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On Quasi Gyrolinear Maps between Möbius Gyrovector Spaces Induced from Finite Matrices

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Abstract: We present some fundamental results concerning to continuous quasi gyrolinear operators between Möbius gyrovector spaces induced by finite matrices. Such mappings are significant like as operators induced by matrices between finite dimensional Hilbert spaces. This gives a novel approach to the study of mappings between Möbius gyrovector spaces that should correspond to bounded linear operators on real Hilbert spaces.

Keywords: gyrogroup; Möbius gyrovector space; matrix; operator

MSC: Primary 47H99; Secondary 20N05; 46T99; 51M10; 83A05

1. Introduction

The theory of gyrogroups and gyrovector spaces was initiated by Ungar in the late 1980s. See also [1] for historical aspects, at least for physically relevant dimensions three and four. The notion of gyrogroups is one of the most natural generalizations of groups, and they form a subclass of loops or quasigroups. Gyrovector spaces are generalized vector spaces, with which they share important analogies. In connection with the special theory of relativity, the ball of Euclidean space $\mathbb{R}^3$ endowed with Einstein's velocity addition was known as the first example of gyrogroups (cf. [2]). The open unit disc in the complex plain endowed with the Möbius addition is another significant example of gyrogroups (cf. [3]). Ungar extended Möbius addition, introduced Möbius scalar multiplication to the balls of arbitrary real inner product spaces and established the concept of gyrovector spaces, which have a vector space-like structure (cf. [4,5]). Although gyro-operations are generally not commutative, associative, or distributive, they enjoy algebraic rules, such as left and right gyroassociative laws, left and right loop properties, gyrocommutative law, scalar distributive law, and scalar associative law, so there exist rich symmetrical structures which we should clarify precisely.

Abe and Hatori [6] introduced the notion of generalized gyrovector spaces (GGVs), which is a generalization of the notion of real inner product gyrovector spaces by Ungar. The set of all positive invertible elements of a unital C^*-algebra is one of the most important examples of GGVs, which is not a real inner product gyrovector space. Hatori [7] showed that the various substructures of positive invertible elements of unital C^*-algebra are actually GGVs. Abe [8] introduced the notion of normed gyrolinear spaces, which is a further generalization of the notion of GGVs. Although we do not deal with them here, they will provide advanced research subjects.

There are remarkable papers on Möbius gyrogroups using Clifford algebra formalism [9–11]. Ferreira and Suksumran [12] introduced the notion of real inner product gyrogroups, which is a generalization of well-known gyrogroups in the literature, and gave a number of interesting results.

One can also consider complex Möbius gyrovector spaces in complex inner product spaces; however, we do not deal with them here. Some basic results on this subject will be published in [13].

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In this article, we concentrate on the Möbius gyrovector spaces. There are the notions of the Einstein gyrovector spaces and the proper velocity (PV) gyrovector spaces by Ungar, and they are isomorphic to the Möbius gyrovector spaces, so most results on each spaces can be directly translated to the other two spaces. In the Möbius gyrovector spaces, it seems easier to consider counterparts to various notions related to Hilbert spaces than in other spaces. In recent years, we have clarified the structure of Möbius gyrovector spaces to some extent, such as the structure of finitely generated gyrovector subspaces, orthogonal gyrodecomposition of any element with respect to any closed gyrovector subspace, orthogonal gyroexpansion of any element with respect to an arbitrary orthogonal basis with weight sequence, Cauchy–Schwarz-type inequalities, and continuous quasi gyrolinear functionals induced by any square summable sequence of real numbers (cf. [14–19]). The purpose of this article is to present a class of continuous maps between Möbius gyrovector spaces induced by finite matrices, which can be regarded as a certain counterpart to bounded linear operators on real Hilbert spaces. The main result is Theorem 8, which is novel, and Theorem 9 as well.

2. Preliminaries

Let us briefly recall some of the most basic definitions and facts of the Möbius gyrovector space. For standard definitions and results of gyrocommutative gyrogroups and gyrovector spaces, see monograph [20] or [21] by Ungar (and references therein).

Let $\mathbb{V} = (\mathbb{V}, +, \cdot)$ be a real inner product space with a binary operation $+$ and a positive definite inner product $\cdot$, and let $\mathbb{V}_s$ be the open s-ball of $\mathbb{V}$,

$$\mathbb{V}_s = \{\boldsymbol{a} \in \mathbb{V} : ||\boldsymbol{a}|| < s\}$$

for any fixed $s > 0$, where $||\boldsymbol{a}|| = (\boldsymbol{a}\cdot\boldsymbol{a})^{\frac{1}{2}}$.

Definition 1. *[21] (Definition 3.40, Definition 6.83) The Möbius addition $\oplus_{\mathrm{M}}$ and the Möbius scalar multiplication $\otimes_{\mathrm{M}}$ are given by the equations*

$$\boldsymbol{a} \oplus_{\mathrm{M}} \boldsymbol{b} = \frac{\left(1 + \frac{2}{s^2}\boldsymbol{a}\cdot\boldsymbol{b} + \frac{1}{s^2}||\boldsymbol{b}||^2\right)\boldsymbol{a} + \left(1 - \frac{1}{s^2}||\boldsymbol{a}||^2\right)\boldsymbol{b}}{1 + \frac{2}{s^2}\boldsymbol{a}\cdot\boldsymbol{b} + \frac{1}{s^4}||\boldsymbol{a}||^2||\boldsymbol{b}||^2}$$

$$r \otimes_{\mathrm{M}} \boldsymbol{a} = s \tanh\left(r \tanh^{-1}\frac{||\boldsymbol{a}||}{s}\right)\frac{\boldsymbol{a}}{||\boldsymbol{a}||} \quad (\text{if } \boldsymbol{a} \neq \boldsymbol{0}), \qquad r \otimes_{\mathrm{M}} \boldsymbol{0} = \boldsymbol{0}$$

for any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{V}_s, r \in \mathbb{R}$. The addition $\oplus_{\mathrm{M}}$ and the scalar multiplication $\otimes_{\mathrm{M}}$ for real numbers are defined by the equations

$$a \oplus_{\mathrm{M}} b = \frac{a + b}{1 + \frac{1}{s^2}ab}$$

$$r \otimes_{\mathrm{M}} a = s \tanh\left(r \tanh^{-1}\frac{a}{s}\right)$$

for any $a, b \in (-s, s), r \in \mathbb{R}$.

The ball $\mathbb{V}_s$ expands to the whole space $\mathbb{V}$ as the parameter $s \to \infty$, and hence, each result in linear functional analysis can be recaptured from the counterpart in gyrolinear analysis.

Proposition 1. *[21] (after Remark 3.41), [5] (p. 1054). The Möbius addition (resp. Möbius scalar multiplication) reduces to the ordinary addition (resp. scalar multiplication) as $s \to \infty$, that is,*

$$\boldsymbol{a} \oplus_{\mathrm{M}} \boldsymbol{b} \to \boldsymbol{a} + \boldsymbol{b}, \qquad r \otimes_{\mathrm{M}} \boldsymbol{a} \to r\boldsymbol{a} \quad (s \to \infty)$$

for any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{V}$ *and* $r \in \mathbb{R}$.

Theorem 1. *[21] (Theorem 6.84), (see also [11,22].)*

(1) $(\mathbb{V}_s, \oplus_{\mathrm{M}})$ *is a gyrocommutative gyrogroup.*

(2) $(\mathbb{V}_s, \oplus_{\mathrm{M}}, \otimes_{\mathrm{M}})$ *is a real inner product gyrovector space.*

Definition 2. *[21] ((6.286), (6.293)). The Möbius gyrodistance function* d_{M} *and the Poincaré distance function* h_{M} *are defined by the equations*

$$d_{\mathrm{M}}(\boldsymbol{a}, \boldsymbol{b}) = ||\boldsymbol{b} \ominus_{\mathrm{M}} \boldsymbol{a}||, \qquad h_{\mathrm{M}}(\boldsymbol{a}, \boldsymbol{b}) = \tanh^{-1} \frac{d_{\mathrm{M}}(\boldsymbol{a}, \boldsymbol{b})}{s}$$

for any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{V}_s$.

Theorem 2. *[21] (6.294) (see also [23,24], [15] (Theorem 26), [25] (Proposition 2).)* h_{M} *satisfies the triangle inequality, so that* $(\mathbb{V}_s, h_{\mathrm{M}})$ *is a metric space. In addition, if* $\mathbb{V}$ *is a real Hilbert space, then* $(\mathbb{V}_s, h_{\mathrm{M}})$ *is complete as a metric space.*

We simply denote $\oplus_{\mathrm{M}}, \otimes_{\mathrm{M}}, d_{\mathrm{M}}, h_{\mathrm{M}}$ by $\oplus, \otimes, d, h$, respectively. We also use $\oplus_s, \otimes_s$ in order to indicate the parameter $s > 0$. For the sake of simplicity, we sometimes state results only for the case of $s = 1$. In this paper, one can immediately obtain results for general $s > 0$ via Proposition 2 (ii) and (iii) below. If several kinds of operations appear in a formula simultaneously, we always give priority to the following order: (1) ordinary scalar multiplication, (2) gyroscalar multiplication $\otimes_s$, and (3) gyroaddition $\oplus_s$, that is,

$$r_1 \otimes_s w_1 \boldsymbol{a}_1 \oplus_s r_2 \otimes_s w_2 \boldsymbol{a}_2 = \{r_1 \otimes_s (w_1 \boldsymbol{a}_1)\} \oplus_s \{r_2 \otimes_s (w_2 \boldsymbol{a}_2)\},$$

and the parentheses are omitted in such cases.

The following identities are an easy consequence of the definition, and frequently used. One can refer to [15] (Lemma 12, Lemma 14 (i)).

Proposition 2. *Let* $s > 0$. *The following formulae hold:*

(i) $||\boldsymbol{a} \oplus_s \boldsymbol{b}||^2 = \dfrac{||\boldsymbol{a}||^2 + 2\boldsymbol{a}\cdot\boldsymbol{b} + ||\boldsymbol{b}||^2}{1 + \frac{2}{s^2}\boldsymbol{a}\cdot\boldsymbol{b} + \frac{1}{s^4}||\boldsymbol{a}||^2||\boldsymbol{b}||^2}$

(ii) $\dfrac{\boldsymbol{a}}{s} \oplus_1 \dfrac{\boldsymbol{b}}{s} = \dfrac{\boldsymbol{a} \oplus_s \boldsymbol{b}}{s}$

(iii) $r \otimes_1 \dfrac{\boldsymbol{a}}{s} = \dfrac{r \otimes_s \boldsymbol{a}}{s}$

for any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{V}_s$ *and* $r \in \mathbb{R}$.

The following lemma is just a consequence of formulae [21] ((3.147), (3.148)).

Lemma 1. *[15] (Lemma 31). If* $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ *is an orthogonal set in* $\mathbb{V}_s$, *then the associative law holds; that is,*

$$\boldsymbol{u} \oplus (\boldsymbol{v} \oplus \boldsymbol{w}) = (\boldsymbol{u} \oplus \boldsymbol{v}) \oplus \boldsymbol{w}.$$

Definition 3. *[15] (Definition 32). Let* $\{\boldsymbol{a}_n\}_n$ *be a sequence in* $\mathbb{V}_s$. *One says that a series*

$$\Big(\big((\boldsymbol{a}_1 \oplus \boldsymbol{a}_2) \oplus \boldsymbol{a}_3\big) \oplus \cdots \oplus \boldsymbol{a}_n\Big) \oplus \cdots$$

converges if there exists an element $\boldsymbol{x} \in \mathbb{V}_s$*, such that* $h(\boldsymbol{x}, \boldsymbol{x}_n) \to 0 \ (n \to \infty)$*, where the sequence* $\{\boldsymbol{x}_n\}_n$ *is defined recursively by* $\boldsymbol{x}_1 = \boldsymbol{a}_1$ *and* $\boldsymbol{x}_n = \boldsymbol{x}_{n-1} \oplus \boldsymbol{a}_n$*. In this case, we say the series converges to* $\boldsymbol{x}$ *and denotes*

$$\boldsymbol{x} = \Big(\big((\boldsymbol{a}_1 \oplus \boldsymbol{a}_2) \oplus \boldsymbol{a}_3 \big) \oplus \cdots \oplus \boldsymbol{a}_n \Big) \oplus \cdots .$$

The following theorem can be considered as a counterpart to the orthogonal Fourier expansion in Hilbert spaces and the Parseval identity.

Theorem 3. *[15] (Theorem 35, Theorem 36). Let* $\{\boldsymbol{e}_n\}_{n=1}^{\infty}$ *be a complete orthonormal sequence in a real Hilbert space* $\mathbb{V}$*. Let* $\{w_n\}_{n=1}^{\infty}$ *be a sequence in* $\mathbb{R}$ *such that* $0 < w_n < s$ *for all n. Then, for any* $\boldsymbol{x} \in \mathbb{V}_s$*, we have the orthogonal gyroexpansion*

$$\boldsymbol{x} = r_1 \otimes w_1 \boldsymbol{e}_1 \oplus r_2 \otimes w_2 \boldsymbol{e}_2 \oplus \cdots \oplus r_n \otimes w_n \boldsymbol{e}_n \oplus \cdots ,$$

where the sequence of gyrocoefficients $\{r_n\}_{n=1}^{\infty}$ *is uniquely determined and can be calculated by an explicit procedure. Moreover, we have the following identity:*

$$||\boldsymbol{x}||^2 = \sum_{n=1}^{\infty}{}^{\oplus} \frac{(r_n \otimes w_n)^2}{s}.$$

Now let us see some related preceding research for maps on the Einstein gyrovector space, which preserve the Einstein addition. Let $(\mathbb{B}^n, \oplus_{\mathrm{E}})$ be the n-dimensional Einstein gyrogroup, where $\mathbb{B}^n = \{\boldsymbol{u} \in \mathbb{R}^n;\ ||\boldsymbol{u}|| < 1\}$.

Theorem 4. *[26] (Theorem 1). (see also [27].) Let* $\beta : \mathbb{B}^3 \to \mathbb{B}^3$ *be a continuous map. Then,* β *is an algebraic endomorphism with respect to the operation* $\oplus_{\mathrm{E}}$*, that is,* β *satisfies*

$$\beta(\boldsymbol{u} \oplus_{\mathrm{E}} \boldsymbol{v}) = \beta(\boldsymbol{u}) \oplus_{\mathrm{E}} \beta(\boldsymbol{v}), \quad \boldsymbol{u}, \boldsymbol{v} \in \mathbb{B}^3$$

if, and only if:

(i) *Either there is a* 3×3 *orthogonal matrix O such that*

$$\beta(\boldsymbol{v}) = O\boldsymbol{v}, \ \ \boldsymbol{v} \in \mathbb{B}^3, or$$

(ii) β *is the trivial map,*

$$\beta(\boldsymbol{v}) = 0, \ \ \boldsymbol{v} \in \mathbb{B}^3.$$

Theorem 5. *[28] (Theorem 6). For* $n \geq 2$*, continuous endomorphisms of the Einstein gyrogroup* $(\mathbb{B}^n, \oplus_{\mathrm{E}})$ *are precisely the restrictions to* $\mathbb{B}^n$ *of orthogonal transformations of* $\mathbb{R}^n$ *and the map that sends everything to 0.*

The following theorem shows that a continuous gyrolinear functional on the Möbius gyrovector space $(\mathbb{V}_1, \oplus, \otimes)$ is just the trivial map. The orthogonal gyroexpansion (Theorem 3) is used for the proof.

Theorem 6. *[19] (Theorem 11). Let* $\mathbb{V}$ *be a separable real Hilbert space with* $\dim \mathbb{V} \geq 2$*. Consider the Poincaré metric h on the ball* $\mathbb{V}_1$ *and the interval* $(-1, 1)$*, respectively. If a continuous map* $f : \mathbb{V}_1 \to (-1, 1)$ *satisfies*

$$f(\boldsymbol{x} \oplus \boldsymbol{y}) = f(\boldsymbol{x}) \oplus f(\boldsymbol{y}) \tag{1}$$

for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{V}_1$*, then* $f(\boldsymbol{x}) = 0$ *for all* $\boldsymbol{x} \in \mathbb{V}_1$*.*

Theorems 4–6 suggest that, in a certain sense, the gyroadditivity (1) is too strong for continuous maps between gyrovector spaces. Therefore, it is natural to introduce a suitable notion which corresponds to the linearity of maps between inner product spaces.

Definition 4. *Let $\mathbb{U}$ and $\mathbb{V}$ be two normed spaces. For any map $f : \mathbb{U}_1 \to \mathbb{V}_1$ and for any positive number $s > 0$, we define a family of maps $f_s : \mathbb{U}_s \to \mathbb{V}_s$ by the equation*

$$f_s(\boldsymbol{x}) = sf\left(\frac{\boldsymbol{x}}{s}\right) \tag{2}$$

for any element $\boldsymbol{x} \in \mathbb{U}_s$.

Now we define the notion of quasi-gyrolinearity for maps between two Möbius gyrovector spaces. It seems that [19] (Theorem 15) provides sufficiently reasonable motivation for making the following definitions.

Definition 5. *(cf. [19] (Definition 17)) Let $\mathbb{U}$ and $\mathbb{V}$ be two real inner product spaces, and let $T : \mathbb{U} \to \mathbb{V}$ be a bounded linear operator. A map $f : \mathbb{U}_1 \to \mathbb{V}_1$ is said to be quasi-gyrolinear with respect to T if the family $\{f_s\}$ defined by Formula (2) satisfies the following conditions:*

$$\begin{aligned} f_s(\boldsymbol{x} \oplus_s \boldsymbol{y}) &\to T(\boldsymbol{x} + \boldsymbol{y}) \\ f_s(\boldsymbol{x}) \oplus_s f_s(\boldsymbol{y}) &\to T\boldsymbol{x} + T\boldsymbol{y} \\ f_s(r \otimes_s \boldsymbol{x}) &\to T(r\boldsymbol{x}) \\ r \otimes_s f_s(\boldsymbol{x}) &\to rT\boldsymbol{x}, \end{aligned}$$

as $s \to \infty$, for any element $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{V}$ and any real number $r \in \mathbb{R}$. Note that $\boldsymbol{x} \oplus_s \boldsymbol{y}$, $r \otimes_s \boldsymbol{x}$ can be defined in $\mathbb{U}_s$ for sufficiently large $s > 0$.

The author presented a class of continuous quasi-gyrolinear functionals on the Möbius gyrovector spaces.

Theorem 7. *[19] (Theorem 27). Let $\mathbb{V}$ be a real Hilbert space, let $\{\boldsymbol{e}_j\}_{j=1}^{\infty}$ be a complete orthonormal sequence in $\mathbb{V}$, and let $\{c_j\}_{j=1}^{\infty}$ be a square summable sequence of real numbers. Consider the Poincaré metric h on both the Möbius gyrovector space $\mathbb{V}_1$ and the interval $(-1, 1)$. For an arbitrary element $\boldsymbol{x}$ in $\mathbb{V}_1$, we can apply the orthogonal gyroexpansion (Theorem 3) to get a unique sequence $(r_1, r_2, \cdots)$ of real numbers, such that*

$$\boldsymbol{x} = \sum_{j=1}^{\infty}{}^{\oplus}\, r_j \otimes \frac{\boldsymbol{e}_j}{2}.$$

Then, we can define a map $f : \mathbb{V}_1 \to (-1, 1)$ by the equation

$$f(\boldsymbol{x}) = \left(\sum_{j=1}^{\infty} c_j r_j\right) \otimes \frac{1}{2}.$$

Moreover, f is continuous and quasi-gyrolinear with respect to the bounded linear functional $\boldsymbol{x} \mapsto \boldsymbol{x} \cdot \boldsymbol{c}$, where $\boldsymbol{c}$ is defined by the equation

$$\boldsymbol{c} = \sum_{j=1}^{\infty} c_j \boldsymbol{e}_j.$$

3. Quasi Gyrolinear Maps between the Möbius Gyrovector Spaces Induced from Finite Matrices

In this section, we assume that real Hilbert spaces are finite-dimensional for simplicity. We denote by $M_{m,n}(\mathbb{R})$ the set of all $m \times n$ matrices whose entries are real numbers.

Definition 6. *Suppose that $\mathbb{U}$ and $\mathbb{V}$ are two finite dimensional real Hilbert spaces, and that $\{e_j\}_{j=1}^n$ (resp. $\{f_i\}_{i=1}^m$) is an orthonormal basis in $\mathbb{U}$ (resp. $\mathbb{V}$). Let $A = (a_{ij}) \in M_{m,n}(\mathbb{R})$, which can be regarded as a bounded linear operator from $\mathbb{U}$ to $\mathbb{V}$. Consider the Poincaré metric h on both the Möbius gyrovector spaces $\mathbb{U}_1$ and $\mathbb{V}_1$. For an arbitrary element $\boldsymbol{x}$ in $\mathbb{U}_1$, we can apply the orthogonal gyroexpansion to get a unique n-tuple $(r_1, \cdots, r_n)$ of real numbers, such that*

$$\boldsymbol{x} = r_1 \otimes_1 \frac{\boldsymbol{e}_1}{2} \oplus_1 \cdots \oplus_1 r_n \otimes_1 \frac{\boldsymbol{e}_n}{2}.$$

Then we define a map $f_A : \mathbb{U}_1 \to \mathbb{V}_1$ by the equation

$$f_A(\boldsymbol{x}) = (a_{11}r_1 + \cdots + a_{1n}r_n) \otimes_1 \frac{\boldsymbol{f}_1}{2} \oplus_1 \cdots \oplus_1 (a_{m1}r_1 + \cdots + a_{mn}r_n) \otimes_1 \frac{\boldsymbol{f}_m}{2}.$$

Now we present the main theorem of this paper, which is a new result.

Theorem 8. *The map $f_A : \mathbb{U}_1 \to \mathbb{V}_1$ defined in Definition 6 is continuous and quasi-gyrolinear with respect to A.*

Proof. Take two arbitrary elements

$$\boldsymbol{x} = x_1\boldsymbol{e}_1 + \cdots + x_n\boldsymbol{e}_n, \qquad \boldsymbol{y} = y_1\boldsymbol{e}_1 + \cdots + y_n\boldsymbol{e}_n$$

in $\mathbb{U}$, where x_j, y_j are real numbers for $j = 1, \cdots, n$. Then, for sufficiently large $s > 0$, it follows from the definition of $\oplus_1$ that

$$\frac{\boldsymbol{x}}{s} \oplus_1 \frac{\boldsymbol{y}}{s} = \frac{(1 + 2\frac{\boldsymbol{x}}{s}\cdot\frac{\boldsymbol{y}}{s} + ||\frac{\boldsymbol{y}}{s}||^2)\frac{\boldsymbol{x}}{s} + (1 - ||\frac{\boldsymbol{x}}{s}||^2)\frac{\boldsymbol{y}}{s}}{1 + 2\frac{\boldsymbol{x}}{s}\cdot\frac{\boldsymbol{y}}{s} + ||\frac{\boldsymbol{x}}{s}||^2||\frac{\boldsymbol{y}}{s}||^2}$$

$$= \frac{1}{s} \cdot \frac{(1 + \frac{2}{s^2}\boldsymbol{x}\cdot\boldsymbol{y} + \frac{1}{s^2}||\boldsymbol{y}||^2)\boldsymbol{x} + (1 - \frac{1}{s^2}||\boldsymbol{x}||^2)\boldsymbol{y}}{1 + \frac{2}{s^2}\boldsymbol{x}\cdot\boldsymbol{y} + \frac{1}{s^4}||\boldsymbol{x}||^2||\boldsymbol{y}||^2}$$

and

$$\left(\frac{\boldsymbol{x}}{s} \oplus_1 \frac{\boldsymbol{y}}{s}\right)\cdot\boldsymbol{e}_j = \frac{1}{s} \cdot \frac{(1 + \frac{2}{s^2}\boldsymbol{x}\cdot\boldsymbol{y} + \frac{1}{s^2}||\boldsymbol{y}||^2)x_j + (1 - \frac{1}{s^2}||\boldsymbol{x}||^2)y_j}{1 + \frac{2}{s^2}\boldsymbol{x}\cdot\boldsymbol{y} + \frac{1}{s^4}||\boldsymbol{x}||^2||\boldsymbol{y}||^2}.$$

For each sufficiently large $s > 0$, there exists a unique n-tuple $(r_1(s), \cdots, r_n(s))$ of real numbers, such that

$$\frac{\boldsymbol{x}}{s} \oplus_1 \frac{\boldsymbol{y}}{s} = r_1(s) \otimes_1 \frac{\boldsymbol{e}_1}{2} \oplus_1 \cdots \oplus_1 r_n(s) \otimes_1 \frac{\boldsymbol{e}_n}{2} = c_1(s)\boldsymbol{e}_1 \oplus_1 \cdots \oplus_1 c_n(s)\boldsymbol{e}_n,$$

where we put $c_j(s) = \tanh\left(r_j(s)\tanh^{-1}\frac{1}{2}\right)$. Then, we have

$$\left|\left|\frac{\boldsymbol{x} \oplus_s \boldsymbol{y}}{s}\right|\right|^2 = \left|\left|\frac{\boldsymbol{x}}{s} \oplus_1 \frac{\boldsymbol{y}}{s}\right|\right|^2 = ||c_1(s)\boldsymbol{e}_1 \oplus_1 \cdots \oplus_1 c_n(s)\boldsymbol{e}_n||^2 = c_1(s)^2 \oplus_1 \cdots \oplus_1 c_n(s)^2$$

$$\geq c_j(s)^2.$$

It follows from $\boldsymbol{x} \oplus_s \boldsymbol{y} \to \boldsymbol{x} + \boldsymbol{y}$ that $\left|\left|\frac{\boldsymbol{x} \oplus_s \boldsymbol{y}}{s}\right|\right| \to 0$, and hence, $c_j(s) \to 0$ as $s \to \infty$.

Put $\boldsymbol{z} = c_2(s)\boldsymbol{e}_2 \oplus_1 \cdots \oplus_1 c_n(s)\boldsymbol{e}_n$. Note that $||z|| \leq \left|\left|\frac{\boldsymbol{x} \oplus_s \boldsymbol{y}}{s}\right|\right| \to 0$ as $s \to \infty$.

$$\left(\frac{\boldsymbol{x}}{s} \oplus_1 \frac{\boldsymbol{y}}{s}\right) \cdot \boldsymbol{e}_1 = (c_1\boldsymbol{e}_1 \oplus_1 \boldsymbol{z}) \cdot \boldsymbol{e}_1 = \frac{(1+||\boldsymbol{z}||^2)c_1\boldsymbol{e}_1 + (1-{c_1}^2)\boldsymbol{z}}{1+{c_1}^2||\boldsymbol{z}||^2} \cdot \boldsymbol{e}_1 = \frac{(1+||\boldsymbol{z}||^2)c_1}{1+{c_1}^2||\boldsymbol{z}||^2}$$

$$(\boldsymbol{x} \oplus_s \boldsymbol{y}) \cdot \boldsymbol{e}_1 = \frac{(1+||\boldsymbol{z}||^2)sc_1}{1+{c_1}^2||\boldsymbol{z}||^2}.$$

By letting $s \to \infty$ in the formula above, we have $sc_1 \to (\boldsymbol{x}+\boldsymbol{y}) \cdot \boldsymbol{e}_1 = x_1 + y_1$.

Assume that we have shown $sc_j \to x_j + y_j \ \ (j = 1, \cdots, j_0)$.

$$\left\{\ominus(c_1\boldsymbol{e}_1 \oplus_1 \cdots \oplus_1 c_{j_0}\boldsymbol{e}_{j_0}) \oplus_1 \left(\frac{\boldsymbol{x}}{s} \oplus_1 \frac{\boldsymbol{y}}{s}\right)\right\} \cdot \boldsymbol{e}_{j_0+1} = (c_{j_0+1}\boldsymbol{e}_{j_0+1} \oplus_1 \cdots \oplus_1 c_n\boldsymbol{e}_n) \cdot \boldsymbol{e}_{j_0+1}$$

$$= \frac{(1+||\boldsymbol{z}'||^2)c_{j_0+1}\boldsymbol{e}_{j_0+1} + (1-{c_{j_0+1}}^2)\boldsymbol{z}'}{1+{c_{j_0+1}}^2||\boldsymbol{z}'||^2} \cdot \boldsymbol{e}_{j_0+1} = \frac{(1+||\boldsymbol{z}'||^2)c_{j_0+1}}{1+{c_{j_0+1}}^2||\boldsymbol{z}'||^2},$$

where we put $\boldsymbol{z}' = c_{j_0+2}\boldsymbol{e}_{j_0+2} \oplus_1 \cdots \oplus_1 c_n\boldsymbol{e}_n$. By multiplying s to both sides,

$$\{-(sc_1\boldsymbol{e}_1 \oplus_s \cdots \oplus_s sc_{j_0}\boldsymbol{e}_{j_0}) \oplus_s (\boldsymbol{x} \oplus_s \boldsymbol{y})\} \cdot \boldsymbol{e}_{j_0+1} = \frac{(1+||\boldsymbol{z}'||^2)sc_{j_0+1}}{1+{c_{j_0+1}}^2||\boldsymbol{z}'||^2}$$

$$sc_{j_0+1} \to \{-((x_1+y_1)\boldsymbol{e}_1 + \cdots + (x_{j_0}+y_{j_0})\boldsymbol{e}_{j_0}) + (\boldsymbol{x}+\boldsymbol{y})\} \cdot \boldsymbol{e}_{j_0+1} = x_{j_0+1} + y_{j_0+1}.$$

For a while, we simply denote f_A by f. Now,

$$f\left(\frac{\boldsymbol{x}}{s} \oplus_1 \frac{\boldsymbol{y}}{s}\right) = (a_{11}r_1 + \cdots + a_{1n}r_n) \otimes_1 \frac{\boldsymbol{f}_1}{2} \oplus_1 \cdots \oplus_1 (a_{m1}r_1 + \cdots + a_{mn}r_n) \otimes_1 \frac{\boldsymbol{f}_m}{2}$$

$$= \tanh\left((a_{11}r_1 + \cdots + a_{1n}r_n)\tanh^{-1}\left|\left|\frac{\boldsymbol{f}_1}{2}\right|\right|\right)\frac{\frac{\boldsymbol{f}_1}{2}}{\left|\left|\frac{\boldsymbol{f}_1}{2}\right|\right|}$$

$$\oplus_1 \cdots \oplus_1 \tanh\left((a_{m1}r_1 + \cdots + a_{mn}r_n)\tanh^{-1}\left|\left|\frac{\boldsymbol{f}_m}{2}\right|\right|\right)\frac{\frac{\boldsymbol{f}_m}{2}}{\left|\left|\frac{\boldsymbol{f}_m}{2}\right|\right|}$$

$$= \tanh\left(\left(a_{11}\frac{\tanh^{-1}c_1}{\tanh^{-1}\frac{1}{2}} + \cdots + a_{1n}\frac{\tanh^{-1}c_n}{\tanh^{-1}\frac{1}{2}}\right)\tanh^{-1}\frac{1}{2}\right)\boldsymbol{f}_1$$

$$\oplus_1 \cdots \oplus_1 \tanh\left(\left(a_{m1}\frac{\tanh^{-1}c_1}{\tanh^{-1}\frac{1}{2}} + \cdots + a_{mn}\frac{\tanh^{-1}c_n}{\tanh^{-1}\frac{1}{2}}\right)\tanh^{-1}\frac{1}{2}\right)\boldsymbol{f}_m$$

$$= \tanh\left(a_{11}\tanh^{-1}c_1 + \cdots + a_{1n}\tanh^{-1}c_n\right)\boldsymbol{f}_1$$

$$\oplus_1 \cdots \oplus_1 \tanh\left(a_{m1}\tanh^{-1}c_1 + \cdots + a_{mn}\tanh^{-1}c_n\right)\boldsymbol{f}_m.$$

It follows from $sc_j \to x_j + y_j$ and $c_j \to 0$ as $s \to \infty$ that $s\tanh^{-1}c_j \to x_j + y_j$. By applying [19] (Lemma 21), we can obtain

$$s\tanh\left(a_{i1}\tanh^{-1}c_1 + \cdots + a_{in}\tanh^{-1}c_n\right) = s\tanh\left(\frac{a_{i1}s\tanh^{-1}c_1 + \cdots + a_{in}s\tanh^{-1}c_n}{s}\right)$$

$$\to a_{i1}(x_1+y_1) + \cdots + a_{in}(x_n+y_n) \quad (s \to \infty)$$

for $i = 1 \cdots, m$. Thus, by [19] (Lemma 19), we can conclude that

$$\begin{aligned} f_s(\boldsymbol{x} \oplus_s \boldsymbol{y}) \to \{a_{11}(x_1 + y_1) + \cdots + a_{1n}(x_n + y_n)\}\boldsymbol{f}_1 \\ + \cdots + \{a_{m1}(x_1 + y_1) + \cdots + a_{mn}(x_n + y_n)\}\boldsymbol{f}_m \\ = A(\boldsymbol{x} + \boldsymbol{y}). \end{aligned}$$

By putting $\boldsymbol{y} = \boldsymbol{0}$ in the result just established above, we have $f_s(\boldsymbol{x}) \to A\boldsymbol{x}$,

$$f_s(\boldsymbol{x}) \oplus_s f_s(\boldsymbol{y}) \to A\boldsymbol{x} + A\boldsymbol{y}$$

and

$$r \otimes_s f_s(\boldsymbol{x}) = s \tanh\left(r \tanh^{-1} \frac{||f_s(\boldsymbol{x})||}{s}\right) \frac{f_s(\boldsymbol{x})}{||f_s(\boldsymbol{x})||} \to r||A\boldsymbol{x}|| \cdot \frac{A\boldsymbol{x}}{||A\boldsymbol{x}||} = rA\boldsymbol{x}$$

as $s \to \infty$.

Moreover, for sufficiently large $s > 0$, it follows from the definition of $\otimes_1$ that

$$r \otimes_1 \frac{\boldsymbol{x}}{s} = \tanh\left(r \tanh^{-1} \left|\left|\frac{\boldsymbol{x}}{s}\right|\right|\right) \frac{\frac{\boldsymbol{x}}{s}}{||\frac{\boldsymbol{x}}{s}||} = \tanh\left(r \tanh^{-1} \frac{||\boldsymbol{x}||}{s}\right) \frac{\boldsymbol{x}}{||\boldsymbol{x}||}$$

$$\left(r \otimes_1 \frac{\boldsymbol{x}}{s}\right) \boldsymbol{\cdot} \boldsymbol{e}_j = \tanh\left(r \tanh^{-1} \frac{||\boldsymbol{x}||}{s}\right) \frac{x_j}{||\boldsymbol{x}||}.$$

We can express as

$$r \otimes_1 \frac{\boldsymbol{x}}{s} = r_1 \otimes_1 \frac{\boldsymbol{e}_1}{2} \oplus_1 \cdots \oplus_1 r_n \otimes_1 \frac{\boldsymbol{e}_n}{2} = c_1 \boldsymbol{e}_1 \oplus_1 \cdots \oplus_1 c_n \boldsymbol{e}_n,$$

where we put $c_j = \tanh\left(r_j \tanh^{-1} \frac{1}{2}\right)$. Then, a similar argument to the first part of the proof shows that $sc_j \to rx_j$, and

$$f_s(r \otimes_s \boldsymbol{x}) \to (a_{11} r x_1 + \cdots + a_{1n} r x_n)\boldsymbol{f}_1 + \cdots + (a_{m1} r x_1 + \cdots + a_{mn} r x_n)\boldsymbol{f}_m = A(r\boldsymbol{x})$$

as $s \to \infty$. Thus, we can conclude that f_A is quasi-gyrolinear with respect to A. The continuity of f_A is an easy consequence of [19] (Lemma 26). This completes the proof. □

The following theorem shows a fundamental property of the composition of quasi-gyrolinear mappings of the form f_A.

Theorem 9. *Suppose that $\{\boldsymbol{e}_j\}_{j=1}^n$, $\{\boldsymbol{f}_i\}_{i=1}^m$, $\{\boldsymbol{g}_k\}_{k=1}^p$ are orthonormal bases of the respective real Hilbert spaces $\mathbb{U}$, $\mathbb{V}$, $\mathbb{W}$. Let $A = (a_{ij}) \in \mathrm{M}_{m,n}(\mathbb{R})$, $B = (b_{ij}) \in \mathrm{M}_{p,m}(\mathbb{R})$. Then, the composed map $f_B \circ f_A$ is also an induced map from the matrix BA. That is,*

$$f_B \circ f_A = f_{BA}.$$

Proof. Because

$$f_A(\boldsymbol{x}) = \left(\sum_{j=1}^n a_{1j} r_j\right) \otimes_1 \frac{\boldsymbol{f}_1}{2} \oplus_1 \cdots \oplus_1 \left(\sum_{j=1}^n a_{mj} r_j\right) \otimes_1 \frac{\boldsymbol{f}_m}{2},$$

we have

$$f_B(f_A(\boldsymbol{x})) = \left\{ b_{11} \sum_{j=1}^{n} a_{1j} r_j + \cdots + b_{1m} \sum_{j=1}^{n} a_{mj} r_j \right\} \otimes_1 \frac{\boldsymbol{g}_1}{2}$$

$$\oplus_1 \cdots \oplus_1 \left\{ b_{p1} \sum_{j=1}^{n} a_{1j} r_j + \cdots + b_{pm} \sum_{j=1}^{n} a_{mj} r_j \right\} \otimes_1 \frac{\boldsymbol{g}_p}{2}.$$

Then,

$$b_{k1} \sum_{j=1}^{n} a_{1j} r_j + \cdots + b_{km} \sum_{j=1}^{n} a_{mj} r_j = \sum_{j=1}^{n} \left(\sum_{l=1}^{m} b_{kl} a_{lj} \right) r_j$$

and $\sum_{l=1}^{m} b_{kl} a_{lj}$ is the (k, j) entry of the matrix BA; hence, the composed map $f_B \circ f_A$ coincides with the map f_{BA} induced from the matrix BA. □

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