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Local Super Antimagic Total Labeling for Vertex Coloring of Graphs

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Abstract: Let $G = (V, E)$ be a graph with vertex set V and edge set E. A local antimagic total vertex coloring f of a graph G with vertex-set V and edge-set E is an injective map from $V \cup E$ to $\{1, 2, \ldots, |V| + |E|\}$ such that if for each $uv \in E(G)$ then $w(u) \neq w(v)$, where $w(u) = \sum_{uv \in E(G)} f(uv) + f(u)$. If the range set f satisfies $f(V) = \{1, 2, \ldots, |V|\}$, then the labeling is said to be *local super antimagic total labeling*. This labeling generates a proper vertex coloring of the graph G with the color $w(v)$ assigning the vertex v. The local super antimagic total chromatic number of graph G, $\chi_{lsat}(G)$ is defined as the least number of colors that are used for all colorings generated by the local super antimagic total labeling of G. In this paper we investigate the existence of the local super antimagic total chromatic number for some particular classes of graphs such as a tree, path, cycle, helm, wheel, gear, sun, and regular graphs as well as an amalgamation of stars and an amalgamation of wheels.

Keywords: vertex coloring; local super antimagic total labeling; local super antimagic total chromatic number

1. Introduction

Vertex coloring is an assignment of colors to every vertex of graph G such that any two adjacent vertices receive different colors and the number of colors used for such coloring is made as minimal as possible. The vertex coloring problem has many applications such as a scheduling system [1] and frequency allocation.

Formally the vertex coloring in a graph is defined as follows. The k-coloring of graph G is a map of $c : V \to \{1, 2, \ldots, k\}$ where V is the set of vertices of G and $c(v)$ is a color of vertex v such that $c(u) \neq c(v)$ whenever vertices u and v are adjacent. The smallest positive integer of k is such that G satisfies the k-coloring and is called the *chromatic number* of G, denoted by $\chi(G)$ [2].

The vertex coloring of graphs is then approached by using the local antimagic total labeling of a graph. In 2017, Arumugam, Premalatha, Bača, and Semaničová-Feňovčíková [3] introduced this concept and defined it as follows. The *local antimagic total labeling* on a graph G with $|V|$ vertices and $|E|$ edges is defined as a bijection $f : E \to \{1, 2, \cdots, m\}$ such that the weights of any two adjacent vertices are different, that is, $w(u) \neq w(v)$ where $w(u) = \Sigma_{e \in E(u)} f(e)$ and $E(u)$ is the set of edges incident to u. Thus, any local antimagic total labeling of a graph G induces a proper vertex coloring of G where the weight $w(u)$ is assigned as the color of vertex u. The local antimagic chromatic number of G, denoted by $\chi_{la}(G)$, is the minimum number of colors taken over all colorings induced by local antimagic total labelings of G.

The local antimagic chromatic number of some families of graphs are presented in the paper [3]. These include the complete graph K_n for $n \geq 3$, the star $K_{1,n}$ for $n \geq 2$, the path P_n for $n \geq 3$, the cycle on n for $n \geq 3$, the friendship graph F_n for $n \geq 2$, the complete bipartite graph $K_{m,n}$ for $m, n \geq 2$, and the wheel W_n for $n \geq 3$. Nazula, Slamin, and Dafik [4] determined the local antimagic chromatic number of uni cyclic graphs. Arumugam, Yi-Chun, Premalatha, and Tao-Ming [5] discovered the local antimagic chromatic number of the corona product of two graphs such as paths, cycles, and complete graphs with other graphs.

Recently, Putri, Dafik, Agustin, and Alfarisi [6] extended this notion by labeling the vertices and edges of a graph G to establish a vertex coloring. The local vertex antimagic total labeling on a graph G with $|V|$ vertices and $|E|$ edges is defined to be an injective assignment $f : V \cup E \rightarrow \{1, 2, \cdots, |V| + |E|\}$ so that the weights of any two adjacent vertices u and v are distinct, that is, $w(u) \neq w(v)$ where $w(u) = f(u) + \Sigma_{e \in E(u)} f(e)$ and $E(u)$ is the set of edges incident to u. For local antimagic total labeling, any local vertex antimagic total labeling of a graph G induces a proper vertex coloring of G where the vertex u is assigned by the color $w(u)$. The *local antimagic total chromatic number*, denoted by $\chi_{lat}(G)$, is the minimum number of colors taken over all colorings induced by local vertex antimagic total labelings of G. It is easy to see that for any graph G, $\chi_{la}(G) \geq \chi_{lat}(G) \geq \chi(G)$. The local antimagic total chromatic number for some family of graphs have been established by several authors such as prisms and Möbius ladders [7], brooms [8], stars [6], wheels, fans, and friendship graphs [9].

In the local antimagic total labeling of graph G, if the vertices of G receive the smallest labels, that is, $\{1, 2, ..., |V|\}$, then it is called *local super antimagic total labeling*. Any local super antimagic total labeling also induces a proper vertex coloring of G where the vertex u is assigned the color $w(u)$. The *local super antimagic total chromatic number*, denoted by $\chi_{lsat}(G)$, is the minimum number of colors taken over all colorings induced by local super antimagic total labelings of G. Again it is easy to see that for any graph G, $\chi_{lat}(G) \geq \chi_{lsat}(G) \geq \chi(G)$. Pratama, Setiawani, and Slamin [10] determined the local super antimagic total vertex coloring of some wheel-related graphs such as fans, even gear graphs, and sun flower graphs. In this paper, we determine the local super antimagic total chromatic number of some families of graphs such as stars, double stars, paths, cycles, gears, suns, helms, wheels and, some graphs obtained from an amalgamation product.

2. Results

We start this section with the upper bound on the chromatic number of the local super antimagic total of a tree as described in the following theorem.

Theorem 1. *If T is a tree on $n \geq 2$ vertices with k leaves, then $\chi_{lsat}(T) \leq n - k + 1$.*

Proof. Let T be a tree on $n \geq 2$ vertices with k leaves. Let v_i be some vertices in T that are adjacent to $nl(v_i)$ leaves, namely, l_i^j, where $1 \leq i \leq r$, $1 \leq j \leq nl(v_i)$, $nl(v_1) \geq nl(v_2) \geq \ldots \geq nl(v_r)$. Label the leaves l_i^j for $1 \leq j \leq nl(v_i)$ by $f(l_i^j) = j$ if $i = 1$ and $f(l_i^j) = j + \sum_{t=1}^{i-1} nl(v_t)$ if $i = 2, 3, ..., r$ and label the edges $e_i = v_i l_i^j$ with $n+1, n+2, \ldots, n+k$ using formula $f(e_i) = n + k + 1 - f(l_i^j)$ to obtain $w(l_1^j) = w(l_2^j) = \ldots = w(l_r^j)$. The rest of the edges are labeled by $\{n+k+1, n+k+2, \ldots, 2n-1\}$ such that $s_e(v_i) \leq s_e(v_j)$, for each $i \neq j$ and $1 \leq i, j \leq n - k$ where $s_e(v_i)$ is the sum of all labels of edges that are incident to the vertex v_i. Label vertex v_i by x and vertex v_j by y, where $x < y$ for $x, y \in \{k+1, k+2, \ldots, n\}$. The weight of leaves are obtained from the smallest labels of vertices and edges. Thus, the total labeling of f produces $w(l_i^j) < w(v_i) < w(v_j)$ for each $i \neq j$, where $1 \leq i, j \leq n - k$. It is clear that the weight of vertices are different whenever they are adjacent. Therefore the local super antmagic total labeling f gives $\chi_{lsat}(T) \leq n - k + 1$. □

Star, denoted by S_{n+1}, is a special class of tree of order $n+1$ with n leaves. By using Theorem 1 and the fact that $\chi_{lsat}(G) \geq 2$ for any connected graph G of order of at least 2, we obtain the local super antimagic total chromatic number of the star as follows.

Corollary 1. *If S_{n+1} is a star, then $\chi_{lsat}(S_{n+1}) = 2$.*

Double star, denoted by $S_{k,n-k}$, is another special class of tree of order $n+2$ with n leaves and two central vertices that are adjacent to k and $n-k$ leaves, respectively. By using Theorem 1 and a local super antimagic total labeling, we obtain the local super antimagic total chromatic number of the double star as follows.

Corollary 2. *If $S_{k,n-k}$, for $n \geq 2$ and $k \geq 1$, is a double star, then $\chi_{lsat}(S_{k,n-k}) = 3$.*

Proof. Let $S_{k,n-k}$, for $n \geq 2$ and $k \geq 1$, be a double star of order $n+2$ with n leaves. Let u_1 and u_2 be two central vertices of the double star $S_{k,n-k}$, where u_1 is adjacent to k leaves, say $u_1^1, u_1^2, ..., u_1^k$, and u_2 is adjacent to $n-k$ leaves, say $u_2^1, u_2^2, ..., u_2^{n-k}$. For $n = 2$ and $k = 1$, the double star $S_{k,n-k}$ or $S_{1,1}$ is isomorphic to path P_4. It is easy to check that $\chi_{lsat}(P_4) \neq 2$, so $\chi_{lsat}(P_4) \geq 3$. Combined with Theorem 1, it implies that $\chi_{lsat}(S_{1,1}) = \chi_{lsat}(P_4) = 3$. We now suppose that $\chi_{lsat}(S_{k,n-k}) = 2$ for $n \geq 3$ and $k \geq 1$. Then there must be two vertices of distance two with the same weight. Without loss of generality, suppose that $w(u_1) = w(u_2^j)$ for any $j = 1, 2, \ldots, n-k$ and $d(u_1) \geq 3$ where $d(u_1)$ is the degree of central vertex u_1. Since the smallest labels are used for $n+2$ vertices and the three smallest edge labels are $n+3$, $n+4$, and $n+5$, it implies that $w(u_1) \geq 1 + (n+3) + (n+4) + (n+5) = 3n+13$. However, the weight of u_2^j for any $j = 1, 2, \ldots, n-k$ is obtained from the sum of one vertex label and one edge label, which can be the largest ones, that is, $w(u_2^j) \leq (n+2) + (2n+3) = 3n+5$. Thus $w(u_1) > w(u_2^j)$, which is a contradiction. So $\chi_{lsat}(S_{k,n-k}) \geq 3$. Again combined with Theorem 1, we obtain $\chi_{lsat}(S_{k,n-k}) = 3$. □

In the next theorem, we present the local super antimagic total chromatic number of a cycle C_n on $n \geq 3$ vertices.

Theorem 2. *If C_n is a cycle on $n \geq 3$ vertices, then:*

$$\chi_{lsat}(C_n) = \begin{cases} 3, & \text{if } n \text{ is odd or } n = 4 \\ 2, & \text{otherwise.} \end{cases}$$

Proof. Let C_n be a cycle on $n \geq 3$ vertices with vertex set $V(C_n) = \{x_i : 1 \leq i \leq n\}$ and edge set $E(C_n) = \{x_i x_{i+1} : 1 \leq i \leq n-1\} \cup \{x_n x_1\}$. To obtain the upper bound on $\chi_{lsat}(C_n)$, we divide into five cases.

Case i. For $n = 4$.

Let $\{x_1, x_2, x_3, x_4\}$ be the vertex set of C_4. Suppose that $\chi_{lsat}(C_4) = 2$ and the weights are y and z. Then there are two pair of vertices of distance two in C_4 with the same weight, without loss of generality, let $w(x_1) = w(x_3) = y$ and $w(x_2) = w(x_4) = z$. in this total labeling, the labels of vertices are used once, while the labels of edges are used twice. Thus, the total weight of all 4 vertices are $2y + 2z = (1+2+3+4) + 2(5+6+7+8)$, that is, $y + z = 31$. There are 4 possible solutions (pairs of weight) for $\{y, z\}$, namely, {12,19}, {13,18}, {14,17}, and {15,16}. Each weight corresponds to two independent triple labels (one vertex label and two edges labels). Table A1 shows all possible local super antimagic total labelings of C_4. It can be seen that the sum of the first two independent triple labels is 15 and the second one is 16. However, it is also impossible because weight 15 and 16 share two edge labels. Thus $\chi_{lsat}(C_4) \geq 2$. Figure A1 shows the local super antimagic total labeling of C_4 with 3 different weights. This implies that $\chi_{lsat}(C_4) \leq 3$. Therefore $\chi_{lsat}(C_4) = 3$.

Case ii. For odd $n \geq 3$.

Label the vertices and edges of cycle C_n using the following formula.

$$
\begin{aligned}
f_1(x_i) &= \begin{cases} 1, & \text{for } i = 1, \\ i-1, & \text{for odd } i \neq 1, \\ i+1, & \text{for even } i, \end{cases} \\
f_1(x_i x_{i+1}) &= \begin{cases} \frac{4n-i+1}{2}, \text{ for odd } i, \\ \frac{3n-i+1}{2}, \text{ for even } i, \end{cases} \\
f_1(x_n x_1) &= \frac{3n+1}{2}.
\end{aligned}
$$

This labeling gives any two adjacent vertices with different weights, that is,

$$
w(x_i) = \begin{cases} \frac{7n+1}{2}, & \text{for odd } i \geq 3, \\ \frac{7n+3}{2}, & \text{for i} = 1, \\ \frac{7n+5}{2}, & \text{for even } i. \end{cases}
$$

Thus $\chi_{lsat}(C_n) \leq 3$ for odd $n \geq 3$.

Case iii. For $n \equiv 2, 6 \pmod 8$.

Label the vertices and edges of cycle C_n using the following formula.

$$
\begin{aligned}
f_2(x_i) &= \begin{cases} \frac{n}{2} + i, & \text{for odd } i \leq \frac{n}{2}, \\ i, & \text{for even } i \leq \frac{n}{2}, \\ i - \frac{n}{2}, & \text{for even } i > \frac{n}{2}, \\ i, & \text{for odd } i > \frac{n}{2}, \end{cases} \\
f_2(x_i x_{i+1}) &= \begin{cases} \frac{4n-i+1}{2}, & \text{for odd } i, \\ \frac{7n-2i+2}{4}, & \text{for even } i > \frac{n}{2}, \\ \frac{5n-2i+2}{4}, & \text{for even } i < \frac{n}{2}, \end{cases} \\
f_2(x_n x_1) &= \frac{5n+2}{4}.
\end{aligned}
$$

This labeling also gives any two adjacent vertices with different weights, that is,

$$
w(x_i) = \begin{cases} \frac{15n+2}{4} + 1, & \text{for odd } i, \\ \frac{13n+2}{4} + 1, & \text{for even } i. \end{cases}
$$

Therefore $\chi_{lsat}(C_n) \leq 2$ for $n \equiv 2, 6 \pmod 8$.

Case iv. For $n \equiv 0 \pmod 8$.

Label the vertices and edges of cycle C_n using the following formula.

$$f_3(x_i) = \begin{cases} \frac{3n+i+2}{4} - (1+(-1)^{\frac{i}{2}})\frac{n-4}{16}, & \text{for even } \frac{n}{2}+4 \le i \le n, \\ \frac{3n+4}{4}, & \text{for } i = \frac{n}{2}+2 \\ \frac{2n+i+1}{4} - (1-(-1)^{\frac{i+1}{2}})\frac{n-4}{16}, & \text{for odd } \frac{n}{2}+3 \le i \le n-1, \\ \frac{2n-i+8}{4} - (1-(-1)^{\frac{i}{2}})\frac{n+4}{16}, & \text{for even } 2 \le i \le \frac{n}{2}, \\ \frac{n-i+7}{4} - (1-(-1)^{\frac{i+1}{2}})\frac{n+4}{16}, & \text{for odd } 1 \le i \le \frac{n}{2}+1, \end{cases}$$

$$f_3(x_i x_{i+1}) = \begin{cases} \frac{8n-i+1}{4} - (1+(-1)^{\frac{i+1}{2}})\frac{n+4}{16}, & \text{for odd } 1 \le i \le \frac{n}{2}+3, \\ \frac{4n-i+1}{2}, & \text{for odd } \frac{n}{2}+5 \le i \le n-1, \\ \frac{5n+2i}{4}, & \text{for even } 2 \le i \le \frac{n}{2}, \\ \frac{7n+2i+4}{8} + (1+(-1)^{\frac{i}{2}})\frac{n-4}{16}, & \text{for even } \frac{n}{2}+2 \le i \le n-2, \end{cases}$$

$$f_3(x_n x_1) = \frac{5n}{4}.$$

This labeling also gives any two adjacent vertices with different weights, that is,

$$w(x_i) = \begin{cases} \frac{27n}{8}+1, & \text{for odd } i, \\ \frac{29n}{8}+2, & \text{for even } i. \end{cases}$$

Therefore $\chi_{lsat}(C_n) \le 2$ for $n \equiv 0 \pmod 8$.

Case v. For $n \equiv 4 \pmod 8$.

Label the vertices and edges of cycle C_n using the following formula.

$$f_4(x_i) = \begin{cases} \frac{n}{2}, & \text{for } i = 1, \\ n, & \text{for } i = 3, \\ \frac{7n-4}{8}, & \text{for } i = \frac{n}{2}+1, \\ \frac{3n+4}{8}, & \text{for } i = \frac{n}{2}+3, \\ \frac{9n-2i+2}{8}, & \text{for } \frac{n}{2}+5 \le i \le n-1, i \equiv 3 (\mathrm{mod}\ 4), \\ \frac{4n-i+1}{4}, & \text{for } \frac{n}{2}+7 \le i \le n-3, i \equiv 1 (\mathrm{mod}\ 4), \\ \frac{7n-2i+8}{8}, & \text{for } \frac{n}{2}+4 \le i \le n-2, i \equiv 2 (\mathrm{mod}\ 4), \\ \frac{3n-i+4}{4}, & \text{for } \frac{n}{2}+2 \le i \le n, i \equiv 0 (\mathrm{mod}\ 4), \\ \frac{3n+2i-2}{8}, & \text{for } 7 \le i \le \frac{n}{2}-3, i \equiv 3 (\mathrm{mod}\ 4), \\ \frac{2n+2i-2}{8}, & \text{for } 5 \le i \le \frac{n}{2}-1, i \equiv 1 (\mathrm{mod}\ 4), \\ \frac{n+2i+4}{8}, & \text{for } 4 \le i \le \frac{n}{2}-2, i \equiv 0 (\mathrm{mod}\ 4), \\ \frac{i+2}{4}, & \text{for } 2 \le i \le \frac{n}{2}, i \equiv 2 (\mathrm{mod}\ 4), \end{cases}$$

$$f_4(x_i x_{i+1}) = \begin{cases} 2n, & \text{for } i = 1, \\ \frac{11n+4}{8}, & \text{for } i = 2, \\ n+1, & \text{for } i = \frac{n}{2}+1, \\ \frac{4n+2-i}{2}, & \text{for even } 4 \le i \le \frac{n}{2}+2, \\ \frac{5n+2i-2}{4}, & \text{for odd } \frac{n}{2}+3 \le i \le n-1, \\ \frac{11n+2i+2}{8}, & \text{for } 5 \le i \le \frac{n}{2}-1, i \equiv 1 (\mathrm{mod}\ 4), \\ \frac{5n+i+1}{4}, & \text{for } 3 \le i \le \frac{n}{2}-3, i \equiv 3 (\mathrm{mod}\ 4), \\ \frac{11n-2i+12}{8}, & \text{for } \frac{n}{2}+6 \le i \le n-4, i \equiv 0 (\mathrm{mod}\ 4), \\ \frac{5n-i+6}{4}, & \text{for } \frac{n}{2}+4 \le i \le n-2, i \equiv 2 (\mathrm{mod}\ 4), \end{cases}$$

$$f_4(x_n x_1) = \frac{9n+12}{8}.$$

This labeling also gives any two adjacent vertices with different weights, that is,

$$w(x_i) = \begin{cases} \frac{29n+12}{8}, & \text{for odd } i, \\ \frac{29n+12}{8} - \frac{n}{4}, & \text{for even } i. \end{cases}$$

Therefore $\chi_{lsat}(C_n) \leq 2$ for $n \equiv 4 \pmod 8$.

Combining all cases with the fact that $\chi_{lsat}(C_n) \geq \chi(C_n) = 3$ for odd n and $\chi_{lsat}(C_n) \geq \chi(C_n) = 2$ for even $n \neq 4$, we conclude that:

$$\chi_{lsat}(C_n) = \begin{cases} 3, & \text{if } n \text{ is odd or } n = 4, \\ 2, & \text{otherwise.} \end{cases}$$

□

The local super antimagic total labeling of the cycle C_n for even $n \geq 6$ can be used for labeling a path P_n for even $n \geq 6$ by removing an edge of C_n as described in the following corollary.

Corollary 3. *If P_n is a path of order even $n \geq 6$, then $3 \leq \chi_{lsat}(P_n) \leq 4$.*

Proof. Let P_n be the path of order even $n \geq 6$ with the vertex set $\{x_1, x_2, \ldots, x_n\}$. Suppose that there exists a labeling f^* such that $\chi_{lsat}(P_n) = 2$. Then for odd i,

$$w(x_i) \geq \frac{\min\left\{\sum_{i=1}^{\frac{n}{2}} f^*(x_{2i-1})\right\} + \sum_{e\in E(P_n)} f^*(e)}{\frac{n}{2}} = \frac{\binom{\frac{n}{2}+1}{2} + \binom{2n}{2} - \binom{n+1}{2}}{\frac{n}{2}} = \frac{13n-10}{4}.$$

However we know that for an end vertex, without loss of generality say x_1, we have $w(x_1) \leq \max\{f(x_i)\} + \max\{f(e)\} = n + 2n - 1 = 3n - 1$. So, $\frac{13n-10}{4} < 3n - 1$. This implies that $n < 6$, contradiction with $n \geq 6$. Therefore, $\chi_{lsat} \geq 3$.

The upper bound can be obtained by labeling in Teorema 2 for even n by removing edge $uv \in E(C_n)$ that satisfies $f(uv) = 2n$. Then $w(u) < w(v) < w(x_i) < w(x_{i+1})$. Thus, $\chi_{lsat}(P_n) \leq 4$. This concludes the proof that for even $n \geq 6$, $3 \leq \chi_{lsat}(P_n) \leq 4$. □

Although for the even case, the local super antimagic total chromatic number of path is not fixed, we have fixed the value for the odd case as presented in the following theorem.

Theorem 3. *If P_n is a path of order odd $n \geq 5$, then $\chi_{lsat}(P_n) = 3$.*

Proof. Let P_n be the path of order odd $n \geq 5$ with the vertex set $\{x_1, x_2, \ldots, x_n\}$. Suppose that there exists a labeling f^* such that $\chi_{lsat}(P_n) = 2$, that is, $w(x_1) = w(x_3) = \ldots = w(x_n) = a$ and $w(x_2) = w(x_4) = \ldots = w(x_{n-1}) = b$ for some positive integers a and b. Then,

$$\begin{aligned} \sum_{v\in V(P_n)} f^*(v) + 2\sum_{e\in E(P_n)} f^*(e) &= \left(\frac{n+1}{2}\right)a + \left(\frac{n-1}{2}\right)b \\ \frac{n(7n-5)}{2} &= \left(\frac{n+1}{2}\right)a + \left(\frac{n-1}{2}\right)b \end{aligned}$$

which is diophantine equations with the initial condition $a_0 = \frac{n(7n-5)}{2}$ and $b_0 = -\frac{n(7n-5)}{2}$. Thus the general solution is $a = \frac{n(7n-5)}{2} - \left(\frac{n-1}{2}\right)t$ and $b = -\frac{n(7n-5)}{2} + \left(\frac{n+1}{2}\right)t$, where t is an integer. Since a and b are positive integers, then:

$$\begin{aligned} \frac{n(7n-5)}{2} - \left(\frac{n-1}{2}\right)t > 0 \quad &\Rightarrow \quad t < \frac{n(7n-5)}{n-1} = 7n + 2 + \frac{2}{n-1} \\ -\frac{n(7n-5)}{2} + \left(\frac{n+1}{2}\right)t < 0 \quad &\Rightarrow \quad t > \frac{n(7n-5)}{n+1} = 7n - 12 + \frac{12}{n+1} \end{aligned}$$

which imply that $7n-11 \leq t \leq 7n+2$. This solution equivalent to $a = \frac{(14-k)n+(k-12)}{2}$ and $b = \frac{kn+(k-12)}{2}$ for $k = 1,2,\ldots,14$. We know that for even i,

$$w(x_i) \geq \frac{\min\left\{\sum_{i=1}^{\frac{n-1}{2}} f^*(x_{2i})\right\} + \sum_{e\in E(P_n)} f^*(e)}{\frac{n-1}{2}} = \frac{\binom{\frac{n+1}{2}}{2} + \binom{2n}{2} - \binom{n+1}{2}}{\frac{n-1}{2}} = \frac{13n+1}{4}.$$

However, this bound is impossible to be satisfied by $k = 1,2,\ldots,6$. Since such a value of k implies that $b < \frac{13n+1}{4}$. If we consider the upper bound of $w(x_i)$ for even i, that is,

$$\begin{aligned} w(x_{even}) &\leq \frac{\max\left\{\sum_{i=1}^{\frac{n-1}{2}} f^*(x_{2i})\right\} + \sum_{e\in E(P_n)} f^*(e)}{\frac{n-1}{2}} = \frac{\binom{n+1}{2} - \binom{\frac{n+3}{2}}{2} + \binom{2n}{2} - \binom{n+1}{2}}{\frac{n-1}{2}} \\ &= \frac{\binom{2n}{2} - \binom{\frac{n+3}{2}}{2}}{\frac{n-1}{2}} = \frac{15n+3}{4}. \end{aligned}$$

Again, this bound is impossible to be satisfied by $k = 8,9,\ldots,14$ as it implies that $b > \frac{15n+3}{4}$. Furthermore, $k = 7$ is also impossible as it implies $a = b$ which is contradiction for f^* as a local super antimagic total labeling of P_n. Thus, there are no values of a and b that satisfy the condition. So, $\chi_{lsat}(P_n) \geq 3$.

To prove the upper bound, we label the vertices and edges of P_n by using the following formula.

$$f(x_i) = \begin{cases} 1, & \text{for } i = 2, \\ n-1, & \text{for } i = 1, \\ n, & \text{for } i = n, \\ n-i, & \text{for odd } i = 3,5,\ldots,n-2, \\ n+2-i, & \text{for even } i = 4,6,\ldots,n-1, \end{cases}$$

$$f(x_ix_{i+1}) = \begin{cases} 2n-1, & \text{for } i = 1, \\ \frac{2n+i-1}{2}, & \text{for odd } i = 3,5,\ldots,n-2, \\ \frac{3n+i-3}{2}, & \text{for even } i = 2,4,\ldots,n-1. \end{cases}$$

It is easy to see that the labeling f gives different vertex weights, namely, $w(x_1) = w(x_n) = 3n-2$, $w(x_i) = \frac{7n-1}{2}$ for odd $i = 3,5,\ldots,n-2$ and $w(x_i) = \frac{7n-1}{2} - 2$ for even $i = 2,4,\ldots,n-1$. Thus $\chi_{lsat}(P_n) \leq 3$. Combined with the lower bound, we conclude that $\chi_{lsat}(P_n) = 3$ for odd $n \geq 5$. □

Gear, denoted by G_n, is a graph of order $2n+1$ that is obtained from a cycle on $2n$ vertices C_{2n} by adding one central vertex and connecting half of rim vertices alternately to the central vertex. If the vertex set C_n is $\{x_1, x_2, \ldots, x_{2n}\}$ and the central vertex is c, then the vertex set of G_n is $\{c, x_1, x_2, \ldots, x_{2n}\}$ and the edge set of G_n is $\{cx_1, cx_3, \ldots, cx_{2n-1}\} \cup \{x_1x_2, x_2x_3, \ldots, x_{2n}x_1\}$. For the case odd $n \geq 5$, the local super antimagic total chromatic number of the gear G_n can be obtained from the local super antimagic total labeling of a cycle C_{2n} as follows.

Corollary 4. *If G_n for odd $n \geq 5$ is a gear, then $\chi_{lsat}(G_n) = 3$.*

Proof. Let G_n for odd $n \geq 5$ be a gear with the vertex set $\{c, x_1, x_2, \ldots, x_{2n}\}$ and the edge set $\{cx_1, cx_3, \ldots, cx_{2n-1}\} \cup \{x_1x_2, x_2x_3, \ldots, x_{2n}x_1\}$. We will use the local super antimagic total labeling of a cycle C_{2n} given in Theorem 2 and then apply a labeling f^+ with formula $f^+(c) = 2n+1$ for the central vertex of G_n, $f^+(e) = f(e)+1$ for the edges of G_n and $f^+(x_i) = f(x_i)$ for the rim vertices of G_n where i is even. When i is odd, label the rim vertices of G_n using $f^+(x_i)$ by permuting labels

$f(x_1), f(x_3), \ldots, f(x_{2n-1})$ according to a rule given in Table A2. This labeling gives any two adjacent vertices of G_n with different weights, that is,

$$w(x_i) = \begin{cases} 12n+5, & \text{for odd } i, \\ \frac{13n+7}{2}, & \text{for even } i, \end{cases}$$
$$w(c) = \frac{1}{2}(9n^2+7n+2).$$

Thus $\chi_{lsat}(G_n) \leq 3$.

We now prove the lower bound on the local super antimagic total chromatic number of G_n. Suppose that $\chi_{lsat}(G_n) = 2$. Then there must exist x_i for some even $2 \leq i \leq 2n$ such that $w(x_i) = w(c)$. The largest $w(x_i)$ at most is the sum of the largest vertex labels and the two largest edge labels, that is, $w(x_i) \leq (2n+1) + (5n+1) + 5n = 12n+2$. While the weight of the central vertex must be at least the sum of the smallest vertex labels and smallest n edge labels, that is, $w(c) \geq 1 + \sum_{i=1}^{n}(2n+1+i) = \frac{5n^2+3n}{2} + 1$. It is easy to see that $12n+2 < \frac{5n^2+3n}{2} + 1$ for any $n \geq 5$ which is a contradiction. So, $\chi_{lsat}(G_n) \geq 3$. Combined with the upper bound above, we obtain $\chi_{lsat}(G_n) = 3$ for odd $n \geq 5$. □

The following theorem presents the local super antimagic total chromatic number of a special class of cubic graphs, so called a *cubic bipartite graph*. The cubic bipartite graph, denoted by CB_n for even $n \geq 6$, is a regular 2-colorable connected graph of degree 3 that can be constructed from a cycle C_n with vertex set $\{v_1, v_2, \ldots, v_{2k}\}$ by connecting v_i to v_j where i is odd and $j = i + k - \frac{1+(-1)^k}{2} \bmod 2k$.

Theorem 4. *If CB_n for even $n \geq 6$ is a cubic bipartite graph, then $\chi_{lsat}(CB_n) = 2$.*

Proof. Let CB_n for even $n \geq 6$ be the circulant cubic graph with the vertex set $V(CB_n) = \{x_i | 1 \leq i \leq 2k\}$ and the edge set $E(CB_n) = \{x_i x_{i+1} | 1 \leq i \leq 2k-1\} \cup \{x_{2k}x_1\} \cup \{x_i x_j | i$ is odd and $j = i + k - \frac{1+(-1)^k}{2} \bmod 2k\}$. Label the vertices and edges of cycle C_n using the following formula.

$$f(x_i) = \begin{cases} k, & \text{for } i = 1, \\ \frac{i-1}{2}, & \text{for odd } i = 3, ..., 2k-1, \\ \frac{3k+i+1}{2}, & \text{for odd } k \text{ and even } i = 2, ..., k-1, \\ \frac{k+i+1}{2}, & \text{for odd } k \text{ and even } i = k+1, ..., 2k, \\ \frac{3k+i+2}{2}, & \text{for even } k \text{ and even } i = 2, ..., k-2, \\ \frac{k+i+2}{2}, & \text{for even } k \text{ and even } i = k, ..., 2k. \end{cases}$$
$$f(x_i x_{i+1}) = \begin{cases} 2k + \frac{i+1}{2}, & \text{for odd } i, \\ 4k+1-\frac{i}{2}, & \text{for even } i, \end{cases}$$
$$f(x_{2k}x_1) = 3k+1,$$
$$f(x_i x_j) = 5k - \frac{i-1}{2}, \text{ for odd } i,$$

This labeling gives different weights for any two adjacent vertices,

$$w(x_i) = \begin{cases} 11k+2, & \text{for odd } i, \\ 12k+2, & \text{for even } i. \end{cases}$$

Thus $\chi_{lsat}(CB_n) \leq 2$. Since CB_n is 2-colorable, then $\chi_{lsat}(CB_n) \geq \chi(CB_n) = 2$. Combining these two bounds, we conclude that $\chi_{lsat}(CB_n) = 2$. □

The joint product of two graphs generates a graph that consists of the union of the two graphs with additional edges connecting all vertices of the first graph to each vertex of the second graph. In the next theorems, we present the local super antimagic total chromatic number of graphs produced by a joint product of two isomorphic cycles and a joint product of two isomorphic cubic bipartite graphs.

Theorem 5. *If $C_n + C_n$ is a joint product of two isomorphic cycles C_n for $n \geq 3$, then*

$$\chi_{lsat}(C_n + C_n) = \begin{cases} 4, & \text{for even } n, \\ 6, & \text{for odd } n. \end{cases}$$

Proof. Let $C_n + C_n$ be the joint product of two isomorphic cycles C_n for $n \geq 3$ with the vertex set $V(C_n + C_n) = \{x_i, y_i | 1 \leq i \leq n\}$ and edge set $E(C_n + C_n) = \{x_i x_{i+1}, y_i y_{i+1} | 1 \leq i \leq n-1\} \cup \{x_n x_1, y_n y_1\} \cup \{x_i y_j | 1 \leq i, j \leq n\}$. We first show the lower bound of $\chi_{lsat}(C_n + C_n)$. Since x_i and y_i are adjacent, for each $1 \leq i \leq n$, any vertex x_i cannot use the color of vertex y_i. Thus $\chi(C_n + C_n) = 2\chi(C_n)$. We know that $\chi_{lsat}(C_n + C_n) \geq \chi(C_n + C_n)$. Consequently, $\chi_{lsat}(C_n + C_n) \geq 2\chi(C_n)$.
To show the upper bound of $\chi_{lsat}(C_n + C_n)$, we use the following labeling. For $n = 4$, we obtain $\chi_{lsat}(C_n + C_n) \leq 4$ by using the labeling as shown in Figure A2.

While for $n \neq 4$, we will use the labeling f in the proof of Theorem 2 as follows.

$$\begin{aligned} f^+(x_i) &= f(x_i), \\ f^+(y_i) &= f(y_i) + n, \\ f^+(e_i) &= f(e_i) + n, \text{ for } e_i = x_i x_{i+1}, \\ f^+(x_i y_j) &= r(i,j) + 3n, \\ f^+(e_j) &= f(e_j) + (n+2)n, \text{ for } e_j = y_j y_{j+1}. \end{aligned}$$

where $r(i,j)$ is the number in the ith row and the jth column of rectangle $R(m,n)$ given by Hagedorn [11] for $m = n$.

The labeling f^+ gives different weights for any two adjacent vertices in $C_n + C_n$, that is,

$$\begin{aligned} w^+(x_i) &= w(x_i) + 2n + \frac{\sum_i^{n^2}(3n+i)}{n}, \\ w^+(y_i) &= w(y_i) + n + 2n(n+2) + \frac{\sum_i^{n^2}(3n+i)}{n}. \end{aligned}$$

Thus $\chi_{lsat}(C_n + C_n) \leq 2\chi_{lsat}(C_n)$ for $n \neq 4$. Combined with the lower bound, we conclude that $\chi_{lsat}(C_n + C_n) = 4$ for even n and $\chi_{lsat}(C_n + C_n) = 6$ for odd n. □

Using similar arguments as in the proof of Theorem 5, we present the local super antimagic total chromatic number of graphs produced by a joint product of two isomorphic cubic bipartite graphs as follows.

Theorem 6. *If $CB_n + CB_n$ is a joint product of two isomorphic CB_n for $n \geq 6$, then $\chi_{lsat}(CB_n + CB_n) = 4$.*

The following theorems presents the local super antimagic total chromatic number of graphs produced by the corona product of two graphs. The corona product of two graphs G_1 and G_2 generates a graph by taking one copy of G_1 and $|V(G_1)|$ copies of G_2 and joining the ith vertex of G_1 to each vertex in the ith copy of G_2.

Theorem 7. *If H is a regular graph of order $n \geq 2$ and $\overline{K_m}$ is a null graph of order $m \geq 2$, then $\chi_{lsat}(H \odot \overline{K_m}) \leq \chi_{lsat}(H) + 1$, where $(m,n) \neq (odd, even)$.*

Proof. Let H be a regular graph of order $n \geq 2$ and $\overline{K_m}$ be the null graph of order $m \geq 2$. Then $H \odot \overline{K_m}$ has the vertex set $V = \{v_j, x_j^i | v_j \in V(H), 1 \leq j \leq n, 1 \leq i \leq m\}$ and the edge set

$E = \{e, v_jx_j^i | e \in E(H), v_j \in V(H), 1 \leq i \leq m, 1 \leq j \leq n\}$. We assume that f is a local super antimagic total labeling for H such that f provides t different vertex weights. We divide the proof into three cases.

Case i. For $(m, n) = (2, 2)$.

There are two possible subcases to consider, that is, $H = K_2$ and $H = \overline{K_2}$. The labeling which is illustrated in Figure A3 shows that $\chi_{lsat}(H \odot \overline{K_2}) \leq \chi_{lsat}(H) + 1$ for both subcases.

Case ii. For $m \equiv n \mod 2$, $(m, n) \neq (2, 2)$, and $m \geq 2$.

We consider the rectangle $R(m, n)$ that consists of m rows and n columns where $m \equiv n \mod 2$ containing mn first natural numbers such that the sum of entries in each row and each column is the same constant. This condition always occur unless $(m, n) = (2, 2)$ as given by Hagedorn [11]. Label every vertex of $H \odot \overline{K_m}$ by $f^+(v_j) = mn + f(v_j)$ and every edge of $H \odot \overline{K_m}$ by $f^+(e) = 2mn + f(e)$ for all $v_j, e \in H$, where f is the local super antimagic total labeling of H such that $\chi_{lsat}(H) = t$. Let $r(i, j)$ be the number in the ith row and the jth column of rectangle $R(m, n)$. Label the edge $v_jx_j^i$ by $r(i, j) + n + mn$ for $1 \leq i \leq m, 1 \leq j \leq n$ and the pendant vertex by $f^+(x_j^i) = mn + n + 1 - f(v_jx_j^i)$. Thus the weight of vertices are:

$$\begin{aligned} w^+(v_j) &= w(v) + 2kmn + mn + \frac{m(mn+1)}{2} + (mn+n)m, \\ w^+(x_j^i) &= mn + n + 1. \end{aligned}$$

Case iii. For even m and odd n.

Label the pendant edge by $f^+(v_jx_j^i) = n(m + i) + j$ for odd i and $f^+(v_jx_j^i) = n(m + i + 1) - j + 1$ for even i. Label then the pendant vertex by $f^+(x_j^i) = mn + n + 1 - f(v_jx_j^i)$. This labeling also gives the the same weights of vertices in Case ii.

From the three cases, we conclude that $\chi_{lsat}(H \odot \overline{K_m}) \leq \chi_{lsat}(H) + 1$. □

The corona product $C_n \odot \overline{K_m}$ is a generalized sun S_n^m. The generalized sun S_n^m has n vertices of degree $m + 2$ and mn vertices of degree 1. The following corollary present the local super antimagic chromatic number of the generalized sun.

Corollary 5. *If $S_n^m = C_n \odot \overline{K_m}$ for $n \geq 3$ and $m \geq 1$ is a generalized sun, then:*

$$\chi_{lsat}(S_n^m) = \begin{cases} 4, & \text{for odd } n, \\ 3, & \text{for even } n. \end{cases}$$

Proof. Let $S_n^m = C_n \odot \overline{K_m}$ for $n \geq 3$ and $m \geq 1$ be a generalized sun with the vertex set $V(S_n^m) = \{x_i, y_i^j | 1 \leq i \leq n, 1 \leq j \leq m\}$ and the edge set $E(S_n^m) = \{x_ix_{i+1} | 1 \leq i \leq n - 1\} \cup \{x_nx_1\} \cup \{x_iy_i^j | 1 \leq i \leq n, 1 \leq j \leq m\}$. Suppose that $\chi_{lsat}(S_n^m) = \chi(S_n^m)$. Then there is a condition that $w(x_i) = w(y_k^j)$ with $d(x_i) \geq 3$ and $d(y_k^j) = 1$. Of course, $w(y_k^j) \leq 3(m + 1)n$, however, $w(x_i) \geq 3(m + 1)n + 7$. This implies that $w(y_k^j) < w(x_i)$, which is a contradiction as they have the same weight. Thus, $\chi_{lsat}(S_n^m) \geq \chi(S_n^m) + 1$.

To show the upper bound of $\chi_{lsat}(S_n^m)$, we divide the proof into four cases.

Case i. For even m and $n = 4$.

Label the vertices and edges of the generalized sun S_n^m using the following formula.

$$
\begin{aligned}
f_1(x_1, x_2, x_3, x_4) &= (3, 1, 4, 2) \\
f_1(x_1y_1^1, x_2y_2^1, x_3y_3^1, x_4y_4^1) &= (8m+2, 8m-1, 8m, 8m-3) \\
f_1(x_1y_1^2, x_2y_2^2, x_3y_3^2, x_4y_4^2) &= (8m+3, 8m+1, 8m+4, 8m-2) \\
f_1(y_1^1, y_2^1, y_3^1, y_4^1) &= (7, 10, 9, 12) \\
f_1(y_1^2, y_2^2, y_3^2, y_4^2) &= (6, 8, 5, 11) \\
f_1(y_i^j) &= \begin{cases} 4j+1+i, & \text{for even } j \geq 4, 1 \leq i \leq 4, \\ 4j+3-i, & \text{for odd } j \geq 3, 1 \leq i \leq 4, \end{cases} \\
f_1(x_iy_i^j) &= \begin{cases} 8m+8-4j-i, & \text{for even } j \geq 4, 1 \leq i \leq 4, \\ 8m-4-4j+i, & \text{for odd } j \geq 3, 1 \leq i \leq 4. \end{cases}
\end{aligned}
$$

The labeling f_1 gives the vertex weights $w(x_i^j) = 8m+9$ for $1 \leq i \leq 4$, $1 \leq j \leq m$, $w(x_1) = w(x_3) = 32m + 21 + \frac{\sum_{i=1}^{m-2} 8m-5+i}{4}$ and $w(x_2) = w(x_4) = 32m + 12 + \frac{\sum_{i=1}^{m-2} 8m-5+i}{4}$. Thus $\chi_{lsat}(S_4^m) \leq 3$.

Case ii. For odd m and even $n \geq 4$.

Theorems 2 and 7 imply that $\chi_{lsat}(S_n^m) = \chi_{lsat}(C_n \odot \overline{K_m}) \leq 3$ for even n and $\chi_{lsat}(S_n^m) \leq 4$ for odd n.

Case iii. For $m = 1$.

Label the vertices and edges of the generalized sun S_n^m using the following formula.

$$
\begin{aligned}
f_2(y_i^1) &= \begin{cases} n, & \text{for } i = 1, \\ i-1, & \text{for } 2 \leq i \leq n, \end{cases} \\
f_2(x_i) &= \begin{cases} 2n-i, & \text{for } i \equiv n-1 \mod 2, \\ 2n+2-i, & \text{for } i \equiv n \mod 2, \\ 2n,\ i=1, & \text{for } n \equiv 1 \mod 2, \end{cases} \\
f_2(x_iy_i^1) &= \begin{cases} 2n+1, & \text{for } i = 1, \\ 3n+2-i, & \text{for } 2 \leq i \leq n, \end{cases} \\
f_2(x_ix_{i+1}) &= 3n+1, \text{ for } 1 \leq i \leq n-1, \\
f_2(x_nx_1) &= 4n.
\end{aligned}
$$

The labeling f_2 gives the vertex weights $w(y_i^1) = 3n+1$, $w(x_1) = 11n+2$ for odd n, $w(x_i) = 11n+3$ for even i, $w(x_i) = 11n+1$ for odd i and $i \neq n$ where n is odd. Thus, $\chi_{lsat}(S_n^1) \leq 3$ for even n and $\chi_{lsat}(S_n^1) \leq 4$ for odd n.

Case iv. For odd m and even n ≥ 3.

Label the vertices and edges of the generalized sun S_n^m using the following formula.

$$f_3(y_i^j) = \begin{cases} n, & \text{for } i=1,\ j=1, \\ i-1, & \text{for } 2 \le i \le n,\ j=1, \\ (j-1)n+i, & \text{for } 1 \le i \le n,\ 2 \le j \le m-1,\ j \equiv 0 \mod 2, \\ jn+1-i, & \text{for } 1 \le i \le n,\ 3 \le j \le m,\ j \equiv 1 \mod 2, \end{cases}$$

$$f_3(x_i) = \begin{cases} (m+1)n-i, & \text{for } 1 \le i \le n,\ i \equiv 1 \mod 2, \\ (m+1)n+2-i, & \text{for } 1 \le i \le n,\ i \equiv 0 \mod 2, \end{cases}$$

$$f_3(x_iy_i^j) = \begin{cases} (2m+1)n+1, & \text{for } i=1,\ j=1, \\ 2(m+1)n+2-i, & \text{for } 2 \le i \le n,\ j=1, \\ (2m+3-j)n+1-i, & \text{for } 1 \le i \le n,\ 2 \le j \le m-1,\ j \equiv 0 \mod 2, \\ (2m+2-j)n+i, & \text{for } 1 \le i \le n,\ 3 \le j \le m,\ j \equiv 1 \mod 2, \end{cases}$$

$$f_3(x_ix_{i+1}) = (2m+1)n+i, \text{ for } 1 \le i \le n,$$

$$f_3(x_nx_1) = 2(m+1)n.$$

The labeling f_3 gives the vertex weights $w(y_i^j) = 2(m+1)n+1$, $w(x_i) = 11n+3+\frac{\sum_{i=1}^{(m-1)n}(n+i)}{n}$ for even i, $w(x_i) = 11n+1+\frac{\sum_{i=1}^{(m-1)n}(n+i)}{n}$ for odd i. Thus, $\chi_{lsat}(S_n^m) \le 3$ for odd m and even $n \ge 3$. From the four cases above, we conclude that $\chi_{lsat}(S_n^m) = 3$ for even n and $\chi_{lsat}(S_n^m) = 4$ for odd n. □

The next theorem presents the upper bound of the local super antimagic chromatic number of a regular graph H whose each vertex is joined to one central vertex c and to m pendant vertices for $m \ge 1$.

Theorem 8. *If G is a graph obtained from a regular graph H of order $n \ge 2$ by joining each vertex to one center vertex and m pendant vertices for $m \ge 1$, then:*

(i) $\chi_{lsat}(G) \le \chi_{lsat}(H+K_1)+1$, for $m \ge 2$ and $(m,n) \ne (odd, even)$;
(ii) $\chi_{lsat}(G) \le \chi_{lsat}(H)+2$, for $(m,n) = (odd, even)$;
(iii) $\chi_{lsat}(G) \le \chi_{lsat}(H)+2$, for $m=1$.

Proof. Let G be the graph obtained from a regular graph H of order $n \ge 2$ by joining each vertex to one central vertex c and m pendant vertices for $m \ge 1$. Then G has the vertex set $V(G) = \{v_i, x_i | v_i \in V(H), 1 \le i \le n\} \cup \{c\}$ and the edge set $E(G) = \{e | e \in E(H)\} \cup \{cv_i, v_ix_i | v_i \in V(H), 1 \le i \le n\}$. It is easy to prove parts (i) and (ii). We now prove part (iii) as follows. Suppose that $\chi_{lsat}(H) = t$ is obtained from local super antimagic total labeling f for H. Label the vertices of G by $f^+(x_i) = i$ for $1 \le i \le n$, $f^+(v_i) = f(v_i)+n$ for each $v_i \in V(H)$ and $f^+(c) = 2n+1$. Label then the edges of G by $f^+(v_ix_i) = 3n+2-i$ for each $v_i \in V(H)$ and $1 \le i \le n$, $f^+(e) = f(e)+2n+1$ for each $e \in E(H)$, $f^+(cv_i) = 3n+1+kn/2+i$ where $1 \le i \le n$. The labeling f^+ gives $t+2$ different vertex weights, that is, $w^+(x_i) = 3n+2$, $w^+(v) = k(2n+1)+7n+kn/2+3+w(v)$ and $w^+(c) = 2n+1+\sum_{i=1}^{n}(3n+1+kn/2+i)$. Thus, $\chi_{lsat}(G) \le \chi_{lsat}(H)+2$. □

Helm, denoted by H_n, is a graph obtained from a regular graph H where $H = C_n$ by joining each vertex of H or C_n to one vertex c called center and to a pendant vertex. The local super antimagic chromatic number of the helm H_n for $n \ge 3$ is presented in the following corollary.

Corollary 6. *If H_n for $n \ge 3$ is a helm, then:*

$$\chi_{lsat}(H_n) = \begin{cases} 4, & \text{for even } n, \\ 5, & \text{for odd } n. \end{cases}$$

Proof. Let H_n for $n \ge 3$ is the helm. Since H_n is a graph obtained from C_n by joining each vertex of C_n to center c and to a pendant vertex, then H_n has the vertex set $V(H_n) = V(C_n \odot K_1) \cup \{c\} = \{x_i | \le i \le n\} \cup \{y_i | 1 \le i \le n\} \cup \{c\}$ and the edge set $E(H_n) = E(C_n \odot K_1) \cup \{cx_i | x_i \in V(C_n)\} =$

$\{x_i x_{i+1} | 1 \leq i \leq n-1\} \cup \{x_n x_1\} \cup \{cx_i | 1 \leq i \leq n\} \cup \{x_i y_i | 1 \leq i \leq n\}$. For $i = 1, 2, \ldots, n$, the degree of x_i is 4 and the degree of y_i is 1, while the degree of c is n. For $n = 3$, it is easy to see that $w(c) \geq 28$ and $w(y_i) \leq 23$. It is impossible to have $w(c) = w(y_i)$ or $w(x_j) = w(y_i)$. Thus $\chi_{lsat}(H_3) > \chi(H_3)$. For $n > 3$, we have $w(x_i) \geq 8n + 15$ and $w(y_i) \leq 7n + 2$. This also implies that $\chi_{lsat}(H_n) > \chi(H_n)$ for any $n > 3$. The upper bound of $\chi_{lsat}(H_n)$ for $n = 4$ can be obtained from the labeling as shiwn in Figure A4, that is, $\chi_{lsat}(H_4) \leq 4$. While for other n the upper bound of $\chi_{lsat}(H_n)$ is obtained from Theorem 8(iii) and Theorem 2. Combining the lower bound and the upper bound of $\chi_{lsat}(H_n)$, we conclude that:

$$\chi_{lsat}(H_n) = \begin{cases} 4, & \text{for even } n, \\ 5, & \text{for odd } n. \end{cases}$$

□

The vertex amalgamation of n copies of graph G at a fix vertex $v \in V(G)$, denoted $Amal(G, v, n)$ for $n \geq 2$ is a graph obtained by identifying n copies of graph G at the vertex v. Thus $Amal(G, v, n) = \bigcup_{i=1}^{n} G_i$ and $\bigcap_{i=1}^{n} G_i = \{v\}$. The following theorem presents the local super antimagic total labeling of amalgamation of n copies stars S_{m+2}

Theorem 9. *If $Amal(S_{m+2}, v, n)$ for $m, n \geq 2$ is a graph obtained from amalgamation of n copies of stars S_{m+2} at the pendant vertex v, then $\chi_{lsat}(Amal(S_{m+2}, v, n)) = 3$.*

Proof. Let $Amal(S_{m+2}, v, n)$ for $m, n \geq 2$ be the graph obtained from amalgamation of n copies of stars S_{m+2} at the pendant vertex v. Then $Amal(S_{m+2}, v, n)$ has the vertex set $V(Amal(S_{m+2}, v, n)) = \{x_i^j, c_i | 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{c\}$ and the edge set $E(Amal(S_{m+2}, v, n)) = \{cc_i, c_i x_i^j | 1 \leq i \leq n, 1 \leq j \leq m\}$. By Corollary 1 and Theorem 8(i), we have the upper bound $\chi_{lsat}(Amal(S_{m+2}, v, n)) \leq 3$ for $(m, n) \neq (odd, even)$. For $(m, n) = (odd, even)$, label the vertices and edges of $Amal(S_{m+2}, v, n)$ using the following formula:

$$\begin{aligned}
f(x_i^j) &= \begin{cases} (m-j+1)n+1-i, & \text{for odd } j = 1, 3, \ldots, m-2, \\ (m-j)n+i, & \text{for even } j = 2, 4, \ldots, m-1, \\ i, & \text{for } j = m, \end{cases} \\
f(c_i) &= \begin{cases} mn + \frac{n}{2} + 1 - i, & \text{for } i \leq \frac{n}{2}, \\ mn + \frac{3n}{2} + 2 - i, & \text{for } i > \frac{n}{2}, \end{cases} \\
f(c) &= mn + \frac{n}{2} + 1, \\
f(cc_i) &= \begin{cases} (2m+1)n + 2i + 1, & \text{for } i \leq \frac{n}{2}, \\ 2mn + 2i, & \text{for } i > \frac{n}{2}, \end{cases} \\
f(c_i x_i^j) &= \begin{cases} (j+m)n + 1 + i, & \text{for odd } j = 1, 3, \ldots, m-2, \\ (j+m+1)n + 2 - i, & \text{for even } j = 2, 4, \ldots, m-1, \\ (2m+1)n + 2 - i, & \text{for } j = m. \end{cases}
\end{aligned}$$

This labeling gives the local super antimagic total labeling with different vertex weights, namely,

$$\begin{aligned}
w(x_i^j) &= (2n+1)m + 2, \\
w(c_i) &= (5m + \frac{5}{2})n + 2 + \frac{\sum_{i=1}^{(m-1)n} ((m+1)n + 1 + i)}{n}, \\
w(c) &= mn + \frac{n}{2} + 1 + \sum_{i=1}^{n} (2m+1)n + i.
\end{aligned}$$

Thus $\chi_{lsat}(Amal(S_{m+2}, v, n)) \leq 3$. Since the degree of c is at least 3 and the degree of x_i^j is 1 when $n \geq 3$. The possibility are $w(c) \geq 3(m+1)n + 10$ and $w(x_i^j) < 3(m+1)n + 1$. Thus $w(c) \neq w(x_i^j)$. For $n = 2$, suppose that $\chi_{lsat}(Amal(S_{m+2}, v, n)) = 2$, that is, $w(c) = w(x_i^j)$ for each $i \in \{1, 2\}$, $j \in \{1, 2, \ldots, m\}$. We know that $w(c) \geq 4m + 10$ since the degree of c is 2. However, the maximum weighs of c and x_i^j are $\frac{\sum_{i=3}^{4m+5} i}{2m+1} < 4m + 10$, a contradiction. Thus $\chi_{lsat}(Amal(S_{m+2}, v, n)) \geq 3$ for every $n \geq 2$. Combining the lower bound and upper bound, we obtain $\chi_{lsat}(Amal(S_{m+2}, v, n)) = 3$, for every $m, n \geq 2$. □

The following theorem presents the local super antimagic total labeling of the amalgamation of m copies wheels W_n at the central vertex v.

Theorem 10. *If $Amal(W_n, v, m)$ for $n \geq 3$ and $m \geq 2$ is a graph obtained from amalgamation of m copies of wheels W_n at the central vertex v, then:*

$$\chi_{lsat}(Amal(W_n, v, m)) = \begin{cases} 3, & \text{for even } n, \\ 4, & \text{for odd } n. \end{cases}$$

Proof. Let $Amal(W_n, v, m)$ for $n \geq 3$ and $m \geq 2$ be the graph obtained from amalgamation of m copies of wheels W_n at the central vertex v. Then $Amal(W_n, v, m)$ has the vertex set $V(Amal(W_n, v, m)) = \{x_i^j | 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{c\}$ and the edge set $E(Amal(W_n, v, m)) = \{x_i^j x_{i+1}^j, cx_i^j | 1 \leq i \leq n-1, 1 \leq j \leq m\} \cup \{x_n^j x_1^j, cx_n^j | 1 \leq j \leq m\}$. Label the vertices and edges of $Amal(W_n, v, m)$ using the following formula.

$$\begin{aligned} f(x_i^j) &= \begin{cases} m(i-1)+j+1, & \text{for } i \equiv n \pmod 2, \\ m(i-3)+j+1, & \text{for } i \equiv n-1 \pmod 2, \\ 3mn+j+1, & \text{for } i = 2,\ n \equiv 1 \pmod 2, \end{cases} \\ f(c) &= 1, \\ f(x_i^j x_{i+1}^j) &= m(3n-i+1)-j+2, \text{ for } 1 \leq i \leq n-1, 1 \leq j \leq m, \\ f(x_n^j x_1^j) &= m(2n+1)-j+2, \text{ for } 1 \leq j \leq m, \\ f(cx_i^j) &= m(n+i-1)+j+1, \text{ for } 1 \leq i \leq n, 1 \leq j \leq m. \end{aligned}$$

This labeling gives local super antimagic total labeling with different vertex weights, namely,

$$\begin{aligned} w(x_i) &= \begin{cases} m(7n+1)+6, & \text{for } i \equiv n-1 \pmod 2, \\ m(7n-1)+6, & \text{for } i \equiv n \pmod 2, \\ 7mn+6, & \text{for } i = 2,\ n \equiv 1 \pmod 2, \end{cases} \\ w(c) &= \frac{mn(3mn+3)}{2} + 1. \end{aligned}$$

Thus $\chi_{lsat}(Amal(W_n, v, m)) \leq 3$ for even n and $\chi_{lsat}(Amal(W_n, v, m)) \leq 4$ for odd n. We know that $\chi_{lsat}(Amal(W_n, v, m)) \geq 3$ for even n and $\chi_{lsat}(Amal(W_n, v, m)) \geq 4$ for odd n. Therefore, $\chi_{lsat}(Amal(W_n, v, m)) = 3$ for even n and $\chi_{lsat}(Amal(W_n, v, m)) = 4$ for odd n. □

The local super antimagic total chromatic number of a wheel W_n for $n \geq 3$ is implied by Theorem 10 when $m = 1$ as follows.

Corollary 7. *If W_n for $n \geq 3$ is a wheel, then:*

$$\chi_{lsat}(W_n) = \begin{cases} 3, & \text{for even } n, \\ 4, & \text{for odd } n. \end{cases}$$

3. Conclusions

We conclude this paper with some open problems. Some classes of graphs have a condition that $\chi_{lsat}(G) - \chi(G) = k$ where $k = 0$ or $k = 1$. Consequently, we have the following open problem.

Problem 1. *Characterize the family of graphs that satisfies $\chi_{lsat}(G) = \chi(G) + k$, for $k \geq 2$.*

We have determined the local super antimagic total chromatic number of cubic bipartite graph that is a regular 2-colorable connected graph of degree 3. In general, the local super antimagic total chromatic number of cubic graphs have not been discovered. So, we have:

Problem 2. *Determine the local super antimagic total chromatic number of cubic graphs of order $n \geq 8$.*

Some particular classes of tree such as path, star, double star, and amalgamation of stars have determined their local super antimagic total chromatic numbers. However, the local super antimagic total chromatic number of tree in general has not been discovered. Thus,

Problem 3. *Determine the local super antimagic total chromatic number of tree T_n of order $n \geq 5$.*

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Appendix A

Table A1. All possible local super antimagic total labelings of C_4.

Vertex Label	Edge Label 1	Edge Label 2	Weight
1	5	6	12
1	5	7	13
2	5	6	13
1	5	8	14
1	6	7	14
2	5	7	14
3	5	6	14
1	6	8	15
2	5	8	15
2	6	7	15
3	5	7	15
4	5	6	15
1	7	8	16
2	6	8	16
3	5	8	16
3	6	7	16
4	5	7	16

Table A1. *Cont.*

Vertex Label	Edge Label 1	Edge Label 2	Weight
2	7	8	17
3	6	8	17
4	6	7	17
3	7	8	18
4	6	8	18
4	7	8	19

Table A2. Permutation of vertex labels of C_n to obtain rim vertex labels of G_n for odd n.

$f(x_i)$	$n+1$	$n+2$	$n+3$	$\dots$	$\frac{3n}{2}$	...	$2n-2$	$2n-1$	$2n$
$f^+(x_i)$	$n+1$	$n+3$	$n+5$	$\dots$	$2n$	...	$2n-5$	$2n-3$	$2n-1$
$f^+(cx_i)$	$\frac{9n+3}{2}$	$\frac{9n+1}{2}$	$\frac{9n-1}{2}$	$\dots$	$4n+2$	...	$\frac{9n+9}{2}$	$\frac{9n+7}{2}$	$\frac{9n+5}{2}$

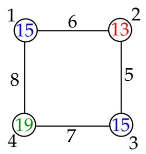

Figure A1. Local super antimagic total labelings of C_4.

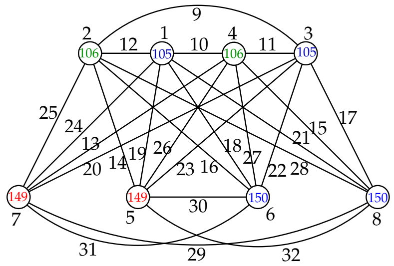

Figure A2. The local super antimagic total labeling of $C_4 + C_4$.

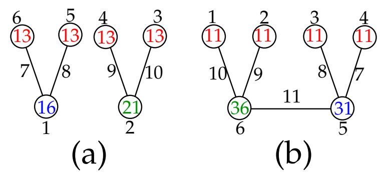

Figure A3. The local super antimagic total labeling of (**a**) $\overline{K_2} \odot \overline{K_2}$ and (**b**) $K_2 \odot \overline{K_2}$.

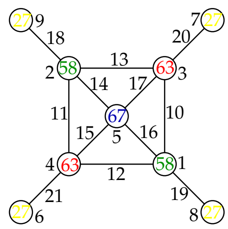

Figure A4. The local super antimagic total labeling of H_4.

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