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Fractional (p, q) -Calculus on Finite Intervals and Some Integral Inequalities

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Abstract: Fractional q -calculus has been investigated and applied in a variety of fields in mathematical areas including fractional q -integral inequalities. In this paper, we study fractional (p, q) -calculus on finite intervals and give some basic properties. In particular, some fractional (p, q) -integral inequalities on finite intervals are proven.

Keywords: quantum calculus; q -shifting operator; (p, q) -calculus; fractional (p, q) -integral; fractional (p, q) -integral inequality

MSC: 05A30, 26A33, 26D10



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1. Introduction

In mathematics, quantum calculus or q -calculus is the study of calculus without limits. In the early Eighteenth Century, the well-known mathematician Leonhard Euler (1707–1783) established q -calculus in the way of Newton's work for infinite series. Yet, q -calculus was known to be initiated by F. H. Jackson in 1910, who introduced the q -derivative and q -integral in [1] (see also [2]).

As a connection between the fields of mathematics and physics, q -calculus has played a significant role in physics phenomena; for instance, Fock [3] studied the symmetry of hydrogen atoms using the q -difference equation. Furthermore, in modern mathematical analysis, q -calculus has many applications such as combinatorics, orthogonal polynomials, basic hypergeometric functions, number theory, quantum theory, mechanics, and the theory of relativity; see also [4–24] and the references cited therein. The book by V. Kac and P. Cheung [25] covers the basic theoretical concept of q -calculus.

In 2013, being one of the most attractive areas, some new researchers are interested in q -calculus; in particular, J. Tariboon and S.K. Ntouyas [26] defined the q_k -calculus and proved some of its significant properties. Next, J. Tariboon and S.K. Ntouyas [27] extended some of the important integral inequalities to q -calculus. Moreover, in 2016, J. Necmettin, Z.S. Mehmet, and İ. İmdat [28] proved the correctness of the left part of the q -Hermite–Hadamard inequality and the generalized q -Hermite–Hadamard inequality. With these results, many researchers have extended some important topics of q -calculus together with applications in many fields, such as q -integral inequalities; see [29–37] for more details.

Fractional calculus is the field of mathematical analysis that deals with the investigation and applications of integrals and derivatives of arbitrary order. In 2015, J. Tariboon, S.K. Ntouyas, and P. Agarwal [38] proposed a new q -shifting operator ${}_a\Phi_q(m) = qm + (1 - q)a$ for studying new concepts of fractional q -calculus. Furthermore, in 2016, since inequalities play a vital role in modern analysis, as well as mathematical analysis depends on

many inequalities, W. Sudsutad, S.K Ntouyas, and J. Tariboon [39] studied some fractional q -integral inequalities such as the fractional q -Hermite–Hadamard integral inequality, the q -Hölder integral inequality, the q -Korkine equality, the q -Grüss integral inequality, the q -Grüss–Chebyshev integral inequality, and the q -Polya–Szeqö integral inequality on finite intervals.

Inspired and motivated by some of the above applications, in 2016, M. Tunç and E. Göv [40,41] defined the (p, q) -derivative and the (p, q) -integral on finite intervals and proved some of its properties. Later on, they also extended some of the new important integral inequalities on finite intervals to (p, q) -calculus. In 2017, M. Kun, İ. İmdat, N. Alp, and M.Z. Sarikaya [42] proved the correctness of the left part of the (p, q) -Hermite–Hadamard inequality and the generalized (p, q) -Hermite–Hadamard inequality. In addition, in 2020, J. Soontharanon and T. Sitthiwirattam [43] introduced the new concept of (p, q) -difference operators on $[0, T]$, where $T > 0$, and studied some fractional (p, q) -calculus properties in the sense of (p, q) -difference operators; especially, they proposed the fractional (p, q) -difference operator of the Riemann–Liouville and Caputo types.

Shortly afterward, many authors generalized and developed the q -calculus theory into a two-parameter (p, q) -integer, which is used efficiently in many fields, and some results on the study of (p, q) -calculus can be found in [44–68].

However, fractional (p, q) -calculus on finite intervals via some integral inequalities has not been studied yet. This gap is the motivation and inspiration for this research. The main purpose of this paper is to study the fractional (p, q) -calculus on finite intervals and to give some of its important properties. Then, we prove many fractional (p, q) -integral inequalities on finite intervals, for instance the fractional (p, q) -Hölder integral inequality, the (p, q) -Hermite–Hadamard integral inequality, the (p, q) -Korkine equality, the (p, q) -Grüss integral inequality, the (p, q) -Grüss–Chebyshev integral inequality, and the (p, q) -Polya–Szeqö integral inequality.

2. Preliminaries

In this section, we would like to recall some well-known facts on fractional (p, q) -calculus, which can be found in [10,11,38,53,55]. Throughout this paper, let $[a, b] \subset \mathbb{R}$ be an interval with $a < b$ and $0 < q < p \leq 1$ be constants,

$$\begin{aligned}
 [k]_{p,q} &= \begin{cases} \frac{p^k - q^k}{p - q}, & k \in \mathbb{N}, \end{cases} & (1) \\
 [k]_{p,q}! &= \begin{cases} [k]_{p,q}[k - 1]_{p,q} \cdots [1]_{p,q} = \prod_{i=1}^k \frac{p^i - q^i}{p - q}, & k \in \mathbb{N}, \\ 1, & k = 0. \end{cases}
 \end{aligned}$$

A q -shifting operator is defined as:

$${}_a\Phi_q(m) = qm + (1 - q)a, \tag{2}$$

where $m \in \mathbb{R}$. For any positive integer k , we have:

$${}_a\Phi_q^k(m) = {}_a\Phi_q^{k-1}({}_a\Phi_q(m)) \quad \text{and} \quad {}_a\Phi_q^0(m) = m. \tag{3}$$

By computing directly, we get the following results.

Property 1. For any $m, n \in \mathbb{R}$ and for all positive integers j, k , we have:

- (i) ${}_a\Phi_q^k(m) = {}_a\Phi_{q^k}(m)$;
- (ii) ${}_a\Phi_q^j({}_a\Phi_q^k(m)) = {}_a\Phi_q^k({}_a\Phi_q^j(m)) = {}_a\Phi_q^{j+k}(m)$;
- (iii) ${}_a\Phi_q(a) = a$;
- (iv) ${}_a\Phi_q^k(m) - a = q^k(m - a)$;
- (v) $m - {}_a\Phi_q^k(m) = (1 - q^k)(m - a)$;

- (vi) ${}_a\Phi_q^k(m) = m_{a/m}\Phi_q^k(1)$, for $m \neq 0$;
- (vii) ${}_a\Phi_q(m) - {}_a\Phi_q^k(n) = q\left(m - {}_a\Phi_q^{k-1}(n)\right)$.

For $m, n \in \mathbb{R}$ and $k \in \mathbb{N} \cup \{0\}$, the q -Pochhammer symbol or the q -shifted factorial is a q -analogue of the Pochhammer symbol, which is defined by:

$$(m; q)_0 = 1, \quad (m; q)_k = \prod_{i=0}^{k-1} (1 - q^i m), \tag{4}$$

and the new power of q -shifting operator is defined by:

$$(n - m)_a^{(0)} = 1, \quad (n - m)_a^{(k)} = \prod_{i=0}^{k-1} (n - {}_a\Phi_q^i(m)). \tag{5}$$

More generally, if $\gamma \in \mathbb{R}$, then:

$$(n - m)^{(\gamma)} = n^\gamma \prod_{i=0}^{\infty} \frac{1 - \left(\frac{m}{n}\right)q^i}{1 - \left(\frac{m}{n}\right)q^{\gamma+i}}, \tag{6}$$

and:

$$(n - m)_a^{(\gamma)} = \prod_{i=0}^{\infty} \frac{n - {}_a\Phi_q^i(m)}{n - {}_a\Phi_q^{\gamma+i}(m)}, \quad n \neq 0. \tag{7}$$

The (p, q) -derivative of function f is defined on $[a, b]$ at $t \in [a, b]$ as follows.

Definition 1 ([40]). Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function. Then:

$${}_aD_{p,q}f(t) = \frac{f(pt + (1 - p)a) - f(qt + (1 - q)a)}{(p - q)(t - a)}, \quad t \neq a; \tag{8}$$

$${}_aD_{p,q}f(a) = \lim_{t \rightarrow a} {}_aD_{p,q}f(t) \tag{9}$$

is called the (p, q) -derivative of a function f at a .

Obviously, f is (p, q) -differentiable on $[a, b]$ if ${}_aD_{p,q}f(t)$ exists for all $t \in [a, b]$. In Definition 1, if $p = 1$, then ${}_aD_{p,q}f = {}_aD_qf$, which is the q -derivative of function f on $[a, b]$, and if $a = 0$, then (8) reduces to the q -derivative of the function f on $[0, b]$; see [25,41] for more details.

Definition 2 ([40]). Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function. Then:

$$\int_a^t f(s)_a d_{p,q}s = (p - q)(t - a) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}}t + \left(1 - \frac{q^n}{p^{n+1}}\right)a\right) \tag{10}$$

is called the (p, q) -integral of f for $t \in [a, b]$.

Furthermore, if $c \in (a, t)$, then the (p, q) -integral is defined by:

$$\int_c^t f(s)_a d_{p,q}s = \int_a^t f(s)_a d_{p,q}s - \int_a^c f(s)_a d_{p,q}s. \tag{11}$$

Note that if $a = 0$ and $p = 1$, then (10) reduces to the q -integral of the function, which can be found in [26].

Theorem 1 ([40]). The following formulas hold for $t \in [a, b]$:

- (i) ${}_a D_{p,q} \int_a^t f(s) {}_a d_{p,q} s = f(t) - f(a);$
- (ii) $\int_a^b {}_a D_{p,q} f(s) {}_a d_{p,q} s = f(t);$
- (iii) $\int_c^t {}_a D_{p,q} f(s) {}_a d_{p,q} s = f(t) - f(c),$ for $c \in (a, t).$

Theorem 2 ([40]). *If $f, g : [a, b] \rightarrow \mathbb{R}$ are continuous functions, $t \in [a, b]$ and $\lambda \in \mathbb{R}$, then the following formulas hold:*

- (i) $\int_a^t [f(s) + g(s)] {}_a d_{p,q} s = \int_a^t f(s) {}_a d_{p,q} s + \int_a^t g(s) {}_a d_{p,q} s;$
- (ii) $\int_a^t \lambda f(s) {}_a d_{p,q} s = \lambda \int_a^t f(s) {}_a d_{p,q} s;$
- (iii) $\int_a^t f(ps + (1-p)s) {}_a D_{p,q} g(s) {}_a d_{p,q} s = (fg)(s)|_a^t - \int_a^t g(qs + (1-q)a) {}_a D_{p,q} (f(s)) {}_a d_{p,q} s.$

Let us define the new (p, q) -analogue of the power function ${}_a(m-n)_{p,q}^k$ with $k \in \mathbb{N} \cup \{0\}$ and $m, n \in \mathbb{R}$ as the following:

$${}_a(m-n)_{p,q}^{(0)} := 1, \quad {}_a(m-n)_{p,q}^{(k)} := \prod_{i=0}^{k-1} \left({}_a\Phi_p^i(m) - {}_a\Phi_q^i(n) \right). \tag{12}$$

It is easy to see that:

$${}_a(m-n)_{p,q}^{(k)} = (m-a)^k \prod_{i=0}^{k-1} p^i \left(1 - \left(\frac{n-a}{m-a} \right) \left(\frac{q}{p} \right)^i \right). \tag{13}$$

More generally, if $\alpha \in \mathbb{R}$, then:

$${}_a(m-n)_{p,q}^{(\alpha)} = (m-a)^\alpha \prod_{i=0}^{\infty} \frac{p^i}{p^{\alpha+i}} \frac{1 - \left(\frac{n-a}{m-a} \right) \left(\frac{q}{p} \right)^i}{1 - \left(\frac{n-a}{m-a} \right) \left(\frac{q}{p} \right)^{(\alpha+i)}}. \tag{14}$$

Property 2. *For $\alpha > 0$, the following formulas hold:*

- (i) ${}_a\Phi_{q/p}^k(m) - a = \left(\frac{q}{p} \right)^k (m-a);$
- (ii) ${}_a(m - {}_a\Phi_{q/p}^k(m))_{p,q}^{(\alpha)} = (m-a)^\alpha \prod_{i=0}^{\infty} \frac{p^i}{p^{\alpha+i}} \frac{1 - \left(\frac{q}{p} \right)^k \left(\frac{q}{p} \right)^i}{1 - \left(\frac{q}{p} \right)^k \left(\frac{q}{p} \right)^{(\alpha+i)}} = (m-a)^\alpha \left(1 - \left(\frac{q}{p} \right)^k \right)_{p,q}^{(\alpha)}.$

Proof. (i) For $\alpha > 0$, we have:

$${}_a\Phi_{q/p}^k(m) - a = \left(\frac{q}{p} \right)^k m + \left(1 - \left(\frac{q}{p} \right)^k \right) a - a = \left(\frac{q}{p} \right)^k (m-a).$$

To prove (ii), we use (i) and let $n = {}_a\Phi_{q/p}^k(m)$ in (14); we have:

$$\begin{aligned} {}_a(m - {}_a\Phi_{q/p}^k(m))_{p,q}^{(\alpha)} &= (m-a)^\alpha \prod_{i=0}^{\infty} \frac{p^i}{p^{\alpha+i}} \frac{1 - \left(\frac{q}{p} \right)^k \left(\frac{q}{p} \right)^i}{1 - \left(\frac{q}{p} \right)^k \left(\frac{q}{p} \right)^{(\alpha+i)}} \\ &= (m-a)^\alpha \left(1 - \left(\frac{q}{p} \right)^k \right)_{p,q}^{(\alpha)}. \end{aligned}$$

□

Lemma 1. *If $f : [a, b] \rightarrow \mathbb{R}$ is continuous at a , then:*

$$\int_a^t \int_a^s f(r) {}_a d_{p,q} r {}_a d_{p,q} s = \int_a^{t/p+(1-1/p)a} \int_{pqr+(1-pq)a}^t f(r) {}_a d_{p,q} s {}_a d_{p,q} r. \tag{15}$$

Proof. By Definition 2, we have:

$$\begin{aligned} \int_a^t \int_a^s f(r) {}_a d_{p,q} r {}_a d_{p,q} s &= \int_a^t (p-q)(s-a) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}}s + \left(1 - \frac{q^n}{p^{n+1}}\right)a\right) {}_a d_{p,q} s \\ &= (p-q) \sum_{n=0}^{\infty} \int_a^t \left(\frac{q^n}{p^{n+1}}s - \frac{q^n}{p^{n+1}}a\right) f\left(\frac{q^n}{p^{n+1}}s + \left(1 - \frac{q^n}{p^{n+1}}\right)a\right) {}_a d_{p,q} s \\ &= (p-q) \sum_{n=0}^{\infty} \int_a^t \left[\left(\frac{q^n}{p^{n+1}}s + \left(1 - \frac{q^n}{p^{n+1}}\right)a\right) f\left(\frac{q^n}{p^{n+1}}s + \left(1 - \frac{q^n}{p^{n+1}}\right)a\right) \right. \\ &\quad \left. - a f\left(\frac{q^n}{p^{n+1}}s + \left(1 - \frac{q^n}{p^{n+1}}\right)a\right)\right] {}_a d_{p,q} s. \end{aligned}$$

By letting $u = \frac{q^n}{p^{n+1}}s + \left(1 - \frac{q^n}{p^{n+1}}\right)a$ and using Definition 2, we get:

$$\begin{aligned} &\int_a^t \int_a^s f(r) {}_a d_{p,q} r {}_a d_{p,q} s \\ &= (p-q) \left(\frac{p^{n+1}}{q^n}\right) \sum_{n=0}^{\infty} \int_a^{\frac{q^n}{p^{n+1}}t + \left(1 - \frac{q^n}{p^{n+1}}\right)a} (u-a) f(u) {}_a d_{p,q} u \\ &= (p-q)^2 (t-a) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{q^m}{p^{m+1}} \left(\frac{q^{m+n}}{p^{m+n+2}}t + \left(1 - \frac{q^{m+n}}{p^{m+n+2}}\right)a - a\right) \\ &\quad \times f\left(\frac{q^{m+n}}{p^{m+n+2}}t + \left(1 - \frac{q^{m+n}}{p^{m+n+2}}\right)a\right) \\ &= (p-q)^2 (t-a)^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{q^{2m+n}}{p^{2m+n+3}} f\left(\frac{q^{m+n}}{p^{m+n+2}}t + \left(1 - \frac{q^{m+n}}{p^{m+n+2}}\right)a\right) \\ &= (p-q)^2 (t-a)^2 \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \frac{q^{m+n}}{p^{m+n+3}} f\left(\frac{q^n}{p^{n+2}}t + \left(1 - \frac{q^n}{p^{n+2}}\right)a\right) \\ &= (p-q)^2 (t-a)^2 \sum_{n=0}^{\infty} \sum_{m=0}^n \frac{q^{m+n}}{p^{m+n+3}} f\left(\frac{q^n}{p^{n+2}}t + \left(1 - \frac{q^n}{p^{n+2}}\right)a\right) \\ &= (p-q)(t-a)^2 \sum_{n=0}^{\infty} \left(\frac{p^{n+1} - q^{n+1}}{p^{n+1}}\right) \frac{q^n}{p^{n+2}} f\left(\frac{q^n}{p^{n+2}}t + \left(1 - \frac{q^n}{p^{n+2}}\right)a\right) \\ &= (p-q)(t-a) \sum_{n=0}^{\infty} \left(1 - \left(\frac{q}{p}\right)^{n+1}\right) \frac{q^n}{p^{n+2}} (t-a) f\left(\frac{q^n}{p^{n+2}}t + \left(1 - \frac{q^n}{p^{n+2}}\right)a\right) \\ &= (p-q) \left(\frac{t}{p} + \left(1 - \frac{1}{p}\right)a - a\right) \sum_{n=0}^{\infty} \left(1 - \left(\frac{q}{p}\right)^{n+1}\right) \frac{q^n}{p^{n+1}} (t-a) \\ &\quad \times f\left(\frac{q^n}{p^{n+1}} \left(\frac{t}{p} + \left(1 - \frac{1}{p}\right)a\right) + \left(1 - \frac{q^n}{p^{n+1}}\right)a\right) \\ &= \int_a^{t/p+(1-1/p)a} (t-pqr - (1-pq)a) f(r) {}_a d_{p,q} r \\ &= \int_a^{t/p+(1-1/p)a} \int_{pqr+(1-pq)a}^t f(r) {}_a d_{p,q} s {}_a d_{p,q} r. \end{aligned}$$

The proof is complete. \square

For $t \in \mathbb{R} \setminus \{0, -1, -2, \dots\}$, the (p, q) -gamma function is defined by:

$$\Gamma_{p,q}(t) = \frac{(p - q)_{p,q}^{(t-1)}}{(p - q)^{t-1}}, \tag{16}$$

and an equivalent definition of (16) was given in [56] as:

$$\Gamma_{p,q}(t) = p^{\frac{t(t-1)}{2}} \int_0^\infty x^{t-1} E_{p,q}^{-qx} d_{p,q}x, \tag{17}$$

where:

$$E_{p,q}^{-qx} = \sum_{n=0}^\infty \frac{q^{\binom{n}{2}}}{[n]_{p,q}} (-qx)^n.$$

Obviously, $\Gamma_{p,q}(t + 1) = [t]_{p,q} \Gamma_{p,q}(t)$. For $s, t > 0$, the definition of the (p, q) -beta function is defined by:

$$B_{p,q}(s, t) = \int_0^1 u^{s-1} {}_0(1 - {}_0\Phi_q(u))_{p,q}^{(t-1)} d_{p,q}u, \tag{18}$$

and (18) can also be written as:

$$B_{p,q}(s, t) = p^{(t-1)(2s+t-2)/2} \frac{\Gamma_{p,q}(s) \Gamma_{p,q}(t)}{\Gamma_{p,q}(s+t)}; \tag{19}$$

see [43,69] for more details.

3. Main Results

From Lemma 1, we shall give that which leads to a definition of the fractional (p, q) -integral of the Riemann–Liouville type with the consideration of the n -time as follows:

$${}_a I_{p,q}^n f(t) = \int_a^t \int_a^{r_{n-1}} \dots \int_a^{r_2} f(r_1) {}_a d_{p,q} r_1 {}_a d_{p,q} r_2 \dots {}_a d_{p,q} r_{n-2} {}_a d_{p,q} r_{n-1}. \tag{20}$$

The function f in (20) will be assumed to be continuous on $[a, b]$. From (15), we have:

$$\begin{aligned} {}_a I_{p,q}^2 f(t) &= \int_a^t \int_a^s f(r) {}_a d_{p,q} r {}_a d_{p,q} s \\ &= \int_a^{t/p+(1-1/p)a} \int_{pqr+(1-pq)a}^t f(r) {}_a d_{p,q} s {}_a d_{p,q} r \\ &= \int_a^{t/p+(1-1/p)a} (t - pqr - (1 - pq)a) f(r) {}_a d_{p,q} r \\ &= \int_a^{t/p+(1-1/p)a} \left[t - q \left((pr + (1 - p)a) - \left(1 - \frac{1}{q}\right)a \right) \right] f(r) {}_a d_{p,q} r \\ &= \frac{1}{p} \int_a^t (t - qr - (1 - q)a) f\left(\frac{r}{p} + \left(1 - \frac{1}{p}\right)a\right) {}_a d_{p,q} r \\ &= \frac{1}{p^{\binom{2}{2}} [1]_{p,q}} \int_a^t {}_a (t - {}_a \Phi_q(r))_{p,q}^{(2-1)} f\left(\frac{r}{p} + \left(1 - \frac{1}{p}\right)a\right) {}_a d_{p,q} r. \end{aligned}$$

Integrating n -times reduces (20) to a single (p, q) -integral on $[a, b]$ as follows.

$${}_a I_{p,q}^n f(t) = \frac{1}{p^{\binom{n}{2}} [n-1]_{p,q}} \int_a^t {}_a (t - {}_a \Phi_q(r))_{p,q}^{(n-1)} f\left(\frac{r}{p^{n-1}} + \left(1 - \frac{1}{p^{n-1}}\right)a\right) {}_a d_{p,q} r. \tag{21}$$

Definition 3. Let f be defined on $[a, b]$, and let $\alpha > 0$. The Riemann–Liouville fractional (p, q) -integral is defined by:

$$\begin{aligned} {}_a I_{p,q}^\alpha f(t) &= \frac{1}{p^{(\frac{\alpha}{2})} \Gamma_{p,q}(\alpha)} \int_a^t {}_a(t - {}_a\Phi_q(s))_{p,q}^{(\alpha-1)} f\left(\frac{s}{p^{\alpha-1}} + \left(1 - \frac{1}{p^{\alpha-1}}\right)a\right) {}_a d_{p,q} s \\ &= \frac{(p-q)(t-a)}{p^{(\frac{\alpha}{2})} \Gamma_{p,q}(\alpha)} \sum_{n=0}^\infty \frac{q^n}{p^{n+1}} {}_a(t - {}_a\Phi_{q/p}^{n+1}(t))_{p,q}^{(\alpha-1)} f\left(\frac{q^n}{p^{\alpha+n}}t + \left(1 - \frac{q^n}{p^{\alpha+n}}\right)a\right). \end{aligned} \tag{22}$$

Definition 4. The fractional (p, q) -derivative of the Riemann–Liouville type of order $\alpha > 0$ of a continuous function f on $[a, b]$ is defined by $({}_a D_{p,q}^0 f)(t) = f(t)$ and:

$$({}_a D_{p,q}^\alpha f)(t) = ({}_a D_{p,q}^v {}_a I_{p,q}^{v-\alpha})(t), \tag{23}$$

where $\alpha > 0$ and v is the smallest integer greater than or equal to α .

The basic q -hypergeometric function is defined in [70] as:

$${}_r F_s(a_1, \dots, a_r; b_1, \dots, b_s; x) = \sum_{n=0}^\infty \frac{(a_1; q)_n \dots (a_r; q)_n}{(q; q)_n (b_1; q)_n \dots (b_s; q)_n} x^n, \tag{24}$$

and q -Vandermonde reversing the order of summation is defined as:

$${}_2 F_1\left[q^{-n}, b; c; \frac{cq^n}{b}\right] = \frac{(c/b; q)_n}{(c; q)_n}. \tag{25}$$

Theorem 3. If f is a continuous function on $[a, b]$ and $\alpha, \beta > 0$, then the Riemann–Liouville fractional (p, q) -integral has the following semi-group property:

$${}_a I_{p,q}^\beta ({}_a I_{p,q}^\alpha f(t)) = {}_a I_{p,q}^\alpha ({}_a I_{p,q}^\beta f(t)) = ({}_a I_{p,q}^{\alpha+\beta} f(t)). \tag{26}$$

Proof. For $t \in [a, b]$, we have:

$$\begin{aligned} &{}_a I_{p,q}^\beta ({}_a I_{p,q}^\alpha f(t)) \\ &= \frac{1}{p^{(\frac{\alpha}{2})+(\frac{\beta}{2})} \Gamma_{p,q}(\alpha) \Gamma_{p,q}(\beta)} \int_a^t \int_a^{\frac{x}{p^{\alpha-1}} + \left(1 - \frac{1}{p^{\alpha-1}}\right)a} {}_a(t - {}_a\Phi_q(x))_{p,q}^{(\alpha-1)} {}_a\left(\frac{x}{p^{\alpha-1}} + \left(1 - \frac{1}{p^{\alpha-1}}\right)a - {}_a\Phi_q(s)\right)_{p,q}^{(\beta-1)} \\ &\quad \times f\left(\frac{s}{p^{\beta-1}} + \left(1 - \frac{1}{p^{\beta-1}}\right)a\right) {}_a d_{p,q} s {}_a d_{p,q} x \\ &= \frac{1}{p^{\alpha-1} p^{(\frac{\alpha}{2})+(\frac{\beta}{2})} \Gamma_{p,q}(\alpha) \Gamma_{p,q}(\beta)} \int_a^t (p-q)(x-a) \sum_{j=0}^\infty \frac{q^j}{p^{j+1}} \left(\frac{1}{p^{\alpha-1}}\right)^{\beta-1} (x-a)^{\beta-1} \\ &\quad \times \left(1 - \left(\frac{q}{p}\right)^{j+1}\right)_{p,q}^{\beta-1} {}_a(t - {}_a\Phi_q(x))_{p,q}^{(\alpha-1)} f\left(\frac{q^j}{p^{\alpha+\beta+j-1}}(x-a) + \left(1 - \frac{1}{p^{\alpha+\beta+j-1}}\right)a\right) {}_a d_{p,q} x \\ &= \frac{(p-q)^2 (t-a)^{\alpha+\beta}}{p^{(\frac{\alpha+\beta}{2})+2} \Gamma_{p,q}(\alpha) \Gamma_{p,q}(\beta)} \sum_{i=0}^\infty \sum_{j=0}^\infty \left(\frac{q}{p}\right)^{i+\beta} \left(\frac{q}{p}\right)^j \left(1 - \left(\frac{q}{p}\right)^{j+1}\right)_{p,q}^{\beta-1} \left(1 - \left(\frac{q}{p}\right)^{i+1}\right)_{p,q}^{\alpha-1} \\ &\quad \times f\left(\frac{q^{i+j}}{p^{\alpha+\beta+i+j}}(t-a) + \left(1 - \frac{1}{p^{\alpha+\beta+i+j}}\right)a\right). \end{aligned}$$

Applying the (p, q) -gamma function in (16), we obtain:

$${}_a I_{p,q}^\beta ({}_a I_{p,q}^\alpha f(t)) = \frac{(p-q)^{\alpha+\beta} (t-a)^{\alpha+\beta}}{p^{(\frac{\alpha+\beta}{2})+2}} \sum_{i=0}^\infty \sum_{j=0}^\infty \left(\frac{q}{p}\right)^{i+\beta} \left(\frac{q}{p}\right)^j \frac{\left(1 - \left(\frac{q}{p}\right)^{j+1}\right)_{p,q}^{\beta-1}}{(p-q)_{p,q}^{(\beta-1)}} \frac{\left(1 - \left(\frac{q}{p}\right)^{i+1}\right)_{p,q}^{(\alpha-1)}}{(p-q)_{p,q}^{(\alpha-1)}}$$

$$\begin{aligned} & \times f\left(\frac{q^{i+j}}{p^{\alpha+\beta+i+j}}(t-a) + \left(1 - \frac{1}{p^{\alpha+\beta+i+j}}\right)a\right) \\ &= \frac{(p-q)^{\alpha+\beta}(t-a)^{\alpha+\beta}}{p^{\binom{\alpha+\beta}{2}}p^{\alpha+\beta}} \sum_{i=0}^{\infty} \binom{q}{p}^{i+i\beta} \frac{\left(\left(\frac{q}{p}\right)^{\alpha}; \frac{q}{p}\right)_i}{\left(\frac{q}{p}; \frac{q}{p}\right)_i} \sum_{j=0}^{\infty} \binom{q}{p}^j \frac{\left(\left(\frac{q}{p}\right)^{\beta}; \frac{q}{p}\right)_j}{\left(\frac{q}{p}; \frac{q}{p}\right)_j} \\ & \times f\left(\frac{q^{i+j}}{p^{\alpha+\beta+i+j}}(t-a) + \left(1 - \frac{1}{p^{\alpha+\beta+i+j}}\right)a\right). \end{aligned}$$

Taking $m = i + j$ and interchanging the order of summation, we obtain:

$$\begin{aligned} {}_a I_{p,q}^{\beta} ({}_a I_{p,q}^{\alpha} f(t)) &= \frac{(p-q)^{\alpha+\beta}(t-a)^{\alpha+\beta}}{p^{\binom{\alpha+\beta}{2}}p^{\alpha+\beta}} \sum_{i=0}^{\infty} \binom{q}{p}^{i+i\beta} \frac{\left(\left(\frac{q}{p}\right)^{\alpha}; \frac{q}{p}\right)_i}{\left(\frac{q}{p}; \frac{q}{p}\right)_i} \sum_{m=i}^{\infty} \binom{q}{p}^{m-i} \frac{\left(\left(\frac{q}{p}\right)^{\beta}; \frac{q}{p}\right)_{m-i}}{\left(\frac{q}{p}; \frac{q}{p}\right)_{m-i}} \\ & \times f\left(\frac{q^m}{p^{\alpha+\beta+m}}(t-a) + \left(1 - \frac{1}{p^{\alpha+\beta+m}}\right)a\right) \\ &= \frac{(p-q)^{\alpha+\beta}(t-a)^{\alpha+\beta}}{p^{\binom{\alpha+\beta}{2}}p^{\alpha+\beta}} \sum_{m=0}^{\infty} \binom{q}{p}^{i+i\beta} \frac{\left(\left(\frac{q}{p}\right)^{\alpha}; \frac{q}{p}\right)_i}{\left(\frac{q}{p}; \frac{q}{p}\right)_i} \sum_{i=0}^m \binom{q}{p}^{m-i} \frac{\left(\left(\frac{q}{p}\right)^{\beta}; \frac{q}{p}\right)_{m-i}}{\left(\frac{q}{p}; \frac{q}{p}\right)_{m-i}} \\ & \times f\left(\frac{q^m}{p^{\alpha+\beta+m}}(t-a) + \left(1 - \frac{1}{p^{\alpha+\beta+m}}\right)a\right) \\ &= \frac{(p-q)^{\alpha+\beta}(t-a)^{\alpha+\beta}}{p^{\binom{\alpha+\beta}{2}}p^{\alpha+\beta}} \sum_{m=0}^{\infty} \binom{q}{p}^m \sum_{i=0}^m \binom{q}{p}^{i\beta} \frac{\left(\left(\frac{q}{p}\right)^{\alpha}; \frac{q}{p}\right)_i}{\left(\frac{q}{p}; \frac{q}{p}\right)_i} \frac{\left(\left(\frac{q}{p}\right)^{\beta}; \frac{q}{p}\right)_{m-i}}{\left(\frac{q}{p}; \frac{q}{p}\right)_{m-i}} \\ & \times f\left(\frac{q^m}{p^{\alpha+\beta+m}}(t-a) + \left(1 - \frac{1}{p^{\alpha+\beta+m}}\right)a\right) \\ &= \frac{(p-q)^{\alpha+\beta}(t-a)^{\alpha+\beta}}{p^{\binom{\alpha+\beta}{2}}p^{\alpha+\beta}} \sum_{m=0}^{\infty} \binom{q}{p}^m f\left(\frac{q^m}{p^{\alpha+\beta+m}}(t-a) + \left(1 - \frac{1}{p^{\alpha+\beta+m}}\right)a\right) \\ & \times \sum_{i=0}^m \binom{q}{p}^{i\beta} \frac{\left(\left(\frac{q}{p}\right)^{\alpha}; \frac{q}{p}\right)_i}{\left(\frac{q}{p}; \frac{q}{p}\right)_i} \frac{\left(\left(\frac{q}{p}\right)^{\beta}; \frac{q}{p}\right)_{m-i}}{\left(\frac{q}{p}; \frac{q}{p}\right)_{m-i}}. \end{aligned}$$

On the other hand,

$$\frac{\left(\left(\frac{q}{p}\right)^{\beta}; \frac{q}{p}\right)_{m-i}}{\left(\frac{q}{p}; \frac{q}{p}\right)_{m-i}} = \frac{\left(1 - \left(\frac{q}{p}\right)^{\beta}\right)_{\frac{q}{p}}^{m-i}}{\left(1 - \left(\frac{q}{p}\right)\right)_{\frac{q}{p}}^{m-i}} = \frac{\left(1 - \left(\frac{q}{p}\right)^{\beta}\right)_{\frac{q}{p}}^m \left(1 - \left(\frac{q}{p}\right)^{\beta+m}\right)_{\frac{q}{p}}^{-i}}{\left(1 - \left(\frac{q}{p}\right)\right)_{\frac{q}{p}}^m \left(1 - \left(\frac{q}{p}\right)^{m+1}\right)_{\frac{q}{p}}^{-i}}, \tag{27}$$

and:

$$\begin{aligned} \frac{\left(1 - \left(\frac{q}{p}\right)^{\beta+m}\right)_{\frac{q}{p}}^{-i}}{\left(1 - \left(\frac{q}{p}\right)^{m+1}\right)_{\frac{q}{p}}^{-i}} &= \frac{\left(1 - \left(\frac{q}{p}\right)^{m-i+1}\right)_{\frac{q}{p}}^i}{\left(1 - \left(\frac{q}{p}\right)^{\beta+m-i}\right)_{\frac{q}{p}}^i} \\ &= \frac{\prod_{k=0}^{i-1} \left(1 - \left(\frac{q}{p}\right)^{m-i+1} \left(\frac{q}{p}\right)^k\right)}{\prod_{k=0}^{i-1} \left(1 - \left(\frac{q}{p}\right)^{\beta+m-i} \left(\frac{q}{p}\right)^k\right)} \end{aligned}$$

$$\begin{aligned}
 &= \left(\frac{q}{p}\right)^{i-i\beta} \prod_{k=0}^{i-1} \frac{\left(1 - \left(\frac{q}{p}\right)^{i-1+m} \left(\frac{q}{p}\right)^{-k}\right)}{\left(1 - \left(\frac{q}{p}\right)^{-\beta-m+i} \left(\frac{q}{p}\right)^{-k}\right)} \\
 &= \frac{\left(\left(\frac{q}{p}\right)^{-m}; \frac{q}{p}\right)_i \left(\frac{q}{p}\right)^{i-i\beta}}{\left(\left(\frac{q}{p}\right)^{1-\beta-m}; \frac{q}{p}\right)_i}.
 \end{aligned} \tag{28}$$

Substituting (28) into (27), we get:

$$\frac{\left(\left(\frac{q}{p}\right)^\beta; \frac{q}{p}\right)_{m-i}}{\left(\frac{q}{p}, \frac{q}{p}\right)_{m-i}} = \frac{\left(\left(\frac{q}{p}\right)^\beta; \frac{q}{p}\right)_m}{\left(\frac{q}{p}, \frac{q}{p}\right)_m} \frac{\left(\left(\frac{q}{p}\right)^{-m}; \frac{q}{p}\right)_i}{\left(\left(\frac{q}{p}\right)^{1-\beta-m}; \frac{q}{p}\right)_i} \left(\frac{q}{p}\right)^{i-i\beta}. \tag{29}$$

Therefore, we obtain:

$$\begin{aligned}
 {}_a I_{p,q}^\beta ({}_a I_{p,q}^\alpha f(t)) &= \frac{(p-q)^{\alpha+\beta} (t-a)^{\alpha+\beta}}{p^{\binom{\alpha+\beta}{2}} p^{\alpha+\beta}} \sum_{m=0}^\infty \left(\frac{q}{p}\right)^m f\left(\frac{q^m}{p^{\alpha+\beta+m}} (t-a) + \left(1 - \frac{1}{p^{\alpha+\beta+m}}\right) a\right) \\
 &\quad \times \frac{\left(\left(\frac{q}{p}\right)^\beta; \frac{q}{p}\right)_m}{\left(\frac{q}{p}, \frac{q}{p}\right)_m} \sum_{i=0}^m \frac{\left(\left(\frac{q}{p}\right)^\alpha; \frac{q}{p}\right)_i}{\left(\frac{q}{p}, \frac{q}{p}\right)_i} \frac{\left(\left(\frac{q}{p}\right)^{-m}; \frac{q}{p}\right)_i}{\left(\left(\frac{q}{p}\right)^{1-\beta-m}; \frac{q}{p}\right)_i} \left(\frac{q}{p}\right)^i.
 \end{aligned} \tag{30}$$

By (24), we have:

$$\begin{aligned}
 {}_a I_{p,q}^\beta ({}_a I_{p,q}^\alpha f(t)) &= \frac{(p-q)^{\alpha+\beta} (t-a)^{\alpha+\beta}}{p^{\binom{\alpha+\beta}{2}} p^{\alpha+\beta}} \sum_{m=0}^\infty \left(\frac{q}{p}\right)^m f\left(\frac{q^m}{p^{\alpha+\beta+m}} (t-a) + \left(1 - \frac{1}{p^{\alpha+\beta+m}}\right) a\right) \\
 &\quad \times \frac{\left(\left(\frac{q}{p}\right)^\beta; \frac{q}{p}\right)_m}{\left(\frac{q}{p}, \frac{q}{p}\right)_m} {}_2F_1\left[\left(\frac{q}{p}\right)^{-m}, \left(\frac{q}{p}\right)^\alpha; \left(\frac{q}{p}\right)^{1-\beta-m}; \frac{q}{p}\right].
 \end{aligned} \tag{31}$$

From (25), we obtain:

$$\begin{aligned}
 {}_2F_1\left[\left(\frac{q}{p}\right)^{-m}, \left(\frac{q}{p}\right)^\alpha; \left(\frac{q}{p}\right)^{1-\beta-m}; \frac{q}{p}\right] &= \frac{\left(\left(\frac{q}{p}\right)^{1-\beta-m-\alpha}; \left(\frac{q}{p}\right)\right)_m}{\left(\left(\frac{q}{p}\right)^{1-\beta-m}; \left(\frac{q}{p}\right)\right)_m} \left(\frac{q}{p}\right)^{m\alpha} \\
 &= \frac{\left(\left(\frac{q}{p}\right)^{\alpha+\beta}; \left(\frac{q}{p}\right)\right)_m}{\left(\left(\frac{q}{p}\right)^\beta; \frac{q}{p}\right)_m}.
 \end{aligned} \tag{32}$$

Substituting (32) into (31), we get:

$$\begin{aligned}
 {}_a I_{p,q}^\beta ({}_a I_{p,q}^\alpha f(t)) &= \frac{(p-q)^{\alpha+\beta} (t-a)^{\alpha+\beta}}{p^{\binom{\alpha+\beta}{2}} p^{\alpha+\beta-1}} \sum_{m=0}^\infty \frac{q^m}{p^{m+1}} \frac{\left(\left(\frac{q}{p}\right)^{\alpha+\beta}; \left(\frac{q}{p}\right)\right)_m}{\left(\frac{q}{p}, \frac{q}{p}\right)_m} \\
 &\quad \times f\left(\frac{q^m}{p^{\alpha+\beta+m}} (t-a) + \left(1 - \frac{1}{p^{\alpha+\beta+m}}\right) a\right),
 \end{aligned} \tag{33}$$

which is the series representation of ${}_a I_{p,q}^{\alpha+\beta} f(t)$. Therefore, (26) holds. \square

Throughout this paper, the variable s is shown inside the fractional integral, which is denoted as:

$$\left({}_a I_{p,q}^\alpha f(s)\right)(t) = \frac{1}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} \int_a^t {}_a(t - {}_a\Phi_q(s))_{p,q}^{(\alpha-1)} f\left(\frac{s}{p^{\alpha-1}} + \left(1 - \frac{1}{p^{\alpha-1}}\right)a\right) {}_a d_{p,q}s.$$

Lemma 2. If $\bar{s} = p^{\alpha-1}s + (1 - p^{\alpha-1})a, \alpha, \beta > 0$, and $t \in [a, p^\alpha b + (1 - p^\alpha)a]$, then the following formula holds:

$$\left({}_a I_{p,q}^\alpha (\bar{s} - a)^\beta\right)(t) = \frac{p^{\beta(\alpha-1)} \Gamma_{p,q}(\beta + 1)}{\Gamma_{p,q}(\beta + \alpha + 1)} (t - a)^{\beta + \alpha}. \tag{34}$$

Proof. Applying Definition 2, Property 2, and (18) and (19), we have:

$$\begin{aligned} \left({}_a I_{p,q}^\alpha (\bar{s} - a)^\beta\right)(t) &= \frac{1}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} \int_a^t {}_a(t - {}_a\Phi_q(s))_{p,q}^{(\alpha-1)} (s - a)^\beta {}_a d_{p,q}s \\ &= \frac{(p - a)(t - a)}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} \sum_{i=0}^\infty \frac{q^i}{p^{i+1}} {}_a \left((t - a) - \frac{q^{i+1}}{p^{i+1}}(t - a) \right)_{p,q}^{(\alpha-1)} \left(\frac{q^i}{p^{i+1}}(t - a) \right)^\beta \\ &= \frac{(p - a)(t - a)^{\beta+1}}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} \sum_{i=0}^\infty \frac{q^i}{p^{i+1}} \prod_{i=0}^\infty \frac{1 - \left(\frac{q}{p}\right)^{i+1} \left(\frac{q}{p}\right)^i}{1 - \left(\frac{q}{p}\right)^{i+1} \left(\frac{q}{p}\right)^{\alpha-1+i}} \left(\frac{q^i}{p^{i+1}}\right)^\beta \\ &= \frac{(p - a)(t - a)^{\alpha+\beta}}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} \sum_{i=0}^\infty \frac{q^i}{p^{i+1}} \left(1 - \left(\frac{q}{p}\right)^{i+1}\right)^{(\alpha-1)} \left(\frac{q^i}{p^{i+1}}\right)^\beta \\ &= \frac{(t - a)^{\alpha+\beta}}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} \int_0^1 s^\beta (1 - {}_0\Phi_q(s))_{p,q}^{(\alpha-1)} {}_0 d_{p,q}s \\ &= \frac{(t - a)^{\alpha+\beta}}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} B_{p,q}(\beta + 1, \alpha) \\ &= \frac{p^{\beta(\alpha-1)} \Gamma_{p,q}(\beta + 1)}{\Gamma_{p,q}(\beta + \alpha + 1)} (t - a)^{\beta + \alpha}. \end{aligned}$$

The proof of Lemma 2 is complete. \square

Remark 1. Define functions $f, g : [a, b] \rightarrow \mathbb{R}$ by $f(t) = t$ and $g(t) = t^2$. For $\alpha > 0$, we get:

- (i) $\left({}_a I_{p,q}^\alpha f(\bar{s})\right)(t) = \frac{(t-a)^\alpha}{\Gamma_{p,q}(\alpha+2)} [p^{\alpha-1}(t - a) + ([\alpha + 1]_{p,q})a];$
- (ii) $\left({}_a I_{p,q}^\alpha g(\bar{s})\right)(t) = \frac{(t-a)^\alpha}{\Gamma_{p,q}(\alpha+3)} [p^{2(\alpha-1)}(p + q)(t - a)^2 + 2ap^{\alpha-1}(t - a)[\alpha + 2]_{p,q} + a^2[\alpha + 2]_{p,q}[\alpha + 1]_{p,q}].$

Next, we study some fractional (p, q) integral inequalities on finite intervals.

- The fractional (p, q) -Hölder inequality on $[a, p^\alpha b + (1 - p^\alpha)a]$:

Theorem 4. Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function, $\bar{s} = p^{\alpha-1}s + (1 - p^{\alpha-1})a$, and $\alpha > 0$. If $p_1, p_2 \geq 0$, such that $1/p_1 + 1/p_2 = 1$, then we have:

$$\left({}_a I_{p,q}^\alpha |f(\bar{s})||g(\bar{s})|\right)(t) \leq \left({}_a I_{p,q}^\alpha |f(\bar{s})|^{p_1}(t)\right)^{1/p_1} \left({}_a I_{p,q}^\alpha |g(\bar{s})|^{p_2}(t)\right)^{1/p_2}. \tag{35}$$

Proof. Using Definition 3 and the discrete Hölder inequality, we have:

$$\begin{aligned} &\left({}_a I_{p,q}^\alpha |f(\bar{s})||g(\bar{s})|\right)(t) \\ &= \frac{1}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} \int_a^t (t - {}_a\Phi_q(s))_{p,q}^{(\alpha-1)} |f(s)||g(s)| {}_a d_{p,q}s \\ &= \frac{(p - q)(t - a)}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} \sum_{n=0}^\infty \frac{q^n}{p^{n+1}} (t - {}_a\Phi_{q/p}^{n+1}(t))_{p,q}^{(\alpha-1)} \left| f\left(\frac{q^n}{p^{n+1}}t + \left(1 - \frac{q^n}{p^{n+1}}\right)a\right) \right| \left| g\left(\frac{q^n}{p^{n+1}}t + \left(1 - \frac{q^n}{p^{n+1}}\right)a\right) \right| \end{aligned}$$

$$\begin{aligned}
&= \frac{(p-q)(t-a)}{p^{(\frac{\alpha}{2})}\Gamma_{p,q}(\alpha)} \sum_{n=0}^{\infty} (t-{}_a\Phi_{q/p}^{n+1}(t))_{p,q}^{(\alpha-1)} \left(\frac{q^n}{p^{n+1}}\right)^{\frac{1}{p_1}} \left|f\left(\frac{q^n}{p^{n+1}}t + \left(1 - \frac{q^n}{p^{n+1}}\right)a\right)\right| \\
&\quad \times \left(\frac{q^n}{p^{n+1}}\right)^{\frac{1}{p_2}} \left|g\left(\frac{q^n}{p^{n+1}}t + \left(1 - \frac{q^n}{p^{n+1}}\right)a\right)\right| \\
&\leq \left(\frac{(p-q)(t-a)}{p^{(\frac{\alpha}{2})}\Gamma_{p,q}(\alpha)} \sum_{n=0}^{\infty} (t-{}_a\Phi_{q/p}^{n+1}(t))_{p,q}^{(\alpha-1)} \frac{q^n}{p^{n+1}} \left|f\left(\frac{q^n}{p^{n+1}}t + \left(1 - \frac{q^n}{p^{n+1}}\right)a\right)\right|^{p_1}\right)^{1/p_1} \\
&\quad \times \left(\frac{(p-q)(t-a)}{p^{(\frac{\alpha}{2})}\Gamma_{p,q}(\alpha)} \sum_{n=0}^{\infty} (t-{}_a\Phi_{q/p}^{n+1}(t))_{p,q}^{(\alpha-1)} \frac{q^n}{p^{n+1}} \left|g\left(\frac{q^n}{p^{n+1}}t + \left(1 - \frac{q^n}{p^{n+1}}\right)a\right)\right|^{p_2}\right)^{1/p_2} \\
&= ({}_aI_{p,q}^{\alpha}|f(\bar{s})|^{p_1}(t))^{1/p_1} ({}_aI_{p,q}^{\alpha}|g(\bar{s})|^{p_2}(t))^{1/p_2}.
\end{aligned}$$

Therefore, the proof of Theorem 4 is complete. \square

Remark 2. If $p = 1$, then (35) reduces to the fractional q -Hölder inequality on $[a, b]$ as:

$$({}_aI_q^{\alpha}|f(s)||g(s)|)(t) \leq ({}_aI_q^{\alpha}|f(s)|^{p_1}(t))^{1/p_1} ({}_aI_q^{\alpha}|g(s)|^{p_2}(t))^{1/p_2}, \quad (36)$$

which appeared in [39]. However, if $p = 1$ and $a = 0$, then (36) reduces to the q -Hölder inequality as:

$$\int_0^t f(s)g(s)d_qs \leq \left(\int_0^t |f(s)|^{p_1}d_qs\right)^{1/p_1} \left(\int_0^t |g(s)|^{p_2}d_qs\right)^{1/p_2},$$

which can be found in [71].

- The fractional (p, q) -Hermite–Hadamard integral inequalities on $[a, p^{\alpha}b + (1 - p^{\alpha})a]$:

Theorem 5. If $f : [a, b] \rightarrow \mathbb{R}$ is a convex differentiable function and $\alpha > 0$, then we have:

$$\begin{aligned}
f\left(\frac{([\alpha + 1]_{p,q} - p^{\alpha})a + p^{\alpha}b}{[\alpha + 1]_{p,q}}\right) &\leq \frac{\Gamma_{p,q}(\alpha + 1)}{p^{\alpha^2}(b-a)^{\alpha}} ({}_aI_{p,q}^{\alpha}f(s))(p^{\alpha}b + (1 - p^{\alpha})a) \\
&\leq \frac{([\alpha + 1]_{p,q} - p^{\alpha})f(a) + p^{\alpha}f(b)}{[\alpha + 1]_{p,q}}.
\end{aligned} \quad (37)$$

Proof. From the left-hand side of the proof in Theorem 3 and Theorem 5.1 in [15,42], respectively, there is a one line support:

$$\begin{aligned}
&f\left(\frac{([\alpha + 1]_{p,q} - p^{\alpha})a + p^{\alpha}b}{[\alpha + 1]_{p,q}}\right) + m\left(\frac{s}{p^{\alpha-1}} + \left(1 - \frac{1}{p^{\alpha-1}}\right)a - \frac{([\alpha + 1]_{p,q} - p^{\alpha})f(a) + p^{\alpha}f(b)}{[\alpha + 1]_{p,q}}\right) \\
&\leq f\left(\frac{s}{p^{\alpha-1}} + \left(1 - \frac{1}{p^{\alpha-1}}\right)a\right),
\end{aligned} \quad (38)$$

for all $s \in [a, b]$ and $m \in \left[f'_- \left(\frac{([\alpha + 1]_{p,q} - p^{\alpha})a + p^{\alpha}b}{[\alpha + 1]_{p,q}}\right), f'_+ \left(\frac{([\alpha + 1]_{p,q} - p^{\alpha})a + p^{\alpha}b}{[\alpha + 1]_{p,q}}\right)\right]$.

Multiplying by $({}_a\Phi_{p^{\alpha}}(b) - {}_a\Phi_q(s))_{p,q}^{(\alpha-1)} / p^{(\frac{\alpha}{2})}\Gamma_{p,q}(\alpha)$ on both sides of (38), we obtain:

$$\begin{aligned}
&\frac{1}{p^{(\frac{\alpha}{2})}\Gamma_{p,q}(\alpha)} ({}_a\Phi_{p^{\alpha}}(b) - {}_a\Phi_q(s))_{p,q}^{(\alpha-1)} f\left(\frac{([\alpha + 1]_{p,q} - p^{\alpha})a + p^{\alpha}b}{[\alpha + 1]_{p,q}}\right) \\
&\quad + m\left[\frac{1}{p^{(\frac{\alpha}{2})}\Gamma_{p,q}(\alpha)} ({}_a\Phi_{p^{\alpha}}(b) - {}_a\Phi_q(s))_{p,q}^{\alpha-1} \left(\frac{s}{p^{\alpha-1}} + \left(1 - \frac{1}{p^{\alpha-1}}\right)a - \left(\frac{([\alpha + 1]_{p,q} - p^{\alpha})a + p^{\alpha}b}{[\alpha + 1]_{p,q}}\right)\right)\right] \\
&\leq \frac{1}{p^{(\frac{\alpha}{2})}\Gamma_{p,q}(\alpha)} ({}_a\Phi_{p^{\alpha}}(b) - {}_a\Phi_q(s))_{p,q}^{(\alpha-1)} f\left(\frac{s}{p^{\alpha-1}} + \left(1 - \frac{1}{p^{\alpha-1}}\right)a\right).
\end{aligned} \quad (39)$$

Taking the fractional (p, q) -integration of order $\alpha > 0$ with respect to s on (39), where $s \in (a, p^\alpha b + (1 - p^\alpha)a)$, we have:

$$\begin{aligned} & \frac{1}{p^{(\frac{\alpha}{2})}\Gamma_{p,q}(\alpha)} \int_a^{p^\alpha b + (1-p^\alpha)a} {}_a(\Phi_{p^\alpha}(b) - {}_a\Phi_q(s))_{p,q}^{(\alpha-1)} f\left(\frac{([\alpha+1]_{p,q} - p^\alpha)a + p^\alpha b}{[\alpha+1]_{p,q}}\right) {}_a d_{p,q} s \\ & + m \left[\frac{1}{p^{(\frac{\alpha}{2})}\Gamma_{p,q}(\alpha)} \int_a^{p^\alpha b + (1-p^\alpha)a} {}_a(\Phi_{p^\alpha}(b) - {}_a\Phi_q(s))_{p,q}^{(\alpha-1)} \right. \\ & \quad \left. \times \left(\frac{s}{p^{\alpha-1}} + \left(1 - \frac{1}{p^{\alpha-1}}\right)a - \left(\frac{([\alpha+1]_{p,q} - p^\alpha)a + p^\alpha b}{[\alpha+1]_{p,q}}\right) \right) \right] {}_a d_{p,q} s \\ & \leq \frac{1}{p^{(\frac{\alpha}{2})}\Gamma_{p,q}(\alpha)} \int_a^{p^\alpha b + (1-p^\alpha)a} {}_a(\Phi_{p^\alpha}(b) - {}_a\Phi_q(s))_{p,q}^{(\alpha-1)} f\left(\frac{s}{p^{\alpha-1}} + \left(1 - \frac{1}{p^{\alpha-1}}\right)a\right) {}_a d_{p,q} s \\ & = ({}_a I_{p,q}^\alpha f(s))(p^\alpha b + (1 - p^\alpha)a). \end{aligned} \quad (40)$$

Moreover, from the left-hand side of (40), we have:

$$\begin{aligned} & \frac{1}{p^{(\frac{\alpha}{2})}\Gamma_{p,q}(\alpha)} \int_a^{p^\alpha b + (1-p^\alpha)a} ({}_a\Phi_{p^\alpha}(b) - {}_a\Phi_q(s))_{p,q}^{(\alpha-1)} f\left(\frac{([\alpha+1]_{p,q} - p^\alpha)a + p^\alpha b}{[\alpha+1]_{p,q}}\right) {}_a d_{p,q} s \\ & = \frac{(p-q)p^\alpha(b-a)}{p^{(\frac{\alpha}{2})}\Gamma_{p,q}(\alpha)} \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left({}_a\Phi_{p^\alpha}(b) - \left(\frac{q^{n+1}}{p^n}(b-a) + a\right) \right)_{p,q}^{(\alpha-1)} f\left(\frac{([\alpha+1]_{p,q} - p^\alpha)a + p^\alpha b}{[\alpha+1]_{p,q}}\right) \\ & = \frac{(p-q)p^{\alpha^2}(b-a)^\alpha}{p^{(\frac{\alpha}{2})}\Gamma_{p,q}(\alpha)} f\left(\frac{([\alpha+1]_{p,q} - p^\alpha)a + p^\alpha b}{[\alpha+1]_{p,q}}\right) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left(1 - \left(\frac{q}{p}\right)^{n+1}\right)_{p,q}^{(\alpha-1)} \\ & = \frac{p^{\alpha^2}(b-a)^\alpha}{p^{(\frac{\alpha}{2})}\Gamma_{p,q}(\alpha)} f\left(\frac{([\alpha+1]_{p,q} - p^\alpha)a + p^\alpha b}{[\alpha+1]_{p,q}}\right) \int_0^1 (1 - {}_0\Phi_q(s))_{p,q}^{(\alpha-1)} {}_0 d_{p,q} s \\ & = \frac{p^{\alpha^2}(b-a)^\alpha}{p^{(\frac{\alpha}{2})}\Gamma_{p,q}(\alpha)} f\left(\frac{([\alpha+1]_{p,q} - p^\alpha)a + p^\alpha b}{[\alpha+1]_{p,q}}\right) B_{p,q}(1, \alpha) \\ & = \frac{p^{\alpha^2}(b-a)^\alpha}{\Gamma_{p,q}(\alpha+1)} f\left(\frac{([\alpha+1]_{p,q} - p^\alpha)a + p^\alpha b}{[\alpha+1]_{p,q}}\right) \end{aligned} \quad (41)$$

and similar to the computation of (41) above, we also get:

$$\begin{aligned} & \frac{1}{p^{(\frac{\alpha}{2})}\Gamma_{p,q}(\alpha)} \int_a^{p^\alpha b + (1-p^\alpha)a} ({}_a\Phi_{p^\alpha}(b) - {}_a\Phi_q(s))_{p,q}^{(\alpha-1)} \\ & \quad \times \left[\frac{s}{p^{\alpha-1}} + \left(1 - \frac{1}{p^{\alpha-1}}\right)a - \left(\frac{([\alpha+1]_{p,q} - p^\alpha)a + p^\alpha b}{[\alpha+1]_{p,q}}\right) \right] {}_a d_{p,q} s \\ & = \frac{(p-q)p^{\alpha^2+1}(b-a)^{\alpha+1}}{p^{(\frac{\alpha}{2})}\Gamma_{p,q}(\alpha)} \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left(1 - \left(\frac{q}{p}\right)^{n+1}\right)_{p,q}^{(\alpha-1)} \frac{q^i}{p^{i+1}} \\ & \quad + \frac{(p-q)p^{\alpha^2}(b-a)^\alpha}{p^{(\frac{\alpha}{2})}\Gamma_{p,q}(\alpha)} \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left(1 - \left(\frac{q}{p}\right)^{n+1}\right)_{p,q}^{(\alpha-1)} \left(a - \left(\frac{([\alpha+1]_{p,q} - p^\alpha)a + p^\alpha b}{[\alpha+1]_{p,q}}\right)\right) \\ & = \frac{p^{\alpha^2}(b-a)^\alpha}{p^{(\frac{\alpha}{2})}\Gamma_{p,q}(\alpha)} \left[p(b-a)B_{p,q}(2, \alpha) + aB_{p,q}(1, \alpha) - \frac{([\alpha+1]_{p,q} - p^\alpha)a + p^\alpha b}{[\alpha+1]_{p,q}} B_{p,q}(1, \alpha) \right] \\ & = p^{\alpha^2}(b-a)^\alpha \left[\frac{([\alpha+1]_{p,q} - p^\alpha)a + p^\alpha b}{\Gamma_{p,q}(\alpha+2)} - \frac{([\alpha+1]_{p,q} - p^\alpha)a + p^\alpha b}{\Gamma_{p,q}(\alpha+2)} \right] \\ & = 0. \end{aligned} \quad (42)$$

By substituting (42) and (41) into (40), we obtain the first part of (37). On the other hand, from the proof of the right-hand side of Theorem 3 in [42], we have:

$$f\left(\frac{s}{p^{\alpha-1}} + \left(1 - \frac{1}{p^{\alpha-1}}\right)a\right) \leq f(a) + \frac{f(b) - f(a)}{b - a} \left(\frac{s}{p^{\alpha-1}} + \left(1 - \frac{1}{p^{\alpha-1}}\right)a - a\right). \tag{43}$$

Multiplying by $({}_a\Phi_{p^\alpha}(b) - {}_a\Phi_q(s))_{p,q}^{(\alpha-1)} / p^{(\frac{\alpha}{2})}\Gamma_{p,q}(\alpha)$ on both sides of (43) and taking the (p, q) -integral of order $\alpha > 0$ with respect to $s \in [a, p^\alpha b + (1 - p^\alpha)a]$, we obtain:

$$\begin{aligned} & \frac{1}{p^{(\frac{\alpha}{2})}\Gamma_{p,q}(\alpha)} \int_a^{p^\alpha b + (1-p^\alpha)a} {}_a({}_a\Phi_{p^\alpha}(b) - {}_a\Phi_q(s))_{p,q}^{(\alpha-1)} f\left(\frac{s}{p^{\alpha-1}} + \left(1 - \frac{1}{p^{\alpha-1}}\right)a\right) {}_a d_{p,q} s \\ & \leq \frac{1}{p^{(\frac{\alpha}{2})}\Gamma_{p,q}(\alpha)} \int_a^{p^\alpha b + (1-p^\alpha)a} {}_a({}_a\Phi_{p^\alpha}(b) - {}_a\Phi_q(s))_{p,q}^{(\alpha-1)} \\ & \quad \times \left[f(a) + \frac{f(b) - f(a)}{b - a} \left(\frac{s}{p^{\alpha-1}} + \left(1 - \frac{1}{p^{\alpha-1}}\right)a - a\right) \right] {}_a d_{p,q} s. \end{aligned} \tag{44}$$

By using the same computation as in (41) and (42) for the left-hand side of (44), we obtain:

$$\begin{aligned} & \frac{1}{p^{(\frac{\alpha}{2})}\Gamma_{p,q}(\alpha)} \int_a^{p^\alpha b + (1-p^\alpha)a} {}_a({}_a\Phi_{p^\alpha}(b) - {}_a\Phi_q(s))_{p,q}^{(\alpha-1)} \\ & \quad \times \left[f(a) + \frac{f(b) - f(a)}{b - a} \left(\frac{s}{p^{\alpha-1}} + \left(1 - \frac{1}{p^{\alpha-1}}\right)a - a\right) \right] {}_a d_{p,q} s \\ & \leq \frac{p^{\alpha^2}(b - a)^\alpha}{\Gamma_{p,q}(\alpha + 1)} f(a) + \frac{p^{\alpha^2 + \alpha}(b - a)^\alpha}{\Gamma_{p,q}(\alpha + 2)} (f(b) - f(a)) \end{aligned} \tag{45}$$

$$= \frac{p^{\alpha^2}(b - a)^\alpha}{\Gamma_{p,q}(\alpha + 1)} \frac{([\alpha + 1]_{p,q} - p^\alpha) f(a) + p^\alpha f(b)}{[\alpha + 1]_{p,q}}. \tag{46}$$

Substituting (45) into (44), we derive the second part of (37). Therefore, the proof of Theorem 5 is complete. \square

Remark 3. If $p = 1$, then (37) reduces to the fractional q -Hermite–Hadamard integral inequality as:

$$f\left(\frac{([\alpha + 1]_q - 1)a + b}{[\alpha + 1]_q}\right) \leq \frac{\Gamma_q(\alpha + 1)}{(b - a)^\alpha} ({}_a I_q^\alpha f(s))(b) \leq \frac{([\alpha + 1]_q - 1)f(a) + f(b)}{[\alpha + 1]_q},$$

which appeared in [15].

If $\alpha = 1$, then (37) reduces to the (p, q) -Hermite–Hadamard integral inequality on $[a, b]$ as:

$$f\left(\frac{qa + pb}{p + q}\right) \leq \frac{1}{p(b - a)} \int_a^{pb + (1-p)a} f(s) {}_a d_{p,q} s \leq \frac{qf(a) + pf(b)}{p + q}, \tag{47}$$

which appeared in [42]. Moreover, if $p = 1$ and $q \rightarrow 1$, then (47) reduces to the classical Hermite–Hadamard integral inequality as:

$$f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b f(s) ds \leq \frac{f(a) + f(b)}{2},$$

which can be found in [72,73].

- The fractional (p, q) -Korkine equality on $[a, p^\alpha b + (1 - p^\alpha)a]$:

Theorem 6. If $f, g: [a, b] \rightarrow \mathbb{R}$ are continuous functions, $\bar{s} = p^{\alpha-1}s + (1 - p^{\alpha-1})a$, $\bar{r} = p^{\alpha-1}r + (1 - p^{\alpha-1})a$, and $\alpha > 0$, then we have:

$$\begin{aligned} & \frac{1}{2} \left({}_a I_{p,q}^{2\alpha} (f(\bar{s}) - f(\bar{r}))(g(\bar{s}) - g(\bar{r})) \right) (p^\alpha b + (1 - p^\alpha)a) \\ &= \frac{p^\alpha (b - a)^\alpha}{\Gamma_{p,q}(\alpha + 1)} \left({}_a I_{p,q}^\alpha (f(\bar{s})g(\bar{s})) \right) (p^\alpha b + (1 - p^\alpha)a) \\ & \quad - \left({}_a I_{p,q}^\alpha (f(\bar{s})) \right) (p^\alpha b + (1 - p^\alpha)a) \left({}_a I_{p,q}^\alpha (g(\bar{s})) \right) (p^\alpha b + (1 - p^\alpha)a). \end{aligned} \tag{48}$$

Proof. Using Definition 3 and applying (18), we have:

$$\begin{aligned} & \left({}_a I_{p,q}^{2\alpha} (f(\bar{s}) - f(\bar{r}))(g(\bar{s}) - g(\bar{r})) \right) (t) \\ &= \frac{1}{p^{(\frac{\alpha}{2})} p^{(\frac{\alpha}{2})} \Gamma_{p,q}^{(2)}(\alpha)} \int_a^t \int_a^t {}_a (t - {}_a \Phi_q(s))_{p,q}^{\alpha-1} {}_a (t - {}_a \Phi_q(r))_{p,q}^{\alpha-1} \\ & \quad \times (f(s)g(s) - f(s)g(r) - f(r)g(s) + f(r)g(r)) {}_a d_{p,q} s {}_a d_{p,q} r \\ &= \frac{(t - a)^\alpha}{\Gamma_{p,q}(\alpha + 1)} \left[\frac{(p - q)(t - a)}{p^{(\frac{\alpha}{2})} \Gamma_{p,q}(\alpha)} \sum_{n=0}^\infty \frac{q^n}{p^{n+1}} {}_a (t - {}_a \Phi_{q/p}^{n+1}(t))_{p,q}^{\alpha-1} \right. \\ & \quad \times f \left(\frac{q^n}{p^{n+1}} t + \left(1 - \frac{q^n}{p^{n+1}} \right) a \right) g \left(\frac{q^n}{p^{n+1}} t + \left(1 - \frac{q^n}{p^{n+1}} \right) a \right) \Big] \\ & \quad - \left[\frac{(p - q)(t - a)}{p^{(\frac{\alpha}{2})} \Gamma_{p,q}(\alpha)} \sum_{n=0}^\infty \frac{q^n}{p^{n+1}} {}_a (t - {}_a \Phi_{q/p}^{n+1}(t))_{p,q}^{\alpha-1} f \left(\frac{q^n}{p^{n+1}} t + \left(1 - \frac{q^n}{p^{n+1}} \right) a \right) \right] \\ & \quad \times \left[\frac{(p - q)(t - a)}{p^{(\frac{\alpha}{2})} \Gamma_{p,q}(\alpha)} \sum_{n=0}^\infty \frac{q^n}{p^{n+1}} {}_a (t - {}_a \Phi_{q/p}^{n+1}(t))_{p,q}^{\alpha-1} g \left(\frac{q^n}{p^{n+1}} t + \left(1 - \frac{q^n}{p^{n+1}} \right) a \right) \right] \\ & \quad - \left[\frac{(p - q)(t - a)}{p^{(\frac{\alpha}{2})} \Gamma_{p,q}(\alpha)} \sum_{n=0}^\infty \frac{q^n}{p^{n+1}} {}_a (t - {}_a \Phi_{q/p}^{n+1}(t))_{p,q}^{\alpha-1} f \left(\frac{q^n}{p^{n+1}} t + \left(1 - \frac{q^n}{p^{n+1}} \right) a \right) \right] \\ & \quad \times \left[\frac{(p - q)(t - a)}{p^{(\frac{\alpha}{2})} \Gamma_{p,q}(\alpha)} \sum_{n=0}^\infty \frac{q^n}{p^{n+1}} {}_a (t - {}_a \Phi_{q/p}^{n+1}(t))_{p,q}^{\alpha-1} g \left(\frac{q^n}{p^{n+1}} t + \left(1 - \frac{q^n}{p^{n+1}} \right) a \right) \right] \\ & \quad + \frac{(t - a)^\alpha}{\Gamma_{p,q}(\alpha + 1)} \left[\frac{(p - q)(t - a)}{p^{(\frac{\alpha}{2})} \Gamma_{p,q}(\alpha)} \sum_{n=0}^\infty \frac{q^n}{p^{n+1}} {}_a (t - {}_a \Phi_{q/p}^{n+1}(t))_{p,q}^{\alpha-1} \right. \\ & \quad \times f \left(\frac{q^n}{p^{n+1}} t + \left(1 - \frac{q^n}{p^{n+1}} \right) a \right) g \left(\frac{q^n}{p^{n+1}} t + \left(1 - \frac{q^n}{p^{n+1}} \right) a \right) \Big] \\ &= \frac{2(t - a)^\alpha}{\Gamma_{p,q}(\alpha + 1)} \left[\frac{(p - q)(t - a)}{p^{(\frac{\alpha}{2})} \Gamma_{p,q}(\alpha)} \sum_{n=0}^\infty \frac{q^n}{p^{n+1}} {}_a (t - {}_a \Phi_{q/p}^{n+1}(t))_{p,q}^{\alpha-1} \right. \\ & \quad \times f \left(\frac{q^n}{p^{n+1}} t + \left(1 - \frac{q^n}{p^{n+1}} \right) a \right) g \left(\frac{q^n}{p^{n+1}} t + \left(1 - \frac{q^n}{p^{n+1}} \right) a \right) \Big] \\ & \quad - 2 \left[\frac{(p - q)(t - a)}{p^{(\frac{\alpha}{2})} \Gamma_{p,q}(\alpha)} \sum_{n=0}^\infty \frac{q^n}{p^{n+1}} {}_a (t - {}_a \Phi_{q/p}^{n+1}(t))_{p,q}^{\alpha-1} f \left(\frac{q^n}{p^{n+1}} t + \left(1 - \frac{q^n}{p^{n+1}} \right) a \right) \right] \\ & \quad \times \left[\frac{(p - q)(t - a)}{p^{(\frac{\alpha}{2})} \Gamma_{p,q}(\alpha)} \sum_{n=0}^\infty \frac{q^n}{p^{n+1}} {}_a (t - {}_a \Phi_{q/p}^{n+1}(t))_{p,q}^{\alpha-1} g \left(\frac{q^n}{p^{n+1}} t + \left(1 - \frac{q^n}{p^{n+1}} \right) a \right) \right] \\ &= \frac{2(t - a)^\alpha}{\Gamma_{p,q}(\alpha + 1)} \left[\frac{1}{p^{(\frac{\alpha}{2})} \Gamma_{p,q}(\alpha)} \int_a^t {}_a (t - {}_a \Phi_q(s))_{p,q}^{\alpha-1} f(s)g(s) {}_a d_{p,q} s \right] \\ & \quad - 2 \left[\frac{1}{p^{(\frac{\alpha}{2})} \Gamma_{p,q}(\alpha)} \int_a^t {}_a (t - {}_a \Phi_q(s))_{p,q}^{\alpha-1} f(s) {}_a d_{p,q} s \right] \end{aligned}$$

$$\begin{aligned}
 & \times \left[\frac{1}{p^{(\alpha)} \Gamma_{p,q}(\alpha)} \int_a^t a(t - {}_a\Phi_q(s))_{p,q}^{\alpha-1} g(s) {}_a d_{p,q} s \right] \\
 &= \frac{2(t-a)^\alpha}{\Gamma_{p,q}(\alpha+1)} \left({}_a I_{p,q}^\alpha (f(\bar{s})g(\bar{s})) \right) (t) - 2 \left({}_a I_{p,q}^\alpha (f(\bar{s})) \right) (t) \left({}_a I_{p,q}^\alpha (g(\bar{r})) \right) (t) \\
 &= \frac{p^\alpha (b-a)^\alpha}{\Gamma_{p,q}(\alpha+1)} \left({}_a I_{p,q}^\alpha (f(\bar{s})g(\bar{s})) \right) (p^\alpha b + (1-p^\alpha)a) \\
 &\quad - \left({}_a I_{p,q}^\alpha (f(\bar{s})) \right) (p^\alpha b + (1-p^\alpha)a) \left({}_a I_{p,q}^\alpha (g(\bar{s})) \right) (p^\alpha b + (1-p^\alpha)a).
 \end{aligned}$$

Therefore, the proof is complete. \square

Remark 4. If $p = 1$, then (48) reduces to the fractional q -Korkine equality on $[a, b]$ as:

$$\begin{aligned}
 & \frac{1}{2} \left({}_a I_q^{2\alpha} (f(s) - f(r))(g(s) - g(r)) \right) (b) \\
 &= \frac{(b-a)^\alpha}{\Gamma_q(\alpha+1)} \left({}_a I_q^\alpha (f(s)g(s)) \right) (b) - \left({}_a I_q^\alpha (f(s)) \right) (b) \left({}_a I_q^\alpha (g(s)) \right) (b),
 \end{aligned}$$

which appeared in [39]. Moreover, if $\alpha = 1$, then (48) reduces to the q -Korkine equality as:

$$\begin{aligned}
 & \frac{1}{2} \int_a^b \int_a^b (f(s) - g(r))(g(s) - g(r)) {}_a d_q s {}_a d_q r \\
 &= (b-a) \int_a^b f(s)g(s) {}_a d_q s - \left(\int_a^b f(s) {}_a d_q s \right) \left(\int_a^b g(s) {}_a d_q s \right),
 \end{aligned}$$

which appeared in [27].

- The fractional (p, q) -Cauchy–Bunyakovsky–Schwarz integral inequality on $[a, p^\alpha b + (1-p^\alpha)a]$:

Theorem 7. If $f, g: [a, b] \rightarrow \mathbb{R}$ are continuous functions, $\bar{s} = p^{\alpha-1}s + (1-p^{\alpha-1})a$, $\bar{r} = p^{\beta-1}r + (1-p^{\beta-1})a$, and $\alpha, \beta > 0$, then we have:

$$\begin{aligned}
 & \left| {}_a I_{p,q}^{\alpha+\beta} f(\bar{s}, \bar{r}) g(\bar{s}, \bar{r}) (p^\alpha b + (1-p^\alpha)a) \right| \\
 & \leq \sqrt{\left({}_a I_{p,q}^{\alpha+\beta} f^2(\bar{s}, \bar{r}) \right) (p^\alpha b + (1-p^\alpha)a)} \sqrt{\left({}_a I_{p,q}^{\alpha+\beta} g^2(\bar{s}, \bar{r}) \right) (p^\alpha b + (1-p^\alpha)a)}. \tag{49}
 \end{aligned}$$

Proof. Using Theorem 3 and Definition 3, we have:

$$\begin{aligned}
 & \left| {}_a I_{p,q}^{\alpha+\beta} f(\bar{s}, \bar{r}) g(\bar{s}, \bar{r}) (t) \right| \\
 &= \frac{1}{p^{(\alpha)} p^{(\beta)} \Gamma_{p,q}^2(\alpha)} \int_a^t \int_a^t a(t - {}_a\Phi_q(s))_{p,q}^{\alpha-1} a(t - {}_a\Phi_q(r))_{p,q}^{\beta-1} f(s, r) {}_a d_{p,q} s {}_a d_{p,q} r \\
 &= \frac{(p-q)^2 (t-a)^2}{p^{(\alpha)} p^{(\beta)} \Gamma_{p,q}(\alpha) \Gamma_{p,q}(\beta)} \sum_{i=0}^\infty \sum_{j=0}^\infty \frac{q^{i+j}}{p^{i+j+2}} a \left(t - {}_a\Phi_{q/p}^{i+1}(t) \right)_{p,q}^{\alpha-1} a \left(t - {}_a\Phi_{q/p}^{j+1}(t) \right)_{p,q}^{\beta-1} \\
 & \quad \times f \left(\frac{q^i}{p^{i+1}} t + \left(1 - \frac{q^i}{p^{i+1}} \right) a, \frac{q^j}{p^{j+1}} t + \left(1 - \frac{q^j}{p^{j+1}} \right) a \right).
 \end{aligned}$$

Applying the classical discrete Cauchy–Schwarz inequality, we have:

$$\left[\left({}_a I_{p,q}^{\alpha+\beta} f(\bar{s}, \bar{r}) g(\bar{s}, \bar{r}) \right) (t) \right]^2$$

$$\begin{aligned}
 &= \left[\frac{(p-q)^2(t-a)^2}{p^{\binom{\alpha}{2}} p^{\binom{\beta}{2}} \Gamma_{p,q}(\alpha) \Gamma_{p,q}(\beta)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{q^{i+j}}{p^{i+j+2}} a \left(t - {}_a\Phi_{q/p}^{i+1}(t)\right)_{p,q}^{\alpha-1} \left(t - {}_a\Phi_{q/p}^{j+1}(t)\right)_{p,q}^{\beta-1} \right. \\
 &\quad \times f\left(\frac{q^i}{p^{i+1}}t + \left(1 - \frac{q^i}{p^{i+1}}\right)a, \frac{q^j}{p^{j+1}}t + \left(1 - \frac{q^j}{p^{j+1}}\right)a\right) \\
 &\quad \left. \times g\left(\frac{q^i}{p^{i+1}}t + \left(1 - \frac{q^i}{p^{i+1}}\right)a, \frac{q^j}{p^{j+1}}t + \left(1 - \frac{q^j}{p^{j+1}}\right)a\right) \right]^2 \\
 &\leq \left(\frac{(p-q)^2(t-a)^2}{p^{\binom{\alpha}{2}} p^{\binom{\beta}{2}} \Gamma_{p,q}(\alpha) \Gamma_{p,q}(\beta)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{q^{i+j}}{p^{i+j+2}} a \left(t - {}_a\Phi_{q/p}^{i+1}(t)\right)_{p,q}^{\alpha-1} \left(t - {}_a\Phi_{q/p}^{j+1}(t)\right)_{p,q}^{\beta-1} \right. \\
 &\quad \times f^2\left(\frac{q^i}{p^{i+1}}t + \left(1 - \frac{q^i}{p^{i+1}}\right)a, \frac{q^j}{p^{j+1}}t + \left(1 - \frac{q^j}{p^{j+1}}\right)a\right) \\
 &\quad \times \left(\frac{(p-q)^2(t-a)^2}{p^{\binom{\alpha}{2}} p^{\binom{\beta}{2}} \Gamma_{p,q}(\alpha) \Gamma_{p,q}(\beta)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{q^{i+j}}{p^{i+j+2}} a \left(t - {}_a\Phi_{q/p}^{i+1}(t)\right)_{p,q}^{\alpha-1} \left(t - {}_a\Phi_{q/p}^{j+1}(t)\right)_{p,q}^{\beta-1} \right. \\
 &\quad \left. \times g^2\left(\frac{q^i}{p^{i+1}}t + \left(1 - \frac{q^i}{p^{i+1}}\right)a, \frac{q^j}{p^{j+1}}t + \left(1 - \frac{q^j}{p^{j+1}}\right)a\right) \right) \\
 &= \left({}_aI_{p,q}^{\alpha+\beta} f^2(\bar{s}, \bar{r})\right)(t) \left({}_aI_{p,q}^{\alpha+\beta} f^2(\bar{s}, \bar{r})\right)(t) \\
 &= \left({}_aI_{p,q}^{\alpha+\beta} f^2(\bar{s}, \bar{r})\right)(p^\alpha b + (1 - p^\alpha)a) \left({}_aI_{p,q}^{\alpha+\beta} f^2(\bar{s}, \bar{r})\right)(p^\alpha b + (1 - p^\alpha)a).
 \end{aligned}$$

Consequently, the proof of (49) is finished. \square

Remark 5. If $p = 1$, then (49) reduces to the fractional q -Cauchy–Bunyakovsky–Schwarz integral inequality on $[a, b]$ as:

$$\left| {}_aI_q^{\alpha+\beta} f(s, r)g(s, r)(b) \right| \leq \sqrt{\left({}_aI_q^{\alpha+\beta} f^2(s, r)\right)(b)} \sqrt{\left({}_aI_q^{\alpha+\beta} g^2(s, r)\right)(b)},$$

which appeared in [39]. Moreover, if $\alpha = 1$, then (49) reduces to the q -Cauchy–Bunyakovsky–Schwarz integral inequality as:

$$\left| \int_a^b \int_a^b f(s, r)g(s, r) {}_a d_q s {}_a d_q r \right| \leq \left[\int_a^b \int_a^b f^2(s, r) {}_a d_q s {}_a d_q r \right]^{1/2} \left[\int_a^b \int_a^b g^2(s, r) {}_a d_q s {}_a d_q r \right]^{1/2},$$

which can be found in [27].

- The fractional (p, q) -Grüss integral inequality on $[a, p^\alpha b + (1 - p^\alpha)a]$:

Theorem 8. If $f, g : [a, b] \rightarrow \mathbb{R}$ are continuous functions satisfying:

$$\phi \leq f(s) \leq \Phi \quad \text{and} \quad \psi \leq g(s) \leq \Psi, \tag{50}$$

for all $s \in [a, b]$, $\phi, \Phi, \psi, \Psi \in \mathbb{R}$, $\bar{s} = p^{\alpha-1}s + (1 - p^{\alpha-1})a$, $\bar{r} = p^{\alpha-1}r + (1 - p^{\alpha-1})a$, and $\alpha > 0$, then we have:

$$\begin{aligned}
 &\left| \frac{\Gamma_{p,q}(\alpha+1)}{p^\alpha(b-a)^\alpha} \left({}_aI_{p,q}^\alpha f(\bar{s})g(\bar{s})\right)(p^\alpha b + (1 - p^\alpha)a) - \frac{\Gamma_{p,q}(\alpha+1)}{p^\alpha(b-a)^\alpha} \left({}_aI_{p,q}^\alpha f(\bar{s})\right)(p^\alpha b + (1 - p^\alpha)a) \right. \\
 &\quad \left. \times \frac{\Gamma_{p,q}(\alpha+1)}{p^\alpha(b-a)^\alpha} \left({}_aI_{p,q}^\alpha g(\bar{r})\right)(p^\alpha b + (1 - p^\alpha)a) \right| \leq \frac{1}{4}(\Phi - \phi)(\Psi - \psi). \tag{51}
 \end{aligned}$$

Proof. Applying Theorem 7, we have:

$$\begin{aligned} \left| \left({}_a I_{p,q}^{2\alpha} (f(\bar{s}) - f(\bar{r})) (g(\bar{s}) - g(\bar{r})) \right) (t) \right| &\leq \sqrt{\left({}_a I_{p,q}^{2\alpha} (f(\bar{s}) - f(\bar{r}))^2 \right) (t)} \\ &\quad \times \sqrt{\left({}_a I_{p,q}^{2\alpha} (g(\bar{s}) - g(\bar{r}))^2 \right) (t)}. \end{aligned} \tag{52}$$

By Theorem 6, we obtain:

$$\begin{aligned} &\frac{1}{2} \left(\frac{\Gamma_{p,q}(\alpha + 1)}{(t - a)^\alpha} \right)^2 \left({}_a I_{p,q}^{2\alpha} (f(\bar{s}) - f(\bar{r}))^2 \right) (t) \\ &= \frac{\Gamma_{p,q}(\alpha + 1)}{(t - a)^\alpha} \left({}_a I_{p,q}^\alpha f^2(\bar{s}) \right) (t) - \left(\frac{\Gamma_{p,q}(\alpha + 1)}{(t - a)^\alpha} \left({}_a I_{p,q}^\alpha f(\bar{r}) \right) (t) \right)^2. \end{aligned} \tag{53}$$

On the other hand, we have:

$$\begin{aligned} &\frac{\Gamma_{p,q}(\alpha + 1)}{(t - a)^\alpha} \left({}_a I_{p,q}^\alpha f^2(\bar{s}) \right) (t) - \left(\frac{\Gamma_{p,q}(\alpha + 1)}{(t - a)^\alpha} \left({}_a I_{p,q}^\alpha f(\bar{r}) \right) (t) \right)^2 \\ &= \left(\Phi - \frac{\Gamma_{p,q}(\alpha + 1)}{(t - a)^\alpha} \left({}_a I_{p,q}^\alpha f(\bar{s}) \right) (t) \right) \left(\frac{\Gamma_{p,q}(\alpha + 1)}{(t - a)^\alpha} \left({}_a I_{p,q}^\alpha f(\bar{s}) \right) (t) - \Phi \right) \\ &\quad - \frac{\Gamma_{p,q}(\alpha + 1)}{(t - a)^\alpha} \left({}_a I_{p,q}^\alpha (f(\bar{s}) - \Phi) (\Phi - f(\bar{s})) \right) (t). \end{aligned} \tag{54}$$

From (50), we have $(f(s) - \Phi)(\Phi - f(s)) \geq 0$. It follows that:

$$\left({}_a I_{p,q}^\alpha (f(\bar{s}) - \Phi) (\Phi - f(\bar{s})) \right) (t) \geq 0.$$

Applying (52) and using the truth that $\left(\frac{K+K'}{2} \right)^2 \geq KK'$, where $K, K' \in \mathbb{R}$, we have:

$$\begin{aligned} &\frac{\Gamma_{p,q}(\alpha + 1)}{(b - a)^\alpha} \left({}_a I_{p,q}^\alpha f^2(\bar{s}) \right) (t) - \left(\frac{\Gamma_{p,q}(\alpha + 1)}{(t - a)^\alpha} \left({}_a I_{p,q}^\alpha f(\bar{r}) \right) (b) \right)^2 \\ &\leq \left(\Phi - \frac{\Gamma_{p,q}(\alpha + 1)}{(t - a)^\alpha} \left({}_a I_{p,q}^\alpha f(\bar{s}) \right) (t) \right) \left(\frac{\Gamma_{p,q}(\alpha + 1)}{(t - a)^\alpha} \left({}_a I_{p,q}^\alpha f(\bar{s}) \right) (t) - \Phi \right) \\ &\leq \frac{1}{4} \left[\left(\Phi - \frac{\Gamma_{p,q}(\alpha + 1)}{(t - a)^\alpha} \left({}_a I_{p,q}^\alpha f(\bar{s}) \right) (t) \right) + \left(\frac{\Gamma_{p,q}(\alpha + 1)}{(t - a)^\alpha} \left({}_a I_{p,q}^\alpha f(\bar{s}) \right) (t) - \Phi \right) \right]^2 \\ &= \frac{1}{4} (\Phi - \Phi)^2. \end{aligned} \tag{55}$$

Similarly,

$$\frac{1}{2} \left(\frac{\Gamma_{p,q}(\alpha + 1)}{(t - a)^\alpha} \right)^2 \left({}_a I_{p,q}^{2\alpha} (g(\bar{s}) - g(\bar{r}))^2 \right) (t) \leq \frac{1}{4} (\Psi - \psi)^2. \tag{56}$$

From (50), it follows that:

$$\begin{aligned} &\left| \left({}_a I_{p,q}^{2\alpha} (f(\bar{s}) - f(\bar{r})) (g(\bar{s}) - g(\bar{r})) \right) (t) \right| \leq \frac{1}{2} \left(\frac{(t - a)^\alpha}{\Gamma_{p,q}(\alpha + 1)} \right)^2 (\Phi - \Phi)(\Psi - \psi) \\ &= \left| \left({}_a I_{p,q}^{2\alpha} (f(\bar{s}) - f(\bar{r})) (g(\bar{s}) - g(\bar{r})) \right) (p^\alpha b + (1 - p^\alpha) a) \right| \leq \frac{1}{2} \left(\frac{p^\alpha (b - a)^\alpha}{\Gamma_{p,q}(\alpha + 1)} \right)^2 (\Phi - \Phi)(\Psi - \psi) \end{aligned} \tag{57}$$

Applying Theorem 6 again for (57), we have that Theorem 8 holds. \square

Remark 6. If $p = 1$, then (51) reduces to the fractional q -Grüss integral inequality as:

$$\left| \frac{\Gamma_q(\alpha + 1)}{(b - a)^\alpha} \left({}_a I_q^\alpha f(s)g(s) \right)(b) - \left(\frac{\Gamma_q(\alpha + 1)}{(b - a)^\alpha} \left({}_a I_q^\alpha f(s) \right)(b) \right) \right. \\ \left. \times \left(\frac{\Gamma_q(\alpha + 1)}{(b - a)^\alpha} \left({}_a I_q^\alpha g(s) \right)(b) \right) \right| \leq \frac{1}{4}(\Phi - \phi)(\Psi - \psi),$$

which appeared in [39]. Moreover, if $\alpha = 1$, $p = 1$, and $q \rightarrow 1$, then (51) reduces to the classical Grüss integral as:

$$\left| \frac{1}{b - a} \int_a^b f(s)g(s)ds - \left(\frac{1}{b - a} \int_a^b f(s)ds \right) \left(\frac{1}{b - a} \int_a^b g(s)ds \right) \right| \leq \frac{1}{4}(\Phi - \phi)(\Psi - \psi),$$

which can be found in [72,73].

- The fractional (p, q) -Grüss–Chebyshev integral inequality on $[a, p^\alpha b + (1 - p^\alpha)a]$:

Theorem 9. If $f, g : [a, b] \rightarrow \mathbb{R}$ are L_1 -, L_2 -Lipschitzian continuous functions such that:

$$|f(s) - f(r)| \leq L_1|s - r|, \quad |g(s) - g(r)| \leq L_2|s - r|, \quad \text{for all } s, r \in [a, b], \quad (58)$$

$\bar{s} = p^{\alpha-1}s + (1 - p^{\alpha-1})a$, $\bar{r} = p^{\alpha-1}r + (1 - p^{\alpha-1})a$, $L_1, L_2 \geq 0$, and $\alpha \geq 0$, then we have:

$$\left| \frac{p^\alpha(b - a)^\alpha}{\Gamma_{p,q}(\alpha + 1)} \left({}_a I_{p,q}^\alpha (f(\bar{s})g(\bar{s})) \right)(p^\alpha b + (1 - p^\alpha)a) \right. \\ \left. - \left({}_a I_{p,q}^\alpha (f(\bar{s})) \right)(p^\alpha b + (1 - p^\alpha)a) \left({}_a I_{p,q}^\alpha (g(\bar{s})) \right)(p^\alpha b + (1 - p^\alpha)a) \right| \\ \leq \frac{L_1 L_2 p^{2\alpha+2} (b - a)^{2\alpha+2} p^{2(\alpha-1)}}{\Gamma_{p,q}(\alpha + 2) \Gamma_{p,q}(\alpha + 3)} ((p + q)[\alpha + 1]_{p,q} - [\alpha + 2]_{p,q}). \quad (59)$$

Proof. By Theorem 6, we have:

$$\frac{(t - a)^\alpha}{\Gamma_{p,q}(\alpha + 1)} \left({}_a I_{p,q}^\alpha (f(\bar{s})g(\bar{s})) \right)(t) - \left({}_a I_{p,q}^\alpha (f(\bar{s})) \right)(t) \left({}_a I_{p,q}^\alpha (g(\bar{s})) \right)(t) \\ = \frac{1}{2} \left({}_a I_{p,q}^{2\alpha} (f(\bar{s}) - f(\bar{r}))(g(\bar{s}) - g(\bar{r})) \right)(t). \quad (60)$$

From the condition in (58), we obtain:

$$|(f(s) - f(r))(g(s) - g(r))| \leq L_1 L_2 (s - r)^2, \quad \text{for all } s, r \in [a, b]. \quad (61)$$

Taking double fractional (p, q) -integration of order $\alpha > 0$ with respect to $s, r \in [a, t]$ with $t = p^\alpha b + (1 - p^\alpha)a$, we have:

$$\left({}_a I_{p,q}^{2\alpha} (f(\bar{s}) - f(\bar{r}))(g(\bar{s}) - g(\bar{r})) \right)(t) \\ = \frac{1}{p^{(\frac{\alpha}{2})} p^{(\frac{\alpha}{2})} \Gamma_{p,q}^2(\alpha)} \int_a^t \int_a^t a(t - {}_a \Phi_q(s))^{\alpha-1} a(t - {}_a \Phi_q(r))^{\alpha-1} \\ \times (f(s)g(s) - f(s)g(r) - f(r)g(s) + f(s)g(s)) {}_a d_{p,q} s {}_a d_{p,q} r \\ \leq \frac{L_1 L_2}{p^{(\frac{\alpha}{2})} p^{(\frac{\alpha}{2})} \Gamma_{p,q}^2(\alpha)} \int_a^t \int_a^t a(t - {}_a \Phi_q(s))^{\alpha-1} a(t - {}_a \Phi_q(r))^{\alpha-1} (s - r)^2 {}_a d_{p,q} s {}_a d_{p,q} r \\ = \frac{L_1 L_2}{p^{(\frac{\alpha}{2})} p^{(\frac{\alpha}{2})} \Gamma_{p,q}^2(\alpha)} \int_a^t \int_a^t a(t - {}_a \Phi_q(s))^{\alpha-1} a(t - {}_a \Phi_q(r))^{\alpha-1} s^2 {}_a d_{p,q} s {}_a d_{p,q} r \\ - \frac{2L_1 L_2}{p^{(\frac{\alpha}{2})} p^{(\frac{\alpha}{2})} \Gamma_{p,q}^2(\alpha)} \int_a^t \int_a^t a(t - {}_a \Phi_q(s))^{\alpha-1} a(t - {}_a \Phi_q(r))^{\alpha-1} sr {}_a d_{p,q} s {}_a d_{p,q} r$$

$$\begin{aligned}
& + \frac{L_1 L_2}{p^{(2)} p^{(\alpha)} \Gamma_{p,q}^2(\alpha)} \int_a^t \int_a^t {}_a(t - {}_a\Phi_q(s))^{\alpha-1} {}_a(t - {}_a\Phi_q(r))^{\alpha-1} r^2 {}_a d_{p,q} s {}_a d_{p,q} r \\
& = 2L_1 L_2 \left(\frac{(t-a)^\alpha}{\Gamma_{p,q}(\alpha+1)} ({}_a I_{p,q}^\alpha \bar{s}^2)(t) - \left(({}_a I_{p,q}^\alpha \bar{s})(t) \right)^2 \right). \tag{62}
\end{aligned}$$

From Remark 1, we get:

$$({}_a I_{p,q}^\alpha \bar{s})(t) = \frac{(t-a)^\alpha}{\Gamma_{p,q}(\alpha+2)} \left[p^{\alpha-1}(t-a) + a[\alpha+1]_{p,q} \right]$$

and:

$$({}_a I_{p,q}^\alpha (\bar{s}^2))(t) = \frac{(t-a)^\alpha}{\Gamma_{p,q}(\alpha+3)} \left[p^{2(\alpha-1)}(p+q)(t-a)^2 + 2ap^{\alpha-1}(t-a)[\alpha+2]_{p,q} + a^2[\alpha+2]_{p,q}[\alpha+1]_{p,q} \right].$$

Moreover, we have:

$$\begin{aligned}
& \frac{(t-a)^\alpha}{\Gamma_{p,q}(\alpha+1)} ({}_a I_{p,q}^\alpha \bar{s}^2)(t) - \left(({}_a I_{p,q}^\alpha \bar{s})(t) \right)^2 \\
& = \frac{(t-a)^{2\alpha}}{\Gamma_{p,q}(\alpha+1)\Gamma_{p,q}(\alpha+3)} \left(p^{2(\alpha-1)}(p+q)(t-a)^2 + 2ap^{\alpha-1}(t-a)[\alpha+2]_{p,q} + a^2[\alpha+2]_{p,q}[\alpha+1]_{p,q} \right) \\
& \quad - \frac{(t-a)^{2\alpha}}{\Gamma_{p,q}^2(\alpha+2)} \left(p^{\alpha-1}(t-a) + a[\alpha+1]_{p,q} \right)^2 \\
& = \frac{(t-a)^{2\alpha+2} p^{2(\alpha-1)}}{\Gamma_{p,q}(\alpha+2)\Gamma_{p,q}(\alpha+3)} \left((p+q)[\alpha+1]_{p,q} - [\alpha+2]_{p,q} \right) \\
& = \frac{p^{2\alpha+2}(b-a)^{2\alpha+2} p^{2(\alpha-1)}}{\Gamma_{p,q}(\alpha+2)\Gamma_{p,q}(\alpha+3)} \left((p+q)[\alpha+1]_{p,q} - [\alpha+2]_{p,q} \right). \tag{63}
\end{aligned}$$

Substituting (63) into (62) and applying to (60), we complete the proof. \square

Remark 7. If $p = 1$, then (59) reduces to the fractional q -Grüss–Chebyshev integral inequality on $[a, b]$ as:

$$\begin{aligned}
& \left| \frac{(b-a)^\alpha}{\Gamma_q(\alpha+1)} ({}_a I_q^\alpha (f(s)g(s)))(b) - ({}_a I_q^\alpha (f(s)))(b) ({}_a I_q^\alpha (g(s)))(b) \right| \\
& \leq \frac{L_1 L_2 (b-a)^{2\alpha+2}}{\Gamma_q(\alpha+2)\Gamma_q(\alpha+3)} \left((1+q)[\alpha+1]_q - [\alpha+2]_q \right),
\end{aligned}$$

which appeared in [39]. Moreover, if $\alpha = 1$, $p = 1$ and $q \rightarrow 1$, then (59) reduces to the classical Grüss–Chebyshev integral as:

$$\left| \frac{1}{b-a} \int_a^b f(s)g(s)ds - \left(\frac{1}{b-a} \int_a^b f(s)ds \right) \left(\frac{1}{b-a} \int_a^b g(s)ds \right) \right| \leq \frac{L_1 L_2}{12} (b-a)^2,$$

which can be found in [72,73].

- The fractional (p, q) -Polya–Szeqö integral inequality on $[a, p^\alpha b + (1-p^\alpha)a]$:

Theorem 10. If $f, g : [a, b] \rightarrow \mathbb{R}$ are two positive integrable functions satisfying:

$$0 < \phi \leq f(s) \leq \Phi \quad \text{and} \quad 0 < \psi \leq g(s) \leq \Psi, \tag{64}$$

for all $s \in [a, b]$, $\phi, \Phi, \psi, \Psi \in \mathbb{R}^+$, and $\alpha > 0$, then we have:

$$\frac{({}_a I_{p,q}^\alpha f^2(p^{\alpha-1}s + (1-p^{\alpha-1})a))(p^\alpha b + (1-p^\alpha)a) ({}_a I_{p,q}^\alpha g^2(p^{\alpha-1}s + (1-p^{\alpha-1})a))(p^\alpha b + (1-p^\alpha)a)}{\left(({}_a I_{p,q}^\alpha f(p^{\alpha-1}s + (1-p^{\alpha-1})a)g(p^{\alpha-1}s + (1-p^{\alpha-1})a))(p^\alpha b + (1-p^\alpha)a) \right)^2}$$

$$\leq \frac{1}{4} \left(\sqrt{\frac{\phi\psi}{\Phi\Psi}} + \sqrt{\frac{\Phi\Psi}{\phi\psi}} \right). \tag{65}$$

Proof. For $s \in [a, b]$, from (64), we obtain:

$$\frac{\phi}{\Psi} \leq \frac{f(s)}{g(s)} \leq \frac{\Phi}{\psi},$$

which yields:

$$\left(\frac{\Phi}{\psi} - \frac{f(s)}{g(s)} \right) \geq 0 \tag{66}$$

and:

$$\left(\frac{f(s)}{g(s)} - \frac{\phi}{\Psi} \right) \geq 0. \tag{67}$$

Multiplying (66) and (67), we get:

$$\left(\frac{\Phi}{\psi} - \frac{f(s)}{g(s)} \right) \left(\frac{f(s)}{g(s)} - \frac{\phi}{\Psi} \right) \geq 0 \tag{68}$$

or:

$$\left(\frac{\Phi}{\psi} + \frac{\phi}{\Psi} \right) \frac{f(s)}{g(s)} \geq \frac{f^2(s)}{g^2(s)} + \frac{\phi\Phi}{\psi\Psi}. \tag{69}$$

Therefore, (69) can be written as:

$$(\phi\psi + \Phi\Psi)f(s)g(s) \geq \psi\Psi f^2(s) + \phi\Phi g^2(s). \tag{70}$$

Multiplying ${}_a(b - {}_a\Phi_{p,q}(s))_q^{\alpha-1} / p^{(\alpha)}\Gamma_{p,q}(\alpha)$ and integrating of order $\alpha \geq 0$ on both sides of (70), where $s \in [a, t]$ with $t = p^\alpha b + (1 - p^\alpha)a$, we obtain:

$$\begin{aligned} & (\phi\psi + \Phi\Psi) \left({}_aI_{p,q}^\alpha f \left(p^{\alpha-1}s + (1 - p^{\alpha-1})a \right) g \left(p^{\alpha-1}s + (1 - p^{\alpha-1})a \right) \right) (t) \\ & \geq \psi\Psi \left({}_aI_{p,q}^\alpha f^2 \left(p^{\alpha-1}s + (1 - p^{\alpha-1})a \right) \right) (t) + \phi\Phi \left({}_aI_{p,q}^\alpha g^2 \left(p^{\alpha-1}s + (1 - p^{\alpha-1})a \right) \right) (t). \end{aligned}$$

Applying the AM-GM inequality, we get:

$$\begin{aligned} & (\phi\psi + \Phi\Psi) \left({}_aI_{p,q}^\alpha f \left(p^{\alpha-1}s + (1 - p^{\alpha-1})a \right) g \left(p^{\alpha-1}s + (1 - p^{\alpha-1})a \right) \right) (t) \\ & \geq 2\sqrt{\phi\psi\Phi\Psi \left({}_aI_{p,q}^\alpha f^2 \left(p^{\alpha-1}s + (1 - p^{\alpha-1})a \right) \right) (t) \left({}_aI_{p,q}^\alpha g^2 \left(p^{\alpha-1}s + (1 - p^{\alpha-1})a \right) \right) (t)}, \end{aligned}$$

which turns into:

$$\begin{aligned} & \phi\psi\Phi\Psi \left({}_aI_{p,q}^\alpha f^2 \left(p^{\alpha-1}s + (1 - p^{\alpha-1})a \right) \right) (t) \left({}_aI_{p,q}^\alpha g^2 \left(p^{\alpha-1}s + (1 - p^{\alpha-1})a \right) \right) (t) \\ & \leq \frac{1}{4} \left((\phi\psi + \Phi\Psi) \left({}_aI_{p,q}^\alpha f \left(p^{\alpha-1}s + (1 - p^{\alpha-1})a \right) \right) (t) \right)^2. \end{aligned}$$

Therefore, Theorem 10 is proven. \square

Remark 8. If $p = 1$, then (65) reduces to the fractional q -Polya–Szeqö integral inequality on $[a, b]$ as:

$$\frac{\left({}_aI_q^\alpha f^2(s) \right) (b) \left({}_aI_q^\alpha g^2(s) \right) (b)}{\left(\left({}_aI_q^\alpha f(s)g(s) \right) (b) \right)^2} \leq \frac{1}{4} \left(\sqrt{\frac{\phi\psi}{\Phi\Psi}} + \sqrt{\frac{\Phi\Psi}{\phi\psi}} \right),$$

which appeared in [39]. Moreover, if $\alpha = 1$, $p = 1$, and $q \rightarrow 1$, then (65) reduces to the classical Polya–Szeqö integral inequality as:

$$\frac{\int_a^b f^2(s)ds \int_a^b g^2(s)ds}{\left(\int_a^b f(s)g(s)ds\right)^2} \leq \frac{1}{4}(\phi\psi + \Phi\Psi)^2,$$

which can be found in [74].

4. Conclusions

In this work, we studied the fractional (p, q) -calculus on finite intervals. We also gave some of its significant properties. Furthermore, we proved some fractional (p, q) -integral inequalities on finite intervals. For the ideas, as well as the techniques of this paper, we hope that it will inspire interested readers working in this field.

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