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On the Hybrid Fractional Differential Equations with Fractional Proportional Derivatives of a Function with Respect to a Certain Function

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Abstract: This paper deals with a new class of hybrid fractional differential equations with fractional proportional derivatives of a function with respect to a certain continuously differentiable and increasing function ϑ. By means of a hybrid fixed point theorem for a product of two operators, an existence result is proved. Furthermore, the sufficient conditions of the continuous dependence on the given parameters are investigated. Finally, a simulative example is given to highlight the acquired outcomes.

Keywords: hybrid fractional differential equations; proportional fractional derivatives; continuous dependence; hybrid fixed point theorem

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1. Introduction

The theory of fractional differential equations has recently acquired plentiful circulation and great significance because of its rife applications in the fields of science and engineering as a mathematical model. For instance, see the books [1–7].

Within the past years, there have propounded various notions about fractional derivatives. Here, we point out to the most famous kinds, including Liouville, Caputo, Hadamard, Caputo-Fabrizio derivatives and etc. In consequence, this has led to different structures of arbitrary order differential equations formulated by several fractional operators. However, it has been understood that the most efficient procedure to discuss such a variety of fractional operators is to accommodate generalized structures of fractional operators that involve many other operators (see [8–12]).

In [13], Khalil et al. introduced a new interesting fractional derivative definition called conformable derivative. This new fractional derivative is not a fractional derivative, but it is simply a first derivative multiplied by an additional simple factor. Therefore, this new definition seems to be a natural extension of the classical derivative. More properties and a modified type of this derivative were explored in [14]. Anderson and Ulness [15] proposed a modified conformable derivative by utilizing proportional derivatives. In fact, they proposed the modified conformable (proportional) differential operator of order ϱ as

$$^{P}\mathfrak{D}_t^{\varrho}\phi(t) = \kappa_1(\varrho, t)\phi(t) + \kappa_0(\varrho, t)\phi^{'}(t),$$

where the function ϕ is differentiable at t, $\phi^{'} = \frac{d\phi}{dt}$ and $\kappa_0, \kappa_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ are continuous functions of the variable t and the parameter $\varrho \in [0,1]$ which satisfy the following conditions for all $t \in \mathbb{R}$:

$$\lim_{\varrho\to 0^+} \kappa_0(\varrho,t) = 0, \quad \lim_{\varrho\to 1^-} \kappa_0(\varrho,t) = 1, \quad \kappa_0(\varrho,t) \neq 0, \; \varrho \in (0,1], \tag{1}$$

$$\lim_{\varrho\to 0^+} \kappa_1(\varrho,t) = 1, \quad \lim_{\varrho\to 1^-} \kappa_1(\varrho,t) = 0, \quad \kappa_1(\kappa,t) \neq 0, \; \varrho \in [0,1). \tag{2}$$

This newly defined local derivative tends to the original function as the order ϱ tends to zero. Thus, they were able to improve the conformable derivatives.

In [16], Jarad et al. exhibited a new type of fractional operators produced from the modified conformable derivatives. Later, Jarad et al. [17,18] proposed a new more general form of the proportional derivative of a function ϕ with respect to a certain continuously differentiable and increasing function ϑ. The kernel obtained in their investigation contains an exponential function and is function dependent (more details about the newly proportional derivative can be seen in Section 2). For the interest of readers, we attract their attention to some recent papers [19–21].

Recently, a new class of mathematical modelings based on hybrid fractional differential equations with hybrid or non-hybrid boundary value conditions have accomplished a large inquisitiveness of many researchers using different techniques (see, for example, [22–25]). Fractional hybrid differential equations can be employed in modeling and describing non-homogeneous physical phenomena that take place in their form. The importance of hybrid differential equations lies in the fact that they include various dynamical systems as particular cases. This class of differential equations includes the derivative of unknown function hybrid with the nonlinearity depending on it.

Furthermore, hybrid differential equations arise from a variety of different areas of applied mathematics and physics, e.g., in the deflection of a curved beam having a constant or varying cross section, a three-layer beam, electromagnetic waves or gravity driven flows and so on (see [26–29]).

Dhage and Lakshmikantham [30] had precedence in dealing with first-order hybrid differential equations, namely

$$\begin{cases} \frac{d}{dt}\left(\frac{x(t)}{f(t,x(t))}\right) = & g(t,x(t)), \; a.e. \;\; t \in J_0, \\ x(t_0) = x_0 \in \mathbb{R}, \end{cases}$$

where $f : J_0 \times \mathbb{R} \to \mathbb{R}\backslash\{0\}$, $g : J_0 \times \mathbb{R} \to \mathbb{R}$ are continuous functions and $J_0 = [t_0, t_0 + a]$ is a bounded interval in $\mathbb{R}$ for some t_0 and $a \in \mathbb{R}$ with $a > 0$. Under mixed Lipschitz and Carathèodory conditions, they established some fundamental hybrid differential inequalities that are useful for the existence of extremal solutions.

Soon after that, Zhao et al. [31] studied the fractional hybrid differential equations with Riemann–Liouville differential operator

$$\begin{cases} \mathfrak{D}^q_{0^+}\left(\frac{x(t)}{f(t,x(t))}\right) = & g(t,x(t)), \; t \in [0,T], \\ x(0) = 0, \end{cases}$$

where $\mathfrak{D}^q_{0^+}$ is the Riemann-Liouville fractional derivative of order $0 < q < 1$, $f : [0,T] \times \mathbb{R} \to \mathbb{R}\backslash\{0\}$ and $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Many articles have been devoted to the hybrid fractional differential equations and inclusions. Before we proceed, we assemble some works in this regard. Baleanu et al. [32]

investigated the results on the existence of solutions for the fractional hybrid differential equations and inclusions:

$$\begin{cases} {}^{C}\mathfrak{D}_0^r\left(\frac{z(\tau)}{\rho(\tau,z(\tau))}\right) = & \kappa(\tau,z(\tau)),\ \tau\in[0,1],\\ {}^{C}\mathfrak{D}_0^r\left(\frac{z(\tau)}{\rho(\tau,z(\tau))}\right) \in & \mathcal{H}(\tau,z(\tau)),\ \tau\in[0,1], \end{cases}$$

supplemented by the three-point integral boundary conditions

$$\begin{cases} z(0)=0,\\ \left(\frac{z(\tau)}{\rho(\tau,z(\tau))}\right)_{\tau=0} + \mathfrak{I}_0^d\left(\frac{z(\tau)}{\rho(\tau,z(\tau))}\right)_{\tau=\eta} = & 0, \quad \eta\in(0,1),\\ \left(\frac{z(\tau)}{\rho(\tau,z(\tau))}\right)_{\tau=0} + \mathfrak{I}_0^d\left(\frac{z(\tau)}{\rho(\tau,z(\tau))}\right)_{\tau=1} = & 0, \end{cases}$$

where ${}^{C}\mathfrak{D}_0^r$ is the Caputo derivative operator of the fractional order $r\in(2,3]$, $\mathfrak{I}_0^d$ is the Riemann–Liouville integral operator of the fractional order $d>0$, $\rho:[0,1]\times\mathbb{R}\to\mathbb{R}\backslash\{0\}$, $\kappa:[0,1]\times\mathbb{R}\to\mathbb{R}$ are continuous functions and $\mathcal{H}:[0,1]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is a multivalued map ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$). Authors established the existence results for above hybrid problems by means of Dhage's nonlinear alternative of Schaefer type.

In [33], Sitho et al. studied the initial value problems for hybrid fractional integro-differential equations:

$$\begin{cases} \mathfrak{D}_{0^+}^{\alpha}\left(\frac{x(t)-\sum_{i=1}^{m}\mathfrak{I}^{\beta_i}h_i(t,x(t))}{f(t,x(t))}\right) = & g(t,x(t)),\ \tau\in[0,T],\\ x(0)=0, \end{cases}$$

where $\mathfrak{D}_{0^+}^{\alpha}$ denotes the Riemann–Liouville fractional derivative of order $0<\alpha\le 1$, and $\mathfrak{I}^{\beta_i}$ is the Riemann–Liouville fractional integral of order $\beta_i>0$, $f:J\times\mathbb{R}\to\mathbb{R}\backslash\{0\}$ and $g:J\times\mathbb{R}\to\mathbb{R}$ are continuous functions. Based on hybrid fixed point theorems for the sum of three operators, the authors proved the main results.

Stimulated by the above papers, we study a new class of hybrid fractional differential equation with fractional proportional derivatives of a function with respect to a certain continuously differentiable and increasing function. Indeed, we consider the following hybrid fractional problem:

$$\begin{cases} {}_{a}\mathfrak{D}^{\delta,\varrho,\vartheta}\left(\frac{u(t)}{\Psi(t,u(t))}\right) = \Phi(t,u(t)),\ \ t\in J:=[a,b],\\ {}_{a}\mathfrak{I}^{1-\delta,\varrho,\vartheta}\left(\frac{u(t)}{\Psi(t,u(t))}\right)_{t=a} = \lambda\in\mathbb{R}, \end{cases} \tag{3}$$

where $0<\delta\le 1$, $\varrho\in(0,1]$, ${}_{a}\mathfrak{D}^{\delta,\varrho,\vartheta}$ is the proportional fractional derivative of order δ with respect to a certain continuously differentiable and increasing function ϑ with $\vartheta'(t)>0$ for all $t\in J$, ${}_{a}\mathfrak{I}^{1-\delta,\varrho,\vartheta}$ is the left proportional fractional integral of order $(1-\delta)$ with respect to a continuously differentiable and increasing function ϑ, $\Psi:J\times\mathbb{R}\to\mathbb{R}\backslash\{0\}$ and $\Phi:J\times\mathbb{R}\to\mathbb{R}$ are continuous functions.

Furthermore, the study of fractional differential equations in terms of their inputs (fractional orders, associated parameters, and appropriate function) has attracted interested researchers due to its significance in experimental process (see [34,35]). Based on this, the topic of continuity of solution of the hybrid fractional problem (3) with respect to inputs is important and worth considering.

The rest of the paper is organized as follows: In Section 2, we recall some useful preliminaries. In Section 3, we give equivalent fractional integral equation to the linear issue of the hybrid fractional differential Equation (3), while in Section 4, we prove the main existence result in this paper. In Section 5, we establish the sufficient conditions under which solutions of the hybrid fractional differential Equation (3) depend continuously on initial conditions and other parameters. In order to confirm the validity of the theoretical findings, a simulative numerical example is given in Section 6.

2. Preliminaries

In this section, we recall some basic definitions, lemmas, and properties of the fractional proportional derivative and integral of a function with respect to a certain function. The terms and notations are taken from [17,18].

Definition 1. *(The proportional derivative of a function with respect to a certain function) Take $\varrho \in [0,1]$ and let the functions $\kappa_0, \kappa_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ be continuous such that for all $t \in \mathbb{R}$ we have*

$$\lim_{\varrho \to 0^+} \kappa_1(\varrho,t) = 1, \quad \lim_{\varrho \to 0^+} \kappa_0(\varrho,t) = 0, \quad \lim_{\varrho \to 1^-} \kappa_1(\varrho,t) = 0, \quad \lim_{\varrho \to 1^-} \kappa_0(\varrho,t) = 1,$$

and $\kappa_1(\varrho,t) \neq 0, \varrho \in [0,1)$, $\kappa_0(\varrho,t) \neq 0, \varrho \in (0,1]$. Let $\vartheta(t)$ be a continuously differentiable and increasing function. Then, the proportional differential operator of order ϱ of ϕ with respect to ϑ is defined by

$$\mathfrak{D}^{\varrho,\vartheta}\phi(t) = \kappa_1(\varrho,t)\phi(t) + \kappa_0(\varrho,t)\frac{\phi'(t)}{\vartheta'(t)}. \tag{4}$$

For the restricted case when $\kappa_1(\varrho,t) = 1 - \varrho$ and $\kappa_0(\varrho,t) = \varrho$, (4) becomes

$$\mathfrak{D}^{\varrho,\vartheta}\phi(t) = (1-\varrho)\phi(t) + \varrho\frac{\phi'(t)}{\vartheta'(t)}. \tag{5}$$

Remark 1. *It is useful to mention that the derivative of the function ϕ in Equation (4) is with respect to another function ϑ. So, one can be sure that*

$$\frac{d\phi}{d\vartheta} = \frac{d\phi/dt}{d\vartheta/dt} = \frac{\phi'}{\vartheta'}.$$

The corresponding integral of (5) is

$${}_a\mathfrak{I}^{1,\varrho,\vartheta}\phi(t) = \frac{1}{\varrho}\int_a^t e^{\frac{\varrho-1}{\varrho}(\vartheta(t)-\vartheta(s))}\phi(s)\vartheta'(s)ds. \tag{6}$$

where we confess that ${}_a\mathfrak{I}^{0,\varrho}\phi(t) = \phi(t)$. For more details, see [17].

Definition 2. *(The proportional integral of a function with respect to a certain function) Take $\varrho \in (0,1]$, $\delta \in \mathbb{C}$, $\Re(\delta) > 0$, $\vartheta \in C^1[a,b]$, $\vartheta'(t) > 0$. The left and right fractional integrals of the function $\phi \in L^1[a,b]$ with respect to another function ϑ are defined by*

$$\begin{aligned} {}_a\mathfrak{I}^{\delta,\varrho,\vartheta}\phi(t) &= \frac{1}{\varrho^\delta\Gamma(\delta)}\int_a^t e^{\frac{\varrho-1}{\varrho}(\vartheta(t)-\vartheta(s))}(\vartheta(t)-\vartheta(s))^{\delta-1}\phi(s)\vartheta'(s)ds, && (7)\\ \mathfrak{I}_b^{\delta,\varrho,\vartheta}\phi(t) &= \frac{1}{\varrho^\delta\Gamma(\delta)}\int_t^b e^{\frac{\varrho-1}{\varrho}(\vartheta(s)-\vartheta(t))}(\vartheta(s)-\vartheta(t))^{\delta-1}\phi(s)\vartheta'(s)ds, && (8) \end{aligned}$$

respectively.

Definition 3. *Take $\varrho \in (0,1]$, $\delta \in \mathbb{C}$, $\Re(\delta) > 0$, $\vartheta \in C[a,b]$, $\vartheta'(t) > 0$. The left fractional derivative of the function $\phi \in C^n[a,b]$ with respect to another function ϑ is defined by*

$$\begin{aligned} {}_a\mathfrak{D}^{\delta,\varrho,\vartheta}\phi(t) &= \mathfrak{D}^{n,\varrho,\vartheta}\,{}_a\mathfrak{I}^{n-\delta,\varrho,\vartheta}\phi(t) \\ &= \frac{\mathfrak{D}_t^{n,\varrho,\vartheta}}{\varrho^{n-\delta}\Gamma(n-\delta)}\int_a^t e^{\frac{\varrho-1}{\varrho}(\vartheta(t)-\vartheta(s))}(\vartheta(t)-\vartheta(s))^{n-\delta-1}\phi(s)\vartheta'(s)ds, \quad (9) \end{aligned}$$

and the right fractional derivative of ϕ with respect to ϑ as

$$\begin{aligned}
\mathfrak{D}_b^{\delta,\varrho,\vartheta}\phi(t) &= {}_\ominus\mathfrak{D}^{n,\varrho,\vartheta}\mathfrak{I}_b^{n-\delta,\varrho,\vartheta}\phi(t) \\
&= \frac{{}_\ominus\mathfrak{D}^{n,\varrho,\vartheta}}{\varrho^{n-\delta}\Gamma(n-\delta)}\int_t^b e^{\frac{\varrho-1}{\varrho}(\vartheta(s)-\vartheta(t))}(\vartheta(s)-\vartheta(t))^{n-\delta-1}\phi(s)\vartheta'(s)ds, \quad (10)
\end{aligned}$$

where $n = [\Re(\delta)] + 1, \quad \mathfrak{D}^{n,\varrho,\vartheta} = \underbrace{\mathfrak{D}^{\varrho,\vartheta}\mathfrak{D}^{\varrho,\vartheta}\cdots\mathfrak{D}^{\varrho,\vartheta}}_{n\ times}$ *and*

$$ {}_\ominus\mathfrak{D}^{\varrho,\vartheta}\phi(t) := (1-\varrho)\phi(t) - \varrho\frac{\phi'(t)}{\vartheta'(t)}, \quad {}_\ominus\mathfrak{D}^{n,\varrho,\vartheta} = \underbrace{{}_\ominus\mathfrak{D}^{\varrho,\vartheta}{}_\ominus\mathfrak{D}^{\varrho,\vartheta}\cdots{}_\ominus\mathfrak{D}^{\varrho,\vartheta}}_{n\ times}.$$

Lemma 1. *[17] If $\varrho \in (0,1]$, $\Re(\delta) > 0$ and $\Re(\nu) > 0$. Then, for ϕ is continuous and defined for $t \geq a$, we have*

$$\begin{aligned}
{}_a\mathfrak{I}^{\delta,\varrho,\vartheta}\left({}_a\mathfrak{I}^{\nu,\varrho,\vartheta}\phi\right)(t) &= {}_a\mathfrak{I}^{\nu,\varrho,\vartheta}\left({}_a\mathfrak{I}^{\delta,\varrho,\vartheta}\phi\right)(t) = \left({}_a\mathfrak{I}^{\delta+\nu,\varrho,\vartheta}\phi\right)(t), \quad (11) \\
\mathfrak{I}_b^{\delta,\varrho,\vartheta}\left(\mathfrak{I}_b^{\nu,\varrho,\vartheta}\phi\right)(t) &= \mathfrak{I}_b^{\nu,\varrho,\vartheta}\left(\mathfrak{I}_b^{\delta,\varrho,\vartheta}\phi\right)(t) = \left(\mathfrak{I}_b^{\delta+\nu,\varrho,\vartheta}\phi\right)(t). \quad (12)
\end{aligned}$$

Lemma 2. *[17] If $\varrho \in (0,1]$, $\Re(\delta) > 0$ and $n = [\Re(\delta)] + 1$. Then, for ϕ is integrable on $t \geq a$ or $t \leq b$, we have*

$$\begin{aligned}
{}_a\mathfrak{D}^{\delta,\varrho,\vartheta}\,{}_a\mathfrak{I}^{\delta,\varrho,\vartheta}\phi(t) &= \phi(t), \quad (13) \\
\mathfrak{D}_b^{\delta,\varrho,\vartheta}\mathfrak{I}_b^{\delta,\varrho,\vartheta}\phi(t) &= \phi(t). \quad (14)
\end{aligned}$$

Lemma 3. *[18] Let $\Re[\delta] > 0$, $n = -[-\Re(\delta)]$, $\phi \in L^1[a,b]$ and $({}_a\mathfrak{I}^{\delta,\varrho,\vartheta}\phi)(t) \in AC^n[a,b]$. Then*

$$ {}_a\mathfrak{I}^{\delta,\varrho,\vartheta}\,{}_a\mathfrak{D}^{\delta,\varrho,\vartheta}\phi(t) = \phi(t) - e^{\frac{\varrho-1}{\varrho}(\vartheta(t)-\vartheta(a))}\sum_{j=1}^{n}({}_a\mathfrak{I}^{j-\delta,\varrho,\vartheta}\phi)(a^+)\frac{(\vartheta(t)-\vartheta(a))^{\delta-j}}{\varrho^{\delta-j}\Gamma(\delta+1-j)}. \quad (15)$$

For $0 < \delta \leq 1$, we have

$$ {}_a\mathfrak{I}^{\delta,\varrho,\vartheta}\,{}_a\mathfrak{D}^{\delta,\varrho,\vartheta}\phi(t) = \phi(t) - e^{\frac{\varrho-1}{\varrho}(\vartheta(t)-\vartheta(a))}({}_a\mathfrak{I}^{1-\delta,\varrho,\vartheta}\phi)(a^+)\frac{(\vartheta(t)-\vartheta(a))^{\delta-1}}{\varrho^{\delta-1}\Gamma(\delta)}. \quad (16)$$

Lemma 4. *[18] Let $\delta, \nu \in \mathbb{C}$ be such that $\Re(\delta) \geq 0$, $\Re(\nu) > 0$ and $n = [\Re(\delta)] + 1$. Then, for any $\varrho > 0$, we have*

1. $\left({}_a\mathfrak{I}^{\delta,\varrho,\vartheta}e^{\frac{\varrho-1}{\varrho}\vartheta(x)}(\vartheta(x)-\vartheta(a))^{\nu-1}\right)(t) = \frac{\Gamma(\nu)\, e^{\frac{\varrho-1}{\varrho}\vartheta(t)}}{\varrho^\delta\Gamma(\nu+\delta)}(\vartheta(t)-\vartheta(a))^{\delta+\nu-1}, \Re(\delta) > 0.$

2. $\left(\mathfrak{I}_b^{\delta,\varrho,\vartheta}e^{-\frac{\varrho-1}{\varrho}\vartheta(x)}(\vartheta(b)-\vartheta(x))^{\nu-1}\right)(t) = \frac{\Gamma(\nu)\, e^{-\frac{\varrho-1}{\varrho}\vartheta(t)}}{\varrho^\delta\Gamma(\nu+\delta)}(\vartheta(b)-\vartheta(t))^{\delta+\nu-1}, \Re(\delta) > 0.$

3. $\left({}_a\mathfrak{D}^{\delta,\varrho,\vartheta}e^{\frac{\varrho-1}{\varrho}\vartheta(x)}(\vartheta(x)-\vartheta(a))^{\nu-1}\right)(t) = \frac{\varrho^\delta\Gamma(\nu)\, e^{\frac{\varrho-1}{\varrho}\vartheta(t)}}{\Gamma(\nu-\delta)}(\vartheta(t)-\vartheta(a))^{\nu-1-\delta}, \Re(\delta) \geq 0.$

4. $\left(\mathfrak{D}_b^{\delta,\varrho,\vartheta}e^{-\frac{\varrho-1}{\varrho}\vartheta(x)}(\vartheta(b)-\vartheta(x))^{\nu-1}\right)(t) = \frac{\varrho^\delta\Gamma(\nu)\, e^{-\frac{\varrho-1}{\varrho}\vartheta(t)}}{\Gamma(\nu-\delta)}(\vartheta(b)-\vartheta(t))^{\nu-1-\delta},$
$\Re(\delta) \geq 0.$

Remark 2. *In view of Definition 3 and for $0 < \delta \leq 1$, it is noted that*

$$\left({}_a\mathfrak{D}^{\delta,\varrho,\vartheta}e^{\frac{\varrho-1}{\varrho}\vartheta(x)}(\vartheta(x)-\vartheta(a))^{\delta-1}\right)(t) = 0.$$

To end this section, we define the supremum norm $\|\cdot\|$ in $\mathbb{E} := C(J,\mathbb{R})$ by $\|u\| = \sup_{t\in J}|u(t)|$ and the multiplication in $\mathbb{E}$ by

$$(uv)(t) = u(t)v(t), \ \ u, v \in \mathbb{E}, \ t \in J.$$

Plainly, $\mathbb{E}$ is a Banach algebra with respect to the supremum norm and multiplication in it.

Lemma 5. *[36] Let Ω be a nonempty, closed convex and bounded subset of a Banach algebra $\mathbb{E}$ and Let $\mathcal{A} : \mathbb{E} \to \mathbb{E}$ and $\mathcal{B} : \Omega \to \mathbb{E}$ be two operators satisfying:*

(i) *$\mathcal{A}$ is Lipschitzian with Lipschitz constant μ,*

(ii) *$\mathcal{B}$ is completely continuous,*

(iii) *$u = \mathcal{A}u\mathcal{B}v \Rightarrow u \in \Omega$ for all $v \in \Omega$,*

(iv) *$\mu L < 1$, where $L = \|\mathcal{B}(\Omega)\| = \sup_{u\in\Omega}\|\mathcal{B}(u)\|$.*

Then, the operator equation $u = \mathcal{A}u\mathcal{B}u$ has a solution in Ω.

3. Existence Results

In this section, we show the existence results for the hybrid fractional problem (3) by virtue of hybrid fixed point theorem for a product of two operators in a Banach algebra due to Dhage [36] (see Lemma 5).

We begin this section by the following essential definition of the mild solution of the hybrid fractional problem (3).

Definition 4. *A function $u \in C(J,\mathbb{R})$ is said to be a mild solution of the hybrid fractional problem (3) if the function $t \mapsto \frac{u}{\Psi(t,u)}$ is continuous for each $u \in \mathbb{R}$ and u satisfies the fractional integral equation*

$$\begin{aligned} u(t) =& \Psi(t,u(t))\left[\frac{\lambda e^{\frac{\varrho-1}{\varrho}(\vartheta(t)-\vartheta(a))}}{\varrho^{\delta-1}\Gamma(\delta)}(\vartheta(t)-\vartheta(a))^{\delta-1}\right. \\ &\left.+\frac{1}{\varrho^{\delta}\Gamma(\delta)}\int_a^t e^{\frac{\varrho-1}{\varrho}(\vartheta(t)-\vartheta(s))}(\vartheta(t)-\vartheta(s))^{\delta-1}w(s)\vartheta'(s)ds\right]. \end{aligned} \tag{17}$$

Now, we consider the following linear issue of the hybrid fractional problem (3):

$$\begin{cases} {}_a\mathfrak{D}^{\delta,\varrho,\vartheta}\left(\frac{u(t)}{\Psi(t,u(t))}\right) = w(t), \ \ t \in J, \\ {}_a\mathfrak{I}^{1-\delta,\varrho,\vartheta}\left(\frac{u(t)}{\Psi(t,u(t))}\right)_{t=a} = \lambda, \end{cases} \tag{18}$$

where w is a continuous real-valued function defined on J.

Lemma 6. *Let $0 < \delta \le 1$ and $w \in C(J)$. The linear hybrid fractional problem (18) has a solution $u \in C(J,\mathbb{R})$, if and only if the fractional integral Equation (17) is solvable, and their solutions coincide.*

Proof. $\Rightarrow$ Assume that u satisfies (18). Then, $\left(\frac{u(t)}{\Psi(t,u(t))}\right)$ is continuous and we get that ${}_a\mathfrak{D}^{\delta,\varrho,\vartheta}\left(\frac{u(t)}{\Psi(t,u(t))}\right)$ exists.

Applying the proportional fractional integral ${}_a\mathfrak{I}^{\delta,\varrho,\vartheta}(\cdot)$ to both sides of (18) and using Lemma 3, one has

$$\frac{u(t)}{\Psi(t,u(t))} = {}_a\mathfrak{I}^{\delta,\varrho,\vartheta} w(t) + \frac{e^{\frac{\varrho-1}{\varrho}(\vartheta(t)-\vartheta(a))}}{\varrho^{\delta-1}\Gamma(\delta)} (\vartheta(t)-\vartheta(a))^{\delta-1} {}_a\mathfrak{I}^{1-\delta,\varrho,\vartheta} \left(\frac{u(t)}{\Psi(t,u(t))}\right)_{t=a}.$$

In virtue of the boundary condition ${}_a\mathfrak{I}^{1-\delta,\varrho,\vartheta}\left(\frac{u(t)}{\Psi(t,u(t))}\right)_{t=a} = \lambda$, the fractional integral Equation (17) is obtained.

$\Leftarrow$ Conversely, assume that u satisfies (17). By definition, the function $t \mapsto \frac{u(t)}{\Psi(t,u(t))}$ is continuous for each $u \in C(J, \mathbb{R}^+)$ and hence almost everywhere differential on J. Then dividing by $\Psi(t, u(t))$, one has

$$\begin{aligned} \frac{u(t)}{\Psi(t,u(t))} =& \frac{\lambda e^{\frac{\varrho-1}{\varrho}(\vartheta(t)-\vartheta(a))}}{\varrho^{\delta-1}\Gamma(\delta)} (\vartheta(t)-\vartheta(a))^{\delta-1} \\ &+ \frac{1}{\varrho^{\delta}\Gamma(\delta)} \int_a^t e^{\frac{\varrho-1}{\varrho}(\vartheta(t)-\vartheta(s))} (\vartheta(t)-\vartheta(s))^{\delta-1} w(s) \vartheta'(s) ds. \end{aligned} \tag{19}$$

Operating the proportional fractional derivative ${}_a\mathfrak{D}^{\delta,\varrho,\vartheta}(\cdot)$ on both sides of (19) and using Lemma 2 with Remark 2, one obtain that

$$\begin{aligned} {}_a\mathfrak{D}^{\delta,\varrho,\vartheta}\left(\frac{u(t)}{\Psi(t,u(t))}\right) =& {}_a\mathfrak{D}^{\delta,\varrho,\vartheta}\left(\frac{\lambda e^{\frac{\varrho-1}{\varrho}(\vartheta(t)-\vartheta(a))}}{\varrho^{\delta-1}\Gamma(\delta)} (\vartheta(t)-\vartheta(a))^{\delta-1}\right) + {}_a\mathfrak{D}^{\delta,\varrho,\vartheta} {}_a\mathfrak{I}^{\delta,\varrho,\vartheta} w(t) \\ =& w(t). \end{aligned}$$

Thus, (18) is satisfied. Furthermore, using (6) and the results in Lemmas 1 and 4, one has

$$\begin{aligned} {}_a\mathfrak{I}^{1-\delta,\varrho,\vartheta}\left(\frac{u(t)}{\Psi(t,u(t))}\right) =& {}_a\mathfrak{I}^{1-\delta,\varrho,\vartheta}\left(\frac{\lambda e^{\frac{\varrho-1}{\varrho}(\vartheta(t)-\vartheta(a))}}{\varrho^{\delta-1}\Gamma(\delta)} (\vartheta(t)-\vartheta(a))^{\delta-1}\right) + {}_a\mathfrak{I}^{1-\delta,\varrho,\vartheta} {}_a\mathfrak{I}^{\delta,\varrho,\vartheta} w(t) \\ =& \lambda e^{\frac{\varrho-1}{\varrho}(\vartheta(t)-\vartheta(a))} + {}_a\mathfrak{I}^{1,\varrho,\vartheta} w(t). \end{aligned}$$

Substitution $t = a$ leads to ${}_a\mathfrak{I}^{1-\delta,\varrho,\vartheta}\left(\frac{u(t)}{\Psi(t,u(t))}\right)_{t=a} = \lambda$. This finishes the proof. □

For investigating the main results, the following assumptions will be imposed.

(A1) The functions $\Psi : J \times \mathbb{R} \to \mathbb{R}\backslash\{0\}$ and $\Phi : J \times \mathbb{R} \to \mathbb{R}$ are continuous.

(A2) For $u, v \in \mathbb{R}$, for all $t \in J$, there exists a bounded function $p : J \to \mathbb{R}^+$ such that

$$|\Psi(t,u) - \Psi(t,v)| \leq p(t)|u - v|,$$

with $p^* = \sup_{t \in J} |p(t)|$.

(A3) For $u \in \mathbb{R}$, for all $t \in J$, there exist a function $q \in C(J, \mathbb{R}^+)$ and a continuous non-decreasing function $\mathrm{H} : [0, \infty) \to [0, \infty)$ such that

$$|\Phi(t,u)| \leq q(t)\mathrm{H}(|u|),$$

with $q^* = \sup_{t \in J} |q(t)|$.

Theorem 1. *Assume that the assumptions* ***(A1)–(A3)*** *are satisfied. Then, the hybrid fractional problem (3) has a mild solution on J, provided that*

$$p^*\Lambda = p^*\left(\frac{|\lambda|(\vartheta(b)-\vartheta(a))^{\delta-1}}{\varrho^{\delta-1}\Gamma(\delta)} + \frac{q^*(\vartheta(b)-\vartheta(a))^{\delta}}{\varrho^{\delta}\Gamma(\delta+1)}\mathrm{H}(k)\right) < 1. \tag{20}$$

Proof. Define,

$$k = \Psi_0\Lambda(1-p^*\Lambda)^{-1}, \tag{21}$$

where $\Psi_0 = \sup_{t\in J}|\Psi(t,0)|$. In view of condition (20), $k > 0$.

Set $\mathbb{E} = C(J,\mathbb{R})$ and define a subset Ω of $\mathbb{E}$ by

$$\Omega = \{u \in \mathbb{E} : \|u\| \le k\}.$$

Clearly, Ω is a closed convex bounded subset of the Banach algebra $\mathbb{E}$.

In view of Lemma 6, we deduce that the mild solution of the hybrid fractional problem (3) is equivalent to the fractional integral equation

$$\begin{aligned} u(t) =& \Psi(t,u(t))\Bigg[\frac{\lambda e^{\frac{\varrho-1}{\varrho}(\vartheta(t)-\vartheta(a))}}{\varrho^{\delta-1}\Gamma(\delta)}(\vartheta(t)-\vartheta(a))^{\delta-1} \\ &+ \frac{1}{\varrho^{\delta}\Gamma(\delta)}\int_a^t e^{\frac{\varrho-1}{\varrho}(\vartheta(t)-\vartheta(s))}(\vartheta(t)-\vartheta(s))^{\delta-1}\Phi(s,u(s))\vartheta'(s)ds\Bigg], \quad t \in J. \end{aligned} \tag{22}$$

Define two operators $\mathcal{A} : \mathbb{E} \to \mathbb{E}$ and $\mathcal{B} : \Omega \to \mathbb{E}$ by

$$\begin{aligned} \mathcal{A}u(t) =& \Psi(t,u(t)), \quad t \in J, \\ \mathcal{B}u(t) =& \frac{\lambda e^{\frac{\varrho-1}{\varrho}(\vartheta(t)-\vartheta(a))}}{\varrho^{\delta-1}\Gamma(\delta)}(\vartheta(t)-\vartheta(a))^{\delta-1} + \frac{1}{\varrho^{\delta}\Gamma(\delta)}\int_a^t e^{\frac{\varrho-1}{\varrho}(\vartheta(t)-\vartheta(s))}(\vartheta(t)-\vartheta(s))^{\delta-1}\Phi(s,u(s))\vartheta'(s)ds, \quad t \in J. \end{aligned}$$

Therefore, the equivalent fraction integral Equation (22) to the hybrid fractional problem (3) can be transformed into the following operator equation:

$$u = \mathcal{A}u\mathcal{B}u, \quad u \in \mathbb{E}.$$

We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ fulfill all stipulation of Lemma 5. The proof will be given in the following steps.

Step 1. The operator $\mathcal{A}$ is Lipschitzian on $\mathbb{E}$.

For any $u, v \in \mathbb{E}$ and each $t \in J$, using **(A2)**, one has

$$\begin{aligned} |\mathcal{A}u(t) - \mathcal{A}v(t)| =& |\Psi(t,u(t)) - \Psi(t,v(t))| \\ \le& p(t)|u(t) - v(t)| \\ \le& p^*\|u - v\|. \end{aligned}$$

This leads to $\|\mathcal{A}u - \mathcal{A}v\| \le p^*\|u - v\|$. Thus, the operator $\mathcal{A}$ is Lipschitzian on $\mathbb{E}$.

Step 2. The operator $\mathcal{B}$ is continuous on Ω.

Take a sequence $\{u_n\} \subset \Omega$ and $u \in \Omega$ such that $\|u_n - u\| \to 0$ as $n \to \infty$. For each $t \in J$ and $\varrho \in (0,1]$, one has

$$
\begin{aligned}
|\mathcal{B}u_n(t)-\mathcal{B}u(t)| \leq & \frac{1}{\varrho^{\delta}\Gamma(\delta)}\int_a^t\left|e^{\frac{\varrho-1}{\varrho}(\vartheta(t)-\vartheta(s))}\right|(\vartheta(t)-\vartheta(s))^{\delta-1}|\Phi(s,u_n(s))-\Phi(s,u(s))|\vartheta'(s)ds\\
\leq & \frac{1}{\varrho^{\delta}\Gamma(\delta)}\int_a^t(\vartheta(t)-\vartheta(s))^{\delta-1}\|\Phi(\cdot,u_n(\cdot))-\Phi(\cdot,u(\cdot))\|\vartheta'(s)ds\\
\leq & \frac{(\vartheta(b)-\vartheta(a))^{\delta}}{\varrho^{\delta}\Gamma(\delta+1)}\|\Phi(\cdot,u_n(\cdot))-\Phi(\cdot,u(\cdot))\|.
\end{aligned}
$$

Therefore, the continuity of Φ implies that the operator $\mathcal{B}$ is continuous on Ω.
Step 3. The operator $\mathcal{B}$ is uniformly bounded in Ω.

For each $t \in J$, $u \in \Omega$ and $\varrho \in (0,1]$, using **(A3)**, one has

$$
\begin{aligned}
|\mathcal{B}u(t)| \leq & \frac{|\lambda|}{\varrho^{\delta-1}\Gamma(\delta)}\left|e^{\frac{\varrho-1}{\varrho}(\vartheta(t)-\vartheta(a))}\right|(\vartheta(t)-\vartheta(a))^{\delta-1}\\
& +\frac{1}{\varrho^{\delta}\Gamma(\delta)}\int_a^t\left|e^{\frac{\varrho-1}{\varrho}(\vartheta(t)-\vartheta(s))}\right|(\vartheta(t)-\vartheta(s))^{\delta-1}|\Phi(s,u(s))|\vartheta'(s)ds\\
\leq & \frac{|\lambda|}{\varrho^{\delta-1}\Gamma(\delta)}(\vartheta(t)-\vartheta(a))^{\delta-1}+\frac{1}{\varrho^{\delta}\Gamma(\delta)}\int_a^t(\vartheta(t)-\vartheta(s))^{\delta-1}q(s)\mathrm{H}(|u(s)|)\vartheta'(s)ds\\
\leq & \frac{|\lambda|(\vartheta(b)-\vartheta(a))^{\delta-1}}{\varrho^{\delta-1}\Gamma(\delta)}+\frac{q^*(\vartheta(b)-\vartheta(a))^{\delta}}{\varrho^{\delta}\Gamma(\delta+1)}\mathrm{H}(k):=\Lambda.
\end{aligned}
$$

Thus, we get $\|\mathcal{B}u\| \leq \Lambda$, for all $u \in \Omega$. This proves that $\mathcal{B}$ is uniformly bounded in Ω.

Step 4. $\mathcal{B}(\Omega)$ is equicontinuous in $\mathbb{E}$.

For $t_1, t_2 \in J,\ t_1 < t_2$ and $u \in \Omega$, using **(A3)**, we have

$$
\begin{aligned}
|\mathcal{B}u(t_2)-\mathcal{B}u(t_1)| \leq & \frac{|\lambda|}{\varrho^{\delta-1}\Gamma(\delta)}\left|e^{\frac{\varrho-1}{\varrho}(\vartheta(t_2)-\vartheta(a))}(\vartheta(t_2)-\vartheta(a))^{\delta-1}-e^{\frac{\varrho-1}{\varrho}(\vartheta(t_1)-\vartheta(a))}(\vartheta(t_1)-\vartheta(a))^{\delta-1}\right|\\
& +\frac{1}{\varrho^{\delta}\Gamma(\delta)}\left|\int_a^{t_2}e^{\frac{\varrho-1}{\varrho}(\vartheta(t_2)-\vartheta(s))}(\vartheta(t_2)-\vartheta(s))^{\delta-1}\Phi(s,u(s))\vartheta'(s)ds\right.\\
& \left.-\int_a^{t_1}e^{\frac{\varrho-1}{\varrho}(\vartheta(t_1)-\vartheta(s))}(\vartheta(t_1)-\vartheta(s))^{\delta-1}\Phi(s,u(s))\vartheta'(s)ds\right|\\
\leq & \frac{|\lambda|}{\varrho^{\delta-1}\Gamma(\delta)}\left|e^{\frac{\varrho-1}{\varrho}(\vartheta(t_2)-\vartheta(a))}(\vartheta(t_2)-\vartheta(a))^{\delta-1}-e^{\frac{\varrho-1}{\varrho}(\vartheta(t_1)-\vartheta(a))}(\vartheta(t_1)-\vartheta(a))^{\delta-1}\right|\\
& +\frac{1}{\varrho^{\delta}\Gamma(\delta)}\left[\int_a^{t_1}\left|e^{\frac{\varrho-1}{\varrho}(\vartheta(t_2)-\vartheta(s))}(\vartheta(t_2)-\vartheta(s))^{\delta-1}-e^{\frac{\varrho-1}{\varrho}(\vartheta(t_1)-\vartheta(s))}(\vartheta(t_1)-\vartheta(s))^{\delta-1}\right|\right.\\
& \left.\times|\Phi(s,u(s))|\vartheta'(s)ds+\int_{t_1}^{t_2}\left|e^{\frac{\varrho-1}{\varrho}(\vartheta(t_2)-\vartheta(s))}\right|(\vartheta(t_2)-\vartheta(s))^{\delta-1}|\Phi(s,u(s))|\vartheta'(s)ds\right]\\
\leq & \frac{|\lambda|}{\varrho^{\delta-1}\Gamma(\delta)}\left|e^{\frac{\varrho-1}{\varrho}(\vartheta(t_2)-\vartheta(a))}(\vartheta(t_2)-\vartheta(a))^{\delta-1}-e^{\frac{\varrho-1}{\varrho}(\vartheta(t_1)-\vartheta(a))}(\vartheta(t_1)-\vartheta(a))^{\delta-1}\right|\\
& +\frac{q^*\mathrm{H}(k)}{\varrho^{\delta}\Gamma(\delta)}\left[\int_a^{t_1}\left|e^{\frac{\varrho-1}{\varrho}(\vartheta(t_2)-\vartheta(s))}(\vartheta(t_2)-\vartheta(s))^{\delta-1}-e^{\frac{\varrho-1}{\varrho}(\vartheta(t_1)-\vartheta(s))}(\vartheta(t_1)-\vartheta(s))^{\delta-1}\right|\vartheta'(s)ds\right.\\
& \left.+\frac{1}{\delta}(\vartheta(t_2)-\vartheta(t_1))^{\delta}\right].
\end{aligned}
$$

Thus, the right-hand side of the above inequality tends to zero independently of $u \in \Omega$ as $t_2 \to t_1$. Hence, from the Steps $2-4$ and the Ascoli–Arzelà theorem, we conclude that the operator $\mathcal{B}$ is a completely continuous on Ω.

Step 5. The condition **(iii)** of Lemma 5 holds.
Let $u \in \mathbb{E}$ and $v \in \Omega$ be arbitrary elements such that $u = \mathcal{A}u\mathcal{B}v$. Then, one obtains that

$$
\begin{aligned}
|u(t)| \leq & |\mathcal{A}u(t)||\mathcal{B}v(t)| \\
\leq & |\Psi(t,u(t))| \left(\frac{|\lambda| \left| e^{\frac{\varrho-1}{\varrho}(\vartheta(t)-\vartheta(a))} \right|}{\varrho^{\delta-1}\Gamma(\delta)} (\vartheta(t)-\vartheta(a))^{\delta-1} \right. \\
& \left. + \frac{1}{\varrho^{\delta}\Gamma(\delta)} \int_a^t \left| e^{\frac{\varrho-1}{\varrho}(\vartheta(t)-\vartheta(s))} \right| (\vartheta(t)-\vartheta(s))^{\delta-1} |\Phi(s,v(s))| \vartheta'(s) ds \right) \\
\leq & |(\Psi(t,u(t)) - \Psi(t,0)) + \Psi(t,0)| \left(\frac{|\lambda|}{\varrho^{\delta-1}\Gamma(\delta)} (\vartheta(t)-\vartheta(a))^{\delta-1} \right. \\
& \left. + \frac{1}{\varrho^{\delta}\Gamma(\delta)} \int_a^t (\vartheta(t)-\vartheta(s))^{\delta-1} q(s) \mathrm{H}(|v(s)|) \vartheta'(s) ds \right) \\
\leq & (p^*|u(t)| + \Psi_0) \left(\frac{|\lambda|(\vartheta(b)-\vartheta(a))^{\delta-1}}{\varrho^{\delta-1}\Gamma(\delta)} + \frac{q^*(\vartheta(b)-\vartheta(a))^{\delta}}{\varrho^{\delta}\Gamma(\delta+1)} \mathrm{H}(k) \right) \\
= & (p^*|u(t)| + \Psi_0)\Lambda,
\end{aligned}
$$

which leads to

$$|u(t)| \leq \frac{\Psi_0 \Lambda}{1 - p^* \Lambda}.$$

Hence, by (21), we get

$$\|u\| \leq \Psi_0 \Lambda (1 - p^* \Lambda)^{-1} = k.$$

Step 6. The condition **(iv)** of Lemma 5 holds.

We shall show that $\mu L < 1$, where $\mu = p^*$ and $L = \|\mathcal{B}(\Omega)\|$.

Since, $L = \|\mathcal{B}(\Omega)\| = \sup_{u \in \Omega} \left\{ \sup_{t \in J} |\mathcal{B}u(t)| \right\} \leq \Lambda$. Then, $\mu L \leq p^* \Lambda < 1$. Thus, all the conditions of Lemma 5 hold true, and hence the operator equation $u = \mathcal{A}u\mathcal{B}u$ possesses at least one solution in Ω. Consequently, the hybrid fractional problem (3) has at least one solution in J. This finishes the proof. □

4. Continuous Dependence on Parameters

In this section, our major intent is to set up sufficient conditions under which solutions of the hybrid fractional problem (3) depend continuously on initial conditions and other parameters. Let us consider the parameterized hybrid fractional problem

$$
\begin{cases}
{}_a\mathfrak{D}^{\delta_m,\varrho_m,\vartheta_m} \left(\frac{u_m(t)}{\Psi(t,u_m(t))} \right) = \Phi(t,u_m(t)), \quad t \in J, \\
{}_a\mathfrak{J}^{1-\delta_m,\varrho_m,\vartheta_m} \left(\frac{u_m(t)}{\Psi(t,u_m(t))} \right)_{t=a} = \lambda_m,
\end{cases}
\tag{23}
$$

where $0 < \delta_m \leq 1$, $\varrho_m \in (0,1]$, $\lambda_m \in \mathbb{R}$ and $\vartheta_m \in C(J,\mathbb{R})$ with $\vartheta'_m > 0$.

In view of Lemma 6, we infer that the solution of the parameterized hybrid fractional problem (23) is given by

$$\begin{aligned} u_m(t) =& \Psi(t, u_m(t)) \left[\frac{\lambda_m e^{\frac{\varrho_m - 1}{\varrho_m}(\vartheta_m(t) - \vartheta_m(a))}}{\varrho_m^{\delta_m - 1} \Gamma(\delta_m)} (\vartheta_m(t) - \vartheta_m(a))^{\delta_m - 1} \right. \\ &+ \left. \frac{1}{\varrho_m^{\delta_m} \Gamma(\delta_m)} \int_a^t e^{\frac{\varrho_m - 1}{\varrho_m}(\vartheta_m(t) - \vartheta_m(s))} (\vartheta_m(t) - \vartheta_m(s))^{\delta_m - 1} \Phi(s, u_m(s)) \vartheta_m'(s) ds \right], \quad t \in J. \end{aligned} \tag{24}$$

To achieve the desired goal in this section, we will impose the following assumption:

(A4) For $u, v \in \mathbb{R}$, for all $t \in J$, there exists a constant $L_\Phi > 0$ such that

$$|\Phi(t, u(t)) - \Phi(t, v(t))| \leq L_\Phi |u(t) - v(t)|.$$

We set

$$\Lambda_m := \frac{|\lambda_m| (\vartheta_m(b) - \vartheta_m(a))^{\delta_m - 1}}{\varrho_m^{\delta_m - 1} \Gamma(\delta_m)} + \frac{q^* (\vartheta_m(b) - \vartheta_m(a))^{\delta_m}}{\varrho_m^{\delta_m} \Gamma(\delta_m + 1)} \mathrm{H}(\|u_m\|),$$

and

$$\Xi := \frac{L_\Phi (\vartheta(b) - \vartheta(a))^{\delta}}{\varrho^{\delta} \Gamma(\delta + 1)}.$$

Theorem 2. *Let $0 < \delta_m \leq 1$, $\varrho_m \in (0, 1]$, $\lambda_m \in \mathbb{R}$ and $\vartheta_m \in C(J, \mathbb{R})$ with $\vartheta_m' > 0$. Assume that the assumptions **(A1)–(A4)** are fulfilled. If*

$$(\delta_m, \varrho_m, \lambda_m) \to (\delta, \varrho, \lambda), \text{ as } m \to \infty, \tag{25}$$

$$\|\vartheta_m - \vartheta\| \to 0, \text{ as } m \to \infty, \tag{26}$$

$$\|\vartheta_m' - \vartheta'\|_{L^1[a,b]} \to 0, \text{ as } m \to \infty, \tag{27}$$

$$p^* \Lambda_m + \Xi < 1. \tag{28}$$

Then the parameterized hybrid fractional problem (23) has at least one solution $u_m \in C(J, \mathbb{R})$ such that

$$\|u_m - u\| \to 0, \text{ as } m \to \infty.$$

Proof.

$$
\begin{aligned}
|u_m(t)-u(t)| =& \Bigg|\Psi(t,u_m(t))\Bigg[\frac{\lambda_m e^{\frac{\varrho_m-1}{\varrho_m}(\vartheta_m(t)-\vartheta_m(a))}}{\varrho_m^{\delta_m-1}\Gamma(\delta_m)}(\vartheta_m(t)-\vartheta_m(a))^{\delta_m-1} \\
&+\frac{1}{\varrho_m^{\delta_m}\Gamma(\delta_m)}\int_a^t e^{\frac{\varrho_m-1}{\varrho_m}(\vartheta_m(t)-\vartheta_m(s))}(\vartheta_m(t)-\vartheta_m(s))^{\delta_m-1}\Phi(s,u_m(s))\vartheta_m'(s)ds\Bigg] \\
&-\Psi(t,u(t))\Bigg[\frac{\lambda e^{\frac{\varrho-1}{\varrho}(\vartheta(t)-\vartheta(a))}}{\varrho^{\delta-1}\Gamma(\delta)}(\vartheta(t)-\vartheta(a))^{\delta-1} \\
&+\frac{1}{\varrho^{\delta}\Gamma(\delta)}\int_a^t e^{\frac{\varrho-1}{\varrho}(\vartheta(t)-\vartheta(s))}(\vartheta(t)-\vartheta(s))^{\delta-1}\Phi(s,u(s))\vartheta'(s)ds\Bigg]\Bigg| \\
\leq& \Big|\Psi(t,u_m(t))-\Psi(t,u(t))\Big|\Bigg\{\frac{|\lambda_m|(\vartheta_m(t)-\vartheta_m(a))^{\delta_m-1}}{\varrho_m^{\delta_m-1}\Gamma(\delta_m)} \\
&+\frac{1}{\varrho_m^{\delta_m}\Gamma(\delta_m)}\int_a^t(\vartheta_m(t)-\vartheta_m(s))^{\delta_m-1}\big|\Phi(s,u_m(s))\big|\vartheta_m'(s)ds\Bigg\} \\
&+\Big(|\Psi(t,u(t))-\Psi(t,0)|+|\Psi(t,0)|\Big)\Bigg\{\Bigg|\frac{\lambda_m e^{\frac{\varrho_m-1}{\varrho_m}(\vartheta_m(t)-\vartheta_m(a))}}{\varrho_m^{\delta_m-1}\Gamma(\delta_m)}(\vartheta_m(t)-\vartheta_m(a))^{\delta_m-1} \\
&-\frac{\lambda e^{\frac{\varrho-1}{\varrho}(\vartheta(t)-\vartheta(a))}}{\varrho^{\delta-1}\Gamma(\delta)}(\vartheta(t)-\vartheta(a))^{\delta-1}\Bigg| \\
&+\Bigg|\frac{1}{\varrho_m^{\delta_m}\Gamma(\delta_m)}\int_a^t e^{\frac{\varrho_m-1}{\varrho_m}(\vartheta_m(t)-\vartheta_m(s))}(\vartheta_m(t)-\vartheta_m(s))^{\delta_m-1}\Phi(s,u_m(s))\vartheta_m'(s)ds \\
&-\frac{1}{\varrho^{\delta}\Gamma(\delta)}\int_a^t e^{\frac{\varrho-1}{\varrho}(\vartheta(t)-\vartheta(s))}(\vartheta(t)-\vartheta(s))^{\delta-1}\Phi(s,u(s))\vartheta'(s)ds\Bigg|\Bigg\} \\
\leq& p(t)|u_m(t)-u(t)|\Bigg\{\frac{|\lambda_m|(\vartheta_m(t)-\vartheta_m(a))^{\delta_m-1}}{\varrho_m^{\delta_m-1}\Gamma(\delta_m)} \\
&+\frac{1}{\varrho_m^{\delta_m}\Gamma(\delta_m)}\int_a^t(\vartheta_m(t)-\vartheta_m(s))^{\delta_m-1}q(s)\mathrm{H}(|u_m(s)|)\vartheta_m'(s)ds\Bigg\} \\
&+\Bigg(p(t)|u(t)|+|\Psi(t,0)|\Bigg)\Bigg\{\Bigg|\frac{\lambda_m e^{\frac{\varrho_m-1}{\varrho_m}(\vartheta_m(t)-\vartheta_m(a))}}{\varrho_m^{\delta_m-1}\Gamma(\delta_m)}(\vartheta_m(t)-\vartheta_m(a))^{\delta_m-1} \\
&-\frac{\lambda e^{\frac{\varrho-1}{\varrho}(\vartheta(t)-\vartheta(a))}}{\varrho^{\delta-1}\Gamma(\delta)}(\vartheta(t)-\vartheta(a))^{\delta-1}\Bigg| \\
&+\Bigg|\frac{1}{\varrho^{\delta}\Gamma(\delta)}\int_a^t e^{\frac{\varrho-1}{\varrho}(\vartheta(t)-\vartheta(s))}(\vartheta(t)-\vartheta(s))^{\delta-1}\vartheta'(s)\Bigg||\Phi(s,u_m(s))-\Phi(s,u(s))|ds
\end{aligned}
$$

$$+\left|\frac{1}{\varrho_m^{\delta_m}\Gamma(\delta_m)}\int_a^t e^{\frac{\varrho_m-1}{\varrho_m}(\vartheta_m(t)-\vartheta_m(s))}(\vartheta_m(t)-\vartheta_m(s))^{\delta_m-1}\vartheta_m'(s)\right.$$
$$\left.-\frac{1}{\varrho^{\delta}\Gamma(\delta)}\int_a^t e^{\frac{\varrho-1}{\varrho}(\vartheta(t)-\vartheta(s))}(\vartheta(t)-\vartheta(s))^{\delta-1}\vartheta'(s)\right||\Phi(s,u_m(s))|ds\Bigg\}$$
$$\leq p^*\|u_m-u\|\left\{\frac{|\lambda_m|(\vartheta_m(b)-\vartheta_m(a))^{\delta_m-1}}{\varrho_m^{\delta_m-1}\Gamma(\delta_m)}+\frac{q^*(\vartheta_m(b)-\vartheta_m(a))^{\delta_m}}{\varrho_m^{\delta_m}\Gamma(\delta_m+1)}\mathrm{H}(\|u_m\|)\right\}$$
$$+\Big(p^*\|u\|+\Psi_0\Big)\left\{\left|\frac{\lambda_m e^{\frac{\varrho_m-1}{\varrho_m}(\vartheta_m(t)-\vartheta_m(a))}}{\varrho_m^{\delta_m-1}\Gamma(\delta_m)}(\vartheta_m(t)-\vartheta_m(a))^{\delta_m-1}\right.\right.$$
$$\left.-\frac{\lambda e^{\frac{\varrho-1}{\varrho}(\vartheta(t)-\vartheta(a))}}{\varrho^{\delta-1}\Gamma(\delta)}(\vartheta(t)-\vartheta(a))^{\delta-1}\right|+\frac{L_\Phi(\vartheta(b)-\vartheta(a))^{\delta}}{\varrho^{\delta}\Gamma(\delta+1)}\|u_m-u\|$$
$$+q^*\mathrm{H}(\|u_m\|)\left|\frac{1}{\varrho_m^{\delta_m}\Gamma(\delta_m)}\int_a^t e^{\frac{\varrho_m-1}{\varrho_m}(\vartheta_m(t)-\vartheta_m(s))}(\vartheta_m(t)-\vartheta_m(s))^{\delta_m-1}\vartheta_m'(s)\right.$$
$$\left.\left.-\frac{1}{\varrho^{\delta}\Gamma(\delta)}\int_a^t e^{\frac{\varrho-1}{\varrho}(\vartheta(t)-\vartheta(s))}(\vartheta(t)-\vartheta(s))^{\delta-1}\vartheta'(s)\right|ds\right\}.$$

Therefore, we get

$$\Big(1-p^*\Lambda_m-\Xi\Big)\|u_m-u\|$$
$$\leq\Big(p^*\|u\|+\Psi_0\Big)\left\{\left|\frac{\lambda_m e^{\frac{\varrho_m-1}{\varrho_m}(\vartheta_m(t)-\vartheta_m(a))}}{\varrho_m^{\delta_m-1}\Gamma(\delta_m)}(\vartheta_m(t)-\vartheta_m(a))^{\delta_m-1}-\frac{\lambda e^{\frac{\varrho-1}{\varrho}(\vartheta(t)-\vartheta(a))}}{\varrho^{\delta-1}\Gamma(\delta)}(\vartheta(t)-\vartheta(a))^{\delta-1}\right|\right.$$
$$+q^*\mathrm{H}(\|u_m\|)\left|\frac{1}{\varrho_m^{\delta_m}\Gamma(\delta_m)}\int_a^t e^{\frac{\varrho_m-1}{\varrho_m}(\vartheta_m(t)-\vartheta_m(s))}(\vartheta_m(t)-\vartheta_m(s))^{\delta_m-1}\vartheta_m'(s)\right.$$
$$\left.\left.-\frac{1}{\varrho^{\delta}\Gamma(\delta)}\int_a^t e^{\frac{\varrho-1}{\varrho}(\vartheta(t)-\vartheta(s))}(\vartheta(t)-\vartheta(s))^{\delta-1}\vartheta'(s)\right|ds\right\}$$
$$\leq\Big(p^*\|u\|+\Psi_0\Big)\left\{\left|\frac{\lambda_m e^{\frac{\varrho_m-1}{\varrho_m}(\vartheta_m(t)-\vartheta_m(a))}}{\varrho_m^{\delta_m-1}\Gamma(\delta_m)}(\vartheta_m(t)-\vartheta_m(a))^{\delta_m-1}-\frac{\lambda e^{\frac{\varrho-1}{\varrho}(\vartheta(t)-\vartheta(a))}}{\varrho^{\delta-1}\Gamma(\delta)}(\vartheta(t)-\vartheta(a))^{\delta-1}\right|\right.$$
$$+q^*\mathrm{H}(\|u_m\|)\left[\left|\frac{1}{\varrho_m^{\delta_m}\Gamma(\delta_m)}-\frac{1}{\varrho^{\delta}\Gamma(\delta)}\right|\frac{(\vartheta_m(b)-\vartheta_m(a))^{\delta_m}}{\delta_m}\right.$$
$$\left.+\frac{1}{\varrho^{\delta}\Gamma(\delta)}\int_a^b\left|e^{\frac{\varrho_m-1}{\varrho_m}(\vartheta_m(t)-\vartheta_m(s))}(\vartheta_m(t)-\vartheta_m(s))^{\delta_m-1}\vartheta_m'(s)-e^{\frac{\varrho-1}{\varrho}(\vartheta(t)-\vartheta(s))}(\vartheta(t)-\vartheta(s))^{\delta-1}\vartheta'(s)\right|ds\right].$$

Hence, according to the conditions (25)–(28), we deduce that the above inequality tends to zero as $m\to\infty$. This proves that $\|u_m-u\|\to 0$ as $m\to\infty$. This completes the proof. □

5. A Simulative Example

In this position, we prepare a simulative example with known constants and parameters to illustrate the effectiveness of the obtained analytical findings.

Example 1. *Consider the following hybrid fractional problem*

$$\begin{cases} {}_0\mathfrak{D}^{\frac{3}{4},\frac{1}{2},t^2}\left(\frac{u(t)}{\Psi(t,u(t))}\right)=\Phi(t,u(t)), & t\in[0,1], \\ {}_0\mathfrak{I}^{\frac{1}{4},\frac{1}{2},t^2}\left(\frac{u(t)}{\Psi(t,u(t))}\right)_{t=0}=1. \end{cases}\tag{29}$$

Here, $\delta = \frac{3}{4}, \varrho = \frac{1}{2}, a = 0, b = 1$ and $\vartheta(t) = t^2$.

Set $\Psi(t, u(t)) = \frac{e^{-2t}}{1+7e^t} \frac{|u(t)|}{1+|u(t)|}$ and $\Phi(t, u(t)) = \frac{t^2}{25} \cos u(t)$.

It is clear that the assumption **(A1)** is satisfied and $\Psi_0 = \frac{1}{8}$.
Let $u, v \in \mathbb{R}$ and $t \in [0, 1]$. Then, we get

$$\begin{aligned} |\Psi(t,u) - \Psi(t,v)| &= \left|\frac{e^{-2t}}{1+7e^t}\right| \left|\frac{|u(t)|}{1+|u(t)|} - \frac{|v(t)|}{1+|v(t)|}\right| \\ &\leq \frac{e^{-2t}}{1+7e^t} \left|\frac{|u(t)| - |v(t)|}{(1+|u(t)|)(1+|v(t)|)}\right| \\ &\leq \frac{e^{-2t}}{1+7e^t} |u(t) - v(t)|. \end{aligned}$$

Thus, the assumption **(A2)** holds true with $p(t) = \frac{e^{-2t}}{1+7e^t}$ and $p^* = \frac{1}{8}$.
Moreover, for $u \in \mathbb{R}$ and $t \in [0, 1]$, we get

$$|\Phi(t, u(t))| = \left|\frac{t^2}{25} \cos u(t)\right| \leq \frac{t^2}{25}.$$

This implies that the assumption **(A3)** is fulfilled with $q(t) = \frac{t^2}{25}$, $q^* = \frac{1}{25}$ and $\mathrm{H}(|u|) = 1$.

By the above data, we get $\Lambda \approx 0.759408$ and $p^*\Lambda \approx 0.094926 < 1$. Thus, we can choose $k > 0.104882$. Accordingly, all the conditions of Theorem 1 are fulfilled, the hybrid fractional problem (29) has at least one solution on $[0, 1]$.

6. Conclusions

In this current research paper, we intend to check the existence aspects of solutions for a category of a new class of hybrid fractional differential equations within generalized fractional derivatives depending on another function. With the aid of a hybrid fixed point theorem for a product of two operators, the desired results are verified. Moreover, the continuity of solutions in terms of inputs (fractional orders, associated parameters, and appropriate function) has attracted interested researchers due to its significance in the experimental process. Based on this, the topic of continuity of solution of the equation of the class under consideration with respect to inputs is important and worth considering. A simulative example is prepared to demonstrate some applicability aspects of the obtained results.

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