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Connected Fundamental Groups and Homotopy Contacts in Fibered Topological (*C*, *R*) Space

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Abstract: The algebraic as well as geometric topological constructions of manifold embeddings and homotopy offer interesting insights about spaces and symmetry. This paper proposes the construction of 2-quasinormed variants of locally dense p-normed 2-spheres within a non-uniformly scalable quasinormed topological (*C*, *R*) space. The fibered space is dense and the 2-spheres are equivalent to the category of 3-dimensional manifolds or three-manifolds with simply connected boundary surfaces. However, the disjoint and proper embeddings of covering three-manifolds within the convex subspaces generates separations of p-normed 2-spheres. The 2-quasinormed variants of p-normed 2-spheres are compact and path-connected varieties within the dense space. The path-connection is further extended by introducing the concept of bi-connectedness, preserving Urysohn separation of closed subspaces. The local fundamental groups are constructed from the discrete variety of path-homotopies, which are interior to the respective 2-spheres. The simple connected boundaries of p-normed 2-spheres generate finite and countable sets of homotopy contacts of the fundamental groups. Interestingly, a compact fibre can prepare a homotopy loop in the fundamental group within the fibered topological (*C*, *R*) space. It is shown that the holomorphic condition is a requirement in the topological (*C*, *R*) space to preserve a convex path-component. However, the topological projections of p-normed 2-spheres on the disjoint holomorphic complex subspaces retain the path-connection property irrespective of the projective points on real subspace. The local fundamental groups of discrete-loop variety support the formation of a homotopically Hausdorff (*C*, *R*) space.

Keywords: topological spaces; homotopy; fundamental group; projection; norm

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1. Introduction

In general, a path-connected topological space is considered to be *locally* path-connected within a path-component maintaining the equivalence relation. A topological space X is termed as homotopically Hausdorff if there is an open neighbourhood at a base point $x_0 \in X$ such that any element of a non-trivial homotopy class of the fundamental group $\pi(X, x_0)$ does not belong to the corresponding open neighbourhood [1]. A first countable path-connected topological space admits countable fundamental groups if the space is a homotopically Hausdorff variety [1]. Interestingly, a homotopically Hausdorff topological space containing countable fundamental groups has universal cover. However, the nature of a fundamental group is different in the lower dimensional topological spaces as compared to the higher dimensional spaces. For example, in a one-dimensional topological space X the fundamental group $\pi(X)$ becomes a free group if the space is a simply connected type [1]. In this case the topological space successfully admits a suitable metric structure. A regular and separable topological space can be uniquely generated from a given regular as well as separable topological space [2]. For example, suppose X is a regular and separable topological space. If we consider that $A \subset X$ and U is a neighbourhood of A then a unique topological space can be generated from X if A is closed and

$U \backslash A$ is a countable or finite sum of disjoint open sets. Note that the uniquely generated topological space is also a regular and separable topological space. This paper proposes the topological construction and analysis of 2-quasinormed variants of p-normed 2-spheres, path-connected fundamental groups and associated homotopy contacts in a fibered as well as quasinormed topological (C, R) space [3]. In this paper the 2-quasinormed variants of p-normed 2-spheres in X are generically denoted as $CRS(X)$. The space is non-uniformly scalable and the fundamental groups are interior to dense subspaces of 2-quasinormed variant of p-normed 2-spheres generating a set of homotopy contacts. First, the brief descriptions about various contact structures, fundamental group varieties and associated homotopies are presented to establish introductory concepts (Sections 1.1 and 1.2). Next, the motivation for this work is illustrated in Section 1.3. In this paper, the symbols R, C, N and Z represent sets of extended real numbers, complex numbers, natural numbers and integers, respectively. Moreover, for clarity, in this paper a 3D manifold is called a three-manifold category in the proposed constructions and topological analysis. Furthermore, the surfaces of three-manifolds and 2-spheres are often alternatively named as respective boundaries for the simplicity of presentation.

1.1. Contact Structures and Fundamental Groups

The constructions of geometric contact structures and the analysis of their topological properties on manifolds are required to understand the characteristics of associated group algebraic varieties. The contact structure on a manifold M is a hyperplane field in the corresponding tangent subbundle. A $2n + 1$ dimensional contact manifold structure is essentially a Hausdorff topological space, which is in the C^{∞} class [4]. In general, the topological analysis considers that a contact manifold is in the compact category and the contact form ω is regular. As a consequence, the integral curves on such contact manifolds are homeomorphic to S^1. It is shown that if a contact structure A is constructed on a three-manifold T^3 then the fundamental group $\pi_1(T^3, A)$ includes an infinite cyclic group [5]. However, a similar variety of results can also be extended on T^2-bundles generated over S^1.

The topological contact structures on three-manifolds can be further generalized towards higher dimensions. However, in case of n-manifolds ($n > 3$) the theory of contact homology plays an important role. Note that if we consider θ as a contact structure and M_n as a n-manifold then the contact homology $HC*(M_n, \theta)$ is invariant of the corresponding contact structure [6]. In this case the contact homology is defined as a chain complex. Interestingly, the higher dimensional manifolds and contact homology can be useful to prove some topological results in the lower dimensional contact structures. For example, the formulations of fundamental group $\pi_1(M_n, \theta)$ for the n-manifold ($n > 3$) and the associated higher order homotopy groups $\pi_k(M_n, \theta)$ are successfully realized by employing the higher dimensional contact homology [7]. The analytic and geometric properties of the higher dimensional contact structures in n-manifold show some very interesting observations. A 2-torus can be generated by attaching a projection of J-holomorphic cylinder to a n-manifold M_n along with homotopy pairs, which results in the preparation of a $H_2(M_n, Z)$ homology class [7].

1.2. Homotopy and Twisting

The contact structures can be twisted and can also be classified. According to the Eliashberg definition, a contact structure θ on a three-manifold M_3 is called overtwisted if it can successfully allow embedding of an overtwisted disc [8]. There is a relationship between the homotopy theory of algebraic topology and the corresponding twisted contact structures. It is shown by Eliashberg that all oriented 2-plane fields on a M_3 structure are essentially homotopic to a contact structure in the overtwisted category. The Haefliger classifications of foliations in the contact manifolds are in a generalized form considering the open manifold variety [9]. The Haefliger categories are further extended by constructing homotopy classifications of foliations on the open contact manifolds [10]. However, in this case the leaves are the open contact submanifolds in the topological space. The contact structures, twisting and manifolds are often viewed in geometric perspectives. The construction and analysis of holomorphic curves on the symplectic manifolds are proposed by Gromov [11]. Note that the contact geometry is an odd dimensional variety of symplectic geometry.

1.3. Motivation and Contributions

The anomalous behaviour in homotopy theory is observed when the uniform limit of a map from a nullhomotopic loop is the essential homotopy loop, which is not nullhomotopic in nature [12]. Moreover, the Baire categorizations of a topological subspace influence the properties of structural embedding within the space. Suppose we consider a path-connected subset A of S^2, where the topological space A^c (complement of A) is a dense subspace. It is shown that the fundamental group of A successfully embeds the fundamental group of Sierpinski curve [1]. In this case, the nullhomotopic loop in the topological space X given by $f : S^1 \to X$ factors through a surjective map on the *planar* topological subspace. Interestingly, in view of algebraic topology one can construct a fundamental group $\pi_1(M, x)$ from a set of equivalence classes of paths on a manifold M [13]. As a result, the covering map given by $p : X \to Y$ between the two topological spaces induces another map given by $p^* : \pi_1(X, x) \to \pi_1(Y, p(x))$, which is injective. Interestingly, the *fibre* over a topological space (Y, τ_Y) is homeomorphic to the discrete $\pi_1^{top}(Y)$ fundamental group [13].

This paper proposes the topological construction and analysis of multiple path-connected fundamental groups of discrete variety within the non-uniformly scaled as well as quasinormed topological (C, R) space. The topological (C, R) space supports fibrations in two varieties, such as compact fibres and non-compact fibres. It is considered that the fundamental groups generating homotopy contacts are interior to the 2-quasinormed variants of p-normed 2-spheres within the topological (C, R) space. This paper addresses two broad questions in the relevant topological contexts such as, (1) what the topological properties of the resulting structures are if the space is dense and, (2) how the homotopy contacts, covering manifold embeddings and path-connections interplay within the topological (C, R) space. Moreover, the question is: how the concept of homotopically Hausdorff fundamental groups influences the proposed structures. The presented construction and analysis employ the combined standpoints of general topology as well as algebraic topology as required. The elements of geometric topology are often used whenever necessary.

The main contributions made in this paper can be summarized as follows. The construction of multiple locally dense p-normed 2-spheres within the dense and fibered non-uniformly scalable topological (C, R) space is proposed in this paper. The three-manifold embeddings and the corresponding formation of covering separation of $CRS(X)$ are analysed. The generation of path-connected components in a holomorphic convex subspace is formulated and the concept of bi-connectedness is introduced. This paper illustrates that the local and discrete variety of fundamental groups interior to the $CRS(X)$ generate the finite and countable sets of homotopy contacts with the simply connected boundaries of $CRS(X)$. Interestingly, a compact fibre in the topological (C, R) space may prepare a homotopy loop. It is shown that the holomorphic condition is required to be maintained in the convex subspace topological (C, R) space to support the respective convex path-component. However, it is observed that the path-connected homotopy loops are not always guaranteed to be bi-connected as an implication.

The rest of the paper is organized as follows. The preliminary concepts are presented Section 2 in brief. The definitions and descriptions of $CRS(X)$, homotopy contacts and fundamental groups are presented in Section 3. The analyses of topological properties are presented in Section 4 in details. Finally, Section 5 concludes the paper.

2. Preliminary Concepts

In this section, the introduction to topological (C, R) space, manifolds and homotopy theory are presented in brief. The topological (C, R) space is a quasinormed topological space constructed on the Cartesian product $C \times R$ resulting in the formation of a three-dimensional topological space in continua. The topological (C, R) space is a non-uniformly scalable space where the set of open cylinders forms the basis. The space successfully admits cylindrically symmetric continuous functions as well as the topological group structure. The identity element of the topological group in the (C, R) space is located on the corresponding real planar subspace. The space can be fibered and the respective fibre space generates an associative magma. The topological (C, R) space can be equipped with various forms of linear operations T_C, T_R, T within the space and the composite algebraic operations involving translations exhibit a set of interesting algebraic as well as topological properties. The topological (C, R) space is suitable for the construction of manifold embeddings. A Hausdorff topological space M_n is an n-dimensional smooth *manifold* if the space can be covered by a set of charts given by $\Psi = \{(U_\alpha, f_\alpha)\}_{\alpha \in \Lambda}$ where Λ is an index set, U_α is an open set and $f_\alpha : U_\alpha \to R^n$ is a homeomorphism. In general, the topological space on M_n represented by (M_n, τ_M) is considered to be equipped with a countable base. It is interesting to note that every paracompact Hausdroff manifold is metrizable as well as second countable and it preserves local topological properties, such as local compactness and local metrizability [14]. Moreover, every paracompact manifold of connected variety is Lindelof and separable. The smoothness of (M_n, τ_M) is maintained by the condition that a function on it is in the $C^r - class$ where $r \in N \cup \{+\infty\}$. Note that a *diffeomorphism* between two smooth manifolds M_n and N_n is a bijection with a smooth inverse. According to Whitney embedding theorem, a smooth as well as compact (M_n, τ_M) can be embedded into m-dimensional Euclidean space if the dimension is sufficiently large as compared to n (i.e., $m > n$ for R^m). Moreover, if $f : M_n \to V_m$ is a map between two differentiable manifolds then it forms another *regular* map F if $m \geq 2n$ [15]. A *complex manifold* is defined in n-dimensional complex space C^n with a restriction that the coordinate chart maps are required to be holomorphic in nature. A *Riemann sphere* with one-point compactification given by $C \cup \{\infty\}$ is essentially a complex manifold such that it is homeomorphic to S^2. Let (X, τ_X) and (Y, τ_Y) be two topological spaces and the functions $f, g : (X, \tau_X) \to (Y, \tau_Y)$ be continuous. The functions f, g are *homotopic* if there exists a continuous function given by $F : X \times [0,1] \to Y$ such that it maintains two conditions: (I) $F(x, 0) = f(x)$ and (II) $F(x, 1) = g(x)$. Suppose $\{p_i | p_i : [0,1] \to X\}$ is a set of continuous functions with two base points $p_i(0) \in X, p_i(1) \in X$ in the space for some $i \in Z^+$. If we consider two continuous functions, $p_1(.)$ and $p_2(.)$ then the continuous function $F_p : [0,1]^2 \to X$ is a *path-homotopy* if it satisfies four conditions given as:

(I) $F_p(s \in [0,1], 0) = p_1(s)$, (II) $F_p(s \in [0,1], 1) = p_2(s)$, (III) $F_p(0, t \in [0,1]) = p_1(0) = p_2(0)$ and (IV) $F_p(1, t \in [0,1]) = p_1(1) = p_2(1)$.

A *fundamental group* $\pi_1(X, b)$ is generated in a topological space (X, τ_X) at the base point $b \in X$ if $\{p_i | p_i : [0,1] \to X\}$ represents a path-homotopy and additionally it supports the condition that: $\forall p_i, p_i(0) = p_i(1) = b$. It indicates that $\pi_1(X, b)$ is formed by a set of homotopic loops based at the base point $b \in X$. A homotopy loop $p_i(.)$ in $\pi_1(X, b)$ is called *simple* if it is an injective type and it is *simple-closed* if it is closed as well as injective except at the points $\{0, 1\}$. If p_i, p_k are two homotopy loops in (X, τ_X) then a *free homotopy* between them is a continuous map $F_\sigma : [0,1] \times S^1 \to X$ such that the restriction to the boundary components are the given loops. A topological space (X, τ_X) is $\pi_1 - shape\ injective$ if the absolute retract Φ containing topologically closed subspace X maintains the property that if p_i is an essential (i.e., not nullhomotopic) closed curve in X then there always exists an open neighbourhood V_i of X in Φ such that p_i is also essential in V_i.

3. Fundamental Groups and Homotopy Contacts

In this section, the construction of 2-quasinormed variants of p-normed 2-spheres and the associated definitions of connected fundamental groups as well as homotopy contacts are presented. The constructions consider that the underlying space is a quasinormed as well as non-uniformly scalable topological (C, R) space. In this paper a 2-quasinormed variant of p-normed 2-sphere centred at point x_c in the topological (C, R) space is algebraically represented as S_c^2 and it is generically termed as $CRS(X)$ without specifying any prefixed centre as indicated earlier. Note that an arbitrary point x_p in the quasinormed topological (C, R) space (X, τ_X) is represented as $x_p = (z_p, r_p)$. The origin of a topological space (X, τ_X) is denoted as $x_0 = (z_0, 0)$, where z_0 is the Gauss origin. In this paper A^o and $\overline{A}$ represent interior and closure of an arbitrary set A such that $\overline{A} = A^o \cup \partial A$. Moreover, if A is homeomorphic to B then it is denoted as $\hom(A, B)$ and $A \cong B$ if they are equivalent (i.e., identified by following the equivalence relation or quotient). Furthermore, the homotopic path equivalence between A and B is denoted as $A \cong_H B$, whereas the homotopic path joining them is algebraically denoted by $A * B$ maintaining the respective sequence. In the remainder of this paper, the *category* of 3D manifold is termed as a three-manifold whereas the surfaces of a three-manifold category M_3 and a $CRS(X)$ given by S_c^2 are denoted as ∂M_3 and ∂S_c^2 respectively (and alternatively called as boundaries). If the interior of a three-manifold category M_3 in the topological (C, R) space (X, τ_X) is denoted as $A \equiv (M_3)^o$ then $A \subset Y$ is locally dense in convex $Y \subset X$ (by following Baire category) as well as open such that $A \cup \partial M_3 = \overline{A}$ in (X, τ_X).

Let (X, τ_X) be a quasinormed topological (C, R) space and the corresponding 2-quasinorm of a point $x_i \in X$ within the space be denoted as $||x_i||_{CR|2}$. This results in the formation of a 2-quasinormed space represented by $\left(X, \left|\left|x_i\right|\right|_{CR|2}\right)$. However, it is known that for every quasinormed space there exists a $0 < p \leq 1$ such that $\left(X, ||x_i||_p\right)$ becomes a respective p-normed space generating a topology, where the corresponding quasinorm function $\left|\left|x_i\right|\right|_p^{1/p}$ also admits a topology in X [16]. First we define a p-normed 2-sphere within the topological (C, R) space (X, τ_X) such that $\left(X, \left|\left|x_i\right|\right|_{CR|2}\right)$ remains a 2-quasinormed topological space.

3.1. Topological $CRS(X)$

A unit p-normed 2-sphere $CRS(X)$ of 2-quasinorm variant centred at $x_c = (z_c, r_c) \in X$ is defined as:

$$S_c^2 = \left\{x_i \in X : \left|\left|x_i - x_c\right|\right|_p \leq 1; \left|\left|x_i\right|\right|_{CR|2} \equiv \left|\left|x_i\right|\right|_p\right\}. \tag{1}$$

Note that, in general a $CRS(X)$ is a closed and locally dense subspace in the 3-dimensional topological (C, R) space (X, τ_X). In an alternative view, a unit $CRS(X)$ S_c^2 is equivalent to a compact three-manifold M_3 homeomorphically embedded in the topological (C, R) space such that $S_c^2 \cong M_3$ in view of category. It indicates that the closed subspace S_c^2 is locally dense in a convex subspace within the topological (C, R) space. We consider that the surface ∂M_3 of the topological three-manifold (M_3, τ_M) is a simply connected variety enabling the existence of a finite number of homotopy contacts on ∂S_c^2.

3.2. Topologically Bi-Connected Subspaces

Let $A \subset X$ and $B \subset X$ be two locally dense (i.e., locally dense in respective convex subspaces) as well as disjoint such that $\overline{A} \cap \overline{B} = \phi$ and $\overline{A} \cup \overline{B} \subset X$. If we consider two continuous functions $f : [0, 1] \to X$ and $g : [0, 1] \to X$ then $\overline{A}, \overline{B}$ are called bi-connected topological subspaces if the following properties are maintained.

$$\begin{aligned} &f(0) \in A^o, f(1) \in B^o, \\ &g(0) \in \partial A, g(1) \in \partial B, \\ &g([0, 1]) \cap (\overline{A} \cup \overline{B}) = \{g(i) : i \in \{0, 1\}\}. \end{aligned} \tag{2}$$

Remark 1. *If $\overline{A}$ and $\overline{B}$ are bi-connected then they are also path-connected subspaces in a dense topological space. Moreover, it is possible to formulate an Urysohn separation of $\overline{A}$ and $\overline{B}$ under continuous $v : Y \to [0, 1]$ such that $(\overline{A} \cup \overline{B}) \subset Y \subset X$ and $\forall x_a \in \overline{A}, \forall x_b \in \overline{B}$ the function maintains $v(X_a)$ = 0 and $v(x_b) = 1$. Note that the boundaries ∂S_c^2 and ∂S_d^2 of two respective $CRS(X)$ are homotopically simply connected Hausdorff and can preserve Urysohn separation of every points on them.*

In general, a path-homotopy $H : [0,1]^2 \to X$ can be constructed in (X, τ_X) by considering continuous functions $f : [0,1] \to X$ and $g : [0,1] \to X$ signifying continuous deformation of $f(.)$ into $g(.)$ in the corresponding path-homotopy. However, in this paper we define a discrete variety of path-homotopy $H_d : [0,1]^2 \to X$ such that it follows three restrictions as mentioned below.

$$\begin{aligned} &H_d([0,1]^2) \subset H([0,1]^2), \\ &H_d((0,0)) = H((0,0)), \\ &H_d((1,1)) = H((1,1)). \end{aligned} \tag{3}$$

The main reason for such construction is to generate a set of homotopy contacts as defined in Section 3.5. First we define the discrete variety of path-homotopy loops and associated homotopy class within the topological (C, R) space.

3.3. Discrete-Loop Homotopy Class

Let $S_c^2 \subset X$ be a dense $CRS(X)$ centred at $x_c \in X$. If a continuous function is given by $f_a : [0,1] \to X$ then a finite sequence of such functions $a \in Z^+, \langle f_a \rangle_{a=1}^n$ generates a discrete variety of path-homotopy loops through $H_d : [0,1]^2 \to X$ in (X, τ_X) if the following conditions are maintained.

$$\begin{aligned} &m, n \in Z^+, 1 < m < n, \\ &\forall a \in [1,n], f_a : [0,1] \to X, \\ &\forall a \in [1,n], f_a(0) = f_a(1) = x_c, \\ &H_d(t \in [0,1], 0) = f_1(t), \\ &H_d(t \in [0,1], 1) = f_n(t), \\ &\forall y \in (0,1), \exists m : H_d(t \in [0,1], y) = f_m(t). \end{aligned} \tag{4}$$

Note that effectively the path-homotopy loops as defined above give rise to the formation of a discrete variety of fundamental group $\pi_1(X, x_c)$ within the topological space at the base point, which is the centre of corresponding $CRS(X)$. In other words, a set of discrete homotopy loops can be constructed from the path-homotopy loops at a base point centred within $CRS(X)$.

Remark 2. *Interestingly, there is a relationship between a compact fibre and a homotopy loop in the fibered topological (C, R) space (X, τ_X). If we consider a compact fibre $\mu_{c \times I}$ at $x_c \in X$ such that $I = \overline{I} = [r_a, r_b]$ then a continuous function $w : \mu_{c \times I} \to X$ would transform a compact fibre into a homotopy loop at the base point $x_c \in X$ if and only if the function preserves following conditions.*

$$\begin{aligned} &w((z_c, r_a)) \cong w((z_c, r_b)), \\ &w((\pi_r \circ \sigma)(x_c)) = (z_c, r = r_c), \\ &\text{hom}(w(\mu_{c \times I}), S^1). \end{aligned} \tag{5}$$

It is relatively straightforward to observe that in this case the fibration maintains $I = I^o \cup \partial I$ and the function $w : \mu_{c \times I} \to X$ also preserves $\forall a \in [1,n], \text{hom}(w(\mu_{c \times I}), f_a([0,1]))$ property under the above-mentioned conditions. Note that the function sequence $\langle f_a \rangle_{a=1}^n$ prepares the discrete loops of a homotopy class at the base point $x_c \in X$, which is denoted as $[h_a]_c$. Moreover, the homotopic loops in a homotopy class $[h_a]_c$ are finitely countable. The corresponding locality of admitted fundamental group in (X, τ_X) is defined below.

3.4. Local Fundamental Group

A fundamental group $\pi_1(X, x_c)$ generated by $\langle f_a \rangle_{a=1}^n$ through the path-homotopy loops $H_d : [0,1]^2 \to X$ is called local if and only if $\forall a \in [1,n], \{f_a(t \in [0,1])\} \subset S_c^2$ and $f_a \in [h_a]_c$.

Note that the discrete variety of a local fundamental group preserves the concept of homotopically Hausdroff property. This is because $\forall x_c \in S_c^2 \subset X, \exists N_c \subset X$ such that

$x_c \in N_c$ (i.e., N_c is an open neighbourhood of x_c) and $\forall a \in [1,n], \{f_a(t \in [0,1])\} \cap N_c \subset \{x_p : x_p \in f_a([0,1])\}$.

Once a local fundamental group is prepared within the dense subspace of a topological space (X, τ_X), the set of homotopy contacts generated by the local fundamental group can be formulated. Recall that a topological space X is defined as simply connected if every continuous function $f : S^1 \to X$ is homotopic to a constant function. It is important to note that the homotopically simple connectedness of $\partial S_c^2 \cong \partial M_3 \subset X$ facilitates the existence of finite as well as countable homotopy contacts.

3.5. Homotopy Contacts

Let $\pi_1(X, x_c)$ be a local fundamental group in the corresponding subspace $S_c^2 \cong M_3$ in (X, τ_X). If we consider a homotopy loop $f_b \in [h_a]_c$ in $\pi_1(X, x_c)$ then $x_b \in X$ is a homotopy contact of $f_b(.)$ if the following condition is satisfied.

$$\begin{aligned} &\forall x_k \in f_b \in [h_a]_c, x_k \neq x_b, \\ &\{x_k\} \subset (S_c^2)^o, \\ &x_b \in f_b([0,1]) \cap \partial S_c^2. \end{aligned} \tag{6}$$

Remark 3. *A set of contacts of a homotopy class $[h_a]_c$ of $\pi_1(X, x_c)$ in the topological (C, R) space (X, τ_X) is given by $\Delta(\pi_1(X, x_c)) = \underset{1 \le b \le n}{\cup} \{x_b\}$.*

4. Main Results

This section presents the analysis and a set of topological properties related to the constructed homotopy contacts and the associated fundamental groups of connected variety. The holomorphic condition on the topological space is not imposed as a precondition to maintain generality and it is later established that holomorphic condition should be maintained within a convex path-connected component. It is shown that the bi-connected functions between subspaces and their extensions preserve holomorphic condition. Moreover, the homotopy contacts maintain simple connectedness of the boundary of a $CRS(X)$, which are essentially dense three-manifolds. First we show that a continuous bi-connection between two $CRS(X)$ is two-points compact in the respective sets of homotopy contacts.

Theorem 1. *If $S_c^2 \subset X$ and $S_d^2 \subset X$ are two bi-connected $CRS(X)$ then $\exists x_a \in \Delta(\pi_1(X, x_c))$ and $\exists x_b \in \Delta(\pi_1(X, x_d))$ such that $g(0) = x_a, g(1) = x_b$ preserves two-points compactness.*

Proof. Let $S_c^2 \subset X$ and $S_d^2 \subset X$ be two bi-connected $CRS(X)$ in a topological (C, R) space (X, τ_X) with the corresponding local fundamental groups $\pi_1(X, x_c)$ and $\pi_1(X, x_d)$, respectively. Let the function $g : [0,1] \to (A \subset X)$ be continuous such that $S_c^2 \subset X$ and $S_d^2 \subset X$ are bi-connected by $g(.)$ along with $f : [0,1] \to (A \subset X)$. This indicates that $g([0,1]) \cap \Delta(\pi_1(X, x_c)) \neq g([0,1]) \cap \Delta(\pi_1(X, x_d)) \neq \phi$ within the topological space if and only if $(S_c^2 \cup S_d^2) \subset A$. According to the definition of topologically bi-connected subspaces, $\exists x_a \in \partial S_c^2$ and $\exists x_b \in \partial S_d^2$ such that $g([0,1]) \cap \Delta(\pi_1(X, x_c)) = \{x_a\}$ and $g([0,1]) \cap \Delta(\pi_1(X, x_d)) = \{x_b\}$. Note that the two $CRS(X)$ are disjoint in (X, τ_X) indicating $x_c \neq x_d$. Moreover, as $g : [0,1] \to (A \subset X)$ is continuous so the function maintains the condition that $g([0,1]) \backslash \{x_a, x_b\} \subset A$, where $g(.)$ is holomorphic (and bounded) in A. Hence, we can conclude that if $g(0) = x_a, g(1) = x_b$, where $g(t \in (0,1)) = g([0,1]) \backslash \{x_a, x_b\}$ then it is a two-points compactification of $g(.)$ on $\partial S_c^2 \cup \partial S_d^2$. □

Note that the continuous function $g : [0,1] \to X$ between any two $CRS(X)$ in the topological (C, R) space is essentially a two-point compactification of a path-connection involving the sets of respective homotopy contacts. Interestingly, the two-point compactification can be performed by employing axiom of choice if the fundamental group is not a trivial variety. In any case, a two-point compact bi-connection between two $CRS(X)$ and its extension are holomorphic in (X, τ_X). The following theorem presents this observation.

Theorem 2. *If a function* $m : [0,1] \to (A \subset X)$ *is an extended bi-connection of* $S^2_c \subset A$ *and* $S^2_d \subset A$ *in* (X, τ_X) *such that the restriction preserves* $m\big|_g = g$ *then* $m(.)$ *is holomorphic in convex* A*.*

Proof. Let $S^2_c \subset A$ and $S^2_d \subset A$ be two $CRS(X)$ in (X, τ_X) and $g : [0,1] \to (A \subset X)$ be a bi-connection. Suppose $m : [0,1] \to A$ is a function extending $g(.)$ such that $m\big|_g = g$. Let us consider two intervals $E_1 \subset [0,1]$ and $E_2 \subset [0,1]$ such that $E_1 \cap E_2 = \phi$ and the extended function maintains the following two conditions: $\Delta(\pi_1(X, x_c)) \subset m(E_1)$ and $\Delta(\pi_1(X, x_d)) \subset m(E_2)$ in (X, τ_X). If $A \subset X$ is a convex topological subspace then $A \subset X$ is a path-connected subspace. Thus the function $m : [0,1] \to A$ is a topological path-connection in $A \subset X$. This indicates further that $\forall t \in [0,1], m(t) = x_t \in A$ where $\pi_C(x_t) \in V \subset C$ (V is compactible) and $\pi_R(x_t) \in R\backslash\{-\infty, +\infty\}$ in (X, τ_X). Hence, the extended bi-connection $m : [0,1] \to (A \subset X)$ is holomorphic in $A \subset X$. □

Corollary 1. *The above theorem indicates that the* $CRS(X)$ *bi-connections are holomorphic in topological* (C, R) *space and as a result the restriction* $m\big|_g = g$ *is also holomorphic in convex* $A \subset X$*.*

The location of existence of centre of a $CRS(X)$ within the topological space often facilitates the generation of connected components and the determination of separation of multiple $CRS(X)$ within the topological space. It is illustrated in the following theorem that the placement of centres of multiple $CRS(X)$ in one-dimensional projective subspaces prepares path-connected $CRS(X)$ components within the space and it can be transformed into a bi-connected form by a bounded continuous function.

Theorem 3. *If* $g : [0,1] \to (V \supset (\partial S^2_\alpha \cup \partial S^2_\beta))$ *is a bounded continuous function in* (X, τ_X) *such that* $\{x_\alpha, x_\beta\} \subset \pi_R(X) \cup Re(\pi_C(X)) \cup Im(\pi_C(X))$ *then* S^2_α, S^2_β *are bi-connected* $CRS(X)$*.*

Proof. Let (X, τ_X) be a topological (C, R) space and the topological projections in one-dimension are given as $E_1 = \pi_R(X), E_2 = \mathrm{Re}(\pi_C(X))$ and $E_3 = \mathrm{Im}(\pi_C(X))$ where $\mathrm{Re}(.)$ and $\mathrm{Im}(.)$ represent the real and imaginary components of a complex projective subspace. Suppose the entire 1D topological projective spaces are given by $W = \underset{k\in[1,3]}{\cup} E_k$ in (X, τ_X). Let S^2_α and S^2_β be two $CRS(X)$ such that $\{x_\alpha, x_\beta\} \subset W$ within the topological space. Thus there exist a set of continuous functions $k \in [1,m], f_k : [0,1] \to W$ such that $\{x_\alpha, x_\beta\} \subset \underset{k\in[1,m]}{\cup} f_k([0,1])$ where $m \in Z^+, m < +\infty$. If we consider that $S^2_\alpha \cap S^2_\beta = \phi$ indicating two distinctly embedded M_3 in (X, τ_X) then we can conclude $x_\alpha \neq x_\beta$ and S^2_α, S^2_β are at least path-connected in W. However, if we consider that $g : [0,1] \to (V \supset (\partial S^2_\alpha \cup \partial S^2_\beta))$ is a holomorphic continuous function then $\exists t_a, t_b \in [0,1], g(t_a) \neq g(t_b)$ such that $g(t_a) \in \partial S^2_\alpha, g(t_b) \in \partial S^2_\beta$. Moreover the function $g(.)$ is two-point compact and bounded in (X, τ_X). Suppose we choose $t_a, t_b \in \{0,1\}$ where $t_a \neq t_b$ representing distinct points. Hence, this results into the conclusion that S^2_α, S^2_β are bi-connected $CRS(X)$ by functions $f_k(.)$ and $g(.)$ within the topological space (X, τ_X). □

Interestingly, the bi-connectedness of two homotopy loops cannot always be guaranteed by the path-connected fundamental groups within multiple $CRS(X)$. The locality of existence of $CRS(X)$ within the topological space is an important parameter in determining the bi-connectedness implication derived from the path-connectedness. This observation is presented in the next lemma.

Lemma 1. *If* $h_i \in [h_a]_\alpha$ *and* $h_k \in [h_b]_\beta$ *are two homotopy loops in the respective* $CRS(X)$ *given by* S^2_α *and* S^2_β *then* h_i, h_k *are path-connected but not necessarily bi-connected if* $\{x_\alpha, x_\beta\} \subset W$*.*

Proof. Let S^2_α and S^2_β be two $CRS(X)$ such that $x_\alpha \neq x_\beta$ and $S^2_\alpha \cap S^2_\beta = \phi$. If $[h_a]_\alpha$ and $[h_b]_\beta$ are two discrete homotopy classes in the respective fundamental groups $\pi_1(X, x_\alpha)$ and $\pi_1(X, x_\beta)$ in $CRS(X)$ then there is a path $f_k : [0,1] \to (W \subset X)$ such that $f_k(0) = x_\alpha \in (S^2_\alpha)^o$ and $f_k(1) = x_\beta \in (S^2_\beta)^o$ in (X, τ_X). This preserves the condition that $\{x_\alpha, x_\beta\} \subset W$ within the topological space. Recall that a $CRS(X)$ is a dense subspace which supports continuity of $f_k : [0,1] \to W$ because $(S^2_\alpha)^o \neq \phi$ and $(S^2_\beta)^o \neq \phi$. Thus the fundamental groups $\pi_1(X, x_\alpha)$ and $\pi_1(X, x_\beta)$ are path-connected by continuous function $f_k : [0,1] \to W$ within the topological space. Suppose we consider the compact (i.e., bounded and finite) and continuous (i.e., holomorphic) function in the topological (C, R) space given as $g : [0,1] \to X$ in a generalized form (i.e., without any specific restrictions imposed on codomain) such that $\bigcup_{i \in \{0,1\}} \{g(i)\} \subset (\partial S^2_\alpha \backslash \Delta(\pi_1(X, x_\alpha))) \cup (\partial S^2_\beta \backslash \Delta(\pi_1(X, x_\beta)))$. Hence, it can be concluded that in this case S^2_α and S^2_β maintain bi-connectedness if $g(0) \in (\partial S^2_\alpha \backslash \Delta(\pi_1(X, x_\alpha))), g(1) \in (\partial S^2_\beta \backslash \Delta(\pi_1(X, x_\beta)))$ but in this case $h_i \in [h_a]_\alpha$ and $h_k \in [h_b]_\beta$ preserve only path-connectedness (not bi-connectedness). □

The topological separation within a space is an important phenomenon to analyse the connectedness of a space as well as the properties of embedded algebraic and geometric structures. It is important to note that two compact $CRS(X)$ denoted by S^2_α and S^2_β are not necessarily separable even if we simply consider that $x_\alpha \neq x_\beta$ within the (C, R) space. Thus a relatively stronger condition is required involving Riemannian covering manifolds and the corresponding embeddings as presented in the following theorem.

Theorem 4. *If RS is a smooth and compact Riemann complex-sphere with $hom(RS, S^2)$ then there exist two three-manifold embeddings in (X, τ_X) given by $M_3 \subset X$ and $N_3 \subset X$ forming the separations of S^2_α and S^2_β if and only if $S^2_\alpha \subset (M_3)^o$ and $S^2_\beta \subset (N_3)^o$ respectively, where $(M_3)^o \cap (N_3)^o = \phi$.*

Proof. *Let (X, τ_X) be a topological (C, R)* space of path-connected variety. Suppose $RS = C \cup \{\infty\}$ is a Riemannian complex-sphere such that it maintains $\text{hom}(RS, S^2)$ condition. Let us consider two three-manifold category chart-maps $M_3 = \{((U_a \subset RS), (f_a(U_a) \subset X))\}_{a \in \Lambda}$ and $N_3 = \{((U_b \subset RS), (f_b(U_b) \subset X))\}_{b \in \Lambda}$ in (X, τ_X) where Λ is an index set. Note that the open sets U_a, U_b are Hausdorff topological subspaces and $f_a : U_a \to X, f_b : U_b \to X$ are homeomorphisms. First we show that such homeomorphisms exist in (X,τ_X) generating three-manifold embeddings by considering two open sets. If we consider an open disk $U_{a=1} = D(z_m, \varepsilon > 0) \subset RS$ centred at $z_m \in C$ then $f_{a=1}(U_{a=1}) \subset (A \subset X)$ where $f_{a=1}(z_k \in U_{a=1}) = (f_{a=1}(z_k) = z_n, r_n \in R)$ and $A \subset X$ is an open set. Moreover, the inverse preserves the condition given as $\forall x_n \in X, f^{-1}_{a=1}((z_n, r_n)) = f^{-1}_{a=1}(z_n) = z_k$. It directly follows that $\forall N_{xn} \subset X, x_n \in N_{xn}$ open neighbourhood $\exists D(z_m, \varepsilon > 0) \subset RS$ such that $f^{-1}_{a=1}(x_u \in N_{xn}) \in D(z_m, \varepsilon > 0)$. Furthermore, there is a coordinate identification map given as:

$$\begin{aligned} &B = U_{a=1} \cap U_{a=2}, \\ &\theta : f_{a=1}(B) \to f_{a=2}(B), \\ &\theta(x_p = f_{a=1}(z_u \in B)) \cong (x_q = f_{a=2}(z_u \in B)). \end{aligned} \tag{7}$$

Note that it maintains the condition that $\pi_R(f_{a=1}(z_u)) = \pi_R(f_{a=2}(z_u))$ because the projections on real subspace do not directly predetermine the locality of embeddings. Let us consider two such embedded subspaces given as $E_1 \subset X, E_2 \subset X, E_1 \cap E_2 = \phi$ such that $\bigcup_{a \in \Lambda} f_a(U_a) \subseteq E_1$ and $\bigcup_{b \in \Lambda} f_b(U_b) \subseteq E_2$ in (X, τ_X). As a result we can conclude that the embedded three-manifolds maintain $(\partial M_3 \neq \phi) \cap (\partial N_3 \neq \phi) = \phi$ condition within the topological space if M_3 and N_3 are compact preserving the condition that $(M_3)^o \cap (N_3)^o = \phi$. Recall that the topological space (X, τ_X) is dense everywhere. Hence, it can be concluded that $(M_3)^o \neq (N_3)^o \neq \phi$ and as a result the compact M_3, N_3 form the separations of S^2_α and S^2_β if $S^2_\alpha \subset (M_3)^o$ and $S^2_\beta \subset (N_3)^o$ in the topological space. □

Note that the above-mentioned separation property enforces a stronger condition in the multidimensional topological (C, R) space; however it is in line with the Urysohn separation concept. The embeddings of separable three-manifolds within a topological (C, R) space invite the possibility of generation of multiple components. The main reasons are that the topological space (X, τ_X) is dense and the multiple $CRS(X)$ are also separable compact subspaces if they can be covered by disjoint compact three-manifolds. This observation is presented in the next corollary.

Corollary 2. *If* $\Omega = \{S^2_k \subset X : 1 \leq k \leq n; k, n \in Z^+\}$ *is a finite set of separable* $CRS(X)$ *in the dense* (X, τ_X) *then* Ω *generates* $k+1$ *components.*

The separable embeddings of Schoenflies variety in a connected as well as dense topological space invite a set of interesting topological properties in view of the Jordan Curve Theorem (JCT). For example, the interrelationship between connected fundamental groups within the multiple compact $CRS(X)$ and the corresponding homotopy contacts are affected by the connectedness of the topological space. The topological properties related to the interplay between connected fundamental groups, homotopy contacts and manifold embeddings within a dense topological (C, R) space are presented in the following subsection.

Homotopy Contacts and Manifold Embeddings

The embeddings of three-manifolds within the dense topological space ensure that multiple $CRS(X)$ are separable, which affects the bi-connectedness property involving respective homotopy contacts. The following theorem illustrates that if the embedded three-manifolds are dense then the different projections of multiple $CRS(X)$ into the complex subspaces retain path-connectedness.

Theorem 5. *If* M_3 *and* N_3 *are two disjoint covering three-manifolds in path-connected dense* (X, τ_X) *with respective interior embeddings* S^2_a, S^2_b *then* $\pi_C(S^2_a) \subset \pi_C(C \times \{r_a\})$ *and* $\pi_C(S^2_b) \subset \pi_C(C \times \{r_b\})$ *are path-connected where* $r_a \neq r_b$ *in a holomorphic subspace* $B \subset C$.

Proof. Let M_3 and N_3 be two three-manifolds in path-connected dense (X, τ_X) such that $M_3 \subset X \backslash N_3$. Recall that we are considering compact three-manifolds such that $X \backslash M_3$ and $X \backslash N_3$ are open (i.e., $M_3 \cap \overline{(X \backslash M_3)} = \partial M_3, N_3 \cap \overline{(X \backslash N_3)} = \partial N_3$). Suppose the corresponding two $CRS(X)$ interior embeddings are prepared by homeomorphisms $f_a : Y_a \to M_3$ and $f_b : Y_b \to N_3$ where $Y_a \subset C \times R, Y_b \subset C \times R$ are two respective (C, R) spaces maintaining $\hom(S^2_a, f_a(Y_a))$ and $\hom(S^2_b, f_b(Y_b))$. Note that in this case M_3 and N_3 are the two disjoint covering three-manifolds of S^2_a and S^2_b, respectively. If $C \times \{r_a\} \subset X$ and $C \times \{r_b\} \subset X$ are two projective spaces with $r_a \neq r_b$ then $A_a \subset C \times \{r_a\}$ and $A_b \subset C \times \{r_b\}$ are the two respective projective subspaces such that $\pi_C(S^2_a) \subset \pi_C(A_a)$ and $\pi_C(S^2_b) \subset \pi_C(A_b)$. Moreover, the projections maintain the condition that $\pi_C(S^2_a) \cap \pi_C(S^2_b) = \phi$ and there is a $B \subset C$ such that $\pi_C(S^2_a) \cup \pi_C(S^2_b) \subset B$. However, if (X, τ_X) is path-connected and dense then there exists a continuous function $p : [0,1] \to B \times R$ such that $p(0) \in A_a$ and $p(1) \in A_b$ within the topological space and the complex subspace $\pi_C(A_a \cup A_b) \subset C$ is also dense. This indicates that the corresponding projection under composition $(\pi_C \circ p) : [0,1] \to B$ is continuous (i.e., the composition $(\pi_C \circ p)$ is holomorphic). Note that the topologically decomposed subspace $B \subset C$ is dense. Thus there is a continuous function $(\pi_C \circ u) : [0,1] \to B$ extending $(\pi_C \circ p)$ such that the restriction preserves $(\pi_C \circ u)\Big|_{(\pi_C \circ p)} = (\pi_C \circ p)$ in $B \subset C$.

Hence, if we consider that $(\pi_C \circ u)(0) \in \pi_C(S^2_a) \subset B$ and $(\pi_C \circ u)(1) \in \pi_C(S^2_b) \subset B$ then $\pi_C(S^2_a) \subset \pi_C(C \times \{r_a\})$ and $\pi_C(S^2_b) \subset \pi_C(C \times \{r_b\})$ are path-connected in dense (X, τ_X). □

It is important to note that the holomorphic condition is a requirement to maintain the path-connectedness under respective complex projections fixed at different points on the real subspace. Interestingly, if the homotopy contacts are present then the complex projections retain bi-connectedness of disjoint complex holomorphic subspaces. This observation is presented in the following lemma.

Lemma 2. *If there exist the contacts of homotopy classes $\Delta(\pi_1(X, x_a))$ and $\Delta(\pi_1(X, x_b))$ of respective S_a^2 and S_b^2 then $\pi_C(S_a^2) \subset \pi_C(C \times \{r_a\})$ and $\pi_C(S_b^2) \subset \pi_C(C \times \{r_b\})$ preserve bi-connectedness in the holomorphic $B \subset C$ under projections.*

Remark 4. *Interestingly, if we relax the condition of interior embedding further such that $\partial M_3 \cap f_a(Y_a) \subset \Delta(\pi_1(X, x_a))$ and $\partial N_3 \cap f_b(Y_b) \subset \Delta(\pi_1(X, x_b))$ then the continuous function $p : [0, 1] \to B \times R$ is a path-connection between $S_a^2 \subset B \times R$ and $S_b^2 \subset B \times R$ where $p(0) = \{c_a\} \subset \Delta(\pi_1(X, x_a))$ and $p(1) = \{c_b\} \subset \Delta(\pi_1(X, x_b))$. Note that in this case we are considering that the sets of contacts of homotopy classes are not empty.*

The compactness of the manifold embeddings in a subspace of (X, τ_X) exhibits an interesting topological property. It can be observed that a path-component can generally be found such that the fundamental groups within the embedded subspace always remain path-connected. This appears to be a relatively stronger property as compared to the connectedness in the topological (C, R) space.

Theorem 6. *If there exist two three-manifold embeddings in dense (X, τ_X) given by $M_3 \subset X$ and $N_3 \subset X$ such that $S_a^2 \subset (M_3)^o$, $S_b^2 \subset (N_3)^o$ and $(M_3)^o \cap (N_3)^o = \phi$ then $\pi_1(X, x_a)$ and $\pi_1(X, x_b)$ are path-connected if $(M_3 \cup N_3) \subset X_{\triangleright p}$, where $X_{\triangleright p}$ is a compact path-component.*

Proof. Let the topological (C, R) space (X, τ_X) be dense and $M_3 \subset X$, $N_3 \subset X$ be two three-manifolds embedded in the space such that $(M_3)^o \cap (N_3)^o = \phi$. Suppose we consider two $CRS(X)$ in the topological space given by $S_a^2 \subset (M_3)^o$ and $S_b^2 \subset (N_3)^o$ containing the two fundamental groups at respective base points $x_a \in (S_a^2)^o$ and $x_b \in (S_b^2)^o$ represented by $\pi_1(X, x_a)$ and $\pi_1(X, x_b)$. The subspace $Y \subset X$ is dense in (X, τ_X) and consider that $Y = (D \subset C) \times (I \subset R)$ is a compact subspace such that $(\partial M_3 \cup \partial N_3) \subset Y$ (i.e., we are considering $Y = \overline{Y}$). Thus there exists a continuous function $p : [0, 1] \to Y$ such that $p(0) \in \partial M_3$ and $p(1) \in \partial N_3$. As the subspace $Y \subset X$ is dense as well as holomorphic so a continuous extension of $p : [0, 1] \to Y$ can be found, which is given by $g : [0, 1] \to Y$ such that $g(0) \in S_a^2$ and $g(1) \in S_b^2$ while maintaining the restriction that $g\big|_p = p$. If we fix $g(0) = x_a$ and $g(1) = x_b$ then a set of continuous functions given by $F = \{g_e : [0, 1] \to Y, e \in Z^+, g_e\big|_g = g\}$ can be constructed in the topological subspace. Hence, we conclude that if $X_{\triangleright F}$ is a path-component under F then $Y \equiv X_{\triangleright F}$ and as a result $\pi_1(X, x_a)$ and $\pi_1(X, x_b)$ are path-connected in compact $Y \subset X$. □

Remark 5. *The above theorem reveals a property in view of geometric topology. If the base point of a fundamental group $\pi_1(X, x_c)$ is at $(z_c = z_0, r_c \in R)$ and the base point of another fundamental group $\pi_1(X, x_d)$ is at $(z_d, r_d = 0)$ then $\pi_1(X, x_c)$ and $\pi_1(X, x_d)$ are path-connected by a continuous function $p : [0, 1] \to X$ such that $x_0 \in p([0, 1])$.*

Lemma 3. *If $\pi_1(X, x_c)$ and $\pi_1(X, x_d)$ are two local fundamental groups in a $X_{\triangleright q}$ then there is $q\big|_p = p$ such that $[\overline{p}] * [h_a]_c * [p] \cong_H [h_b]_d$.*

Proof. The proof is relatively straightforward. First consider two local fundamental groups $\pi_1(X, x_c)$ and $\pi_1(X, x_d)$ in $X_{\triangleright q}$. Thus there is a continuous function $q : [0, 1] \to X_{\triangleright q}$ and its restriction $p : [0, 1] \to X_{\triangleright q}$ such that $q\big|_p = p$, $p(0) = x_c$ and $p(1) = x_d$. Suppose $[e_c]$ and $[e_d]$ are the left and right identities of the path $q\big|_p = p$ at the respective base points of two corresponding fundamental groups. If we consider that $\forall t \in [0, 1], q(t)\big|_p \equiv p$ and

$q(1-t)\big|_p \equiv \overline{p}$ then $[\overline{p}] * [h_a]_c \cong_H [e_c]$ and $[h_a]_c * [p] \cong_H [e_d]$. Hence, it results in the conclusion that $[\overline{p}] * ([h_a]_c * [p]) \cong_H [h_b]_d * [e_d] \cong_H [h_b]_d$ in $X_{\triangleright q}$. □

The homeomorphisms between two discrete varieties of local fundamental groups can be established once the homotopy equivalences are established. Note that it is considered that the local fundamental groups are path-connected in nature. The condition for formation of a homeomorphism between the two path-connected discrete fundamental groups is presented in the following corollary.

Corollary 3. *If $\pi_1(X, x_c)$ and $\pi_1(X, x_d)$ are two local fundamental groups generated by function sequences $\langle f_a \rangle_{a=1}^{n}$ in S_c^2 and $\langle f_b \rangle_{b=1}^{m}$ in S_d^2 respectively then $g : \pi_1(X, x_c) \to \pi_1(X, x_d)$ is a homeomorphism if and only if $n = m$ in the corresponding discrete homotopy classes $[h_a]_c$ and $[h_b]_d$.*

Proof. Let $\pi_1(X, x_c)$ and $\pi_1(X, x_d)$ be two local fundamental groups in the two closed S_c^2 and S_d^2 generated by function sequences $\langle f_a \rangle_{a=1}^{n}$ and $\langle f_b \rangle_{b=1}^{m}$ respectively within the topological space. As a result the two corresponding discrete homotopy classes are formed denoted by $[h_a]_c$ and $[h_b]_d$. Suppose we consider a function $g : \pi_1(X, x_c) \to \pi_1(X, x_d)$ such that $\forall i \in [1, n], \exists k \in [1, m]$ if $\forall f_i \in [h_a]_c$ then the function maintains the condition given by, $g * f_i = f_b \in [h_b]_d$. If we restrict that $g : \pi_1(X, x_c) \to \pi_1(X, x_d)$ is a bijection then $n = m$ maintaining $g * [f_a] = [f_b]$. Hence, it can be concluded that the bijective function $g : \pi_1(X, x_c) \to \pi_1(X, x_d)$ is a homeomorphism. □

Interestingly there is an interrelationship between the path-connection between the base points of two fundamental groups within the respective two dense $CRS(X)$ and the simple connectedness of the boundaries of corresponding $CRS(X)$ within the topological space. The simple connectedness of boundaries of $CRS(X)$ enables the formation of a path-homotopy involving the sets of homotopy contacts as illustrated in the following theorem.

Theorem 7. *If $\pi_1(X, x_c)$ and $\pi_1(X, x_d)$ are two fundamental groups path-connected by $p : [0,1] \to X$ at the base points in dense (X, τ_X) then there is a path-homotopy equivalence $g([0,1]) \cong_H p([0,1])$ if ∂S_c^2 and ∂S_d^2 are simply connected such that $g([0,1]) \cap \Delta(\pi_1(X, x_c)) \neq g([0,1]) \cap \Delta(\pi_1(X, x_d)) \neq \phi$.*

Proof. Let $\pi_1(X, x_c)$ and $\pi_1(X, x_d)$ be two path-connected fundamental groups by a continuous function $p : [0,1] \to X$ such that $p(0) = x_c$ and $p(1) = x_d$ within the dense topological space (X, τ_X). Let us consider that $p([0,1]) \cap (\Delta(\pi_1(X, x_c)) \cup \Delta(\pi_1(X, x_d))) = \phi$ preserving the generality of $p : [0,1] \to X$. Suppose we consider that ∂S_c^2 and ∂S_d^2 are simply connected surfaces indicating that $\forall x_a \in \partial S_c^2, \forall x_b \in \partial S_d^2$ there exist respective nullhomotopies $H_a : [0,1]^2 \to \{x_a\}$ and $H_b : [0,1]^2 \to \{x_b\}$. Let us further consider that $\{x_{ac}\} \subset \Delta(\pi_1(X, x_c))$ and $\{x_{bd}\} \subset \Delta(\pi_1(X, x_d))$ within the topological space. Thus one can construct a compact continuous function $g : [0,1] \to X$ such that $g(0) = x_c, g(1) = x_d$ and $\exists t_a \in (0,1), \exists t_b \in (0,1)$ maintaining $g(t_a) = x_{ac}$ and $g(t_b) = x_{bd}$. Note that in this case $t_a \neq t_b$ and $\Delta(\pi_1(X, x_c)) \cap \Delta(\pi_1(X, x_d)) = \phi$ within (X, τ_X). Moreover, as S_c^2 and S_d^2 are bi-connected so there is a continuous function $u : [0,1] \to X$ such that $u(0) = x_{ac}, u(1) = x_{bd}$ and $g|_u = u$. Hence, we conclude that $g : [0,1] \to X$ is a path-connection between $\pi_1(X, x_c)$ and $\pi_1(X, x_d)$ at base points preserving path-homotopy equivalence $g([0,1]) \cong_H p([0,1])$. □

Remark 6. *The above theorem leads to the observation further that the following algebraic properties are maintained by the respective path-homotopies.*

$$\begin{aligned} &[f_a \in [h_a]_c] * [p] * [\overline{g}] = [e_c], \\ &[f_b \in [h_b]_d] * [\overline{g}] * [p] = [e_d], \\ &[e_c] * [p] = [e_d], \\ &[e_d] * [\overline{g}] = [e_c]. \end{aligned} \tag{8}$$

Moreover, the simple connectedness property allows inward retraction of boundary of $CRS(X)$ *in the dense topological* (C, R) *space under projection. It means that* $\forall \overline{A} \subset \partial S_c^2$ *it is possible to find an inward continuous retraction function* $\eta_E : \pi_C(\overline{A}) \rightarrow (\overline{B} \subset \pi_C(\overline{A}))$*, where* $\pi_C(\overline{A}) \subset \pi_C(C \times \{r \in R\})$*. Interestingly, the retraction is independent of the influence of real subspace and it can be fixed at any arbitrary point in the real subspace.*

5. Conclusions

A q-quasinormed topological space can equally admit a corresponding topology generated by the respective p-norm function. The resulting structures provide a set of interesting topological properties in view of homotopy theory and fundamental groups. The proposed constructions of 2-quasinormed variety of locally dense p-normed 2-spheres within a non-uniformly scalable quasinormed topological (C, R) space enable the formulation of path-connected fundamental groups interior to it. The space is fibered and, in view of Baire category the topological space is dense, which supports path-connection as well as the concept of bi-connection between multiple p-normed 2-spheres as long as the continuous functions in the respective convex subspace are holomorphic in nature. The 2-quasinormed varieties of p-normed 2-spheres are equivalent to the category of connected three-manifolds with simply connected boundaries in terms of nullhomotopy. The p-normed 2-spheres admit Urysohn separation of the closed subspaces. Moreover, the separations can also be formed by proper embeddings of respective covering three-manifolds within the topological (C, R) space. The homotopically simple connected boundaries of 2-quasinormed varieties of p-normed 2-spheres support a finite and countable set of homotopy contacts generated by a set of discrete-loop local fundamental groups. Interestingly, a compact fibre in the space can prepare a homotopy loop in the local fundamental group within the fibered topological (C, R) space. It is shown that the path-connected homotopy loops are not guaranteed to be bi-connected as an implication. Moreover, the topological projections of 2-quasinormed varieties of p-normed 2-spheres on the disjoint holomorphic complex subspaces successfully retain path-connection irrespective of the projective points on real subspace. The algebraic topological properties, the properties of compactness of holomorphic convex path-components and the homeomorphism between local fundamental groups are analysed in detail. The concepts and topological constructions proposed in this paper may have potential applications in the theory of topological manifolds and the structural (geometric) aspects of cosmology.

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