\sigma)$.

Theorem 7. *Let $h(z)$ be univalent in $\mathbb{U}$. Let $\phi : \mathbb{C}^4 \times \mathbb{U} \longrightarrow \mathbb{C}$. Suppose that the differential equation*

$$\begin{aligned}
&\phi\Big(q(z), (\frac{k}{b})\Big(zq'(z) + \frac{b}{k}q(z)\Big),\\
&\frac{k^2}{b(b+1)}\Big(z^2q''(z) + (\frac{2b+1}{k} + 1)zq'(z) + \frac{b(b+1)}{k^2}q(z)\Big),\\
&\frac{k^3}{b(b+1)(b+2)}\Big(z^3q'''(z) + 3(\frac{b+1}{k} + 1)z^2q''(z) + (\frac{3b^2+6b+2}{k^2} + \frac{3(b+1)}{k} + 1)zq'(z)\\
&\qquad + \frac{b(b+1)(b+2)}{k^3}q(z)\Big); z\Big) = h(z),
\end{aligned} \tag{26}$$

has a solution $q(z)$ with $q(0) = 1$, which satisfies (7). If $f(z) \in \mathcal{H}$ satisfies (24) and

$$\phi\Big(\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+1,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+2,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+3,k}_{\alpha,\beta}(f(z))}{z}; z\Big),$$

is analytic in $\mathbb{U}$, then

$$\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z)) \prec q(z), \tag{27}$$

and $q(z)$ is the best dominant of (27).

Proof. By using Theorem 3 that $q(z)$ is a dominant of (24). Since $q(z)$ satisfies (26), it is also a solution of (24) and therefore $q(z)$ will be dominated by all dominants. Hence, $q(z)$ is the best dominant. □

Moreover, in a similar way, using Theorem 4, we have

Theorem 8. *Let $h(z)$ be univalent in $\mathbb{U}$. Let $\phi : \mathbb{C}^4 \times \mathbb{U} \longrightarrow \mathbb{C}$. Suppose that the differential equation*

$$\begin{aligned}\phi\Big(& q(z), (\frac{\alpha}{c})\Big(zq'(z) + \frac{c}{\alpha}q(z)\Big),\\ &\frac{\alpha^2}{c(c-1)}\Big(z^2q''(z) + (\frac{2c-1}{\alpha}+1)zq'(z) + \frac{c(c-1)}{\alpha^2}q(z)\Big),\\ &\frac{\alpha^3}{c(c-1)(c-2)}\Big(z^3q'''(z) + 3(\frac{c-1}{\alpha}+1)z^2q''(z) + (\frac{3c^2-6c+2}{\alpha^2} + \frac{3(c-1)}{\alpha}+1)zq'(z)\\ &+ \frac{c(c-1)(c-2)}{\alpha^3}q(z)\Big); z\Big) = h(z), \end{aligned} \tag{28}$$

has a solution $q(z)$ with $q(0) = 1$, which satisfies (15). If $f(z) \in \mathcal{H}$ satisfies (25) and

$$\phi\Big(\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta-1}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta-2}(f(z))}{z}; z\Big),$$

is analytic in $\mathbb{U}$, then

$$\mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z)) \prec q(z). \tag{29}$$

and $q(z)$ is the best dominant of (29).

In the case $q(z) = 1 + Mz, (M > 0)$ and in view of the Definition 7, the class of admissible functions $\Psi_\Gamma[\Omega, q]$ denoted by $\Psi_\Gamma[\Omega, M]$ is defined below.

Definition 9. *Let $\Omega \subseteq \mathcal{C}$ and $M > 0$. The class of admissible functions $\Psi_\Gamma[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^4 \times \mathcal{U} \longrightarrow \mathbb{C}$ that satisfy the admissibility condition*

$$\begin{aligned}\phi\Big(& 1 + Me^{i\theta}, 1 + (\frac{k}{b})\Big(\frac{b}{k} + \ell\Big)Me^{i\theta},\\ &1 + \frac{k^2}{b(b+1)}\Big(L + (\frac{b(b+1)}{k^2} + \ell(\frac{2b+1}{k}+1))Me^{i\theta}\Big),\\ &1 + \frac{k^3}{b(b+1)(b+2)}\Big(N + 3L(\frac{b+1}{k}+1) + (\frac{b(b+1)(b+2)}{k^3}\\ &+ \ell(\frac{3b^2+6b+2}{k^2} + \frac{3(b+1)}{k}+1))Me^{i\theta}\Big); z\Big) \notin \Omega,\end{aligned}$$

where $z \in \mathbb{U}$, $Re(Le^{-i\theta}) \geq \ell(\ell-1)M$ and $Re(Ne^{-i\theta}) \geq 0$ for all real θ and $\ell \in \mathbb{N} \setminus \{1\}$.

Corollary 5. *Let $\phi \in \Psi_\Gamma[\Omega, M]$. If $f(z) \in \mathcal{H}$ satisfy the following conditions:*

$$\Big|\frac{1}{z}[\mathcal{H}^{\eta+1,k}_{\alpha,\beta}(f(z)) - \mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))]\Big| \leq |\frac{k}{b}|\ell M, \tag{30}$$

and

$$\phi\Big(\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+1,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+2,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+3,k}_{\alpha,\beta}(f(z))}{z}; z\Big) \in \Omega,$$

then

$$\Big|\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z} - 1\Big| < M.$$

Furthermore, with Definition 8, we can define the following:

Definition 10. *Let $\Omega \subseteq \mathcal{C}$ and $M > 0$. The class of admissible functions $\Psi_\Gamma[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^4 \times \mathbb{U} \longrightarrow \mathbb{C}$ that satisfy the admissibility condition*

$$\phi\Big(1 + Me^{i\theta}, 1 + (\frac{\alpha}{c})\Big(\frac{c}{\alpha} + \ell\Big)Me^{i\theta},$$
$$1 + \frac{\alpha^2}{c(c-1)}\Big(L + (\frac{c(c-1)}{\alpha^2} + \ell(\frac{2c-1}{\alpha} + 1))Me^{i\theta}\Big),$$
$$1 + \frac{\alpha^3}{c(c-1)(c-2)}\Big(N + 3L(\frac{c-1}{\alpha} + 1) + (\frac{c(c-1)(c-2)}{\alpha^3}$$
$$+ \ell(\frac{3c^2 - 6c + 2}{\alpha^2} + \frac{3(c-1)}{\alpha} + 1))Me^{i\theta}\Big); z\Big) \notin \Omega,$$

where $z \in \mathbb{U}$, $Re(Le^{-i\theta}) \geq \ell(\ell - 1)M$ and $Re(Ne^{-i\theta}) \geq 0$ for all real θ and $\ell \in \mathbb{N} \setminus \{1\}$.

Corollary 6. *Let $\phi \in \Psi_\Gamma[\Omega, M]$. If $f(z) \in \mathcal{H}$ satisfy the following conditions:*

$$\Big|\frac{1}{z}[\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z)) - \mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z))]\Big| \leq |\frac{\alpha}{c}|\ell M, \tag{31}$$

and

$$\phi\Big(\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta-1}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta-2}(f(z))}{z}; z\Big) \in \Omega,$$

then

$$\Big|\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z))}{z} - 1\Big| < M.$$

In the case $\Omega = q(\mathbb{U}) = \{\omega : |\omega - 1| < M, (M > 0)\}$, we use notation $\Phi_\Gamma[M]$ to the class $\Phi_\Gamma[\Omega, M]$.

Corollary 7. *Let $\phi \in \Psi_\Gamma[\Omega, M]$. If $f(z) \in \mathcal{H}$ satisfy the conditions (30) and*

$$\Big|\phi\Big(\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+1,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+2,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+3,k}_{\alpha,\beta}(f(z))}{z}; z\Big) - 1\Big| < M,$$

then

$$\Big|\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z} - 1\Big| < M.$$

Putting $\phi(a_1, a_2, a_3, a_4; z) = a_2 = 1 + (\frac{k}{b})\left(\frac{b}{k} + \ell\right)Me^{i\theta}$ in Corollary 7, we have the following corollary:

Corollary 8. *Let $M > 0$ and $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$ with $Re(b) < \frac{-\ell}{2}$, $\ell \in \mathbb{N} \setminus \{1\}$. If $f(z) \in \mathcal{H}$ satisfy the condition (30), and also, if :*

$$\Big|\frac{\mathcal{H}^{\eta+1,k}_{\alpha,\beta}(f(z))}{z} - 1\Big| < M,$$

then

$$\Big|\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z} - 1\Big| < M.$$

Corollary 9. *Let* $\phi \in \Psi_\Gamma[\Omega, M]$. *If* $f(z) \in \mathcal{H}$ *satisfy the conditions (31) and*

$$\Big|\phi\Big(\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta-1}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta-2}(f(z))}{z}; z\Big) - 1\Big| < M,$$

then

$$|\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z))}{z} - 1| < M.$$

Putting $\phi(a_1, a_2, a_3, a_4; z) = a_2 = 1 + (\frac{\alpha}{c})\left(\frac{c}{\alpha} + \ell\right) Me^{i\theta}$ in Corollary 9, we have the following corollary:

Corollary 10. *Let* $M > 0$ *and* $c \in \mathbb{C} \setminus \mathbb{Z}_0^-$ *with* $Re(c) < \frac{-\ell}{2}$, $\ell \in \mathbb{N} \setminus \{1\}$. *If* $f(z) \in \mathcal{H}$ *satisfy the conditions (31) and*

$$|\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z} - 1| < M,$$

then

$$|\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z))}{z} - 1| < M.$$

Corollary 11. *Let* $\ell \in \mathbb{N} \setminus 1$, $M > 0$ *and* $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$. *If* $f(z) \in \mathcal{H}$ *satisfies the condition (30) and*

$$\Big|\frac{1}{z}\big[\mathcal{H}^{\eta+1,k}_{\alpha,\beta}(f(z)) - \mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))\big]\Big| \le |\frac{k}{b}|\ell M, \tag{32}$$

then

$$|\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z} - 1| < M.$$

Proof. Let $\phi(a_1, a_2, a_3, a_4; z) = a_2 - a_1$. Using Corollary 5 with $\Omega = h(\mathbb{U})$ and

$$h(z) = |\frac{k}{b}|\ell M \quad (z \in \mathbb{U}).$$

Now we show that $\phi \in \Psi_\Gamma[\Omega, M]$. Since the condition (30) is satisfied from the condition (32) and

$$\begin{aligned} &\Big|\phi\Big(1 + Me^{i\theta}, 1 + (\frac{k}{b})\Big(\frac{b}{k} + \ell\Big)Me^{i\theta}, \\ &1 + \frac{k^2}{b(b+1)}\Big(L + (\frac{b(b+1)}{k^2} + \ell(\frac{2b+1}{k} + 1))Me^{i\theta}\Big), \\ &1 + \frac{k^3}{b(b+1)(b+2)}\Big(N + 3L(\frac{b+1}{k} + 1) + (\frac{b(b+1)(b+2)}{k^3} \\ &+\ell(\frac{3b^2+6b+2}{k^2} + \frac{3(b+1)}{k} + 1))Me^{i\theta}\Big); z\Big)\Big| = \Big|\frac{k\ell Me^{i\theta}}{b}\Big| = |\frac{k}{b}|\ell M, \end{aligned}$$

then we have the Corollary 11. □

Corollary 12. *Let* $\ell \in \mathbb{N} \setminus \{1\}$, $M > 0$ *and* $c \in \mathbb{C} \setminus \mathbb{Z}_0^-$. *If* $f(z) \in \mathcal{H}$ *satisfies the condition (31) and*

$$\Big|\frac{1}{z}\big[\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z)) - \mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z))\big]\Big| \le |\frac{\alpha}{c}|\ell M, \tag{33}$$

then

$$\left|\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z))}{z}-1\right|<M.$$

Proof. Let $\phi(a_1,a_2,a_3,a_4;z)=a_2-a_1$. Using Corollary 6 with $\Omega=h(\mathbb{U})$ and

$$h(z)=\left|\frac{\alpha}{c}\right|\ell M \quad (z\in\mathbb{U}).$$

Now we show that $\phi\in\Psi_\Gamma[\Omega,M]$. Since the condition (31) is satisfied from the condition (33) and

$$\begin{aligned}&\Bigg|\phi\Bigg(1+Me^{i\theta},1+(\frac{\alpha}{c})\Big(\frac{c}{\alpha}+\ell\Big)Me^{i\theta},\\ &\quad 1+\frac{\alpha^2}{c(c-1)}\Big(L+(\frac{c(c-1)}{\alpha^2}+\ell(\frac{2c-1}{\alpha}+1))Me^{i\theta}\Big),\\ &\quad 1+\frac{\alpha^3}{c(c-1)(c-2)}\Big(N+3L(\frac{c-1}{\alpha}+1)+(\frac{c(c-1)(c-2)}{\alpha^3}\\ &\quad +\ell(\frac{3c^2-6c+2}{\alpha^2}+\frac{3(c-1)}{\alpha}+1))Me^{i\theta}\Big);z\Bigg)\Bigg|=\left|\frac{\alpha\ell Me^{i\theta}}{c}\right|=\left|\frac{\alpha}{c}\right|\ell M,\end{aligned}$$

then the corollary is completed. □

Corollary 13. *Let $\ell\in\mathbb{N}\setminus\{1\}$, $M>0$ and $b\in\mathbb{C}\setminus\mathbb{Z}_0^-$. If $f(z)\in\mathcal{H}$ satisfy the condition (30) and*

$$\begin{aligned}\left|\frac{1}{z}\left[\mathcal{H}^{\eta+3,k}_{\alpha,\beta}(f(z))-\mathcal{H}^{\eta+2,k}_{\alpha,\beta}(f(z))\right]\right|\le 2\left|\frac{k^3}{b(b+1)(b+2)}\right|\\ \left(\left|\frac{2b+1}{k}+3\right|+\left|\frac{b(b+1)}{k^2}+\frac{2b+1}{k}+1\right|\right)M,\end{aligned}$$

then

$$\left|\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z}-1\right|<M.$$

Proof. Let $\phi(a_1,a_2,a_3,a_4;z)=a_4-a_3$. We use Corollary 5 with $\Omega=h(\mathbb{U})$

$$h(z)=2\left|\frac{k^3}{b(b+1)(b+2)}\right|\left(\left|\frac{2b+1}{k}+3\right|+\left|\frac{b(b+1)}{k^2}+\frac{2b+1}{k}+1\right|\right)Mz,\quad (z\in\mathbb{U}).$$

Now we show that $\phi \in \Psi_{\Gamma}[\Omega, M]$. Since

$$\begin{aligned}
&\Big|\phi\Big(1+Me^{i\theta}, 1+(\frac{k}{b})\Big(\frac{b}{k}+\ell\Big)Me^{i\theta},\\
&1+\frac{k^2}{b(b+1)}\Big(L+(\frac{b(b+1)}{k^2}+\ell(\frac{2b+1}{k}+1))Me^{i\theta}\Big),\\
&1+\frac{k^3}{b(b+1)(b+2)}\Big(N+3L(\frac{b+1}{k}+1)+(\frac{b(b+1)(b+2)}{k^3}\\
&\qquad+\ell(\frac{3b^2+6b+2}{k^2}+\frac{3(b+1)}{k}+1))Me^{i\theta}\Big);z\Big)\Big|\\
&=\Big|\frac{k^3}{b(b+1)(b+2)}\Big(N+(\frac{2b+1}{k}+3)L+\ell(\frac{b(b+1)}{k^2}+\frac{2b+1}{k}+1)Me^{i\theta}\Big)\Big|\\
&=\Big|\frac{k^3e^{i\theta}}{b(b+1)(b+2)}\Big(Ne^{-i\theta}+(\frac{2b+1}{k}+3)Le^{-i\theta}+\ell(\frac{b(b+1)}{k^2}+\frac{2b+1}{k}+1)M\Big)\Big|\\
&\geq\Big|\frac{k^3}{b(b+1)(b+2)}\Big|\Big(Re(Ne^{-i\theta})+\Big|\frac{2b+1}{k}+3\Big|Re(Le^{-i\theta})\\
&\qquad+\ell\Big|\frac{b(b+1)}{k^2}+\frac{2b+1}{k}+1\Big|M\Big)\\
&\geq\Big|\frac{k^3}{b(b+1)(b+2)}\Big|\Big(\ell(\ell-1)M\Big|\frac{2b+1}{k}+3\Big|+\ell\Big|\frac{b(b+1)}{k^2}+\frac{2b+1}{k}+1\Big|M\Big)\\
&\geq 2\Big|\frac{k^3}{b(b+1)(b+2)}\Big|\Big(\Big|\frac{2b+1}{k}+3\Big|+\Big|\frac{b(b+1)}{k^2}+\frac{2b+1}{k}+1\Big|\Big)M,
\end{aligned}$$

we complete the proof of Corollary 13. □

Corollary 14. *Let $\ell \in \mathbb{N} \setminus \{1\}$, $M > 0$ and $c \in \mathbb{C} \setminus \mathbb{Z}_0^-$. If $f(z) \in \mathcal{H}$ satisfy the condition (31) and*

$$\begin{aligned}
\Big|\frac{1}{z}\big[\mathcal{H}^{\eta,k}_{\alpha,\beta-2}(f(z))-\mathcal{H}^{\eta,k}_{\alpha,\beta-1}(f(z))\big]\Big| &\leq 2\Big|\frac{\alpha^3}{c(c-1)(c-2)}\Big|\\
&\quad(\Big|\frac{2c-1}{\alpha}+3\Big|+\Big|\frac{c(c-1)}{\alpha^2}+\frac{2c-1}{\alpha}+1\Big|)M,
\end{aligned}$$

then

$$\Big|\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z))}{z}-1\Big| < M.$$

Proof. Let $\phi(a_1, a_2, a_3, a_4; z) = a_4 - a_3$. We use Corollary 6 with $\Omega = h(\mathbb{U})$

$$h(z) = 2\Big|\frac{\alpha^3}{c(c-1)(c-2)}\Big|(\Big|\frac{2c-1}{\alpha}+3\Big|+\Big|\frac{c(c-1)}{\alpha^2}+\frac{2c-1}{\alpha}+1\Big|)Mz, \quad (z \in \mathbb{U}).$$

Now we show that $\phi \in \Psi_\Gamma[\Omega, M]$. Since

$$\begin{aligned}
&\Big|\phi\Big(1+Me^{i\theta}, 1+(\frac{\alpha}{c})\Big(\frac{c}{\alpha}+\ell\Big)Me^{i\theta},\\
&1+\frac{\alpha^2}{c(c-1)}\Big(L+(\frac{c(c-1)}{\alpha^2}+\ell(\frac{2c-1}{\alpha}+1))Me^{i\theta}\Big),\\
&1+\frac{\alpha^3}{c(c-1)(c-2)}\Big(N+3L(\frac{c-1}{\alpha}+1)+(\frac{c(c-1)(c-2)}{\alpha^3}\\
&\qquad\qquad +\ell(\frac{3c^2-6c+2}{\alpha^2}+\frac{3(c-1)}{\alpha}+1))Me^{i\theta}\Big);z\Big)\Big|\\
&=\Big|\frac{\alpha^3}{c(c-1)(c-2)}\Big(N+(\frac{2c-1}{\alpha}+3)L+\ell(\frac{c(c-1)}{\alpha^2}+\frac{2c-1}{\alpha}+1)Me^{i\theta}\Big)\Big|\\
&=\Big|\frac{\alpha^3e^{-i\theta}}{c(c-1)(c-2)}\Big(Ne^{-i\theta}+(\frac{2c-1}{\alpha}+3)Le^{-i\theta}+\ell(\frac{c(c-1)}{\alpha^2}+\frac{2c-1}{\alpha}+1)M\Big)\Big|\\
&\geq\Big|\frac{\alpha^3}{c(c-1)(c-2)}\Big|\Big(Re(Ne^{-i\theta})+\Big|\frac{2c-1}{\alpha}+3\Big|Re(Le^{-i\theta})\\
&\qquad\qquad +\ell\Big|\frac{c(c-1)}{\alpha^2}+\frac{2c-1}{\alpha}+1\Big|M\Big)\\
&\geq\Big|\frac{\alpha^3}{c(c-1)(c-2)}\Big|\Big(\ell(\ell-1)M\Big|\frac{2c-1}{k}+3\Big|+\ell\Big|\frac{c(c-1)}{\alpha^2}+\frac{2c-1}{\alpha}+1\Big|M\Big)\\
&\geq 2\Big|\frac{\alpha^3}{c(c-1)(c-2)}\Big|\Big(\Big|\frac{2c-1}{\alpha}+3\Big|+\Big|\frac{c(c-1)}{\alpha^2}+\frac{2c-1}{\alpha}+1\Big|\Big)M,
\end{aligned}$$

we complete the proof of Corollary 14. □

3. Third Order Differential Supordination with $\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))$

Definition 11. *Let $\Omega \subseteq \mathbb{C}$ and $q(z) \in \mathbb{Q}$. The class of admissible functions $\Psi_\Gamma[\Omega, q(z)]$ consists of those functions $\psi : \mathbb{C}^4 \times \overline{\mathbb{U}} \longrightarrow \mathbb{C}$ that satisfy the admissibility condition*

$$\psi(a_1, a_2, a_3, a_4; z) \notin \Omega,$$

whenever

$$\begin{aligned}
&a_1 = q(\zeta), \quad a_2 = \frac{k\zeta q'(\zeta)+bq(\zeta)}{bm},\\
&Re\Big(\frac{(b+1)(a_3-a_1)}{k(a_2-a_1)}-\frac{2b+1}{k}\Big) \leq \frac{1}{m} \quad Re\Big(\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\Big),\\
&Re\Big(\frac{(b+1)(b+2)(a_4-a_1)-3(b+1)(b+k+1)(a_3-a_1)}{k^2(a_2-a_1)}+\frac{3b(b+1)+1}{k^2}\\
&\qquad\qquad +\frac{6b+3}{k}+2\Big) \leq \frac{1}{m^2} \quad Re\Big(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\Big),
\end{aligned}$$

where $z \in \mathbb{U}$; $\zeta \in \partial\mathbb{U}$, $m \in \mathbb{N} \setminus \{1\}$ and $b = \eta + k$.

Theorem 9. *Let $\phi \in \Psi'_{\Gamma}[\Omega, q(z)]$. If $f(z) \in \mathcal{H}$ and $\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z} \in \mathbb{Q}_1$ satisfy the following conditions:*

$$Re\Big(\frac{zq''(z)}{q'(z)}\Big) \geq 0, \quad \Big|\frac{\frac{1}{z}\big[\mathcal{H}^{\eta+1,k}_{\alpha,\beta}(f(z)) - \mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))\big]}{q'(z)}\Big| \leq |\frac{k}{b}|m, \tag{34}$$

$$\Big\{\phi\Big(\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+1,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+2,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+3,k}_{\alpha,\beta}(f(z))}{z}; z\Big) : z \in \mathbb{U}\Big\},$$

are univalent, and

$$\Omega \subset \Big\{\phi\Big(\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+1,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+2,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+3,k}_{\alpha,\beta}(f(z))}{z}; z\Big) : z \in \mathbb{U}\Big\}, \tag{35}$$

then

$$q(z) \prec \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z}.$$

Proof. Let the functions $p(z)$ and ψ are defined by (9) and (13). Since $\phi \in \Psi'_{\Gamma}[\Omega, q(z)]$, therefore (14) and (35) give

$$\Omega \subset \psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z).$$

The admissible condition for $\phi \in \Psi'_{\Gamma}[\Omega, q(z)]$ is equivalent to the admissible condition for ψ in Definition 6 with $n = 2$. Therefore, $\psi \in \Psi'_{\Gamma}[\Omega, q(z)]$ and by using (34) and Theorem 2, we have $q(z) \prec p(z)$ which yields $q(z) \prec \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z}$. Therefore, we complete the proof of Theorem 9. □

Moreover, in a similar way, we can define the following:

Definition 12. *Let $\Omega \subseteq \mathbb{C}$ and $q(z) \in \mathbb{Q}$. The class of admissible functions $\Psi_{\Gamma}[\Omega, q(z)]$ consists of those functions $\psi : \mathbb{C}^4 \times \overline{\mathbb{U}} \longrightarrow \mathbb{C}$ that satisfy the admissibility condition:*

$$\psi(a_1, a_2, a_3, a_4; z) \notin \Omega,$$

whenever

$$a_1 = q(\zeta), \quad a_2 = \frac{\alpha\zeta q'(\zeta) + cq(\zeta)}{cm},$$
$$Re\Big(\frac{(c-1)(a_3 - a_1)}{\alpha(a_2 - a_1)} - \frac{2c-1}{\alpha}\Big) \leq \frac{1}{m} \quad Re\Big(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\Big),$$
$$Re\Big(\frac{(c-1)(c-2)(a_4 - a_1) - 3(c-1)(c+\alpha-1)(a_3 - a_1)}{\alpha^2(a_2 - a_1)} + \frac{3c(c-1)+1}{\alpha^2} + \frac{6c-3}{\alpha} + 2\Big) \leq \frac{1}{m^2} \quad Re\Big(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\Big),$$

where $z \in \mathbb{U}$; $\zeta \in \partial\mathbb{U}$, $m \in \mathbb{N} \setminus \{1\}$ and $c = \alpha + \beta$.

With the assist of Definition 12 and Theorem 4, we have the following theorem

Theorem 10. *Let $\phi \in \Psi'_{\Gamma}[\Omega, q(z)]$. If $f(z) \in \mathcal{H}$ and $\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z))}{z} \in \mathbb{Q}_1$ satisfy the following conditions:*

$$Re\Big(\frac{zq''(z)}{q'(z)}\Big) \geq 0, \qquad \Big|\frac{\frac{1}{z}\big[\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z)) - \mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z))\big]}{q'(z)}\Big| \leq |\frac{\alpha}{c}|m, \tag{36}$$

$$\Big\{\phi\Big(\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta-1}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta-2}(f(z))}{z}; z\Big) : z \in \mathbb{U}\Big\},$$

are univalent, and

$$\Omega \subset \Big\{\phi\Big(\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta-1}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta-2}(f(z))}{z}; z\Big) : z \in \mathbb{U}\Big\},$$

then

$$q(z) \prec \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z))}{z}.$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω. In this case, the class $\Psi'_{\Gamma}[h(u), q(z)]$ is written as $\Psi'_{\Gamma}[h, q]$. The following theorem is a direct consequence of Theorems 3 and 4.

Theorem 11. *Let $\phi \in \Psi'_{\Gamma}[\Omega, q(z)]$ and $h(z)$ be analytic function in $\mathbb{U}$, and $f(z) \in \mathcal{H}$ and $\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z} \in \mathbb{Q}_1$ satisfy the condition (34). If*

$$\Big\{\phi\Big(\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+1,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+2,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+3,k}_{\alpha,\beta}(f(z))}{z}; z\Big) : z \in \mathbb{U}\Big\},$$

is univalent function in $\mathbb{U}$, and

$$h(z) \prec \phi\Big(\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+1,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+2,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+3,k}_{\alpha,\beta}(f(z))}{z}; z\Big),$$

then

$$q(z) \prec \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z}.$$

Theorem 12. *Let $\phi \in \Psi'_{\Gamma}[\Omega, q(z)]$ and $h(z)$ be analytic function in $\mathbb{U}$. If $f(z) \in \mathcal{H}$ and $\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z))}{z} \in \mathbb{Q}_1$ satisfy the condition (36). If*

$$\Big\{\phi\Big(\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+2,k}_{\alpha,\beta-1}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta-2}(f(z))}{z}; z\Big) : z \in \mathbb{U}\Big\},$$

is univalent function in $\mathbb{U}$ and

$$h(z) \prec \phi\Big(\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta-1}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta-2}(f(z))}{z}; z\Big),$$

then

$$q(z) \prec \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z))}{z}.$$

Theorem 13. *Let $h(z)$ be analytic function in $\mathbb{U}$ and let $\psi : \mathbb{C}^4 \times \bar{\mathbb{U}} \longrightarrow \mathbb{C}$ and ψ is given by (13). Suppose that the differential (26) has a solution $q(z) \in \mathbb{Q}_1$, and $f(z) \in \mathcal{H}$ satisfy the condition (34). If*

$$\Big\{\phi\Big(\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+1,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+2,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+3,k}_{\alpha,\beta}(f(z))}{z}; z\Big) : z \in \mathbb{U}\Big\},$$

is univalent function in $\mathbb{U}$, and

$$h(z) \prec \phi\Big(\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+1,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+2,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+3,k}_{\alpha,\beta}(f(z))}{z}; z\Big), \tag{37}$$

then

$$q(z) \prec \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z}.$$

and $q(z)$ is the best subordinant of relation (36).

Proof. The proof is similar to Theorem 7 and it is being omitted here. □

Combining both Theorems 5 and 11, we have the following sandwich result:

Corollary 15. *Let $h_1(z)$ and $q_1(z)$ be analytic functions in $\mathbb{U}$, also, let $h_2(z)$ be univalent in $\mathbb{U}$, $q_2(z) \in \mathbb{Q}_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Psi_\Gamma[\Omega, q(z)] \cap \Psi'_\Gamma[\Omega, q(z)]$. If $f(z) \in \mathcal{H}$ and $\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z} \in \mathbb{Q}_1 \cap \mathcal{H}$,*

$$\Big\{\phi\Big(\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+1,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+2,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+3,k}_{\alpha,\beta}(f(z))}{z}; z\Big) : z \in \mathbb{U}\Big\},$$

is univalent function in $\mathbb{U}$, and the conditions (22) and (34) are satisfied, also let

$$h_1(z) \prec \phi\Big(\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+1,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+2,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta+3,k}_{\alpha,\beta}(f(z))}{z}; z\Big) \prec h_2(z),$$

then

$$q_1(z) \prec \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z} \prec q_2(z).$$

The proof of the following theorem is similar to Theorem 8; therefore, we omitted it.

Theorem 14. *Let $h(z)$ be analytic function in $\mathbb{U}$ and let $\psi : \mathbb{C}^4 \times \bar{\mathbb{U}} \longrightarrow \mathbb{C}$ and ψ is given by (21). Suppose that the differential (28) has a solution $q(z) \in \mathbb{Q}_1$.If $f(z) \in \mathcal{H}$ satisfy the condition (36). If*

$$\Big\{\phi\Big(\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta-1}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta-2}(f(z))}{z}; z\Big) : z \in \mathbb{U}\Big\},$$

is univalent function in $\mathbb{U}$, and

$$h(z) \prec \phi\Big(\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta-1}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta-2}(f(z))}{z}; z\Big), \tag{38}$$

then

$$q(z) \prec \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z))}{z}.$$

and $q(z)$ is the best subordinant of (38).

By combining Theorems 8 and 12, we obtain the following sandwich type result.

Corollary 16. *Let $h_1(z)$ and $q_1(z)$ be analytic functions in $\mathbb{U}$ and let$h_2(z)$ be univalent in $\mathbb{U}$, $q_2(z) \in \mathbb{Q}_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Psi_\Gamma[\Omega, q(z)] \cap \Psi'_\Gamma[\Omega, q(z)]$. If $f(z) \in \mathcal{H}$ and $\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z))}{z} \in \mathbb{Q}_1 \cap \mathcal{H}$,*

$$\left\{\phi\left(\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta-1}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta-2}(f(z))}{z}; z\right) : z \in \mathbb{U}\right\},$$

is univalent function in $\mathbb{U}$, and the conditions (23) and (36) are satisfied; also let

$$h_1(z) \prec \quad \phi\left(\frac{\mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta-1}(f(z))}{z}, \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta-2}(f(z))}{z}; z\right) \prec h_2(z),$$

then

$$q_1(z) \prec \frac{\mathcal{H}^{\eta,k}_{\alpha,\beta+1}(f(z))}{z} \prec q_2(z).$$

4. Conclusions

By using the method of third-order differential subordination and superordination, we obtained many interesting results concerning the subordination and superordination properties of analytic functions associated with the operator $\mathcal{H}^{\eta,k}_{\alpha,\beta}(f)$.

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