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Toward a Wong–Zakai Approximation for Big Order Generators

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Abstract: We give a new approximation with respect of the traditional parametrix method of the solution of a parabolic equation whose generator is of big order and under the Hoermander form. This generalizes to a higher order generator the traditional approximation of Stratonovitch diffusion which put in relation random ordinary differential equation (the leading process is random and of finite energy. When a trajectory of it is chosen, the solution of the equation is defined) and stochastic differential equation (the leading process is random and only continuous and we cannot choose a path in the solution which is only almost surely defined). We consider simple operators where the computations can be fully performed. This approximation fits with the semi-group only and not for the full path measure in the case of a stochastic differential equation.

Keywords: ordinary differential equation; parabolic equation; big order generator

1. Introduction

Let us consider a compact Riemannian manifold M of dimension d endowed with its normalized Riemannian measure dx ($x \in M$).

Let us consider m smooth vector fields X_i (we will suppose later that they are without divergence). Some times vector fields are considered as one order differential operators acting on the space of smooth functions on the manifold M, sometimes they are considered as smooth sections of the tangent bundle of M. We consider the second order differential operator:

$$L = 1/2 \sum_{i=1}^{m} X_i^2 \tag{1}$$

It generates a Markovian semi-group P_t which acts on continuous function f on M

$$\frac{\partial}{\partial t} P_t f = L P_t f \; ; P_0 f = f \tag{2}$$

$P_t f \geq 0$ if $f \geq 0$. It is represented by a stochastic differential equation in Stratonovitch sense ([1])

$$P_t f(x) = E[f(x_t(x))] \tag{3}$$

where

$$dx_t(x) = \sum_{i=1}^{m} X_i(x_t(x)) dw_t^i \; ; x_0(x) = x \tag{4}$$

where $t \to w_t^i$ is a flat Brownian motion on $\mathbb{R}^m$ Classically, the Stratonovitch diffusion $x_t(x)$ can be approximated by its Wong–Zakai approximation.

Let $w_t^{n,i}$ be the polygonal approximation of the Brownian path $t \to w_t^n$ for a subdivision of $[0,1]$ of length n.

We introduce the random ordinary differential equation

$$dx_t^n(x) = \sum_{i=1}^m X_i(x_t^n(x))dw_t^{n,i}\ ; x_0^n(x) = x \tag{5}$$

Wong–Zakai theorem ([1]) states that if f is continuous

$$E[f(x_t^n(x))] \to E[f(x_t(x))] \tag{6}$$

We are motivated in this paper by an extension of (6) to higher order generators.

Let us consider the generator $L^k = (-1)^k \sum_I^m X_i^{2k}$. We suppose that the vector fields X_i span the tangent space of M in all point of M and that they are divergent free. L^k is an elliptic postive essentially self-adjoint operator [2] which generates a contraction semi-group P_t^k on $L^2(dx)$

Let $L^{f,k}$ be the generator on $\mathbb{R}^m$ $((w_i) \in \mathbb{R}^m)$. By [3], it generates a semi-group $P_t^{f,k}$ on $C(\mathbb{R}^m)$, the space of continuous functions on the flat space endowed with the uniform topology, which is represented by a heat-kernel:

$$P_t^{f,k}[f](w_0) = \int_{\mathbb{R}^m} f(w + w_0)p_t^{f,k}(w) \otimes dw_i \tag{7}$$

where $(w = (w_i))$. In [4], it is noticed that heuristically $P_t^{f,k}$ is represented by a formal path space measure $Q^{f,k}$ such that

$$\int_E f(w_t^k + w_0)dQ^{f,k}(w_.) = P_t^{k,f}(f)(w_0) \tag{8}$$

If we were able to construct a differential equation in the Stratonovitch sense

$$dx_t^k(x) = \sum_{i=1}^m X_i(x_t^k(x))dw_{t,i}^k\ ; x_0^k(x) = x \tag{9}$$

$$P_t^{f,k}(x) = \int f(x_t^k(x))dQ^{f,k} \tag{10}$$

These are formal considerations because in such a case the path measures are not defined. We will give an approach to (9) by showing that some convenient Wong–Zakai approximation converges to the semi-group. We introduce, according to [5] the Wong–Zakai operator

$$Q_t^k[f](x) = \int_{\mathbb{R}^m} f(x_i(w)(x))p_t^{f,k}(w) \otimes dw_i = \int_{\mathbb{R}^m} f(x(t^{1/2k}w)(x))p_1^{f,k}(w)dw \tag{11}$$

where

$$dx_1(w)(x) = \sum_{i=1}^m X_i(x_s(w))w_i ds\ ; x_0(w)(x) = x \tag{12}$$

As a first theorem, we state:

Theorem 1. *(Wong–Zakai) Let us suppose that the vector fields X_i commute. Then $(Q_{1/n}^k)^n(f)$ converge in $L^2(dx)$ to $P_1^k f$ if f is in $L^2(dx)$. This means that if we give f in $L^2(dx)$ that*

$$\|(Q_{1/n}^k)^n f - P_1^k f\|_{L^2(dx)} \to 0 \tag{13}$$

To give another example, we suppose that M is a compact Lie group G endowed with its normalized Haar measure dg and that the vector fields X_i are elements of the Lie algebra of G considered as right invariant vector fields. This means that if we consider the right action on $L^2(dg)$ R_{g_0}

$$f \to (g \to (f(gg_0)) \tag{14}$$

we have

$$R_{g_0}[X_i f](.) = X_i[R_{g_0} f](.) \tag{15}$$

We consider the rightinvariant elliptic differential operator

$$L^k = (-1)^k \sum_{i=1}^{m} X_i^{2k} \tag{16}$$

It is an elliptic positive essentially selfadjoint operator. By elliptic theory ([2]), it has a positive spectrum λ associated to eigenvectors f_λ. $\lambda \geq 0$ if λ belongs to the spectrum.

Theorem 2. *(Wong–Zakai) Let $f = \sum a_\lambda f_\lambda$ such that $\sum_\lambda |a_\lambda|^2 C^\lambda < \infty$ for all $C > 0$. Then $(Q^k_{1/n})^n(f)$ converges in $L^2(dg)$ to $P^k_1 f$. This means that*

$$\|(Q^k_{1/n})^n f - P^k_1 f\|_{L^2(dg)} \to 0 \tag{17}$$

We refer to the reviews [3,6] for the study of stochastic analysis without probability for non-markovian semi-groups.

Let us describe the main difference with the Wong–Zakai approximation of these semi-groups and the the traditional parametrix approximation of these by slicing the time. We work on $\mathbb{R}^d$ to simplify the exposition. Let be

$$L^a = \sum_{|(\alpha)| \leq p} a_{(\alpha)}(x) \frac{\partial^{(\alpha)}}{\partial x^{(\alpha)}} \tag{18}$$

where $(\alpha) = (\alpha_1, ...\alpha_d)$ is a multi-index on the flat space with length $|(\alpha)| = \sum \alpha_i$. We suppose that the function $a_{(\alpha)}(x)$ are smooth with bounded derivatives at each order. We consider if $y = (y_1, ..., y_d)$

$$y^{(\alpha)} = y_1^{\alpha_1} ... y_d^{\alpha_d} \tag{19}$$

We consider the symbol associated to the operator

$$a(x, \xi) = \sum a_{(\alpha)}(x)(i\xi)^{(\alpha)} \tag{20}$$

We suppose that we are in an elliptic situation: for all x

$$|a(x, \xi)| \geq C|\xi|^p - C \tag{21}$$

We suppose that the operator is positive bounded below. We can consider in this case the parabolic equation starting from $f \in L^2(dx)$

$$\frac{\partial}{\partial t} P_t^a f = -L^a P_t^a f \; ; \; P_0^a f = f \tag{22}$$

It has a unique solution. The parametrix method consist to freeze the starting point x by considering the family of operators

$$L_x f(y) = \sum_{|(\alpha)| \leq p} a_{(\alpha)}(x) \frac{\partial^{(\alpha)}}{\partial y^{(\alpha)}} f(y) \tag{23}$$

We consider the family of non-markovian semi-groups P_t^x satisfying the parabolic equation

$$\frac{\partial}{\partial t} P_t^x f(y) = -L_x P_t^x f(y) \; ; \; P_0^x f(y) = f(y) \tag{24}$$

We introduce the kernel

$$f \to Q_t^p f(x) = P_t^x f(x) \tag{25}$$

Parametrix method states that

$$(Q_{1/n}^p)^n f \to P_1^a \tag{26}$$

in $L^2(dx)$ when $n \to \infty$ At the point of view od path integrals, parametrix is related to the formal path integrals of Klauder (see [4] for a rigorous approach). Consider the Fourier transform $\hat{f}$ of a function which belongs to $L^2(dx)$. We get

$$(\hat{L_x f})(\xi) = \sum a_{(\alpha)}(x)(i\xi)^{(\alpha)} \hat{f}(\xi) \tag{27}$$

such that

$$P_t^{\hat{x}} f(\xi) = \exp[-ta(x,\xi)]\hat{f}(\xi) \tag{28}$$

By using the inverse of the Fourier transform,the lefthandside of (28) gives an approximation of Klauder path integral on the phase space.

Hamiltonian path integrals are not well defined as measures. Let us consider the case of an order 2 differential operator

$$L^a = \sum_{0<|(\alpha)|\le 2} a_{(\alpha)}(x) \frac{\partial^{(\alpha)}}{\partial x^{(\alpha)}} \tag{29}$$

We suppose that

$$\sum_{|(\alpha)|=2} a_\alpha(x)(i\xi)^{(\alpha)} = 1/2 \sum_i^m < X_i(x), \xi >^2 \tag{30}$$

where $x \to X_i(x)$ are smooth vector fields. Moreover, the part of order 1 of the operator define a smooth vector field X_0. In such a case, $P_t^a(x)$ is represented by an Itô stochastic differential equation starting from x

$$\delta x_t(x) = \sum_i^m X_i(x_t(x))\delta w_t^i + X_0(x_t(x))dt \tag{31}$$

$t \to (w_t^i)$ is a flat Brownian motion on $\mathbb{R}^m$. We have

$$P_t^a f(s) = E[f(x_t(x))] \tag{32}$$

Itô stochastic differential equations can be approximated by the Euler scheme if we consider a subdivision $[t_{k-1}, t_k]$ of $[0,1]$ of mesh $1/n$. t^n is the biggest t_k smaller than t. The approximation of the Itô equation is

$$x_t^n(x) = x_{t^n}^n(x) = \sum_i^m X_i(x_{t^n}^n)(w_t^i - w_{t^n}^i) + X_0(x_{t^n}^n)(t - t^n) \tag{33}$$

starting from x, by stochastic calculus ([1]), the law of $t \to x_t^n(x)$ for the uniform norm tends to the law of $t \to x_t(x)$. In particular, if f is a bounded continuous function,

$$E[f(x_1^n(x)] \to E[f(x_1(x))] \tag{34}$$

when $n \to \infty$. However, in such a case,

$$E[f(x_1^n(x))] = (Q_{1/n}^p)^n f(x) \tag{35}$$

The Calculus on flat Brownian motion shows

$$P^x_{1/n} f(x) = E[f(x^n_{1/n}(x))] \tag{36}$$

Let $t \to h_t = (h^1_t, ..., h^m_t)$ a finite energy path in $\mathbb{R}^m$ starting from 0. We consider the energy norm

$$\|h\|^2 = \sum_{i=1}^{m} \int_0^1 |d/dth^i_t|^2 dt \tag{37}$$

At the formal path integral point of view, the law of the flat Brownian motion $t \to w_t$ is the Gaussian law

$$1/Z \exp[-\|h\|^2/2] dD(h) \tag{38}$$

where $dD(h)$ is the formal Lebesgue measure on the finite energy paths (which does not exist) and z a normalized constant, called the partition function, which is infinite and not well defined.

We introduce the solution of the ordinary differential equation $x^h_t(x)$ starting from x

$$dx^h_t(x) = \sum_{i=1}^{m} X_i(x^h_t(x)) dh^i_t \tag{39}$$

The Wong–Zakai approximation explains that at a formal point of view, the solution of the Stratonovitch Equation (4) $x_1(x)$ can be seen as $x^h_1(x)$ where h is chosen according the formal Gaussian measure (38). This remark is not suitable for Itô equation.

Bismutian procedure [7] is the use of the implicit function theorem for $h \to x^h_1(x)$ to study the heat-kernel associated to the semi-group P_t. It was translated in semi-group theory in [5] by introducing some Wong–Zakai kernels associated to the semi-group generated by L. The long term motivation of this paper is to implement Bismut procedure in big order operators of Hoermander's type.

Proof of Theorem 1. L^k is an elliptic positive operator. By elliptic theory [2], it has a discete spectrum λ associated to normalized eigenfunctions f_λ. Since $\int_{\mathbb{R}^m} |p^{f,k}_1(w)|^2 dw < \infty$, $Q^k_{1/n}$ is a bounded operator on $L^2(dx)$. Moreover

$$Q^k_{1/n} f = \sum a_\lambda Q^k_{1/n} f_\lambda \tag{40}$$

if

$$f = \sum a_\lambda f_\lambda \tag{41}$$

The main remark is that we can compute explicitely $Q^k_{1/n} f_\lambda$. We put $t = 1/n$. Formaly

$$f_\lambda(x(t^{1/2k} w))(x) = \sum_{n'} 1/n'!(\sum_m X_i w_i t^{1/2k}))^{n'}(f_\lambda)(x) \tag{42}$$

Namely, by ellipticity and because the vector fields X_i commute with L^k, we can conclude that the L^2-norm of $X^{\alpha_1}_{i_1} X^{\alpha_2}_{i_2} ... X^{\alpha_l}_{i_l} f_\lambda$ has a bound in $\lambda^{\sum \alpha_i/2k} C^{\sum \alpha_i}$ in order to deduce that the series in (42) converges.

Let us show how to prove this estimate. We have

$$L^k f_\lambda = \lambda f_\lambda \tag{43}$$

Since X_i commutes with L^k, we have

$$X_i L^k f_\lambda = L^k X_i f_\lambda \tag{44}$$

Therefore $X_i f_\lambda$ is still an eigenfunction associate to λ. Therefore, X_i is a linear operator on the eigenspace E_λ associate to λ which is of finite dimension by elliptic theory. By Garding inequality [2]

$$\|X_i f_\lambda\|_{L^2(G)} \leq C\|f_\lambda\|_{L^2(dx)} + \|(L^k + C)^{1/2k} f_\lambda\|_{L^2(dx)} \tag{45}$$

We use for that $(L^k + C)^{1/2k}$ is an elliptic pseudo-differential operator of order 1 (see the end of this paper). E_λ is an eigenspace for $(L^k + C)^{1/2k}$ associated to the eigenvalue $(\lambda + C)^{1/2k}$. Therefore X_i is a linear operator on E_λ with norm smaller than $C(\lambda^{1/2k} + 1)$.

It is enough to compute

$$1/n'! \int_{\mathbb{R}^m} (\sum X_i w_i t^{1/2k}))^{n'} f_\lambda(x) p_1^{k,f}(w)dw = B_{n'} \tag{46}$$

The main remark (see the end of this paper) is if one of the l_i is not a multiple of $2k$, we have

$$\int_{\mathbb{R}^m} w_1^{l_1} ... w_m^{l_m} p_1^{k,f}(w)dw = 0 \tag{47}$$

On the other hand, by using the semi-group properties of $P_t^{k,f}$, we have

$$\int_{\mathbb{R}^m} w_1^{2kl_1} ... w_m^{2kl_m} p_1^{k,f}(w)dw = \frac{(2kl_1)!}{l_1!} ... \frac{(2kl_m)!}{l_m!} \tag{48}$$

We ignore some immediated problems of signs. Therefore, $B_{n'} = 0$ if n' is not a multiple of $2k$ and is equal because the vector field commute, ,if $n' = 2kl'$ to

$$\frac{1}{(2kl)'!} \sum X_1^{2kl'1} ... X_m^{2kl'_m} \frac{(2kl_1)!}{l_1!} ... \frac{(2kl_m)!}{l_m!} \frac{(2kl')!}{(2kl'_1)! ... (2kl'_m)!} f_\lambda = 1/l'!(L^k)^{l'} f_\lambda \tag{49}$$

We deduce that

$$Q_{1/n}^k f_\lambda = \exp[-1/n\lambda] f_\lambda \tag{50}$$

and that

$$(Q_{1/n}^k)^n f_\lambda = \exp[-\lambda] f_\lambda \tag{51}$$

such that

$$(Q_{1/n}^k)^n f = \exp[-L^k] f \tag{52}$$

if $f = \sum a_\lambda f_\lambda$. □

In such a case, the Wong–Zakai approximation is exact. It is analog to the classic result for diffusions of Doss–Sussmann ([8,9]). The Stratonovitch diffusion in this case satisfy

$$x_1(x) = \exp[\sum_i^m X_i w_1^i](x) \tag{53}$$

The map $y \to \exp[\sum_i^m X_i y_t^i](x)$ is defined as follows. We consider the ordinary differential equation issued from x

$$dx_t^y(x) = \sum_{i=1}^m X_i(x_t^y(x)) y^i dt \tag{54}$$

$y = (y^1, ..., y^m)$ belongs to $\mathbb{R}^m$ and we put

$$x_1^y(x) = \exp[\sum_i^m X_i y_t^i](x) \tag{55}$$

Proof of Theorem 2. Let E_λ be the space of eigenfunctions associated to the eigenvalue λ of L^k. Since L^k commutes with the right action of G, E_λ is a representation for the right action of G ([10]). Therefore rightinvariant vector fields act on E_λ. If Z is a rightinvariant vector field, we can consider the L^2 norm of Zf_λ for f_λ belonging to E_λ. We remark that $(L^k + C)^{1/2k}$ is an elliptic pseudodifferential operator of order 1 (C is strictly positive). By Garding inequality [2],

$$\|Zf_\lambda\|_{L^2(dg)} \leq C\|f_\lambda\|_{L^2(dg)} + \|(L^k + C)^{1/2k} f_\lambda\|_{L^2(dg)} \tag{56}$$

f_λ is an eigenfunction associated to $(L^k + C)^{1/2k}$ and the eigenvalue $(\lambda + C)^{1/2k}$.

Let us consider a polynomial $X_{i_1}^{\alpha_1}...X_{i_l}^{\alpha_l} = Z_l$. It acts on E_λ and is norm is bounded by $((\lambda + C)^{1/2k} + C)^{\sum \alpha_i}$ for the L^2 norm.

From that we deduce that if f_λ is an eigenfunction associated to λ of L^k that the series

$$\sum_l \frac{(X_i t^{1/2k} w_i)^l}{l!} f_\lambda \tag{57}$$

converges and is equal to $f_\lambda(x(t^{1/2k}w)(x))$. By distinguishing if w is big or not and using (46), we see that if $l \neq 2kl'$

$$\int_{\mathbb{R}^m} (\sum_{i=1}^m X_i t^{1/2k} w_i)^{l'} f_\lambda p_1^{f,k}(w)dw = 0 \tag{58}$$

Moreover, by (47) and (48)

$$\frac{1}{(2kl')!}\int_{\mathbb{R}^m} (\sum_{i=1}^m X_i t^{1/2k} w_i)^{2kl'} f_\lambda p_1^{f,k}(x)dw =$$

$$\frac{t^{l'}}{(2kl')!}\int_{\mathbb{R}^m} \sum X_{\alpha_1}...X_{\alpha_{2kl'}} f_\lambda w_1^{2kl'_1}..w_m^{2kl'_m} p_1^{f,k}(w)dw \tag{59}$$

where $2kl'_j$ is the number of of α_i equal to j. By using (47) and (48), we recognize in (59)

$$\frac{t^{l'}}{(2kl')!}\sum_{\alpha_i} X_{\alpha_1}...X_{\alpha_{2kl'}} f_\lambda \frac{(2kl'_1)!}{l'_1!}...\frac{(2kl'_m)!}{l'_m!} \tag{60}$$

For $l' = 1$, we recognize tL. Let us compute the L^2 norm of the previous element. It is bounded by

$$\frac{t^{l'}}{(2kl')!}\sum_{\alpha_i} (\lambda^{1/2k} + C)...(\lambda^{1/2k} + C)\frac{(2kl'_1)!}{l'_1!}...\frac{(2kl'_m)!}{l'_m!} \tag{61}$$

For $l' = 1$, we recognize tL.

We recognize in the previous sum

$$\frac{t^{l'}}{(2kl')!}\sum \frac{(2kl')!}{(2kl'_1)!...(2kl'_m)!)}\frac{(2kl'_1)!}{l'_1!}...\frac{(2kl'_m)!}{l'_m!}(\lambda^{1/2k} + C)^{2kl'} \tag{62}$$

We deduce a bound of the operation given by (59) in $\frac{t^{l'}C^{2kl'}}{(l')!}(\lambda + C)^{l'}$.

By the same argument, we have a bound of $\frac{t^{l'}}{l'!}(L^k)^{l'}$ on E_λ in $\frac{t^{l'}}{l'!}C^{l'}(\lambda + C)^{l'}$.

In order to conclude, we see that on E_λ

$$Q_t^k = \exp[-\lambda t]Id + \sum_{l'>1}\frac{t^{l'}}{l'!}Q_\lambda^{l',t} \tag{63}$$

where $Q_\lambda^{l',t}$ has a bound on E_λ in $C^{l'}(\lambda+C)^{l'}$. We deduce that Q_t^k acts on E_λ by

$$\exp[-\lambda t]Id + t^2 Q_t^\lambda = R_t^\lambda \tag{64}$$

where the norm on E_λ of Q_t^λ is smaller that $C\exp[C\lambda t]$.

However, if $f = \sum a_\lambda f_\lambda$

$$(Q_{1/n}^k)^n f = \sum a_\lambda (R_{1/n}^\lambda)^n f_\lambda \tag{65}$$

Moreover

$$\begin{aligned}\|(Q_t^k)f\|_{L^2(dg)} = \int_G |\int_{\mathbb{R}^m} f(x(t^{1/2k}w)(g)p_1^{f,k}(w)dw|^2 dg \leq \\ C\int_G dg \int_{\mathbb{R}^m} |f(x(t^{1/2k}w)(g)|^2 |p_1^{f,k}(w)|^2 dw \leq \\ C\int_{\mathbb{R}^m} |p_1^{f,k}(w)|^2 dw \int_G |f(x(t^{1/2k}w)(g)|^2 dg\end{aligned} \tag{66}$$

However,

$$\int_G |f(x(t^{1/2k}w)(g)|^2 dg = \|f\|^2_{L^2(dg)} \tag{67}$$

because the vector fields are without divergence.

To conclude, we remark that the series

$$\sum_\lambda |a_\lambda|^2 \|(R_{1/n}^\lambda)^n - \exp[-\lambda])f_\lambda\|^2_{L^2(dg)} \tag{68}$$

tends to 0 when $n \to \infty$. Namely each term is bounded by $|a_\lambda|^2 C^\lambda$ and tends simply to 0. The result arises by the dominated convergence theorem. □

2. Some Results on Linear Operators

We work on functions f with values in $\mathbb{R}$, but it is possible to work in $\mathbb{C}$. We refer to [2] for details.

Let us begin to work on $\mathbb{R}^d$. We consider a smooth function on $\mathbb{R}^d \times \mathbb{R}^d$ called a symbol $a(.,.)$ such that

$$\inf_x |\frac{\partial^{(\alpha)}}{\partial x^{(\alpha)}} \frac{\partial^{(\beta)}}{\partial x^{(\beta)}} a(x,\xi)| \leq C|\xi|^{p-|(\beta)|} + C \tag{69}$$

We define the operator L^a associated to the symbol a by

$$L^a f(x) = \int_{\mathbb{R}^d} a(x,\xi)\hat{f}(\xi)\exp[2\pi i < x,\xi >]d\xi \tag{70}$$

acting on smooth function with bounded derivatives at each order. We suppose

$$\inf_x |a(x,\xi)| \geq C|\xi|^p - C \tag{71}$$

We sat the operator L^a is a pseudodifferential operator of order p. This notion is invariant if we do a diffeomorphism of $\mathbb{R}^d$ with bounded derivatives at each order. This explain that we can define an elliptic operator on a smooth compact manifold M. On each space of the tangent bundle, we introduce a metric strictly positive which depends smoothly on $x \in M$. We say that the manifold is equipped of a Riemannian structure. In such a case, we can introduce the analog of the normalized Lebesgue measure which is called the Riemannian measure dx. We say that L^a is symmetric if

$$\int_M f_1 L^a f_1 dx = \int_M f_2 L^a f_1 dx \tag{72}$$

It is called positive if

$$\int_M f L^a f dx \geq 0 \tag{73}$$

and strictly positive if there exists $C > 0$ such that for all f

$$\int_M f L^a f dx \geq C\|f\|^2_{L^2(dx)} \tag{74}$$

If we consider vector fields X on M as differential operators, we can consider their adjoint:

$$\int_M f_1 X f_2 dx = -\int_M f_2 X f_1 dx + \int_M f_1 f_2 div X dx \tag{75}$$

such that the operators considered in this work are symmetric. Moreover, there are alliptic of order $2k$.

We can consider the eigenvalue problem: for what λ, there exists a $f_\lambda \in L^2(dx)$ not equal to 0 such that

$$L^a f_\lambda = \lambda f_\lambda \tag{76}$$

In the compact case, this problem is solved for a positive symmetric elliptic pseudodifferential operator. λ belongs to a discrete subset of $\mathbb{R}^+$ called the spectrum of L^a. The solutions of (76) constitute a linear subset of finite dimension E_λ which constitutes an orthonormal decomposition of $L^2(dx)$. Each element of E_λ is smooth.

If L^a is a strictly positive elliptic pseudodifferential operator of order p, we can define is power $(L^a)^\alpha$ for any positive real α ([11]). It is still a strictly positive pseudodifferential of order $p\alpha$. The eigenspaces are the same, but associated to the eigenvalue λ^α. Therefore $(L^a)^{1/p}$ is a pseudodifferential operator of order 1.

If L^a is a strictly positive pseudo differential operator of order 1, it satisfy to the Garding inequality

$$\|X_i f_\lambda\|_{L^2(G)} \leq C\|f_\lambda\|_{L^2(dx)} + \|(L^a)^{1/2k} f_\lambda\|_{L^2(dx)} \tag{77}$$

We refer to [2] for this material.

Let us look to the case of non-compact set by looking the driving semi-group $P_t^{k,f}$. First of all, by classic results (see [12] for instance)

$$|p_1^{f,k}(w)| \leq C\exp[-|w|^\alpha] \tag{78}$$

Let us recall that the main difference in this work with respect of the case of diffusion is that the flat heat-kernel $p_1^{f,k}(w)$ can change of sign. Moreover, if f is a polynomial, the series

$$\exp[-L^{f,k}]f(w) = \sum(-1)^{n'}/n'!(L^{f,k})^{n'} f(w) \tag{79}$$

converges and only in fact a finite numbers of terms are different to zero. This shows

$$P_t^{f,k}(f)(0) = \exp[-L^{f,k}]f(0) = \sum(-1)^{n'}/n'!(L^{f,k})^{n'} f(0) = \int_{\mathbb{R}^m} f(w) p_t^{f,k}(w) dw \tag{80}$$

This last formula explains (47) and (48) modulo some minor problems of signs.

3. Conclusions

We continue in this paper our previous works (see [6,11] for reviews) which implement stochastic analysis in non-Markovian semi-groups (they do not preserve positivity). The traditional Wong–Zakai approximation of Stratonovitch diffusion is interpreted in this framework, for the case of higher order elliptic operators under Hoermander's form. Computations are done by using the global property of the generator. This gives a new approximation than the parametrix one, which was done by freezing the starting point in the generator.

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