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Convergence Analysis of a Three-Step Iterative Algorithm for Generalized Set-Valued Mixed-Ordered Variational Inclusion Problem

Praveen Agarwal [1,2,3,*], Doaa Filali [4], M. Akram [5] and M. Dilshad [6]

1 Department of Mathematics, Anand International College of Engineering, Jaipur 303012, India
2 Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman AE 346, United Arab Emirates
3 International Center for Basic and Applied Sciences, Jaipur 302029, India
4 Department of Mathematical Science, College of Sciences, Princess Nourah Bint Abdulrahman University, Riyadh 84428, Saudi Arabia; dkfilali@pnu.edu.sa
5 Department of Mathematics, Faculty of Science, Islamic University of Madinah, Madinah 170, Saudi Arabia; akramkhan_20@rediffmail.com
6 Department of Mathematics, Faculty of Science, University of Tabuk, Tabuk 71491, Saudi Arabia; mdilshaad@gmail.com
* Correspondence: praveen.agarwal@anandice.ac.in

Abstract: This manuscript aims to study a generalized, set-valued, mixed-ordered, variational inclusion problem involving $\mathcal{H}(\cdot,\cdot)$-compression XOR-$\alpha_{\mathcal{M}}$-non-ordinary difference mapping and relaxed cocoercive mapping in real-ordered Hilbert spaces. The resolvent operator associated with $\mathcal{H}(\cdot,\cdot)$-compression XOR-$\alpha_{\mathcal{M}}$-non-ordinary difference mapping is defined, and some of its characteristics are discussed. We prove existence and uniqueness results for the considered generalized, set-valued, mixed-ordered, variational inclusion problem. Further, we put forward a three-step iterative algorithm using a $\oplus$ operator, and analyze the convergence of the suggested iterative algorithm under some mild assumptions. Finally, we reconfirm the existence and convergence results by an illustrative numerical example.

Keywords: $\mathcal{H}(\cdot,\cdot)$-compression mapping; resolvent operator; ordered inclusion problem; three-step iterative algorithm; XOR-operator; XNOR-operator

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1. Introduction

The theory of variational inequalities was studied in the early 1960s to solve a problem which appeared in a mechanical system. In 1964, Stampacchia [1] solved the variational inequality problem, where it was found that $\bar{x} \in K$ such that

$$\langle T(\bar{x}), \tilde{x} - \bar{x} \rangle \geq 0, \text{ for all } \tilde{x} \in K, \tag{1}$$

where $K(\neq \phi)$ is a closed convex subset of a real Hilbert space $\mathcal{H}$ and $T : \mathcal{H} \to \mathcal{H}$ is a single-valued mapping. The author studied the existence and uniqueness of the solution for this proposed problem. This theory is one of the most powerful tools for studying problems arising in nonlinear analysis, including differential equations, mechanics, control problems, equilibrium problems, transportation, and so forth, which has proven to be quite application-oriented and fruitful.

Since its inception, the variational inequalities are being appealed to investigate problems appearing in miscellaneous areas of basic and applied sciences, such as [2–8]. These unusual utilizations motivated researchers to drive and expand the variational inequalities and related optimization problems in different format using advanced and innovatory techniques, such as [9–14] and the references cited therein. One of the most important generalizations of variational inequality due to Verma [15] is the system of variational inequalities, because a number of equilibrium problems, such as the traffic equilibrium, the spatial equilibrium, the Nash equilibrium, and the general equilibrium programming problems can be designed as a system of variational inequalities. The birth of this theory can be observed as the simultaneous acquirement of two different lines of research—namely, it affirms the qualitative aspects of the solution to important classes of problems, and also empowers us to establish influential and fruitful new techniques for the solving of problems.

In 1994, Hassouni and Moudafi [16] evolved a class of mixed-type variational inequalities with single-valued mappings using the technique of a resolvent operator for monotone mapping, namely- variational inclusion problem. They developed a perturbed algorithm to estimate the solution of mixed variational inequalities. Recently, the fixed-point theory is widely applied to study problems appearing in real ordered Banach spaces, see [17–21]. Li [22] studied the nonlinear ordered variational inequalities and developed an iterative algorithm to estimate its solution in real ordered Banach spaces. In 2012, Li [23] planted a new class of variational inclusions for (α, λ)-NODM set-valued mappings in an ordered Hilbert space. Using the technique of resolvent operator, the existence result was proven and the convergence of sequence obtained from iterative algorithm was discussed. In [24], Li et al. investigated the solution of a general nonlinear ordered variational inclusion with (γ_G, λ)-weak-GRD mappings. Recently, Li et al. [25] presented the convergence of an Ishikawa-type iterative method for the general nonlinear ordered variational inclusion with (γ_G, λ)-weak-GRD set-valued mappings, and exhibited the stability of the algorithm. Very recently, ordered variational inclusions with XOR operator are considered in diverse direction—see, for example, [26,27].

Following the facts mentioned above and encouraged by recent investigations in this order, we introduce $H(\cdot,\cdot)$-compression XOR-α_M-non-ordinary difference mapping. A resolvent operator associated to this mapping is defined, and we discuss some of its characteristics. We examine a generalized, set-valued, mixed-ordered, variational inclusion with $H(\cdot,\cdot)$-compression XOR-α_M-non-ordinary difference mapping and relaxed cocoercive mapping in real ordered Hilbert spaces. We validate the existence and uniqueness of the solution for the considered ordered variational inclusion. In addition, we present a three-step iterative algorithm using a $\oplus$ operator and analyze the convergence of the proposed iterative algorithm under suitable assumptions. At last, a numerical example is given to show that the considered three-step iterative algorithm converges to the unique solution of a generalized, set-valued, mixed-ordered, variational inclusion.

2. Preliminaries and Auxiliary Results

We remind some necessary definitions, notions, and auxiliary results which are constructive tools and will be used throughout this paper.

Let $\mathcal{H}_p$ be a real Hilbert space equipped with the norm $\|\cdot\|$, and inner product $\langle\cdot,\cdot\rangle$ where d is the metric induced by the norm $\|\cdot\|$. Let $\mathcal{C}$ be a normal cone in $\mathcal{H}_p$ with normal constant $\lambda_{\mathcal{C}}$, and " $\leq$" denotes the partial ordering defined by $\mathcal{C}$. The Hilbert space $\mathcal{H}_p$ equipped with partial ordering $\leq$ defined by $\mathcal{C}$ is called an ordered Hilbert space. Let $CB(\mathcal{H}_p)$ (respectively, $2^{\mathcal{H}_p}$) be the family of all non-empty closed and bounded subsets (respectively, all non-empty subsets) of $\mathcal{H}_p$. For any arbitrary $\mu, \nu \in \mathcal{H}_p$, glb$\{\mu,\nu\}$ and lub$\{\mu,\nu\}$ represent the greatest lower bound and least upper bound, respectively, for the set $\{\mu,\nu\}$ with partial ordering $\leq$. The operators $\wedge, \vee, \oplus$, and $\odot$ are called AND, OR, XOR, and XNOR operators, respectively, and defined as follows:

(i) $\mu \wedge \nu = \inf\{\mu,\nu\}$,
(ii) $\mu \vee \nu = \sup\{\mu,\nu\}$,
(iii) $\mu \oplus \nu = (\mu - \nu) \vee (\nu - \mu)$,
(iv) $\mu \odot \nu = (\mu - \nu) \wedge (\nu - \mu)$.

Throughout this paper, unless otherwise stated, we denote positive, real ordered Hilbert space by $\mathcal{H}_p$.

Definition 1 ([28]). *A non-empty closed convex subset $\mathcal{C}$ of $\mathcal{H}_p$ is called a cone:*

(i) If $\mu \in \mathcal{C}$ and $\lambda > 0$, then $\lambda\mu \in \mathcal{C}$;
(ii) If $\mu \in \mathcal{C}$ and $-\mu \in \mathcal{C}$, then $\mu = 0$.

Definition 2 ([28,29]). *Let $\mathcal{C}$ be a cone. Then,*

(i) $\mathcal{C}$ is called a normal cone if there exists a normal constant $\lambda_{\mathcal{C}} > 0$ such that $0 \leq \mu \leq \nu$ implies $\|\mu\| \leq \lambda_{\mathcal{C}}\|\nu\|$, for all $\mu, \nu \in \mathcal{H}_p$;
(ii) For arbitrary elements $\mu, \nu \in \mathcal{H}_p$, $\mu \leq \nu$ if, and only if $\mu - \nu \in \mathcal{C}$;
(iii) μ and ν are said to be comparable to each other if, and only if $\mu \leq \nu$ or $\nu \leq \mu$ exists, and we denote it by $\mu \propto \nu$.

Proposition 1 ([29]). *Let $\mathcal{C}$ be a normal cone with normal constant $\lambda_{\mathcal{C}}$ in $\mathcal{H}_p$. Then, for each $\mu, \nu \in \mathcal{H}_p$, the following relations hold:*

(i) $\|0 \oplus 0\| = \|0\| = 0$;
(ii) $\|\mu \vee \nu\| \leq \|\mu\| \vee \|\nu\| \leq \|\mu\| + \|\nu\|$;
(iii) $\|\mu \oplus \nu\| \leq \|\mu - \nu\| \leq \lambda_{\mathcal{C}}\|\mu \oplus \nu\|$;
(iv) If $\mu \propto \nu$, then $\|\mu \oplus \nu\| = \|\mu - \nu\|$.

Lemma 1 ([30]). *Let $\{\omega_n\}$ be a nonnegative real sequence satisfying the following inequality*

$$\omega_{n+1} \leq (1 - \vartheta_n)\omega_n + \varepsilon_n, \quad \forall n \geq n_0,$$

where $\vartheta_n \in [0,1]$, $\sum_{n=0}^{\infty} \vartheta_n = \infty$ and $\varepsilon_n = o(\vartheta_n)$. Then, $\lim_{n\to\infty} \omega_n = 0$.

Definition 3 ([31]). *A mapping $T : \mathcal{H}_p \to \mathcal{H}_p$ is said to be relaxed (μ, γ)-cocoercive if there exist constants $\mu, \gamma > 0$ such that*

$$\langle Tx - Ty, x - y\rangle \geq (-\mu)\|Tx - Ty\|^2 + \gamma\|x - y\|^2, \text{ for all } x, y \in \mathcal{H}_p.$$

Remark 1. *Every γ strongly monotone mapping is relaxed (μ, γ)-cocoercive mapping, and every γ-cocoercive mapping is $\frac{1}{\gamma}$-Lipschitz-continuous.*

Proposition 2 ([23,29]). *Let $\mathcal{C}$ be a cone in $\mathcal{H}_p$ with partial ordering $\leq$ induced by $\mathcal{C}$. Let $\oplus$ be an XOR operation and $\odot$ be an XNOR operation. Then, for any $\vartheta, \upsilon, \mu, \nu, \omega \in \mathcal{H}_p$, the following relations hold:*

(i) $\mu \odot \mu = 0$, $\mu \odot \nu = \nu \odot \mu = -(\mu \oplus \nu) = -(\nu \oplus \mu)$;
(ii) If $\mu \propto 0$, then $-\mu \oplus 0 \leq \mu \leq \mu \oplus 0$;
(iii) $0 \leq \mu \oplus \nu$ if $\mu \propto \nu$;
(iv) $(\rho\mu) \oplus (\rho\nu) = |\rho|(\mu \oplus \nu)$, for any real ρ;
(v) If $\mu \propto \nu$, then $\mu \oplus \nu = 0$ if, and only if $\mu = \nu$;
(vi) $(\vartheta + \upsilon) \odot (\mu + \nu) \geq (\vartheta \odot \mu) + (\upsilon \odot \nu)$;
(vii) $(\vartheta + \upsilon) \odot (\mu + \nu) \geq (\vartheta \odot \nu) + (\upsilon \odot \mu)$;
(viii) If μ, ν, ω are comparable to each other, then $(\mu \oplus \nu) \leq (\mu \oplus \omega) + (\omega \oplus \nu)$;
(ix) $\alpha\vartheta \oplus \beta\vartheta = |\alpha - \beta|\vartheta = (\alpha \oplus \beta)\vartheta$ if $\vartheta \propto 0$, for any real α, β.

Definition 4 ([22,23]). *A single-valued mapping $\mathcal{A} : \mathcal{H}_p \to \mathcal{H}_p$ is said to be:*

(i) Comparison mapping if $\mu \propto \nu$ for any $\mu, \nu \in \mathcal{H}_p$, then $\mathcal{A}(\mu) \propto \mathcal{A}(\nu)$, $\mu \propto \mathcal{A}(\mu)$, $\nu \propto \mathcal{A}(\nu)$;
(ii) strong comparison mapping if $\mathcal{A}$ is comparison mapping and $\mathcal{A}(\mu) \propto \mathcal{A}(\nu)$ if, and only if $\mu \propto \nu$, for any $\mu, \nu \in \mathcal{H}_p$;
(iii) π-ordered compression mapping if $\mathcal{A}$ is a comparison mapping and if there exists a constant $\pi \in (0,1)$ such that

$$\mathcal{A}(\mu) \oplus \mathcal{A}(\nu) \leq \pi(\mu \oplus \nu), \text{ for all } \mu, \nu \in \mathcal{H}_p.$$

Definition 5 ([26]). *Let $\mathcal{A}, \mathcal{B} : \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings. Then, the single-valued mapping $\mathcal{H} : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ is called*

(i) *π_1-ordered compression mapping with respect to $\mathcal{A}$ if there exists a constant $\pi_1 \in (0,1)$ such that*
$$\mathcal{H}(\mathcal{A}(\mu), \cdot) \oplus \mathcal{H}(\mathcal{A}(\nu), \cdot) \leq \pi_1(\mu \oplus \nu), \text{ for all } \mu, \nu \in \mathcal{H}_p;$$

(ii) *π_2-ordered compression mapping with respect to $\mathcal{B}$ if there exists a constant $\pi_2 \in (0,1)$ such that*
$$\mathcal{H}(\cdot, \mathcal{B}(\mu)) \oplus \mathcal{H}(\cdot, \mathcal{B}(\nu)) \leq \pi_2(\mu \oplus \nu), \text{ for all } \mu, \nu \in \mathcal{H}_p;$$

(iii) *Mixed comparison mapping with respect to $\mathcal{A}$ and $\mathcal{B}$ if for all $\mu, \nu \in \mathcal{H}_p, \mu \propto \nu$ then $\mathcal{H}(\mathcal{A}(\mu), \mathcal{B}(\mu)) \propto \mathcal{H}(\mathcal{A}(\nu), \mathcal{B}(\nu)), \mu \propto \mathcal{H}(\mathcal{A}(\mu), \mathcal{B}(\mu))$ and $\nu \propto \mathcal{H}(\mathcal{A}(\nu), \mathcal{B}(\nu))$;*

(iv) *Mixed strong comparison mapping with respect to $\mathcal{A}$ and $\mathcal{B}$ if, for all $\mu, \nu \in \mathcal{H}_p$, $\mathcal{H}(\mathcal{A}(\mu), \mathcal{B}(\mu)) \propto \mathcal{H}(\mathcal{A}(\nu), \mathcal{B}(\nu))$ if, and only if $\mu \propto \nu$, for all $\mu, \nu \in \mathcal{H}_p$.*

Definition 6 ([26]). *Let $\mathcal{A}, \mathcal{B}, \mathcal{F}, \mathcal{G} : \mathcal{H}_p \to \mathcal{H}_p$ and $\mathcal{H} : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings. Then,*

(i) *$\mathcal{H}(\mathcal{A}, \cdot)$ is said to be β_1-ordered compression mapping with respect to $\mathcal{F}$ and $\mathcal{G}$ if there exists a constant $\beta_1 \in (0,1)$ such that*
$$\langle \mathcal{H}(\mathcal{A}(\mu), \mathcal{B}(\mu)) \oplus \mathcal{H}(\mathcal{A}(\nu),\mathcal{B}(\mu)), (\mathcal{F}(\mu), \mathcal{G}(\mu)) \oplus (\mathcal{F}(\nu), \mathcal{G}(\nu)) \rangle$$
$$\leq \beta_1 \|(\mathcal{F}(\mu), \mathcal{G}(\mu)) \oplus (\mathcal{F}(\nu), \mathcal{G}(\nu))\|^2, \text{ for all } \mu, \nu \in \mathcal{H}_p;$$

(ii) *$\mathcal{H}(\cdot, \mathcal{B})$ is called β_2-ordered compression mapping with respect to $\mathcal{F}$ and $\mathcal{G}$ if there exists a constant $\beta_2 \in (0,1)$ such that*
$$\langle \mathcal{H}(\mathcal{A}(\mu), \mathcal{B}(\mu)) \oplus \mathcal{H}(\mathcal{A}(\mu),\mathcal{B}(\nu)), (\mathcal{F}(\mu), \mathcal{G}(\mu)) \oplus (\mathcal{F}(\nu), \mathcal{G}(\nu)) \rangle$$
$$\leq \beta_2 \|(\mathcal{F}(\mu), \mathcal{G}(\mu)) \oplus (\mathcal{F}(\nu), \mathcal{G}(\nu))\|^2, \text{ for all } \mu, \nu \in \mathcal{H}_p.$$

Definition 7 ([23]). *Let $\mathcal{A} : \mathcal{H}_p \to \mathcal{H}_p; \mathcal{T} : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings and $\mathcal{P} : \mathcal{H}_p \to CB(\mathcal{H}_p)$ be a set-valued mapping. Then,*

(i) *$\mathcal{A}$ is called Lipschitz-continuous if for any $\mu, \nu \in \mathcal{H}_p$ and $\mu \propto \nu$, there exists a constant $\delta_{\mathcal{A}} > 0$ such that*
$$\|\mathcal{A}(\mu) \oplus \mathcal{A}(\nu)\| \leq \delta_{\mathcal{A}} \|\mu \oplus \nu\|;$$

(ii) *$\mathcal{P}$ is called $\mathcal{D}$-Lipschitz-continuous if for any $\mu, \nu \in \mathcal{H}_p$ and $\mu \propto \nu$, there exists a constant $\zeta_{\mathcal{P}} > 0$ such that*
$$\mathcal{D}(\mathcal{P}(\mu), \mathcal{P}(\nu)) \leq \zeta_{\mathcal{P}} \|\mu \oplus \nu\|;$$

(iii) *$\mathcal{T}$ is scalled Lipschitz-continuous in the first argument if, for any $\mu, \nu \in \mathcal{H}_p$ and $\mu \propto \nu$, there exists a constant $\delta_{\mathcal{T}_1} > 0$ such that*
$$\|\mathcal{T}(\mu, \cdot) \oplus \mathcal{T}(\nu, \cdot)\| \leq \delta_{\mathcal{T}_1} \|\mu \oplus \nu\|.$$

In the same fashion, the Lipschitz continuity of $\mathcal{T}$ can be defined in the second argument.

Definition 8 ([18,23]). *Let $\mathcal{F}, \mathcal{G} : \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings and $\mathcal{M} : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be a set-valued mapping. Then,*

(i) *$\mathcal{M}$ is called a comparison mapping with respect to $\mathcal{F}$ and $\mathcal{G}$ if, for any $p_\mu \in \mathcal{M}(\mathcal{F}(\mu), \mathcal{G}(\mu))$, $\mu \propto p_\mu$ and if $\mu \propto \nu$, then for any $p_\mu \in \mathcal{M}(\mathcal{F}(\mu), \mathcal{G}(\mu))$ and $p_\nu \in \mathcal{M}(\mathcal{F}(\nu), \mathcal{G}(\nu))$, $p_\mu \propto p_\nu$, for all $\mu, \nu \in \mathcal{H}_p$;*

(ii) *A comparison mapping $\mathcal{M}$ is called an XOR-$\alpha_{\mathcal{M}}$-non-ordinary difference mapping with respect to $\mathcal{A}$ and $\mathcal{B}$ if there exists a constant $\alpha_{\mathcal{M}} > 0$, for each $\mu, \nu \in \mathcal{H}_p$ there exist $p_\mu \in \mathcal{M}(\mathcal{F}(\mu), \mathcal{G}(\mu))$ and $p_\nu \in \mathcal{M}(\mathcal{F}(\nu), \mathcal{G}(\nu))$ such that*

$$(p_\mu \oplus p_\nu) \oplus \alpha_{\mathcal{M}}(\mathcal{H}(\mathcal{A}(\mu), \mathcal{B}(\mu)) \oplus \mathcal{H}(\mathcal{A}(\nu), \mathcal{B}(\nu))) = 0;$$

(iii) *a comparison mapping $\mathcal{M}$ is called $\vartheta_{\mathcal{M}}$-ordered rectangular with respect to $\mathcal{F}$ and $\mathcal{G}$ if there exists a constant $\vartheta_{\mathcal{M}} > 0$, for each $\mu, \nu \in \mathcal{H}_p$ there exist $p_\mu \in M(\mathcal{F}(\mu), \mathcal{G}(\mu))$ and $p_\nu \in M(\mathcal{F}(\nu), \mathcal{G}(\nu))$ such that*

$$\langle p_\mu \odot p_\nu, -[(\mathcal{F}(\mu), \mathcal{G}(\mu)) \oplus (\mathcal{F}(\nu), \mathcal{G}(\nu))] \rangle \geq \vartheta_{\mathcal{M}} \|(\mathcal{F}(\mu), \mathcal{G}(\mu)) \oplus (\mathcal{F}(\nu), \mathcal{G}(\nu))\|^2, \quad \text{for all } \mu, \nu \in \mathcal{H}_p;$$

(iv) *A comparison mapping $\mathcal{M}$ is called ϱ-XOR-ordered strongly monotone mapping with respect to $\mathcal{F}$ and $\mathcal{G}$ if for any $\mu, \nu \in \mathcal{H}_p$, $p_\mu \in \mathcal{M}(\mathcal{F}(\mu), \mathcal{G}(\mu))$ and $p_\nu \in \mathcal{M}(\mathcal{F}(\nu), \mathcal{G}(\nu))$ there exists a constant $\varrho > 0$*

$$\varrho(p_\mu \oplus p_\nu) \geq [(\mathcal{F}(\mu), \mathcal{G}(\mu)) \oplus (\mathcal{F}(\nu), \mathcal{G}(\nu))].$$

Definition 9. *Let $\mathcal{A}, \mathcal{B}, \mathcal{F}, \mathcal{G} : \mathcal{H}_p \to \mathcal{H}_p$ and $\mathcal{H} : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings. The set-valued mapping $\mathcal{M} : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ is said to be $\mathcal{H}(\cdot, \cdot)$-compression XOR-$\alpha_{\mathcal{M}}$-non-ordinary difference mapping, if $\mathcal{H}$ is π_1 and π_2-ordered compression mapping with respect to $\mathcal{A}$ and $\mathcal{B}$, respectively, and $\mathcal{M}$ is XOR-$\alpha_{\mathcal{M}}$-non-ordinary difference mapping with respect to $\mathcal{A}$ and $\mathcal{B}$ such that*

$$[\mathcal{H}(\mathcal{A}, \mathcal{B}) \oplus \rho\mathcal{M}(\mathcal{F}, \mathcal{G})](\mathcal{H}_p) = \mathcal{H}_p, \text{ for every } \rho, \alpha_{\mathcal{M}} > 0 \text{ and } 0 < \pi_1, \pi_2 < 1.$$

Definition 10. *Let $\mathcal{A}, \mathcal{B}, \mathcal{F}, \mathcal{G} : \mathcal{H}_p \to \mathcal{H}_p$ and $\mathcal{H} : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings. Let $\mathcal{M} : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be a set-valued $\mathcal{H}(\cdot, \cdot)$-compression XOR-$\alpha_{\mathcal{M}}$-non-ordinary difference mapping. The resolvent operator $R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})} : \mathcal{H}_p \to \mathcal{H}_p$ associated with $\mathcal{A}, \mathcal{B}, \mathcal{F}$ and $\mathcal{G}$ is defined by*

$$R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\mu) = [\mathcal{H}(\mathcal{A}, \mathcal{B}) \oplus \rho\mathcal{M}(\mathcal{F}, \mathcal{G})]^{-1}(\mu), \text{ for all } \mu \in \mathcal{H}_p, \ \rho, \alpha_{\mathcal{M}} > 0. \tag{2}$$

Lemma 2. *Let $\mathcal{A}, \mathcal{B}, \mathcal{F}, \mathcal{G} : \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings such that $\mathcal{F}$ and $\mathcal{G}$ are one–one mappings; $\mathcal{H} : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be a single-valued mapping such that $\mathcal{H}(\mathcal{A}, \cdot)$ is β_1 and $\mathcal{H}(\cdot, \mathcal{B})$ is β_2-ordered compression mapping with respect to $\mathcal{F}$ and $\mathcal{G}$. Let $\mathcal{M} : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be a set-valued $\theta_{\mathcal{M}}$-ordered rectangular mapping with respect to $\mathcal{F}$ and $\mathcal{G}$ such that $\rho\theta_{\mathcal{M}} > \beta_1 + \beta_2$ and $(\mu \oplus \nu) \propto 0$. Then the resolvent operator $R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})} : \mathcal{H}_p \to \mathcal{H}_p$ associated with $\mathcal{A}, \mathcal{B}, \mathcal{F}$ and $\mathcal{G}$ is a single-valued mapping.*

Proof. For any given $\omega \in \mathcal{H}_p, \mu \propto \nu$ and $\rho > 0$, let $\mu, \nu \in [\mathcal{H}(\mathcal{A}, \mathcal{B}) \oplus \rho\mathcal{M}(\mathcal{F}, \mathcal{G})]^{-1}(\omega)$. Then, we have

$$p_\mu = \frac{1}{\rho}[\omega \oplus \mathcal{H}(\mathcal{A}(\mu), \mathcal{B}(\mu))] \in \mathcal{M}(\mathcal{F}(\mu), \mathcal{G}(\mu)),$$

and

$$p_\nu = \frac{1}{\rho}[\omega \oplus \mathcal{H}(\mathcal{A}(\nu), \mathcal{B}(\nu))] \in \mathcal{M}(\mathcal{F}(\nu), \mathcal{G}(\nu)).$$

By utilizing (i) and (ii) of Proposition 2, we have

$$\begin{aligned} p_\mu \odot p_\nu &= \frac{1}{\rho}[\omega \oplus \mathcal{H}(\mathcal{A}(\mu), \mathcal{B}(\mu))] \odot \frac{1}{\rho}[\omega \oplus \mathcal{H}(\mathcal{A}(\nu), \mathcal{B}(\nu))] \\ &= -\frac{1}{\rho}[(\omega \oplus \mathcal{H}(\mathcal{A}(\mu), \mathcal{B}(\mu))) \oplus (\omega \oplus \mathcal{H}(\mathcal{A}(\nu), \mathcal{B}(\nu)))] \\ &= -\frac{1}{\rho}[(\omega \oplus \omega) \oplus (\mathcal{H}(\mathcal{A}(\mu), \mathcal{B}(\mu)) \oplus \mathcal{H}(\mathcal{A}(\nu), \mathcal{B}(\nu)))] \\ &= -\frac{1}{\rho}[0 \oplus (\mathcal{H}(\mathcal{A}(\mu), \mathcal{B}(\mu)) \oplus \mathcal{H}(\mathcal{A}(\nu), \mathcal{B}(\nu)))] \\ &\leq -\frac{1}{\rho}[\mathcal{H}(\mathcal{A}(\mu), \mathcal{B}(\mu)) \oplus \mathcal{H}(\mathcal{A}(\nu), \mathcal{B}(\nu))]. \end{aligned} \tag{3}$$

Since $\mathcal{M}$ is $\theta_{\mathcal{M}}$-ordered rectangular mapping with respect to $\mathcal{F}$ and $\mathcal{G}$, $\mathcal{H}(\mathcal{A}, \cdot)$ is β_1 and $\mathcal{H}(\cdot, \mathcal{B})$ is β_2-ordered compression mapping with respect to $\mathcal{F}$ and $\mathcal{G}$. Therefore,

$$\begin{aligned} \theta_{\mathcal{M}}\|(\mathcal{F}(\mu), \mathcal{G}(\mu)) \oplus (\mathcal{F}(\nu), \mathcal{G}(\nu))\|^2 &\leq \langle p_\mu \odot p_\nu, -((\mathcal{F}(\mu), \mathcal{G}(\mu)) \oplus (\mathcal{F}(\nu), \mathcal{G}(\nu)))\rangle \\ &\leq \frac{1}{\rho}\langle(\mathcal{H}(\mathcal{A}(\mu), \mathcal{B}(\mu)) \oplus \mathcal{H}(\mathcal{A}(\nu), \mathcal{B}(\nu)), \\ &\quad ((\mathcal{F}(\mu), \mathcal{G}(\mu)) \oplus (\mathcal{F}(\nu), \mathcal{G}(\nu)))\rangle \\ &= \frac{1}{\rho}\langle(\mathcal{H}(\mathcal{A}(\mu), \mathcal{B}(\mu)) \oplus \mathcal{H}(\mathcal{A}(\nu), \mathcal{B}(\mu)), \\ &\quad ((\mathcal{F}(\mu), \mathcal{G}(\mu)) \oplus (\mathcal{F}(\nu), \mathcal{G}(\nu)))\rangle \\ &\quad + \frac{1}{\rho}\langle(\mathcal{H}(\mathcal{A}(\nu), \mathcal{B}(\mu)) \oplus \mathcal{H}(\mathcal{A}(\nu), \mathcal{B}(\nu)), \\ &\quad ((\mathcal{F}(\mu), \mathcal{G}(\mu)) \oplus (\mathcal{F}(\nu), \mathcal{G}(\nu)))\rangle \\ &\leq \frac{\beta_1}{\rho}\|(\mathcal{F}(\mu), \mathcal{G}(\nu)) \oplus (\mathcal{F}(\nu), \mathcal{G}(\nu))\|^2 \\ &\quad + \frac{\beta_2}{\rho}\|(\mathcal{F}(\mu), \mathcal{G}(\mu)) \oplus (\mathcal{F}(\nu), \mathcal{G}(\nu))\|^2 \\ &= \frac{\beta_1 + \beta_2}{\rho}\|(\mathcal{F}(\mu), \mathcal{G}(\mu)) \oplus (\mathcal{F}(\nu), \mathcal{G}(\nu))\|^2, \end{aligned} \tag{4}$$

which implies that

$$\left(\theta_{\mathcal{M}} - \frac{\beta_1 + \beta_2}{\rho}\right)\|(\mathcal{F}(\mu), \mathcal{G}(\mu)) \oplus (\mathcal{F}(\nu), \mathcal{G}(\nu))\|^2 \leq 0.$$

Since $\rho\theta_{\mathcal{M}} > \beta_1 + \beta_2$, therefore, $\|(\mathcal{F}(\mu), \mathcal{G}(\mu)) \oplus (\mathcal{F}(\nu), \mathcal{G}(\nu))\| = 0$. Thus, we have

$$(\mathcal{F}(\mu), \mathcal{G}(\mu)) \oplus (\mathcal{F}(\nu), \mathcal{G}(\nu)) = 0,$$

which in turn becomes

$$(\mathcal{F}(\mu), \mathcal{G}(\mu)) = (\mathcal{F}(\nu), \mathcal{G}(\nu)).$$

That is, $\mathcal{F}(\mu) = \mathcal{F}(\nu)$ and $\mathcal{G}(\mu) = \mathcal{G}(\nu)$. Since the mappings $\mathcal{F}$ and $\mathcal{G}$ are one–one, therefore, $\mu = \nu$. Hence, $R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}$ associated with $\mathcal{A}, \mathcal{B}, \mathcal{F}$ and $\mathcal{G}$ is single-valued. □

Lemma 3. *Let $\mathcal{A}, \mathcal{B}, \mathcal{F}, \mathcal{G} : \mathcal{H}_p \to \mathcal{H}_p$ and $\mathcal{H} : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings such that $\mathcal{H}(\cdot, \cdot)$ is a mixed strong comparison mapping with respect to $\mathcal{A}$ and $\mathcal{B}$. Let $\mathcal{M} : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be a set-valued ρ-XOR-ordered strongly monotone mapping with respect to $\mathcal{F}$ and $\mathcal{G}$ and assume that $(\mathcal{F}(\mu), \mathcal{G}(\mu)) \oplus (\mathcal{F}(\nu), \mathcal{G}(\nu)) \propto \mu \oplus \nu$. Then, the resolvent operator $R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})} : \mathcal{H}_p \to \mathcal{H}_p$ associated with $\mathcal{A}, \mathcal{B}, \mathcal{F}$ and $\mathcal{G}$ is a comparison mapping.*

Proof. For any given $\mu, \nu \in \mathcal{H}_p$, suppose $\mu \propto \nu$,

$$p_\mu = \frac{1}{\rho}[\mu \oplus \mathcal{H}(\mathcal{A}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\mu)), \mathcal{B}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\mu)))] \in \mathcal{M}(\mathcal{F}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\mu)), \mathcal{G}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\mu))), \tag{5}$$

and

$$p_\nu = \frac{1}{\rho}[\nu \oplus \mathcal{H}(\mathcal{A}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\nu)), \mathcal{B}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\nu)))] \in \mathcal{M}(\mathcal{F}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\nu)), \mathcal{G}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\nu))). \tag{6}$$

For the sake of simplicity, we take

$$\kappa(\mu) = R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\mu) \text{ and } \kappa(\nu) = R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\nu).$$

Since $\mathcal{M}$ is a ρ-XOR-ordered strongly monotone mapping with respect to $\mathcal{F}$ and $\mathcal{G}$, and using (5) and (6), we have

$$\begin{aligned}(\mathcal{F}(\mu), \mathcal{G}(\mu)) \oplus (\mathcal{F}(\nu), \mathcal{G}(\nu)) &\leq \rho(p_\mu \oplus p_\nu) \\ &= [\mu \oplus \mathcal{H}(\mathcal{A}(\kappa(\mu)), \mathcal{B}(\kappa(\mu)))] \oplus [\nu \oplus \mathcal{H}(\mathcal{A}(\kappa(\nu)), \mathcal{B}(\kappa(\nu)))] \\ &= (\mu \oplus \nu) \oplus [\mathcal{H}(\mathcal{A}(\kappa(\mu)), \mathcal{B}(\kappa(\mu))) \oplus \mathcal{H}(\mathcal{A}(\kappa(\nu)), \mathcal{B}(\kappa(\nu)))].\end{aligned}$$

Thus, we have

$$0 \leq [(\mathcal{F}(\mu), \mathcal{G}(\mu)) \oplus (\mathcal{F}(\nu), \mathcal{G}(\nu)) \oplus (\mu \oplus \nu)] \oplus [\mathcal{H}(\mathcal{A}(\kappa(\mu)), \mathcal{B}(\kappa(\mu))) \oplus \mathcal{H}(\mathcal{A}(\kappa(\nu)), \mathcal{B}(\kappa(\nu)))].$$

Since $(\mathcal{F}(\mu), \mathcal{G}(\mu)) \oplus (\mathcal{F}(\nu), \mathcal{G}(\nu)) \propto \mu \oplus \nu$, therefore

$$0 \leq [(\mu \oplus \nu) \oplus (\mu \oplus \nu)] \oplus [\mathcal{H}(\mathcal{A}(\kappa(\mu)), \mathcal{B}(\kappa(\mu))) \oplus \mathcal{H}(\mathcal{A}(\kappa(\nu)), \mathcal{B}(\kappa(\nu)))],$$

which implies that

$$0 \leq [\mathcal{H}(\mathcal{A}(\kappa(\mu)), \mathcal{B}(\kappa(\mu))) \oplus \mathcal{H}(\mathcal{A}(\kappa(\nu)), \mathcal{B}(\kappa(\nu)))].$$

The above inequality gives

$$\begin{aligned}0 \leq &[\mathcal{H}(\mathcal{A}(\kappa(\mu)), \mathcal{B}(\kappa(\mu))) - \mathcal{H}(\mathcal{A}(\kappa(\nu)), \mathcal{B}(\kappa(\nu)))] \\ &\vee [\mathcal{H}(\mathcal{A}(\kappa(\nu)), \mathcal{B}(\kappa(\nu))) - \mathcal{H}(\mathcal{A}(\kappa(\mu)), \mathcal{B}(\kappa(\mu)))].\end{aligned}$$

Thus, we have

$$\text{either } 0 \leq [\mathcal{H}(\mathcal{A}(\kappa(\mu)), \mathcal{B}(\kappa(\mu))) - \mathcal{H}(\mathcal{A}(\kappa(\nu)), \mathcal{B}(\kappa(\nu)))]$$

$$\text{or } 0 \leq [\mathcal{H}(\mathcal{A}(\kappa(\nu)), \mathcal{B}(\kappa(\nu))) - \mathcal{H}(\mathcal{A}(\kappa(\mu)), \mathcal{B}(\kappa(\mu)))].$$

That is,

$$\mathcal{H}(\mathcal{A}(\kappa(\nu)), \mathcal{B}(\kappa(\nu))) \leq \mathcal{H}(\mathcal{A}(\kappa(\mu)), \mathcal{B}(\kappa(\mu))) \tag{7}$$

$$\text{or } \mathcal{H}(\mathcal{A}(\kappa(\mu)), \mathcal{B}(\kappa(\mu))) \leq \mathcal{H}(\mathcal{A}(\kappa(\nu)), \mathcal{B}(\kappa(\nu))). \tag{8}$$

It follows from (7) and (8) that

$$\mathcal{H}(\mathcal{A}(\kappa(\mu)), \mathcal{B}(\kappa(\mu))) \propto \mathcal{H}(\mathcal{A}(\kappa(\nu)), \mathcal{B}(\kappa(\nu))).$$

That is,

$$\mathcal{H}(\mathcal{A}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\mu)), \mathcal{B}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\mu))) \propto \mathcal{H}(\mathcal{A}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\nu)), \mathcal{B}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\nu))).$$

Since $\mathcal{H}$ is a mixed strong comparison mapping with respect to $\mathcal{A}$ and $\mathcal{B}$, then we have $R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\mu) \propto R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\nu)$, that is, $R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}$ is a comparison mapping. □

Lemma 4. *Let $\mathcal{A}, \mathcal{B}, \mathcal{F}, \mathcal{G} : \mathcal{H}_p \to \mathcal{H}_p$ and $\mathcal{H} : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings. Let $\mathcal{M} : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be a set-valued $\mathcal{H}(\cdot,\cdot)$-compression XOR-$\alpha_{\mathcal{M}}$-non-ordinary difference mapping, $\alpha_{\mathcal{M}} > 0$. Assume that $[\mathcal{H}(\mathcal{A}(\epsilon), \mathcal{B}(\epsilon)) \oplus \mathcal{H}(\mathcal{A}(\omega), \mathcal{B}(\omega))] \propto \epsilon \oplus \omega$, for all $\epsilon, \omega \in \mathcal{H}_p$.*

Then the resolvent operator $R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})} : \mathcal{H}_p \to \mathcal{H}_p$ *is* $\left(\dfrac{1}{1+\rho\alpha_{\mathcal{M}}}\right)$*-Lipschitz-continuous—that is,*

$$\|R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\mu) \oplus R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\nu)\| \leq \frac{1}{1+\rho\alpha_{\mathcal{M}}}\|\mu \oplus \nu\|, \text{ for all } \mu,\nu \in \mathcal{H}_p.$$

Proof. Let p_μ and p_ν assume the same values as in (5) and (6), respectively. Then,

$$\begin{aligned} p_\mu \oplus p_\nu &= \frac{1}{\rho}[\mu \oplus \mathcal{H}(\mathcal{A}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\mu)), \mathcal{B}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\mu)))] \\ &\quad \oplus \frac{1}{\rho}[\nu \oplus \mathcal{H}(\mathcal{A}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\nu)), \mathcal{B}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\nu)))] \\ &\leq \frac{1}{\rho}\Big[(\mu \oplus \nu) \oplus [\mathcal{H}(\mathcal{A}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\mu)), \mathcal{B}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\mu))) \\ &\quad \oplus \mathcal{H}(\mathcal{A}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\nu)), \mathcal{B}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\nu)))]\Big]. \end{aligned} \tag{9}$$

Since $\mathcal{M}$ is XOR-$\alpha_{\mathcal{M}}$-non-ordinary difference mapping with respect to $\mathcal{A}$ and $\mathcal{B}$, we have

$$\alpha_{\mathcal{M}}[\mathcal{H}(\mathcal{A}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\mu)), \mathcal{B}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\mu))) \oplus \mathcal{H}(\mathcal{A}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\nu)), \mathcal{B}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\nu)))] = p_\mu \oplus p_\nu,$$

which implies that

$$\begin{aligned} &\alpha_{\mathcal{M}}[\mathcal{H}(\mathcal{A}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\mu)), \mathcal{B}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\mu))) \oplus \mathcal{H}(\mathcal{A}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\nu)), \mathcal{B}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\nu)))] \\ &\qquad \leq \frac{1}{\rho}\Big[(\mu \oplus \nu) \oplus [\mathcal{H}(\mathcal{A}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\mu)), \mathcal{B}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\mu))) \\ &\qquad \oplus \mathcal{H}(\mathcal{A}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\nu)), \mathcal{B}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\nu)))]\Big]. \end{aligned}$$

Thus, we have

$$[1+\rho\alpha_{\mathcal{M}}][\mathcal{H}(\mathcal{A}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\mu)), \mathcal{B}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\mu))) \oplus \mathcal{H}(\mathcal{A}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\nu)), \mathcal{B}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\nu)))] \leq (\mu \oplus \nu).$$

It follows from the assumption that

$$\begin{aligned} &[\mathcal{H}(\mathcal{A}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\mu)), \mathcal{B}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\mu))) \oplus \mathcal{H}(\mathcal{A}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\nu)), \mathcal{B}(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\nu)))] \\ &\qquad \propto R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\mu) \oplus R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\nu), \end{aligned}$$

which gives us

$$[1+\rho\alpha_{\mathcal{M}}][R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\mu) \oplus R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\nu)] \leq (\mu \oplus \nu).$$

That is, $R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\mu) \oplus R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\nu) \leq \left(\dfrac{1}{1+\rho\alpha_{\mathcal{M}}}\right)(\mu \oplus \nu)$. Thus, we have

$$\|R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\mu) \oplus R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(\nu)\| \leq \left(\frac{1}{1+\rho\alpha_{\mathcal{M}}}\right)\|\mu \oplus \nu\|,$$

i.e., $R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}$ is $\left(\dfrac{1}{1+\rho\alpha_{\mathcal{M}}}\right)$-Lipschitz-continuous. □

3. Formulation of the Problem and Existence Result

This section begins with the designing of a generalized ordered variational inclusion problem involving $\mathcal{H}(\cdot,\cdot)$-compression XOR-$\alpha_{\mathcal{M}}$-non-ordinary difference mapping and relaxed cocoercive mapping. We discuss the existence of a unique solution for the considered problem.

Let $\mathcal{H}_p$ be a real ordered Hilbert space and $\mathcal{C}$ be a normal cone with normal constant $\lambda_{\mathcal{C}}$. Let $\mathcal{A}, \mathcal{B}, \mathcal{F}, \mathcal{G}, \mathcal{P}, \mathcal{S} : \mathcal{H}_p \to \mathcal{H}_p$ and $\mathcal{H}, \varphi : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings. Let $\mathcal{R}, \mathcal{T} : \mathcal{H}_p \to CB(\mathcal{H}_p)$ be the set-valued mappings and $\mathcal{M} : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be a set-valued $\mathcal{H}(\cdot,\cdot)$-compression XOR-$\alpha_{\mathcal{M}}$-non-ordinary difference mapping. We deal with the following generalized ordered variational inclusion problem:

For some $a \in \mathcal{H}_p$ and any $b \in \mathbb{R}$, find $x \in \mathcal{H}_p, \mu \in \mathcal{R}(x), \nu \in \mathcal{T}(x)$ such that

$$a \in \mathcal{P}(\mu) \oplus \tau\mathcal{M}(\mathcal{F}(x), \mathcal{G}(x)) - b\varphi(x - \mathcal{S}x, \nu), \tag{10}$$

where $\tau > 0$ is a constant. Problem (10) is termed as generalized, set-valued, mixed-ordered, variational inclusion problem with XOR-operator (in short, GSMOVIP).

Note that GSMOVIP (10) is more general, and for appropriate selection of the mappings comprised in the designing, it includes many problems existing in the literature. Some particular cases of GSMOVIP (10) are reported below:

(i) If $\mathcal{F} = I$, the identity mapping, $\mathcal{G}, \mathcal{S} = 0$, $\mathcal{M}(\mathcal{F}(x), \mathcal{G}(x)) = \mathcal{M}(x)$ and $\mathcal{R}, \mathcal{T}$ are the single-valued mappings, and then GSMOVIP (10) becomes an equivalent problem of finding $x \in \mathcal{H}_p$, for some $a \in \mathcal{H}_p$ and any $b \in \mathbb{R}$ such that

$$a \in \mathcal{P}(x) \oplus \tau\mathcal{M}(x) - b\varphi(x, g(x)). \tag{11}$$

Problem (11) was constructed and examined by Ahmad et al. [27], using (γ_R, λ)-weak-RRD mappings.

(ii) If $\tau = 1$ and $\varphi(\cdot,\cdot) = 0$, then problem (11) coincides with the problem of finding $x \in \mathcal{H}_p$ such that

$$a \in \mathcal{P}(x) \oplus \mathcal{M}(x). \tag{12}$$

Problem (12) was investigated by Li et al. [18] in the framework of weak-ANODD set-valued mappings.

(iii) If $a = 0$ and $\mathcal{P} = 0$, then problem (11) coincides to the problem of finding $x \in \mathcal{H}_p$ such that

$$0 \in \tau\mathcal{M}(x) - b\varphi(x, g(x)). \tag{13}$$

Problem (13) was examined by Li et al. [25].

(iv) If $\varphi(\cdot,\cdot) = 0$ and $\mathcal{P} = 0$, then problem (11) becomes an equivalent problem of finding $x \in \mathcal{H}_p$ such that

$$a \in \tau\mathcal{M}(x). \tag{14}$$

Problem (14) was investigated by Li et al. [24] in the framework of (γ_G, λ)-weak-RRD mappings.

In the following Lemma, we set up an equivalence between ordered fixed-point problem and GSMOVIP (10).

Lemma 5. *Let $\mathcal{C}$ be a normal cone with normal constant $\lambda_{\mathcal{C}}$ in $\mathcal{H}_p$. Let $\mathcal{A}, \mathcal{B}, \mathcal{F}, \mathcal{G}, \mathcal{P}, \mathcal{S} : \mathcal{H}_p \to \mathcal{H}_p$ and $\mathcal{H}, \varphi : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings. Let $\mathcal{R}, \mathcal{T} : \mathcal{H}_p \to CB(\mathcal{H}_p)$ be the set-valued mappings, and $\mathcal{M} : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be a set-valued $\mathcal{H}(\cdot,\cdot)$-compression XOR-$\alpha_{\mathcal{M}}$-non-ordinary difference mapping. Then, $(x, \mu, \nu), x \in \mathcal{H}_p, \mu \in \mathcal{R}(x), \nu \in \mathcal{T}(x)$ solves GSMOVIP (10), if and only if*

$$x = R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}[\mathcal{H}(\mathcal{A}x, \mathcal{B}x) + \frac{\rho}{\tau}(\mathcal{P}(\mu) \oplus b\varphi(x - \mathcal{S}x, \nu)) + a]. \tag{$*$}$$

Proof. The proof of the lemma follows immediately by using the definition of a resolvent operator, so we omit the proof. □

Theorem 1. *Let $\mathcal{C}$ be a normal cone in $\mathcal{H}_p$ with normal constant $\lambda_{\mathcal{C}}$. Let $\mathcal{A}, \mathcal{B}, \mathcal{F}, \mathcal{G}, \mathcal{P}, \mathcal{S} : \mathcal{H}_p \to \mathcal{H}_p$ and $\mathcal{H}, \varphi : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings such that $\mathcal{P}$ is a $\delta_{\mathcal{P}}$-Lipschitz-continuous mapping; $\mathcal{S}$ is $\delta_{\mathcal{S}}$-Lipschitz-continuous and relaxed $(\alpha_{\mathcal{S}}, k_{\mathcal{S}})$-cocoercive; $\mathcal{H}(\cdot,\cdot)$ is π_1-ordered compression mapping with respect to $\mathcal{A}$ in the first argument and π_2-ordered compression mapping with respect to $\mathcal{B}$ in the second argument, and $\varphi(\cdot,\cdot)$ is δ_{φ_1} and δ_{φ_2}-Lipschitz-continuous mapping with respect to the first and second argument, respectively. Let $\mathcal{R}, \mathcal{T} : \mathcal{H}_p \to CB(\mathcal{H}_p)$ be $\mathcal{D}$-Lipschitz-continuous mappings with constant $\zeta_{\mathcal{R}}, \zeta_{\mathcal{T}}$, respectively, and $\mathcal{M} : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be a set-valued $\mathcal{H}(\cdot,\cdot)$-compression XOR-$\alpha_{\mathcal{M}}$-non-ordinary difference mapping. If $x_1 \propto x_2$ and the following condition is satisfied:*

$$(\pi_1 + \pi_2)\tau + \rho\lambda_{\mathcal{C}}[\delta_{\mathcal{P}}\zeta_{\mathcal{R}} \oplus |b|(\delta_{\varphi_1}\sqrt{(1-2k_{\mathcal{S}}) + (1+2\alpha_{\mathcal{S}})\delta_{\mathcal{S}}} \oplus \delta_{\varphi_2}\zeta_{\mathcal{T}})] < 1; \rho, \alpha_{\mathcal{M}} > 0. \tag{15}$$

Then, GSMOVIP (10) admits a unique solution.

Proof. For the sake of convenience, we assume $Q(x_i) = \mathcal{P}(\mu_i) \oplus b\varphi(x_i - \mathcal{S}x_i, \nu_i) + a$, for $i = 1, 2$. It follows from Lemma 4 and Proposition 2 that

$$\begin{aligned}
&\|\Big(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}[\mathcal{H}(\mathcal{A},\mathcal{B}) + \frac{\rho}{\tau}Q]\Big)(x_1) \oplus \Big(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}[\mathcal{H}(\mathcal{A},\mathcal{B}) + \frac{\rho}{\tau}Q]\Big)(x_2)\| \\
&\quad \le \frac{1}{1+\rho\alpha_{\mathcal{M}}}\|[\mathcal{H}(\mathcal{A}(x_1),\mathcal{B}(x_1)) + \frac{\rho}{\tau}Q(x_1)] \oplus [\mathcal{H}(\mathcal{A}(x_2),\mathcal{B}(x_2)) + \frac{\rho}{\tau}Q(x_2)]\| \\
&\quad \le \frac{1}{1+\rho\alpha_{\mathcal{M}}}\Big[\|\mathcal{H}(\mathcal{A}(x_1),\mathcal{B}(x_1)) \oplus \mathcal{H}(\mathcal{A}(x_2),\mathcal{B}(x_2))\| + \frac{\rho}{\tau}\|Q(x_1) \oplus Q(x_2)\|\Big].
\end{aligned} \tag{16}$$

Since $\mathcal{H}(\cdot,\cdot)$ is π_1-ordered compression mapping with respect to $\mathcal{A}$ in the first argument and π_2-ordered compression mapping with respect to $\mathcal{B}$ in the second argument,

$$\|\mathcal{H}(\mathcal{A}(x_1),\mathcal{B}(x_1)) \oplus \mathcal{H}(\mathcal{A}(x_2),\mathcal{B}(x_2))\| \le (\pi_1 + \pi_2)\|x_1 \oplus x_2\|. \tag{17}$$

Since $\mathcal{P}$ is $\delta_{\mathcal{P}}$-Lipschitz-continuous mapping and $\mathcal{R}$ is $\mathcal{D}$-Lipschitz-continuous mapping with constant $\zeta_{\mathcal{R}}$, then we have

$$\begin{aligned}
0 \le& Q(x_1) \oplus Q(x_2) \\
=& [\mathcal{P}(\mu_1) \oplus b\varphi(x_1 - \mathcal{S}x_1, \nu_1) + a] \oplus [\mathcal{P}(\mu_2) \oplus b\varphi(x_2 - \mathcal{S}x_2, \nu_2) + a] \\
=& [(\mathcal{P}(\mu_1) \oplus \mathcal{P}(\mu_2)) \oplus ((b\varphi(x_1 - \mathcal{S}x_1, \nu_1) + a) \oplus (b\varphi(x_2 - \mathcal{S}x_2, \nu_2) + a))] \\
\le& \delta_{\mathcal{P}}(\mu_1 \oplus \mu_2) \oplus |b|(\varphi(x_1 - \mathcal{S}x_1, \nu_1) \oplus \varphi(x_2 - \mathcal{S}x_2, \nu_2)) + (a \oplus a) \\
\le& \delta_{\mathcal{P}}\zeta_{\mathcal{R}}(x_1 \oplus x_2) \oplus |b|(\varphi(x_1 - \mathcal{S}x_1, \nu_1) \oplus \varphi(x_2 - \mathcal{S}x_2, \nu_2)).
\end{aligned} \tag{18}$$

Thus, from the definition of a normal cone, utilizing the Lipschitz continuity of φ in the first and second argument and the $\mathcal{D}$-Lipschitz continuity of $\mathcal{T}$, we can write

$$\begin{aligned}
\|Q(x_1) \oplus Q(x_2)\| \le& \lambda_{\mathcal{C}}\delta_{\mathcal{P}}\zeta_{\mathcal{R}}\|x_1 \oplus x_2\| \oplus \lambda_{\mathcal{C}}|b|\|\varphi(x_1 - \mathcal{S}x_1, \nu_1) \oplus \varphi(x_2 - \mathcal{S}x_2, \nu_2)\| \\
\le& \lambda_{\mathcal{C}}\delta_{\mathcal{P}}\zeta_{\mathcal{R}}\|x_1 \oplus x_2\| \oplus \lambda_{\mathcal{C}}|b|\|\varphi(x_1 - \mathcal{S}x_1, \nu_1) \oplus \varphi(x_2 - \mathcal{S}x_2, \nu_1)\| \\
&\oplus \lambda_{\mathcal{C}}|b|\|\varphi(x_2 - \mathcal{S}x_2, \nu_1) \oplus \varphi(x_2 - \mathcal{S}x_2, \nu_2)\| \\
\le& \lambda_{\mathcal{C}}\delta_{\mathcal{P}}\zeta_{\mathcal{R}}\|x_1 \oplus x_2\| \oplus \lambda_{\mathcal{C}}|b|\delta_{\varphi_1}\|(x_1 - \mathcal{S}x_1) \oplus (x_2 - \mathcal{S}x_2)\| \\
&\oplus \lambda_{\mathcal{C}}|b|\delta_{\varphi_2}\|\nu_1 \oplus \nu_2\| \\
\le& \lambda_{\mathcal{C}}\delta_{\mathcal{P}}\zeta_{\mathcal{R}}\|x_1 \oplus x_2\| \oplus \lambda_{\mathcal{C}}|b|\delta_{\varphi_1}\|(x_1 - x_2) - (\mathcal{S}x_1 - \mathcal{S}x_2)\| \\
&\oplus \lambda_{\mathcal{C}}|b|\delta_{\varphi_2}\zeta_{\mathcal{T}}\|x_1 \oplus x_2\|.
\end{aligned} \tag{19}$$

Since $\mathcal{S}$ is $\delta_{\mathcal{S}}$-Lipschitz-continuous and relaxed $(\alpha_{\mathcal{S}}, k_{\mathcal{S}})$-cocoercive mapping, therefore

$$\begin{aligned}
\|(x_1 - x_2) - (\mathcal{S}x_1 - \mathcal{S}x_2)\|^2 =& \|x_1 - x_2\|^2 - 2\langle \mathcal{S}x_1 - \mathcal{S}x_2, x_1 - x_2\rangle + \|\mathcal{S}x_1 - \mathcal{S}x_2\|^2 \\
\le& \|x_1 - x_2\|^2 + 2\alpha_{\mathcal{S}}\|\mathcal{S}x_1 - \mathcal{S}x_2\|^2 - 2k_{\mathcal{S}}\|x_1 - x_2\|^2 + \|\mathcal{S}x_1 - \mathcal{S}x_2\|^2 \\
\le& [(1 - 2k_{\mathcal{S}}) + (1 + 2\alpha_{\mathcal{S}})\delta_{\mathcal{S}}]\|x_1 - x_2\|^2,
\end{aligned}$$

which inferred that

$$\begin{aligned}\|(x_1 - x_2) - (\mathcal{S}x_1 - \mathcal{S}x_2)\| &\leq \sqrt{(1 - 2k_{\mathcal{S}}) + (1 + 2\alpha_{\mathcal{S}})\delta_{\mathcal{S}}}\|x_1 - x_2\| \\ &= \lambda_{\mathcal{C}}\sqrt{(1 - 2k_{\mathcal{S}}) + (1 + 2\alpha_{\mathcal{S}})\delta_{\mathcal{S}}}\|x_1 \oplus x_2\|.\end{aligned} \tag{20}$$

Utilizing (17)–(20), (16) becomes

$$\begin{aligned}&\|\Big(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}[\mathcal{H}(\mathcal{A},\mathcal{B}) + \frac{\rho}{\tau}Q]\Big)(x_1) \oplus \Big(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}[\mathcal{H}(\mathcal{A},\mathcal{B}) + \frac{\rho}{\tau}Q]\Big)(x_2)\| \\ &\leq \frac{1}{1+\rho\alpha_{\mathcal{M}}}\Big[(\pi_1 + \pi_2) + \frac{\rho}{\tau}[\lambda_{\mathcal{C}}\delta_{\mathcal{P}}\zeta_{\mathcal{R}} \oplus \lambda_{\mathcal{C}}|b|\delta_{\varphi_1}\sqrt{(1 - 2k_{\mathcal{S}}) + (1 + 2\alpha_{\mathcal{S}})\delta_{\mathcal{S}}} \\ &\quad \oplus \lambda_{\mathcal{C}}|b|\delta_{\varphi_2}\zeta_{\mathcal{T}}]\Big]\|x_1 \oplus x_2\| \\ &= \frac{1}{\tau(1+\rho\alpha_{\mathcal{M}})}\Big[\pi_1 + \pi_2)\tau + \rho[\lambda_{\mathcal{C}}\delta_{\mathcal{P}}\zeta_{\mathcal{R}} \oplus \lambda_{\mathcal{C}}|b|\delta_{\varphi_1}\sqrt{(1 - 2k_{\mathcal{S}}) + (1 + 2\alpha_{\mathcal{S}})\delta_{\mathcal{S}}} \\ &\quad \oplus \lambda_{\mathcal{C}}|b|\delta_{\varphi_2}\zeta_{\mathcal{T}}]\Big]\|x_1 \oplus x_2\|,\end{aligned}$$

which implies that

$$\|\Big(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}[\mathcal{H}(\mathcal{A},\mathcal{B}) + \frac{\rho}{\tau}Q]\Big)(x_1) \oplus \Big(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}[\mathcal{H}(\mathcal{A},\mathcal{B}) + \frac{\rho}{\tau}Q]\Big)(x_2)\| \leq \Theta\|x_1 \oplus x_2\|, \tag{21}$$

where

$$\Theta = \frac{(\pi_1 + \pi_2)\tau + \rho\lambda_{\mathcal{C}}[\delta_{\mathcal{P}}\zeta_{\mathcal{R}} \oplus |b|(\delta_{\varphi_1}\sqrt{(1 - 2k_{\mathcal{S}}) + (1 + 2\alpha_{\mathcal{S}})\delta_{\mathcal{S}}} \oplus \delta_{\varphi_2}\zeta_{\mathcal{T}})]}{\tau(1+\rho\alpha_{\mathcal{M}})}. \tag{22}$$

From condition (15), we deduce that $\Theta < 1$. Thus, from (21), we can see that the mapping $R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}[\mathcal{H}(\mathcal{A},\mathcal{B}) + \frac{\rho}{\tau}Q]$ is contraction. There exists a unique point $x \in \mathcal{H}_p$ such that

$$x = R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}[\mathcal{H}(\mathcal{A}x, \mathcal{B}x) + \frac{\rho}{\tau}(\mathcal{P}(\mu) \oplus b\varphi(x - \mathcal{S}x, \nu)) + a].$$

Thus, from Lemma 5, one can conclude that $(x, \mu, \nu), x \in \mathcal{H}_p, \mu \in \mathcal{R}(x), \nu \in \mathcal{T}(x)$ is the unique solution of GSMOVIP (10). □

4. Convergence Analysis

This section begins with the construction of a three-step iterative algorithm. Finally, the convergence analysis of the proposed iterative algorithm to the unique solution of GSMOVIP (10) is discussed.

Theorem 2. *Let $\mathcal{C}$ be a normal cone in $\mathcal{H}_p$ with normal constant $\lambda_{\mathcal{C}}$. Let the mappings $\mathcal{A}, \mathcal{B}, \mathcal{F}, \mathcal{G}, \mathcal{P}$, $\mathcal{S}, \mathcal{H}, \varphi, \mathcal{R}, \mathcal{T}$ be identical as in Theorem 1, such that all the suppositions of Theorem 1 are satisfied. Let $\mathcal{M} : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be a set-valued $\mathcal{H}(\cdot,\cdot)$-compression XOR-$\alpha_{\mathcal{M}}$-non-ordinary difference mapping. If $x_{n+1} \propto x$ and the following condition holds:*

$$(\pi_1 + \pi_2)\tau + \rho\lambda_{\mathcal{C}}[\delta_{\mathcal{P}}\zeta_{\mathcal{R}} \oplus |b|(\delta_{\varphi_1}\sqrt{(1 - 2k_{\mathcal{S}}) + (1 + 2\alpha_{\mathcal{S}})\delta_{\mathcal{S}}} \oplus \delta_{\varphi_2}\zeta_{\mathcal{T}})] < 1; \rho, \alpha_{\mathcal{M}} > 0. \tag{23}$$

Then, the approximate sequences $\{x_n\}, \{\mu_n\}$ and $\{\nu_n\}$ generated by Algorithm 1 converge strongly to the unique solution x, μ, and ν, respectively, of GSMOVIP (10).

Algorithm 1 Let $\mathcal{C}$ be a normal cone with normal constant $\lambda_{\mathcal{C}}$ in $\mathcal{H}_p$. Let $\mathcal{A}, \mathcal{B}, \mathcal{F}, \mathcal{G}, \mathcal{P}, \mathcal{S} : \mathcal{H}_p \to \mathcal{H}_p$ and $\mathcal{H}, \varphi : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings.

Let $\mathcal{R}, \mathcal{T} : \mathcal{H}_p \to CB(\mathcal{H}_p)$ be the set-valued mappings and $\mathcal{M} : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be a set-valued $\mathcal{H}(\cdot,\cdot)$-compression XOR-$\alpha_{\mathcal{M}}$-non-ordinary difference mapping. For any initial guesses $x_0 \in \mathcal{H}_p, \mu_0 \in \mathcal{R}(x_0), \nu_0 \in \mathcal{T}(x_0)$, compute the sequences $\{x_n\}, \{\mu_n\}, \{\nu_n\}$ by the following iterative scheme:

$$\begin{aligned}
z_n &= (1 - c_n)x_n + c_n[R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}[\mathcal{H}(\mathcal{A}x_n, \mathcal{B}x_n) + \frac{\rho}{\tau}(\mathcal{P}(\mu_n) \oplus b\varphi(x_n - \mathcal{S}x_n, \nu_n)) + a],\\
y_n &= (1 - b_n)x_n + b_n[R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}[\mathcal{H}(\mathcal{A}z_n, \mathcal{B}z_n) + \frac{\rho}{\tau}(\mathcal{P}(\mu''_n) \oplus b\varphi(z_n - \mathcal{S}z_n, \nu''_n)) + a],\\
x_{n+1} &= (1 - a_n)x_n + a_n[R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}[\mathcal{H}(\mathcal{A}y_n, \mathcal{B}y_n) + \frac{\rho}{\tau}(\mathcal{P}(\mu'_n) \oplus b\varphi(y_n - \mathcal{S}y_n, \nu'_n)) + a],\\
\mu_{n+1} &\in \mathcal{R}(x_{n+1}) : \|\mu_{n+1} \oplus \mu_n\| \le \mathcal{D}(\mathcal{R}(x_{n+1}), \mathcal{R}(x_n)),\\
\nu_{n+1} &\in \mathcal{T}(x_{n+1}) : \|\nu_{n+1} \oplus \nu_n\| \le \mathcal{D}(\mathcal{T}(x_{n+1}), \mathcal{T}(x_n)),
\end{aligned}$$

where $\mu_n \in \mathcal{R}(x_n), \mu'_n \in \mathcal{R}(y_n), \mu''_n \in \mathcal{R}(z_n), \nu_n \in \mathcal{T}(x_n), \nu'_n \in \mathcal{T}(y_n), \nu''_n \in \mathcal{T}(z_n)$ and $0 \le a_n, b_n, c_n \le 1$ satisfying $\sum_{n=0}^{\infty} a_n = \infty$, for all $n = 0, 1, 2, \cdots$. Stop the iteration if the sequences $\{x_n\}, \{\mu_n\}$ and $\{\nu_n\}$ satisfy the fixed-point problem $(*)$, otherwise continue.

Proof. Assume that $x \in \mathcal{H}_p$ is a solution of GSMOVIP (10). Then, from Theorem 1, we have

$$\begin{aligned}
x &= (1 - a_n)x + a_n\Big(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}[\mathcal{H}(\mathcal{A}x, \mathcal{B}x) + \frac{\rho}{\tau}(\mathcal{P}(\mu) \oplus b\varphi(x - \mathcal{S}x, \nu)) + a]\Big)\\
&= (1 - b_n)x + b_n\Big(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}[\mathcal{H}(\mathcal{A}x, \mathcal{B}x) + \frac{\rho}{\tau}(\mathcal{P}(\mu) \oplus b\varphi(x - \mathcal{S}x, \nu)) + a]\Big) \qquad (24)\\
&= (1 - c_n)x + c_n\Big(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}[\mathcal{H}(\mathcal{A}x, \mathcal{B}x) + \frac{\rho}{\tau}(\mathcal{P}(\mu) \oplus b\varphi(x - \mathcal{S}x, \nu)) + a]\Big).
\end{aligned}$$

From Algorithm 1, (22), (24), Lemma 4, and Proposition 2, it follows that

$$\begin{aligned}
0 &\le x_{n+1} \oplus x\\
&= \Big[(1 - a_n)x_n + a_n\Big(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}[\mathcal{H}(\mathcal{A}y_n, \mathcal{B}y_n) + \frac{\rho}{\tau}(\mathcal{P}(\mu'_n) \oplus b\varphi(y_n - \mathcal{S}y_n, \nu'_n)) + a]\Big)\Big]\\
&\quad \oplus \Big[(1 - a_n)x + a_n\Big(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}[\mathcal{H}(\mathcal{A}x, \mathcal{B}x) + \frac{\rho}{\tau}(\mathcal{P}(\mu) \oplus b\varphi(x - \mathcal{S}x, \nu)) + a]\Big)\Big]\\
&\le (1 - a_n)(x_n \oplus x) \qquad (25)\\
&\quad + a_n\Big(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}[\mathcal{H}(\mathcal{A}y_n, \mathcal{B}y_n) + \frac{\rho}{\tau}(\mathcal{P}(\mu'_n) \oplus b\varphi(y_n - \mathcal{S}y_n, \nu'_n)) + a]\\
&\quad \oplus R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}[\mathcal{H}(\mathcal{A}x, \mathcal{B}x) + \frac{\rho}{\tau}(\mathcal{P}(\mu) \oplus b\varphi(x - \mathcal{S}x, \nu)) + a]\Big)\\
&\le (1 - a_n)(x_n \oplus x) + \Theta a_n(y_n \oplus x).
\end{aligned}$$

Again, using the similar arguments, we estimate

$$
\begin{aligned}
0 &\leq y_n \oplus x \\
&= \Big[(1-b_n)x_n + b_n\Big(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}[\mathcal{H}(\mathcal{A}z_n,\mathcal{B}z_n) + \frac{\rho}{\tau}(\mathcal{P}(\mu''_n) \oplus b\varphi(z_n - \mathcal{S}z_n, \nu''_n)) + a]\Big)\Big] \\
&\quad \oplus \Big[(1-b_n)x + b_n\Big(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}[\mathcal{H}(\mathcal{A}x,\mathcal{B}x) + \frac{\rho}{\tau}(\mathcal{P}(\mu) \oplus b\varphi(x - \mathcal{S}x, \nu)) + a]\Big)\Big] \\
&\leq (1-b_n)(x_n \oplus x) \\
&\quad + b_n\Big(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}[\mathcal{H}(\mathcal{A}z_n,\mathcal{B}z_n) + \frac{\rho}{\tau}(\mathcal{P}(\mu''_n) \oplus b\varphi(z_n - \mathcal{S}z_n, \nu''_n)) + a] \\
&\quad \oplus R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}[\mathcal{H}(\mathcal{A}x,\mathcal{B}x) + \frac{\rho}{\tau}(\mathcal{P}(\mu) \oplus b\varphi(x - \mathcal{S}x, \nu)) + a]\Big) \\
&\leq (1-b_n)(x_n \oplus x) + \Theta b_n(z_n \oplus x).
\end{aligned} \tag{26}
$$

Again, using the same facts as mentioned above, we have

$$
\begin{aligned}
0 &\leq z_n \oplus x \\
&= \Big[(1-c_n)x_n + c_n\Big(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}[\mathcal{H}(\mathcal{A}x_n,\mathcal{B}x_n) + \frac{\rho}{\tau}(\mathcal{P}(\mu_n) \oplus b\varphi(x_n - \mathcal{S}x_n, \nu_n)) + a]\Big)\Big] \\
&\quad \oplus\Big[(1-c_n)x + c_n\Big(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}[\mathcal{H}(\mathcal{A}x,\mathcal{B}x) + \frac{\rho}{\tau}(\mathcal{P}(\mu) \oplus b\varphi(x - \mathcal{S}x, \nu)) + a]\Big)\Big] \\
&\leq (1-c_n)(x_n \oplus x) \\
&\quad + c_n\Big(R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}[\mathcal{H}(\mathcal{A}x_n,\mathcal{B}x_n) + \frac{\rho}{\tau}(\mathcal{P}(\mu_n) \oplus b\varphi(x_n - \mathcal{S}x_n, \nu_n)) + a] \\
&\quad \oplus R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}[\mathcal{H}(\mathcal{A}x,\mathcal{B}x) + \frac{\rho}{\tau}(\mathcal{P}(\mu) \oplus b\varphi(x - \mathcal{S}x, \nu)) + a]\Big) \\
&\leq (1-c_n)(x_n \oplus x) + \Theta c_n(x_n \oplus x).
\end{aligned} \tag{27}
$$

It follows from (25)–(27) that

$$
\begin{aligned}
0 &\leq x_{n+1} \oplus x \\
&\leq (1-a_n)(x_n \oplus x) + \Theta a_n[(1-b_n)(x_n \oplus x) \\
&\quad + \Theta b_n((1-c_n)(x_n \oplus x) + \Theta c_n(x_n \oplus x))] \\
&\leq (1 - a_n(1-\Theta))(x_n \oplus x).
\end{aligned} \tag{28}
$$

Utilizing the definition of normal cone and applying Proposition 1, we acquire

$$\|x_{n+1} - x\| \leq (1 - a_n(1-\Theta))\|x_n - x\|. \tag{29}$$

Setting $\omega_n = \|x_n - x\|, \vartheta_n = a_n(1-\Theta)$, we get

$$\omega_{n+1} \leq (1-\vartheta_n)\omega_n. \tag{30}$$

Thus, from Lemma 1 and (30), we get $\omega_n \to 0$ as $n \to \infty$. Hence, $x_n \to x$ as $n \to \infty$. It follows from the Algorithm 1 that

$$\|\mu_{n+1} \oplus \mu\| \leq \mathcal{D}(\mathcal{R}(x_{n+1}), \mathcal{R}(x)) \leq \zeta_{\mathcal{R}}\|x_{n+1} - x\|; \tag{31}$$

$$\|\nu_{n+1} \oplus \nu\| \leq \mathcal{D}(\mathcal{T}(x_{n+1}), \mathcal{T}(x)) \leq \zeta_{\mathcal{T}}\|x_{n+1} - x\|. \tag{32}$$

From (31) and (32), one can see that $\{\mu_n\}$ and $\{\nu_n\}$ are Cauchy sequences in $\mathcal{H}_p$. Therefore, there exist $\mu, \nu \in \mathcal{H}_p$ such that $\mu_n \to \mu$ and $\nu_n \to \nu$, strongly for sufficiently large n. Next, we show that $\mu \in \mathcal{R}(x)$ and $\nu \in \mathcal{T}(x)$. Since $\mu_n \in \mathcal{R}(x_n)$, then we have

$$
\begin{aligned}
d(\mu, \mathcal{R}(x)) &\leq \|\mu - \mu_n\| + d(\mu_n, \mathcal{R}(x)) \\
&\leq \|\mu - \mu_n\| + \mathcal{D}(\mathcal{R}(x_n), \mathcal{R}(x)) \\
&\leq \|\mu - \mu_n\| + \zeta_{\mathcal{R}}\|x_n - x\| \to 0, \text{ as } n \to \infty.
\end{aligned} \tag{33}
$$

Hence, $d(\mu, \mathcal{R}(x)) \to 0$, and therefore $\mu \in \mathcal{R}(x)$ as $\mathcal{R}(x) \in CB(\mathcal{H}_p)$. In a similar fashion, we can show that $\nu \in \mathcal{T}(x)$. Therefore, by Lemma 5, one can come to an conclusion that $\{(x_n, \mu_n, \nu_n)\}$ converges strongly to the unique solution (x, μ, ν) of GSMOVIP (10). □

5. Numerical Example

Example 1. *Let $\mathcal{H}_p = (-\infty, \infty)$ with the usual inner product and norm, and let $\mathcal{C} = [0,1]$ be a normal cone with normal constant $\lambda_{\mathcal{C}} = 1$. Let $\mathcal{A}, \mathcal{B}, \mathcal{F}, \mathcal{G}, \mathcal{P}, \mathcal{S} : \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings defined by*

$$\mathcal{A}(x) = \frac{x}{2}, \mathcal{B}(x) = \frac{x}{3}, \mathcal{F}(x) = \frac{2x}{3}, \mathcal{G}(x) = \frac{2x}{3}, \mathcal{P}(x) = \frac{x}{2}, \mathcal{S}(x) = \frac{x}{4}, \text{ for all } x \in \mathcal{H}_p.$$

Now, for each $x, y \in \mathcal{H}_p, x \propto y$. Then, we can easily verify that $\mathcal{P}$ is $\frac{2}{3}$-Lipschitz-continuous, $\mathcal{S}$ is $\frac{1}{3}$-Lipschitz-continuous and relaxed $(1, \frac{1}{4})$-cocoerecive.

Let $\mathcal{H}, \varphi : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings defined by

$$\mathcal{H}(x,y) = \frac{x \oplus y}{2}, \varphi(x,y) = \frac{x \oplus y}{3}, \text{ for all } x, y \in \mathcal{H}_p.$$

Then for any $u, x, y \in \mathcal{H}_p, x \propto y$, we have

$$\begin{aligned} \mathcal{H}(\mathcal{A}(x), u) \oplus \mathcal{H}(\mathcal{A}(y), u) &= \Big(\frac{\mathcal{A}(x) \oplus u}{2}\Big) \oplus \Big(\frac{\mathcal{A}(y) \oplus u}{2}\Big) \\ &\leq \frac{1}{2}[\mathcal{A}(x) \oplus \mathcal{A}(y)] \\ &= \frac{1}{2}\Big(\frac{x}{2} \oplus \frac{y}{2}\Big) \\ &\leq \frac{1}{3}(x \oplus y). \end{aligned}$$

That is, $\mathcal{H}(\mathcal{A}(x), u) \oplus \mathcal{H}(\mathcal{A}(y), u) \leq \frac{1}{3}(x \oplus y)$. *Hence, $\mathcal{H}(\mathcal{A}, \mathcal{B})$ is $\frac{1}{3}$-ordered compression mapping with respect to $\mathcal{A}$. Similarly, one can show that $\mathcal{H}(\mathcal{A}, \mathcal{B})$ is $\frac{1}{5}$-ordered compression mapping with respect to $\mathcal{B}$. Now for any $u, x, y \in \mathcal{H}_p, x \propto y$, we have*

$$\begin{aligned} \varphi(x - \mathcal{S}(x), \nu) \oplus \varphi(y - \mathcal{S}(y), \nu) &= \Big[\frac{(x - \mathcal{S}(x)) \oplus \nu}{3}\Big] \oplus \Big[\frac{(y - \mathcal{S}(y)) \oplus \nu}{3}\Big] \\ &\leq \frac{1}{3}[(x - \mathcal{S}(x)) \oplus (y - \mathcal{S}(y))] \\ &= \frac{1}{3}[(x - \frac{x}{4}) \oplus (y - \frac{y}{4})] \\ &= \frac{1}{4}\Big(x \oplus y\Big) \\ &\leq \frac{1}{3}(x \oplus y). \end{aligned}$$

Hence, φ is $\frac{1}{3}$-Lipschitz-continuous with respect to the first argument. Similarly, we can show that φ is $\frac{1}{2}$-Lipschitz-continuous with respect to the second argument.

Let $\mathcal{R}, \mathcal{T} : \mathcal{H}_p \to CB(\mathcal{H}_p)$ and $M : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be the set-valued mappings defined by

$$\mathcal{R}(x) = \Big\{\Big(\frac{x}{2}, \frac{1}{3}\Big) : x \in \mathcal{H}_p, 0 \leq x \leq 1\Big\}, \text{ for all } x \in \mathcal{H}_p,$$

$$\mathcal{T}(x) = \Big\{\Big(\frac{x}{3}, \frac{1}{5}\Big) : x \in \mathcal{H}_p, 0 \leq x \leq 1\Big\}, \text{ for all } x \in \mathcal{H}_p,$$

$$\mathcal{M}(x,y) = \{3(x+y)\}, \text{ for all } x, y \in \mathcal{H}_p.$$

Then, it is easy to show that $\mathcal{R}$ and $\mathcal{T}$ are $\mathcal{D}$-Lipschitz-continuous mappings with constants $\frac{1}{2}$ and $\frac{1}{3}$, respectively. Additionally,

$$\begin{aligned}\mathcal{M}(\mathcal{F}(x),\mathcal{G}(x)) &= 3(\mathcal{F}(x)+\mathcal{G}(x))\\ &= 3\Big(\frac{2x}{3}+\frac{2x}{3}\Big) = 4x.\end{aligned}$$

Let $p_x = 4x \in \mathcal{M}(\mathcal{F}(x),\mathcal{G}(x))$ and $p_y = 4y \in \mathcal{M}(\mathcal{F}(y),\mathcal{G}(y))$, then $\mathcal{M}$ is a comparison mapping, and we estimate

$$\begin{aligned}(p_x \oplus p_y) \oplus \alpha_{\mathcal{M}}(\mathcal{H}(\mathcal{A}(x),\mathcal{B}(x)) \oplus \mathcal{H}(\mathcal{A}(y),\mathcal{B}(y))) &= 4(x\oplus y) \oplus \alpha_{\mathcal{M}}\Big(\frac{5}{12}(x\oplus y)\Big)\\ &= 4[(x\oplus y)\oplus(x\oplus y)] = 0.\end{aligned}$$

Hence, $\mathcal{M}$ is a XOR-$\frac{48}{5}$-non-ordinary difference mapping with respect to $\mathcal{A}$ and $\mathcal{B}$. Additionally, for $\rho = 1$,

$$[\mathcal{H}(\mathcal{A},\mathcal{B}) + \rho\mathcal{M}(\mathcal{F},\mathcal{G})](\mathcal{H}_p) = \mathcal{H}_p.$$

Hence, $\mathcal{M}$ is $\mathcal{H}(\cdot,\cdot)$-compression XOR-$\frac{48}{5}$-non-ordinary difference mapping. The resolvent operator $R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})} : \mathcal{H}_p \to \mathcal{H}_p$ associated with $\mathcal{A},\mathcal{B},\mathcal{F}$ and $\mathcal{G}$ is given by

$$R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(x) = \frac{6x}{29}, \text{ for all } x \in \mathcal{H}_p.$$

It is easy to verify that the resolvent operator is single-valued and comparison mapping. Additionally,

$$\begin{aligned}\|R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(x) \oplus R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}(y)\| &= \|\frac{6x}{29} \oplus \frac{6y}{29}\|\\ &= \frac{6}{29}\|x\oplus y\|\\ &\le \frac{1}{5}\|x\oplus y\|, \text{ for all } x,y \in \mathcal{H}_p.\end{aligned}$$

That is, the resolvent operator is $\frac{1}{5}$-Lipschitz-continuous. For $\rho,\tau,b = 1$ and $b = 0$, we have

$$\begin{aligned}&R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}[\mathcal{H}(\mathcal{A}x,\mathcal{B}x) + \frac{\rho}{\tau}(\mathcal{P}(\mu) \oplus b\varphi(x-\mathcal{S}x,\nu)) + a]\\ &= \frac{6}{29}[\mathcal{H}(\mathcal{A}x,\mathcal{B}x) + (\mathcal{P}(\mu) \oplus \varphi(x-\mathcal{S}x,\nu))]\\ &= \frac{6}{29}\Big[\Big(\frac{\mathcal{A}(x)\oplus\mathcal{B}(x)}{2}\Big) + \Big(\frac{x}{4}\Big) \oplus \frac{1}{3}\Big(x - \frac{x}{4} \oplus \frac{x}{3}\Big)\Big]\\ &= \frac{6}{29}\Big[\frac{5x}{12} \oplus \frac{4x}{3}\Big]\\ &= \frac{21x}{58}.\end{aligned}$$

Clearly, 0 is the fixed-point of $R^{\mathcal{H}(\mathcal{A},\mathcal{B})}_{\rho,\mathcal{M}(\mathcal{F},\mathcal{G})}[\mathcal{H}(\mathcal{A}x,\mathcal{B}x) + \frac{\rho}{\tau}(\mathcal{P}(\mu) \oplus b\varphi(x-\mathcal{S}x,\nu)) + a]$.

Let $a_n = 1/(n+1), b_n = 3n/(3n+2)$ and $c_n = n + 2/(n+3)$. It is easily seen that the sequences $\{a_n\}, \{b_n\}$ and $\{c_n\}$ satisfy the condition $0 \le a_n, b_n, c_n \le 1, \sum_{n=0}^{\infty} a_n = \infty$.

Now, we can approximate the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ by the Algorithm 1 as follows:

$$x_{n+1} = \frac{n}{n+1}x_n + \frac{21}{58(n+1)}y_n,$$
$$y_n = \frac{2}{3n+2}x_n + \frac{63n}{58(3n+2)}z_n,$$
$$z_n = \frac{1}{n+1}x_n + \frac{21(n+2)}{58(n+3)}x_n.$$

It is also clear that the condition (23) is satisfied. Hence, all the suppositions of Theorem 2 are verified. Therefore, the sequence $\{(x_n, \mu_n, \nu_n)\}$ converges strongly to the unique solution 0, which is the solution of GSMOVIP (10).

6. Conclusions

We introduced a new class of mapping, namely, $H(\cdot,\cdot)$-compression XOR-α_M-non-ordinary difference mapping. A resolvent operator associated to this mapping was defined, and we discussed some of its characteristics. We examined a generalized, set-valued, mixed-ordered, variational inclusion problem involving $H(\cdot,\cdot)$-compression XOR-α_M-non-ordinary difference mapping and relaxed cocoercive mapping in the setting of real ordered Hilbert spaces. An existence result was discussed for our considered ordered inclusion problem. Further, a three-step iterative algorithm using a $\oplus$ operator was suggested, and a convergence analysis of the proposed iterative algorithm was presented. Finally, a numerical example was given to illustrate the existence and convergence results. Further, the results presented in this paper can be generalized in ordered Banach spaces and ordered uniformly smooth Banach spaces.

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