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Multiple Solutions for a Class of Nonlinear Fourth-Order Boundary Value Problems

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Abstract: This paper is concerned with multiple solutions for a class of nonlinear fourth-order boundary value problems with parameters. By constructing a special cone and applying fixed point index theory, the multiple solutions for the considered systems are obtained under some suitable assumptions. The main feature of obtained solutions $(u(t),\ v(t))$ is that the solution $u(t)$ is positive, and the other solution $v(t)$ may change sign. Finally, two examples with continuous function f_1 being positive and f_2 being semipositone are worked out to illustrate the main results.

Keywords: multiple solutions; fixed point theory; boundary value problems

1. Introduction

It is well known that the subject of the existence of solutions to numerous boundary value problems (BVP) for differential equations such as second-order [1–3], fourth-order [4–6], even fractional order BVP [7–11] has gained considerable attention and popularity. A growing number of outstanding progress has been made in the theory of such BVP in the last decades due mainly to their extensive applications in the fields of hydrodynamics, nuclear physics, biomathematics, chemistry, and control theory. For further details, please see References [12–29] and references therein.

It is noted that fourth-order boundary value problems have an important application in practical problems, that is, they can be used to describe the deformation of elastic beam, see References [30–33] and references therein. For example, in Reference [32], by means of the theory of fixed point index on cone, Y. Li investigated the following boundary value problems of fourth-order ordinary differential equation

$$\begin{cases} u^{(4)}(t)+\beta u''(t)-\alpha u(t)=f(t,u), & 0<t<1; \\ u(0)=u(1)=u''(0)=u''(1)=0, \end{cases}$$

where $f\in C([0,1]\times\mathbb{R}^+,\mathbb{R}^+)$, $\alpha,\beta\in\mathbb{R}$ and satisfy $\beta<2\pi^2, \alpha\geq-\beta^4/4, \alpha/\pi^4+\beta/\pi^2<1$. By constructing a special cone, the existence of at least one positive solution was obtained under some suitable assumptions.

Recently, in Reference [33], Q. Wang and L. Yang studied the following boundary value problems

$$\begin{cases} u^{(4)}(t)+\beta_1 u''(t)-\alpha_1 u(t)=f_1(t,u(t),v(t)), & 0<t<1; \\ v^{(4)}(t)+\beta_2 v''(t)-\alpha_2 v(t)=f_2(t,u(t),v(t)), & 0<t<1; \\ u(0)=u(1)=u''(0)=u''(1)=0; \\ v(0)=v(1)=v''(0)=v''(1)=0, \end{cases} \tag{1}$$

where $f_1, f_2 \in C([0,1] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$, and $\beta_i, \alpha_i \in \mathbb{R}(i = 1,2)$ satisfy the following conditions:

$$\beta_i < 2\pi^2, \ -\beta_i/4 \le \alpha_i, \ \alpha_i/\pi^4 + \beta_i/\pi^2 < 1. \tag{2}$$

These conditions involve a two-parameter non-resonance condition. By constructing two classes of cones and using the fixed point theory, the existence of at least one positive solution was obtained. It is remarkable that the premise of this establishment of the result in Reference [33] is that the nonlinear term f_2 must be positive.

We point out that there are some limitations in those existing results of fourth-order boundary value problems. All solutions obtained in the above references are positive, and moreover, the corresponding conclusions in them are not valid when the nonlinear term is allowed to be non-positive. Considering that two variables u and v in the nonlinear term usually have some connections in many practical problems, there is no description of the relationship between them in the aforementioned papers. It is an interesting problem to seek such solutions for BVP (1) that one variable is positive and the other may be non-positive under the assumptions that nonlinearity may be semipositoned, and some connection will be added between these two variables. As far as we know, there is no paper considering such problem for BVP (1). The purpose of the present paper is to fill this gap.

This paper, motivated by all the above mentioned discussions, investigates the multiple solutions for BVP (1) under the more different conditions compared with Reference [33]. By constructing a very special cone and using the fixed point index theory, the existence and multiplicity results of solutions to (1) are obtained when $\beta_i, \alpha_i \in \mathbb{R}$ $(i = 1,2)$ satisfy the conditions (2), $f_1 \in C([0,1] \times \mathbb{R}^+ \times \mathbb{R}, \mathbb{R}^+)$, and $f_2 \in C([0,1] \times \mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$.

The nonlinear term f_2 is allowed to change sign by contrast, $f_2 \in C([0,1] \times \mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$. A relationship is imposed between two variables u, v in nonlinear terms, which is that the variable v is controlled by u. In obtained solution (u, v), the component u is positive, but the component v is allowed to be negative in comparison with Reference [33].

The rest of this paper is organized as follows—Section 2 contains some background materials and preliminaries. The main results will be given and proved in Section 3. Finally, in Section 4, two examples are given to support our results.

2. Background Materials and Preliminaries

The basic space used in this paper is $E := C[0,1] \times C[0,1]$. It is a Banach space endowed with the norm $\|(u,v)\| = \max\{\|u\|, \|v\|\}$ for $(u,v) \in E$, where $\|u\| = \max_{t\in[0,1]} |u(t)|$, $\|v\| = \max_{t\in[0,1]} |v(t)|$. Under the condition (2), as in Reference [32], let

$$\xi_{i,1} = \frac{-\beta_i + \sqrt{\beta_i^2 + 4\alpha_i}}{2}, \ \xi_{i,2} = \frac{-\beta_i - \sqrt{\beta_i^2 + 4\alpha_i}}{2}, \ (i = 1,2),$$

and let $G_{i,j}(t,s)(i,j = 1,2)$ be the Green's function of the linear boundary value problem

$$\begin{cases} -u_i''(t) + \xi_{i,j} u_i(t) = 0, \ \ 0 < t < 1; \\ u_i(0) = u_i(1) = 0, \ \ i,j = 1,2. \end{cases}$$

Then for $h_i \in C[0,1]$, the solution of the following nonlinear boundary value problem

$$\begin{cases} u_i^{(4)}(t) + \beta_i u_i''(t) - \alpha_i u_i = h_i(t), \ \ 0 < t < 1; \\ u_i(0) = u_i(1) = u_i''(0) = u_i''(1) = 0, \ \ i,j = 1,2 \end{cases}$$

can be expressed as

$$u_i(t) = \int_0^1 \int_0^1 G_{i,1}(t,\tau)G_{i,2}(\tau,s)h_i(s)dsd\tau, \ \ t \in [0,1].$$

Lemma 1. *The function $G_{i,j}(t,s)(i=1,2)$ has the following properties:*
(1) $G_{i,j}(t,s) > 0$ for $t,s \in (0,1)$;
(2) $G_{i,j}(t,s) \le C_{i,j}G_{i,j}(s,s)$ for $t,s \in [0,1]$, where $C_{i,j} > 0$ is a constant;
(3) $G_{i,j}(t,s) \ge \delta_{i,j}G_{i,j}(t,t)G_{i,j}(s,s)$ for $t,s \in [0,1]$, where $\delta_{i,j} > 0$ is a constant;
(4) $G_{2,j}(t,s) \le N_jG_{1,j}(t,s)$ for $t,s \in [0,1]$, where $N_j > 0$ is a constant.

Proof of Lemma 1. (1)–(3) can be seen from Reference [32]. In addition, by careful calculation and Lemma 2.1 in Reference [32], it is not difficult to prove that $N_j := \sup\limits_{0<t,s<1} \frac{G_{2,j}(t,s)}{G_{1,j}(t,s)} < +\infty$. Immediately, (4) is derived. □

The main tool used here is the following fixed-point index theory.

Lemma 2 ([34])**.** *Let E_1 be a Banach space and P be a cone in E_1. Denote $P_r = \{u \in P : \|u\| < r\}$ and $\partial P_r = \{u \in P : \|u\| = r\}$ $(\forall r > 0)$. Let $T : P \to P$ be a complete continuous mapping, then the following conclusions are valid.*

(1) If $\mu Tu \ne u$ for $u \in \partial P_r$ and $\mu \in (0,1]$, then $i(T,P_r,P) = 1$;

(2) If $\inf\limits_{u\in\partial P_r} \|Tu\| > 0$ and $\mu Tu \ne u$ for $u \in \partial P_r$ and $\mu \ge 1$, then $i(T,P_r,P) = 0$.

3. Main Results

In this section, we shall establish the existence and multiplicity results, which is based on the fixed point index theory. For this matter, first we define the mappings $T_1, T_2 : E \to C[0,1]$, and $T : E \to E$ by

$$T_1(u,v)(t) = \int_0^1 \int_0^1 G_{1,1}(t,\tau)G_{1,2}(\tau,s)f_1(s,u(s),v(s))dsd\tau,$$

$$T_2(u,v)(t) = \int_0^1 \int_0^1 G_{2,1}(t,\tau)G_{2,2}(\tau,s)f_2(s,u(s),v(s))dsd\tau,$$

$$T(u,v)(t) = (T_1(u,v)(t), T_2(u,v)(t)), \quad \forall (u,v) \in E.$$

Then, BVP (1) in operator forms becomes

$$(u,v) = T(u,v). \tag{3}$$

By (3), one can easily see that the existence of solutions for BVP (1) is equivalent to the existence of nontrivial fixed point of T. Therefore, we need to find only the nontrivial fixed point of T in the following work.

Subsequently, for simplicity and convenience, set

$$M_{i,j} = \max_{t\in[0,1]} G_{i,j}(t,t), \quad C_i = \int_0^1 G_{i,1}(\tau,\tau)G_{i,2}(\tau,\tau)d\tau, \quad \text{and } \lambda_i = \pi^4 - \beta_i\pi^2 - \alpha_i.$$

Then, $M_{i,j}, C_i$, and $\lambda_i (i,j = 1,2)$ are positive numbers.

Now let us list the following assumptions satisfied throughout the paper.

(H1) $f_1 \in C([0,1] \times \mathbb{R}^+ \times \mathbb{R}, \mathbb{R}^+)$, $f_2 \in C([0,1] \times \mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$, and there exists $N_3 > 0$ such that $|f_2(t,u,v)| \le N_3 f_1(t,u,v)$ for $(t,u,v) \in [0,1] \times \mathbb{R}^+ \times \mathbb{R}$.

(H2) $\lim\limits_{\substack{|v|\le Nu\\ u\to 0^+}} \sup \max\limits_{t\in[0,1]} \frac{f_1(t,u,v)}{u} < \lambda_1 < \lim\limits_{\substack{|v|\le Nu\\ u\to +\infty}} \inf \min\limits_{t\in[0,1]} \frac{f_1(t,u,v)}{u}$.

(H3) $\lim\limits_{\substack{|v|\le Nu\\ u\to 0^+}} \inf \min\limits_{t\in[0,1]} \frac{f_1(t,u,v)}{u} > \lambda_1 > \lim\limits_{\substack{|v|\le Nu\\ u\to +\infty}} \sup \max\limits_{t\in[0,1]} \frac{f_1(t,u,v)}{u}$.

In addition, for the sake of obtaining the nontrivial fixed point of operator T, let

$$P = \{(u,v) \in E : u(t) \ge \sigma(t)\|u\| \text{ and } |v(t)| \le Nu(t), \forall t \in [0,1]\},$$

where $\sigma(t) = \dfrac{\delta_{1,1}\delta_{1,2}C_1}{C_{1,1}C_{1,2}M_{1,1}} G_{1,1}(t,t)$ and $N = N_1N_2N_3$. N_1, N_2, and N_3 are defined in Lemma 1 and (H1), respectively.

Obviously, P is a nonempty, convex, and closed subset of E. Furthermore, one can prove that P is a cone of Banach space E.

For convenience, set

$$\Lambda_Y = \{(u,v) \in \mathbb{R}^+ \times \mathbb{R} : u \in Y \subset \mathbb{R}^+, |v| \le Nu\},$$

$$P_r = \{(u,v) \in P : \|u\| < r\},$$

$$\partial P_r = \{(u,v) \in P : \|u\| = r\},$$

$$\bar{P}_r = \{(u,v) \in P : \|u\| \le r\}.$$

It is not difficult to see that P_r is a relatively open and bounded set of P for each $r > 0$.

Lemma 3. *To calculate the fixed point index of T in P_r, we first need to prove the following result. Assume that (H_1) hold. Then $T : P \to P$ is completely continuous, and $T(P) \subset P$.*

Proof of Lemma 3. For $(u,v) \in P$, by virtue of Lemma 1, one can easily obtain that

$$\begin{aligned} T_1(u,v)(t) &= \int_0^1 \int_0^1 G_{1,1}(t,\tau)G_{1,2}(\tau,s)f_1(s,u(s),v(s))dsd\tau \\ &\ge \frac{\delta_{1,1}\delta_{1,2}C_1}{C_{1,1}C_{1,2}M_{1,1}} G_{1,1}(t,t)\|T_1(u,v)\| = \sigma(t)\|T_1(u,v)\|, \quad \forall t \in [0,1]. \end{aligned}$$

Moreover, (H1) together with Lemma 1 guarantees that

$$\begin{aligned} |T_2(u,v)(t)| &= |\int_0^1 \int_0^1 G_{2,1}(t,\tau)G_{2,2}(\tau,s)f_2(s,u(s),v(s))dsd\tau| \\ &\le N_3 \int_0^1 \int_0^1 G_{2,1}(t,\tau)G_{2,2}(\tau,s)f_1(s,u(s),v(s))dsd\tau \\ &\le N_1N_2N_3 \int_0^1 \int_0^1 G_{1,1}(t,\tau)G_{1,2}(\tau,s)f_1(s,u(s),v(s))dsd\tau \\ &= N|T_1(u,v)(t)|. \end{aligned}$$

Therefore, $T(u,v) \in P$, namely, $T(P) \subset P$. In addition, since f_1, f_2, and $G_{i,j}$ are continuous, one can deduce that T is completely continuous by using normal methods such as Arscoli-Arzela theorem, and so forth. □

Now we are in a position to prove our main results in the following.

Theorem 1. *Under the assumptions (H1) and (H2), the BVP (1) admits at least one nontrivial solution.*

Proof of Theorem 1. To obtain the nontrivial solution for BVP (1), we will choose a bounded open set $P_{R_1} \setminus \bar{P}_{r_1}$ in cone P and calculate the fixed point index $i(T, P_{R_1} \setminus \overline{P_{r_1}}, P)$. For this, the proof of Theorem 1 will be carried out in three steps.

First, notice that by (H2), there exist $\varepsilon \in (0,1)$ and $r_1 > 0$ such that

$$f_1(t,u,v) \le (1-\varepsilon)\lambda_1 u \quad \forall t \in [0,1], (u,v) \in \Lambda_{[0,r_1]}. \tag{4}$$

We claim that

$$\mu T(u,v) \ne (u,v),\ \forall \mu \in (0,1],\ (u,v) \in \partial P_{r_1}. \tag{5}$$

To this end, suppose on the contrary that there exist $\mu_0 \in (0,1]$ and $(u_0,v_0) \in \partial P_{r_1}$ such that

$$\mu_0 T(u_0,v_0) = (u_0,v_0).$$

Therefore, (u_0,v_0) satisfies the following differential equation

$$\begin{cases} u_0^{(4)}(t) + \beta_1 u_0''(t) - \alpha_1 u_0(t) = f_1(t,u(t),v(t)), \quad 0 < t < 1; \\ u_0(0) = u_0(1) = u_0''(0) = u_0''(1) = 0; \end{cases} \tag{6}$$

It follows from (4) and (6) that

$$u_0^{(4)}(t) + \beta_1 u_0''(t) - \alpha_1 u_0(t) \le f_1(t,u_0(t),v_0(t)) \le (1-\varepsilon)\lambda_1 u_0(t).$$

Multiplying the above inequality by $\sin(\pi t)$ and then integrating from 0 to 1, one can easily get

$$\int_0^1 \lambda_1 u_0(t)\sin(\pi t)dt \le (1-\varepsilon)\int_0^1 \lambda_1 u_0(t)\sin(\pi t)dt.$$

Noticing that $\int_0^1 \lambda_1 u_0(t)\sin(\pi t)dt > 0$, we obtain a contradiction.

Second, from (H2), there exist $\varepsilon > 0$ and $m > 0$ such that

$$f_1(t,u,v) \ge (1+\varepsilon)\lambda_1 u \quad \forall t \in [0,1], (u,v) \in \Lambda_{[m,+\infty)}. \tag{7}$$

Set $C := \max\limits_{\substack{t\in[0,1] \\ (u,v)\in\Lambda_{[0,m]}}} |f_1(t,u,v) - (1+\varepsilon)\lambda_1 u| + 1$. Then one can easily find that

$$f_1(t,u,v) \ge (1+\varepsilon)\lambda_1 u - C,\ \forall t \in [0,1],\ (u,v) \in \Lambda_{\mathbb{R}^+}. \tag{8}$$

Now, we will show that there exists $R_1 > r_1$ such that

$$\inf_{(u,v)\in\partial P_{R_1}} \|T(u,v)\| > 0 \ \text{ and } \ \mu T(u,v) \ne (u,v),\ \forall \mu \ge 1,\ (u,v) \in \partial P_{R_1}. \tag{9}$$

Suppose, on the contrary, that there exist $\mu_0 \ge 1$ and $(u_0,v_0) \in \partial P_{R_1}$ such that $\mu_0 T(u_0,v_0) = (u_0,v_0)$. Combining (6) with (8), we immediately get

$$u_0^{(4)}(t) + \beta_1 u_0''(t) - \alpha_1 u_0(t) \ge f_1(t,u_0(t),v_0(t)) \ge (1+\varepsilon)\lambda_1 u_0(t) - C.$$

Hence,

$$\int_0^1 \lambda_1 u_0(t)\sin(\pi t)dt \ge (1+\varepsilon)\int_0^1 \lambda_1 u_0(t)\sin(\pi t)dt - \frac{2C}{\pi},$$

which yields

$$\int_0^1 \lambda_1 u_0(t)\sin(\pi t)dt \le \frac{2C}{\pi\varepsilon\lambda_1}.$$

On the other hand, in view of the definition of cone P, one can easily obtain that

$$\|u_0\| \int_0^1 \sigma(t)\sin(\pi t)dt \le \int_0^1 u_0(t)\sin(\pi t)dt \le \frac{2C}{\pi\varepsilon\lambda_1},$$

which means

$$\|u_0\| \le \frac{2C}{\pi\varepsilon\lambda_1 \int_0^1 \sigma(t)\sin(\pi t)dt} := R_1^*. \tag{10}$$

Therefore, if $R_1 > R_1^*$, immediately, one can get $\mu T(u,v) \neq (u,v)$ for $\mu \ge 1$ and $(u,v) \in \partial P_{R_1}$.

In addition, if $R_1 > \dfrac{m}{\min\limits_{t\in[\frac14,\frac34]} \sigma(t)} := \dfrac{m}{\sigma^*}$, then by the definition of cone P, one can get that for any $t \in [\frac14, \frac34]$ and $(u,v) \in \partial P_{R_1}$,

$$u(t) \ge \min_{t\in[\frac14,\frac34]} u(t) \ge \sigma^* R_1 > m. \tag{11}$$

So, by (7), (11), and Lemma 1, one can get that for all $(u,v) \in \partial P_{R_1}$,

$$\begin{aligned}
\|T(u,v)\| &\ge T_1(u,v)(\frac12) \\
&= \int_0^1 \int_0^1 G_{1,1}(\frac12,\tau)G_{1,2}(\tau,s)f_1(s,u(s),v(s))dsd\tau \\
&\ge \delta_{1,1}\delta_{1,2}G_{1,2}(\frac12,\frac12)C_1 \int_0^1 G_{1,2}(s,s)f_1(s,u(s),v(s))ds \\
&\ge \delta_{1,1}\delta_{1,2}G_{1,2}(\frac12,\frac12)C_1(1+\varepsilon)\lambda_1 \int_{\frac14}^{\frac34} G_{1,2}(s,s)u(s)ds \\
&\ge \delta_{1,1}\delta_{1,2}G_{1,2}(\frac12,\frac12)C_1(1+\varepsilon)\lambda_1 R_1\sigma^* \int_{\frac14}^{\frac34} G_{1,2}(s,s)ds \\
&\ge \delta_{1,1}\delta_{1,2}G_{1,2}(\frac12,\frac12)C_1(1+\varepsilon)\lambda_1 m \int_{\frac14}^{\frac34} G_{1,2}(s,s)ds > 0.
\end{aligned}$$

That is, $\inf\limits_{(u,v)\in\partial P_{R_1}} \|T(u,v)\| > 0$. So, we can ultimately choose $R_1 > \max\{R_1^*, r_1, \frac{m}{\sigma^*}\}$ such that (9) holds.

Based on (5), (9), Lemma 2, and Lemma 3, we have

$$i(T, P_{R_1} \setminus \bar{P_{r_1}}, P) = i(T, P_{R_1}, P) - i(T, P_{r_1}, P) = 0 - 1 = -1.$$

As a result, the conclusion of this theorem follows. □

Theorem 2. *Assume that (H1) and (H3) hold. Then the BVP (1) has at least one nontrivial solution.*

Proof of Theorem 2. In the following, we divide the proof of Theorem 2 into three steps.

Step 1. From condition (H3), there exist $\varepsilon > 0$ and $r_2 > 0$ such that

$$f_1(t,u,v) \ge (1+\varepsilon)\lambda_1 u,\ \forall t \in [0,1],\ (u,v) \in \Lambda_{[0,r_2]}. \tag{12}$$

Subsequently, we claim that

$$\inf_{(u,v)\in\partial P_{r_2}} \|T(u,v)\| > 0 \ \text{ and } \ \mu T(u,v) \neq (u,v),\ \forall \mu \ge 1,\ (u,v) \in \partial P_{r_2}. \tag{13}$$

In fact, if there exist $\mu_0 \geq 1$ and $(u_0, v_0) \in \partial P_{r_2}$ such that $\mu_0 T(u_0, v_0) = (u_0, v_0)$, then by (6) and (12), one can obtain immediately

$$u_0^{(4)}(t) + \beta_1 u_0''(t) - \alpha_1 u_0(t) \geq f_1(t, u_0(t), v_0(t)) \geq (1+\varepsilon)\lambda_1 u_0(t).$$

Hence,

$$\int_0^1 \lambda_1 u_0(t)\sin(\pi t)dt \geq (1+\varepsilon)\int_0^1 \lambda_1 u_0(t)\sin(\pi t)dt.$$

Noticing that $\int_0^1 \lambda_1 u_0(t)\sin(\pi t)dt > 0$, we get a contradiction.

In addition, it follows from Lemma 1 and (12) that for $(u, v) \in \partial P_{r_2}$,

$$\begin{aligned}
\|T(u,v)\| &\geq T_1(u,v)(\frac{1}{2}) \\
&= \int_0^1\int_0^1 G_{1,1}(\frac{1}{2},\tau)G_{1,2}(\tau,s)f_1(s,u(s),v(s))dsd\tau \\
&\geq \delta_{1,1}\delta_{1,2}G_{1,2}(\frac{1}{2},\frac{1}{2})C_1\int_0^1 G_{1,2}(s,s)f_1(s,u(s),v(s))ds \\
&\geq \delta_{1,1}\delta_{1,2}G_{1,2}(\frac{1}{2},\frac{1}{2})C_1(1+\varepsilon)\lambda_1\int_0^1 G_{1,2}(s,s)u(s)ds \\
&\geq \delta_{1,1}\delta_{1,2}G_{1,2}(\frac{1}{2},\frac{1}{2})C_1(1+\varepsilon)\lambda_1 r_2\int_0^1 G_{1,2}(s,s)\sigma(s)ds > 0,
\end{aligned}$$

which yields $\inf\limits_{(u,v)\in\partial P_{r_2}} \|T(u,v)\| > 0$.

Step 2. The assumption (H3) implies that there exist $\varepsilon \in (0,1)$ and $m > 0$ such that

$$f_1(t,u,v) \leq (1-\varepsilon)\lambda_1 u,\ \forall t \in [0,1],\ (u,v) \in \Lambda_{[m,+\infty)}. \tag{14}$$

Moreover, by the continuity of f_1 and f_2, there exists $C^* > 0$ such that

$$f_1(t,u,v) \leq (1-\varepsilon)\lambda_1 u + C^*,\ \forall t \in [0,1],\ (u,v) \in \Lambda_{\mathbb{R}^+}. \tag{15}$$

We claim that there exists a large enough $R_2 > r_2$ such that

$$\mu T(u,v) \neq (u,v),\ \forall \mu \in (0,1],\ (u,v) \in \partial P_{R_2}. \tag{16}$$

Suppose, on the contrary, there exist $\mu_0 \in (0,1]$ and $(u_0, v_0) \in \partial P_{R_2}$ such that $\mu_0 T(u_0, v_0) = (u_0, v_0)$. Then (6) together with (15) guarantees

$$u_0^{(4)}(t) + \beta_1 u_0''(t) - \alpha_1 u_0(t) \leq f_1(t, u_0(t), v_0(t)) \leq (1-\varepsilon)\lambda_1 u_0(t) + C^*.$$

Consequently,

$$\int_0^1 \lambda_1 u_0(t)\sin(\pi t)dt \leq (1-\varepsilon)\int_0^1 \lambda_1 u_0(t)\sin(\pi t)dt + \frac{2C^*}{\pi},$$

namely,

$$\int_0^1 u_0(t)\sin(\pi t)dt \leq \frac{2C^*}{\pi\varepsilon\lambda_1}.$$

Moreover, based on the definition of cone P, we can immediately get

$$\|u_0\|\int_0^1 \sigma(t)\sin(\pi t)dt \leq \int_0^1 u_0(t)\sin(\pi t)dt \leq \frac{2C^*}{\pi\varepsilon\lambda_1},$$

which means

$$\|u_0\| \leq \frac{2C^*}{\pi\varepsilon\lambda_1 \int_0^1 \sigma(t)\sin(\pi t)dt} := R_2^*. \tag{17}$$

So, one can choose $R_2 > \max\{R_2^*, r_2\}$ such that (16) holds.

Step 3. From (13), (16), Lemma 2, and Lemma 3, we deduce that

$$i(T, P_{R_2} \setminus \bar{P_{r_2}}, P) = i(T, P_{R_2}, P) - i(T, P_{r_2}, P) = 1 - 0 = 1.$$

As a result, BVP(1) has at least one nontrivial solution. □

Up to now, some existence results of BVP(1) have been obtained by applying the fixed point index theory. In the following, the multiple solutions will be considered for BVP (1).

Theorem 3. *Assume that (H1) holds. In addition, suppose that*

(1) $\lim\limits_{\substack{|v|\leq Nu \\ u\to 0^+}} \sup \max\limits_{t\in[0,1]} \frac{f_1(t,u,v)}{u} < \lambda_1$, $\lim\limits_{\substack{|v|\leq Nu \\ u\to +\infty}} \sup \max\limits_{t\in[0,1]} \frac{f_1(t,u,v)}{u} < \lambda_1$;

(2) There exists $r > 0$ *and a continuous nonnegative function* Φ_r *such that*

$$f_1(t,u,v) \geq \Phi_r(t), \forall (t,u,v) \in [0,1] \times (\sigma(t)r, r) \times [-Nr, Nr]$$

and

$$\max_{t\in[0,1]} \int_0^1 \int_0^1 G_{1,1}(t,\tau)G_{1,2}(\tau,s)\Phi_r(s)dsd\tau > r.$$

Then the BVP (1) has at least two nontrivial solutions.

Proof of Theorem 3. In order to obtain this conclusion, we firstly claim that

$$\inf_{(u,v)\in\partial P_r} \|T(u,v)\| > 0 \text{ and } \mu T(u,v) \neq (u,v), \forall\, \mu \geq 1,\ (u,v) \in \partial P_r. \tag{18}$$

Suppose, on the contrary, there exist $\mu_0 \geq 1$ and $(u_0, v_0) \in \partial P_r$ such that $\mu_0 T(u_0, v_0) = (u_0, v_0)$. Then,

$$\begin{aligned} \|u_0\| &\geq \|T(u_0,v_0)\| \geq T_1(u_0,v_0)(t) \\ &= \int_0^1 \int_0^1 G_{1,1}(t,\tau)G_{1,2}(\tau,s)f_1(s,u(s),v(s))dsd\tau \\ &\geq \int_0^1 \int_0^1 G_{1,1}(t,\tau)G_{1,2}(\tau,s)\Phi_r(s)dsd\tau. \end{aligned} \tag{19}$$

Taking the maximum for both sides of the above inequality in $t \in [0,1]$, we get that

$$\|u_0\| \geq \max_{t\in[0,1]} \int_0^1 \int_0^1 G_{1,1}(t,\tau)G_{1,2}(\tau,s)\Phi_r(s)dsd\tau > r. \tag{20}$$

This means $(u_0, v_0)\overline{\in} \partial P_r$, which is a contradiction. Moreover, one can easily see that $\inf\limits_{(u,v)\in\partial P_r} \|T(u,v)\| > 0$ holds from (19) and (20).

Next, similar to the process of proving (5) and (16), there exist $r_1 \in (0,r)$ and $R_2 \geq \max\{R_2^*, r_2, r\}$ such that

$$\mu T(u,v) \neq (u,v), \forall \mu \in (0,1],\ \forall (u,v) \in \partial P_{r_1}, \tag{21}$$

$$\mu T(u,v) \neq (u,v),\ \forall \mu \in (0,1],\ \forall (u,v) \in \partial P_{R_2}. \tag{22}$$

Thus, by (18), (21), (22), Lemma 2, and Lemma 3, one can immediately obtain that

$$i(T, P_{R_2} \setminus \bar{P}_r, P) = i(T, P_{R_2}, P) - i(T, P_r, P) = 1 - 0 = 1,$$

$$i(T, P_r \setminus \bar{P_{r_1}}, P) = i(T, P_r, P) - i(T, P_{r_1}, P) = 0 - 1 = -1.$$

Namely, there exist $(u_1, v_1) \in P_r \setminus \bar{P_{r_1}}$ and $(u_2, v_2) \in P_{R_2} \setminus \bar{P}_r$ satisfying $T(u_i, v_i) = (u_i, v_i)(i = 1, 2)$, that is, $(u_i, v_i)(i = 1, 2)$ is the solution of BVP(1).

Finally, we show $(u_1, v_1) \neq (u_2, v_2)$. To see this we need only to prove BVP(1) has no solution on ∂P_r. Suppose on the contrary, there exists $(u^*, v^*) \in \partial P_r$ being a solution of BVP(1). Then $T(u^*, v^*) = (u^*, v^*)$. By a similar process of obtaining (20), one can get $\|u^*\| = \|T_1(u^*, v^*)\| > r$, which is a contradiction. To sum up, Theorem 3 is proved. $\square$

From a process similar to the above, the following conclusion can be obtained.

Theorem 4. *Suppose that (H1) holds. In addition, suppose that*

(1) $\lim\limits_{\substack{|v|\le Nu \\ u\to 0^+}} \inf \min\limits_{t\in[0,1]} \frac{f_1(t,u,v)}{u} > \lambda_1$, $\lim\limits_{\substack{|v|\le Nu \\ u\to +\infty}} \inf \min\limits_{t\in[0,1]} \frac{f_1(t,u,v)}{\lambda_1 u} > \lambda_1$;

(2) There exists $R > 0$, and a continuous nonnegative function Ψ_R such that

$$f_1(t, u, v) \le \Psi_R(t), \forall (t, u, v) \in [0, 1] \times [\sigma(t)R, R] \times [-NR, NR]$$

and

$$\max_{t\in[0,1]} \int_0^1 \int_0^1 G_{1,1}(t, \tau) G_{1,2}(\tau, s) \Psi_R(s) ds d\tau < R.$$

Then the BVP (1) has at least two nontrivial solutions.

Proof of Theorem 4. We firstly prove that

$$\mu T(u, v) \neq (u, v),\ \forall \mu \in (0, 1],\ (u, v) \in \partial P_R. \tag{23}$$

To this end, suppose on the contrary that there exist $\mu_0 \in (0, 1]$ and $(u_0, v_0) \in \partial P_R$ such that $\mu_0 T(u_0, v_0) = (u_0, v_0)$. Hence, we get $u_0 = \mu_0 T_1(u_0, v_0)$, that is

$$u_0(t) \le T_1(u_0, v_0)(t) \quad \le \int_0^1 \int_0^1 G_{1,1}(t, \tau) G_{1,2}(\tau, s) \Psi_R(s) ds d\tau < R. \tag{24}$$

Noticing that $(u_0, v_0) \in \partial P_R$, this is a contradiction.

Next, from a process similar to (9) and (13), there exist $R_1 > \max\{R, R_1^*, r_1, \frac{m}{\sigma^*}\}$ and $r_2 \in (0, R)$ such that

$$\inf_{(u,v)\in\partial P_{R_1}} \|T(u, v)\| > 0 \ \text{ and } \ \mu T(u, v) \neq (u, v),\ \forall \mu \ge 1,\ (u, v) \in \partial P_{R_1}, \tag{25}$$

$$\inf_{(u,v)\in\partial P_{r_2}} \|T(u, v)\| > 0 \ \text{ and } \ \mu T(u, v) \neq (u, v),\ \forall \mu \ge 1,\ (u, v) \in \partial P_{r_2}. \tag{26}$$

So, by (23)–(26), Lemma 2, and Lemma 3, one can get

$$i(T, P_{R_1} \setminus \bar{P_R}, P) = i(T, P_{R_1}, P) - i(T, P_R, P) = 0 - 1 = -1,$$

$$i(T, P_R \setminus \bar{P_{r_2}}, P) = i(T, P_R, P) - i(T, P_{r_2}, P) = 1 - 0 = 1.$$

Finally, from a process similar to the end of proof of Theorem 3, BVP(1) has at least two nontrivial solutions. As a result, the conclusion of this theorem follows. $\square$

4. Examples

In this section, two illustrative examples are worked out to show the effectiveness of the obtained results.

Example 1. *Consider the following BVP of fourth-order ordinary differential systems*

$$\begin{cases} u^{(4)}(t) + u''(t) - \pi^2 u(t) = f_1(t,u,v), \ \ 0 < t < 1; \\ v^{(4)}(t) + \dfrac{1}{2}v''(t) - \dfrac{\pi^2}{2}v(t) = f_2(t,u,v), \ \ 0 < t < 1; \\ u(0) = u(1) = u''(0) = u''(1) = 0; \\ v(0) = v(1) = v''(0) = v''(1) = 0, \end{cases} \tag{27}$$

where

$$f_1(t,u,v) = \begin{cases} (\pi^4 - 2\pi^2)(1+\sin(\pi t))uv^{\frac{1}{4}} & \text{if } 0 < u < 1, \ |v| < u; \\ (\pi^4 - 2\pi^2)(1+\sin(\pi t))v^{\frac{1}{4}} & \text{if } u = 1, \quad |v| < u; \\ (\pi^4 - 2\pi^2)(1+\sin(\pi t))u^{\frac{1}{4}}v^{\frac{1}{4}} & \text{if } u > 1, \quad |v| < u, \end{cases}$$

$$f_2(t,u,v) = \begin{cases} (\pi^4 - 2\pi^2)(1+\cos(\pi t))uv^{\frac{1}{4}} & \text{if } 0 < u < 1, \ |v| < u; \\ (\pi^4 - 2\pi^2)(1+\cos(\pi t))v^{\frac{1}{4}} & \text{if } u = 1, \quad |v| < u; \\ (\pi^4 - 2\pi^2)(1+\cos(\pi t))u^{\frac{1}{4}}v^{\frac{1}{4}} & \text{if } u > 1, \quad |v| < u, \end{cases}$$

Then, BVP (27) has at least two nontrivial solutions.

Proof of Example 1 . BVP (27) can be regarded as a BVP of the form (1). Choosing $\alpha_1 = \pi^2$, $\beta_1 = 1$, and $\lambda_1 = \pi^4 - 2\pi^2 > 0$, then we have

$$\xi_{1,1} = \frac{-\beta_1 + \sqrt{\beta_1^2 + 4\alpha_1}}{2} = \frac{-1 + \sqrt{1+4\pi^2}}{2}, \ \xi_{1,2} = \frac{-\beta_1 - \sqrt{\beta_1^2 + 4\alpha_1}}{2} = \frac{-1 - \sqrt{1+4\pi^2}}{2}.$$

Clearly, α_1 and β_1 satisfy the condition (2). Moreover, by careful calculation and Lemma 2.1 in Reference [32], one can obtain that

$$G_{1,1}(t,s) = \begin{cases} \dfrac{\sinh w_{1,1}t \sinh w_{1,1}(1-s)}{w_{1,1}\sinh w_{1,1}} & 0 \le t \le s \le 1; \\ \dfrac{\sinh w_{1,1}s \sinh w_{1,1}(1-t)}{w_{1,1}\sinh w_{1,1}} & 0 \le s \le t \le 1, \end{cases}$$

$$G_{1,2}(t,s) = \begin{cases} \dfrac{\sin w_{1,2}t \sin w_{1,2}(1-s)}{w_{1,2}\sin w_{1,2}} & 0 \le t \le s \le 1; \\ \dfrac{\sin w_{1,2}s \sin w_{1,2}(1-t)}{w_{1,2}\sin w_{1,2}} & 0 \le s \le t \le 1, \end{cases}$$

where $w_{1,i} = \sqrt{|\xi_{1,i}|}(i = 1,2)$.

Now, $|v| \le 2u$, $|f_2(t,u,v)| \le 2|f_1(t,u,v)|$, and $N = N_1N_2N_3$. Thus, one can easily get that (H1) holds by choosing $N_3 \ge \max\{2, \dfrac{2}{N_1N_2}\}$, where $N_j = \sup\limits_{0<t,s<1} \frac{G_{2,j}(t,s)}{G_{1,j}(t,s)}$, $j = 1,2$.

In addition, by calculation, we get that

$$\lim_{\substack{|v|\le Nu \\ u\to 0^+}} \sup \max_{t\in[0,1]} \frac{f_1(t,u,v)}{u} = \lim_{\substack{|v|\le Nu \\ u\to 0^+}} \sup \max_{t\in[0,1]} \frac{(\pi^4 - 2\pi^2)(1+\sin(\pi t))uv^{\frac{1}{4}}}{u} = 0 < \lambda_1,$$

$$\lim_{\substack{|v|\le Nu \\ u\to +\infty}} \sup \max_{t\in[0,1]} \frac{f_1(t,u,v)}{u} = \lim_{\substack{|v|\le Nu \\ u\to +\infty}} \sup \max_{t\in[0,1]} \frac{(\pi^4 - 2\pi^2)(1+\sin(\pi t))u^{\frac{1}{4}}v^{\frac{1}{4}}}{u} = 0 < \lambda_1.$$

Choose

$$r = \min\{1, [\delta_{1,1}\delta_{1,2}\int_0^1 \sqrt{\sigma(t)}\sin(\pi t)dt \min_{t\in[\frac{1}{4},\frac{3}{4}]}(G_{1,1}(t,t)G_{1,2}(t,t))^2]^2\} > 0$$

and

$$\Phi_r(t) = \sqrt{r\sigma(t)}\sin(\pi t).$$

Then, it is not difficult to obtain that the condition (2) in Theorem 3 holds. Hence, our conclusion follows from Theorem 3. □

Example 2. *Consider the following BVP of fourth-order ordinary differential systems*

$$\begin{cases} u^{(4)}(t) + u''(t) - \pi^2 u(t) = f_1(t,u,v), \ \ 0 < t < 1; \\ v^{(4)}(t) + \dfrac{1}{2}v''(t) - \dfrac{\pi^2}{2}v(t) = f_2(t,u,v), \ \ 0 < t < 1; \\ u(0) = u(1) = u''(0) = u''(1) = 0; \\ v(0) = v(1) = v''(0) = v''(1) = 0. \end{cases} \tag{28}$$

where

$$f_1(t,u,v) = \begin{cases} (\pi^4 - 2\pi^2)(2+t)u^{\frac{1}{2}}v^{\frac{1}{3}} & \text{if } 0 < u < 1, \ 0 < v < 1; \\ (\pi^4 - 2\pi^2)(2+t)v^{\frac{1}{3}} & \text{if } u = 1, \quad 0 < v < 1; \\ (\pi^4 - 2\pi^2)(2+t)u^2 v^{\frac{1}{3}} & \text{if } u > 1, \quad 0 < v < 1, \end{cases}$$

$$f_2(t,u,v) = \begin{cases} (\pi^4 - 2\pi^2)(1+\cos(\pi t))u^{\frac{1}{2}}v^{\frac{1}{3}} & \text{if } 0 < u < 1, \ 0 < v < 1; \\ (\pi^4 - 2\pi^2)(1+\cos(\pi t))v^{\frac{1}{3}} & \text{if } u = 1, \quad 0 < v < 1; \\ (\pi^4 - 2\pi^2)(1+\cos(\pi t))u^2 v^{\frac{1}{3}} & \text{if } u > 1, \quad 0 < v < 1, \end{cases}$$

Then, BVP (28) has at least two nontrivial solutions.

Proof of Example 2 . BVP (28) can be regarded as a BVP of the form (1). Using a similar process of the proof of Example 1, one can easily obtain that

$$\liminf_{\substack{|v|\le Nu \\ u\to 0^+}} \min_{t\in[0,1]} \frac{f_1(t,u,v)}{u} = \liminf_{\substack{|v|\le Nu \\ u\to 0^+}} \min_{t\in[0,1]} \frac{(\pi^4 - 2\pi^2)(2+t)u^{\frac{1}{2}}v^{\frac{1}{3}}}{u} = +\infty > \pi^4 - 2\pi^2 = \lambda_1,$$

$$\liminf_{\substack{|v|\le Nu \\ u\to +\infty}} \min_{t\in[0,1]} \frac{f_1(t,u,v)}{u} = \liminf_{\substack{|v|\le Nu \\ u\to +\infty}} \min_{t\in[0,1]} \frac{(\pi^4 - 2\pi^2)(2+t)u^2 v^{\frac{1}{3}}}{u} + \infty > \pi^4 - 2\pi^2 = \lambda_1.$$

In addition, it is obvious that (H1) holds by choosing $N_3 = 2$. In the following, set

$$R = \max\{1, \frac{2}{5\pi^2 C_{1,1}C_{1,2}\max_{t\in[0,1]}[G_{1,1}(t,t)G_{1,2}(t,t)]}\} > 0$$

and

$$\Psi_R(t) = \pi^4 R^2 (2+t).$$

Then, it is trivial to verify that assumption (2) of Theorem 3 is true.

As a result, by Theorem 4, system (28) has at least two nontrivial solutions. □

5. Conclusions

In this paper, we have obtained some appropriate results corresponding to multiple solutions for a class of nonlinear fourth-order boundary value problems with parameters. The multiple solutions for the considered systems are obtained under some suitable assumptions via fixed point index theory. The whole theoretical results has been demonstrated by providing two interesting examples. Hence, we claim that fixed point index theory can be used as a strong technique to study nonlinear fourth-order boundary value problems with parameters.

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