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Graphs with Minimal Strength

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Abstract: For any graph G of order p, a bijection $f : V(G) \to \{1, 2, \ldots, p\}$ is called a numbering of G. The strength $str_f(G)$ of a numbering f of G is defined by $str_f(G) = \max\{f(u) + f(v) \mid uv \in E(G)\}$, and the strength $str(G)$ of a graph G is $str(G) = \min\{str_f(G) \mid f \text{ is a numbering of } G\}$. In this paper, many open problems are solved, and the strengths of new families of graphs are determined.

Keywords: strength; minimum degree; δ-sequence; independence number; 2-regular

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1. Introduction

We only consider simple and loopless graph $G = (V, E)$ with order $|V(G)| = p$ and size $|E(G)| = q$. If $uv \in E(G)$, we say that v is a neighbor of u. The degree of a vertex v in a graph G is the number of neighbors of v in G, denoted as $\deg_G(v)$. The minimum degree (and maximum degree) of G is the minimum (and maximum) degree among the vertices of G, denoted as $\delta(G)$ (and $\Delta(G)$). A vertex of degree 0 is called an isolated vertex and a vertex of degree 1 is called a pendant vertex; its incident edge is called a pendant edge. The set of all neighbors of u is denoted as $N_G(u)$. For $S \subset V(G)$, let $N_G(S)$ be the set of all neighbors of the vertices in S. We shall drop the subscript G if there is no ambiguity. For $a < b$, the set of integers from a to b is denoted as $[a, b]_{\mathbb{Z}}$. For notations not defined in this paper, we refer to [1].

The magic square [2] is one of the oldest problems on symmetry [3]. It was later extended to the idea of magic graphs [4], super magic graphs [5], and super edge-magic graphs [6]. The notion of the strength of a graph G was then introduced by Ichishima et al. [7] as a generalization of super magic strength [8], which is effectively defined only for super edge-magic graphs (also called strong vertex-graceful [9] and strongly indexable [10]), to any nonempty graphs as follows.

A bijection $f : V(G) \to [1, p]_{\mathbb{Z}}$ is called a numbering of the graph G of order p.

Definition 1. *The strength $str_f(G)$ of a numbering $f : V(G) \to [1, p]_{\mathbb{Z}}$ of G is defined by*

$$str_f(G) = \max\{f(u) + f(v) \mid uv \in E(G)\},$$

and the strength $str(G)$ of a graph G itself is

$$str(G) = \min\{str_f(G) \mid f \text{ is a numbering of } G\}.$$

A numbering f is called a strength labeling of G if $str_f(G) = str(G)$.

The super magic strength, $sm(G)$, of a graph G is defined as the minimum of all magic constants over all super edge-magic labelings of G. A necessary and sufficient condition for a graph to be super edge-magic [11] then gives rise to the concept of the consecutive strength labeling of a graph, which is equivalent to super edge-magic labeling. In [7] (Section 4), Ichishima et al. showed that a graph G of order p has $sm(G) \geq str(G) + p + 1$. Several new lower bounds of $sm(G)$ in terms of other graph parameters were also obtained. Moreover, all of the bounds are sharp. Thus, one may use the strength of graphs to establish the super magic strength of the corresponding graphs.

Several lower and upper bounds for $str(G)$ were obtained in [7]. The following two are needed in what follows.

Lemma 1. *If H is a subgraph of a graph G, then*

$$str(H) \leq str(G).$$

Lemma 2. *For every graph G of order p with $\delta(G) \geq 1$,*

$$str(G) \geq p + \delta(G).$$

Let $G + H$ be the disjoint union of G and H with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H)$. The disjoint union of m copies of G is denoted as mG. We first extend Lemma 2 to graphs with isolated vertices.

Lemma 3. *Let G be a graph with $\delta(G) \geq 1$. If $m \geq 1$, then*

$$str(G + mK_1) = str(G).$$

Proof. Let f be a strength labeling of the graph G. We extend f to a numbering of $G + mK_1$ by assigning all m isolated vertices by labels in $[p + 1, p + m]_{\mathbb{Z}}$. Clearly, $str_f(G + mK_1) = str_f(G) = str(G)$. Hence $str(G + mK_1) \leq str(G)$. Combining with Lemma 1, we have the lemma. □

Thus, from now on, we only consider graphs without isolated vertices. In [7,12], the authors showed that $str(G) = p + \delta(G)$ if G is a path, cycle, complete graph, complete bipartite graph, ladder graph, prism graph, Möbius ladder, book graph, or $K_{m,n} \times K_2$, each of which has order p. Moreover, if $str(G) = p + \delta(G)$ for a graph G of order p with $\delta(G) \geq 1$, then $str(G \odot nK_1) = (n + 1)p + 1$, where $G \odot H$ is the corona product of G and H. The following problems are posed.

Problem 1. *Find sufficient conditions for a graph G of order p with $\delta(G) \geq 1$ to ensure $str(G) = p + \delta(G)$.*

Problem 2. *Find good bounds for the strength of a graph.*

Problem 3. *For every lobster T, determine the exact value of $str(T)$.*

Problem 4. *For every integer $n \geq 3$, determine the strength of Q_n, the n-dimensional hypercube.*

In this paper, we obtained a sufficient condition for a graph to have $str(G) = |V(G)| + \delta(G)$. Moreover, we showed that every graph G either has $str(G) = |V(G)| + \delta(G)$ or is a proper subgraph of a graph H that has $str(H) = |V(H)| + \delta(H)$ with $\delta(H) = \delta(G)$. Further, new good lower bounds of $str(G)$ are obtained. Consequently, Problems 1 to 3 are solved. Moreover, we completely determine the strength of 2-regular graphs and give a partial solution to Problem 4.

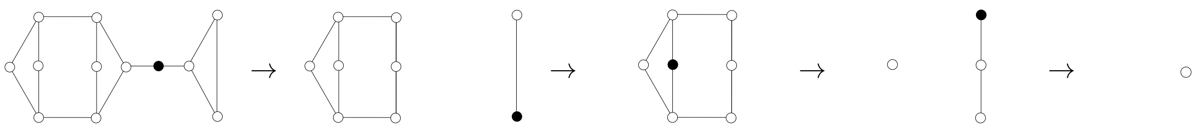
2. Sufficient Condition

Let $\mathcal{G}_1 = G_1 = G$ be a graph of order p with $p - 2 \geq \delta(G) = \delta_1 \geq 1$. Suppose that $\mathcal{G}_i$, $i \geq 1$, is not m_iK_1 for $m_i \geq 1$ nor $m_iK_1 + K_r$ with $m_i \geq 0$ for some $r \geq 2$. We may denote

$\mathcal{G}_i$ by $m_iK_1 + G_i$, where $m_i \geq 0$ and $\delta_i = \delta(G_i) \geq 1$. Let $\mathcal{G}_{i+1}$ be a graph obtained from G_i by deleting the $m_i \geq 1$ isolated vertices that exist in $\mathcal{G}_i$, and a vertex of degree δ_i together with all its neighbors in G_i. Continue the procedure until $\mathcal{G}_s$ is either m_sK_1 with $m_s \geq 1$ or $m_sK_1 + K_r$ with $m_s \geq 0$ for some $r, s \geq 2$. This sequence $\{\mathcal{G}_i\}_{i=1}^{s}$ of subgraphs is called a δ-sequence of G. When $\mathcal{G}_s = m_sK_1$, we let $\delta_s = 0$ by convention. Let $\widetilde{y}_j(G) = m_j + 1 - \delta_j$ for $1 \leq j \leq s$.

Example 1. *The following are two examples to illustrate the above construction. The black vertex is the chosen vertex that will be deleted at each stage.*

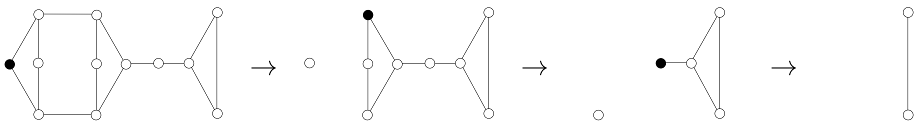
1.



$\mathcal{G}_1 = G_1$, $\delta_1 = 2$; $\mathcal{G}_2 = G_2$, $\delta_2 = 1, m_2 = 0$; $\mathcal{G}_3 = G_3$, $\delta_3 = 2, m_3 = 0$; $\mathcal{G}_4 = K_1 + G_4$, $\delta_4 = 1, m_4 = 1$; $\mathcal{G}_5 = K_1$, $\delta_5 = 0, m_5 = 1$

Now, $(m_2 + 1 - \delta_2) + (m_3 + 1 - \delta_3) = (0) + (-1) = -1$. *So, this δ-sequence of the graph G_1 does not satisfy the condition of* (1) *mentioned below.*

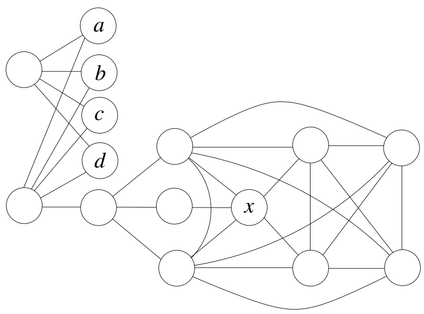
2.



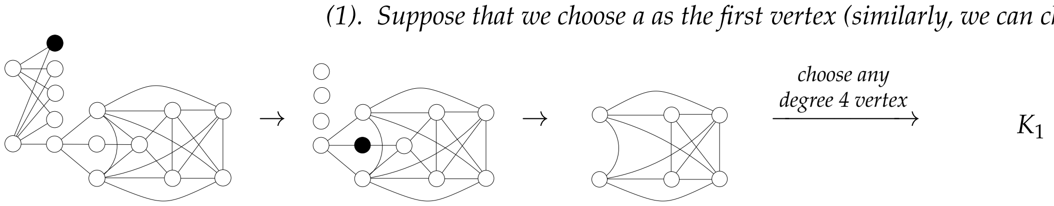
$\mathcal{G}_1 = G_1$, $\delta_1 = 2$; $\mathcal{G}_2 = K_1 + G_2$, $\delta_2 = 2, m_2 = 1$; $\mathcal{G}_3 = K_1 + G_3$, $\delta_3 = 1, m_3 = 1$; $\mathcal{G}_4 = K_2$, $\delta_4 = 1, m_4 = 0$

This δ-sequence of the graph G_1 satisfies the condition of (1).

Example 2. *Consider the following graph G:*



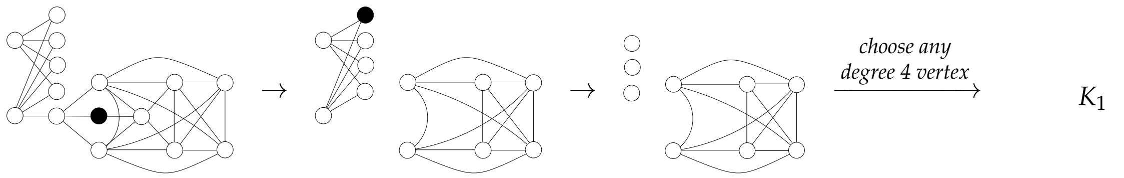
(1). Suppose that we choose a as the first vertex (similarly, we can choose b, c, or d). So, we have



$\mathcal{G}_1 = G$, $\delta_1 = 2$; $\mathcal{G}_2 = 3K_1 + G_2$, $\delta_2 = 2, m_2 = 3$; $\mathcal{G}_3 = G_3$, $\delta_3 = 4, m_3 = 0$; $\mathcal{G}_4 = K_1$, $\delta_4 = 0, m_4 = 1$

Now, $(m_2 + 1 - \delta_2) + (m_3 + 1 - \delta_3) = (2) + (-3) = -1$.

(2). Suppose that we choose x as the first vertex. So, we have



$\mathcal{G}_1 = G$, $\delta_1 = 2$ $\quad$ $\mathcal{G}_2 = G_2$, $\delta_2 = 2, m_2 = 0$ $\quad$ $\mathcal{G}_3 = 3K_3 + G_3$, $\delta_3 = 4, m_3 = 3$ $\quad$ $\mathcal{G}_4 = K_1$, $\delta_4 = 0, m_4 = 1$

Now, $(m_2 + 1 - \delta_2) = -1$.
So, every δ*-sequence of* G *does not satisfy condition* (1).

The following theorem gives a sufficient condition for a graph to have minimal strength.

Theorem 1. *For a graph* G *of order* p *with* $p - 2 \geq \delta(G) = \delta_1 \geq 1$, *if there is a* δ*-sequence* $\{\mathcal{G}_i\}_{i=1}^{s}$ *of* G *such that*

$$\widetilde{z}_i(G) = \sum_{j=2}^{i} \widetilde{y}_j(G) \geq 0 \text{ for } 2 \leq i \leq s, \tag{1}$$

then $str(G) = p + \delta(G)$.

Note that a sum with an empty term is treated as zero, as usual. If there is no ambiguity, we will write $\widetilde{y}_j(G)$ as $\widetilde{y}_j$ and $\widetilde{z}_i(G)$ as $\widetilde{z}_i$.

Proof. Let $\{\mathcal{G}_i\}_{i=1}^{s}$ be a δ-sequence of G satisfying condition (1). Let u_i be a vertex in G_i of degree δ_i, which is deleted from G_i to obtain $\mathcal{G}_{i+1}$, $1 \leq i \leq s-1$. Now, $\mathcal{G}_1 = G_1 = G$. We shall construct a numbering f of G such that $str_f(G) = p + \delta_1$.

Label u_1 by p and all its neighbors by 1 to δ_1 in arbitrary order. This guarantees that the largest induced edge label is $p + \delta_1$ at this stage.

Suppose that we have labeled vertices in $V(G) \setminus V(\mathcal{G}_{i+1})$ by using the labels in $[1, \sum\limits_{j=1}^{i} \delta_j] \cup [p + 1 - \sum\limits_{j=1}^{i}(m_j + 1), p]_{\mathbb{Z}}$, where $1 \leq i \leq s - 1$. Moreover, the neighbors of u_i are labeled with labels in $[1 + \sum\limits_{j=1}^{i-1} \delta_j, \sum\limits_{j=1}^{i} \delta_j]_{\mathbb{Z}}$, and all induced edge labels are at most $p + \delta_1$, up to now. Note that a sum with an empty term is treated as zero.

Now, we consider the graph $\mathcal{G}_{i+1}$.

(a) Suppose that $2 \leq i + 1 < s$. We label the m_{i+1} isolated vertices of $\mathcal{G}_{i+1}$ with labels in $[p + 2 - \sum\limits_{j=1}^{i+1}(m_j + 1), p - \sum\limits_{j=1}^{i}(m_j + 1)]_{\mathbb{Z}}$, respectively (if $m_{i+1} = 0$, then this process does not exist), and u_{i+1} by $p + 1 - \sum\limits_{j=1}^{i+1}(m_j + 1)$ and its neighbors with labels in $[1 + \sum\limits_{j=1}^{i} \delta_j, \sum\limits_{j=1}^{i+1} \delta_j]_{\mathbb{Z}}$, respectively.

Now, the vertices of $V(G) \setminus V(\mathcal{G}_{i+2})$ are labeled by using the labels in $[1, \sum\limits_{j=1}^{i+1} \delta_j] \cup [p + 1 - \sum\limits_{j=1}^{i+1}(m_j + 1), p]_{\mathbb{Z}}$.

Since each isolated vertex of $\mathcal{G}_{i+1}$ is only adjacent to some neighbors of u_i, and u_{i+1} may be adjacent with some neighbors of u_i, the largest new induced edge label related to these vertices is

$$\begin{aligned} p + 1 - \sum_{j=1}^{i+1}(m_j + 1) + \sum_{j=1}^{i} \delta_j &= p + 1 - \sum_{j=2}^{i}(m_j + 1 - \delta_j) - (m_{i+1} + 1) - (m_1 + 1) + \delta_1 \\ &= p - z_i - m_{i+1} - 1 + \delta_1 < p + \delta_1. \qquad \text{(since } m_1 = 0\text{)} \end{aligned}$$

The largest new induced edge label related to u_{i+1} and its neighbors in $\mathcal{G}_{i+1}$ is

$$p+1-\sum_{j=1}^{i+1}(m_j+1)+\sum_{j=1}^{i+1}\delta_j = p+1-\sum_{j=2}^{i+1}(m_j+1-\delta_j)-(m_1+1-\delta_1)$$
$$= p+1-\widetilde{z}_{i+1}-(m_1+1-\delta_1) \le p+1-(1-\delta_1) = p+\delta_1.$$

Repeat this process until $i+1=s$.

(b) Suppose that $i+1=s$. Now, $\mathcal{G}_s = m_sK_1$ with $m_s \ge 1$ or $m_sK_1+K_r$ for some $r \ge 2$ and $m_s \ge 0$. In this case, the set of unused labels is $[1+\sum_{j=1}^{s-1}\delta_j, p-\sum_{j=1}^{s-1}(m_j+1)]_{\mathbb{Z}}$. That is, $m_s = p-\sum_{j=1}^{s-1}(m_j+1+\delta_j)$ or $m_s+r = p-\sum_{j=1}^{s-1}(m_j+1+\delta_j)$.

When $\mathcal{G}_s = m_sK_1$, the process is the same as in the above case. Hence, we have a numbering for G with the strength $p+\delta_1$.

When $\mathcal{G}_s = m_sK_1+K_r$, where $\delta_s+1 = r = p-m_s-\sum_{j=1}^{s-1}(m_j+1+\delta_j)$, we label the m_s isolated vertices of $\mathcal{G}_s$ with labels in $[p-m_s+1-\sum_{j=1}^{s-1}(m_j+1), p-\sum_{j=1}^{s-1}(m_j+1)]_{\mathbb{Z}}$, respectively (if $m_s=0$, then this process is not performed). Finally, label the vertices of K_r with labels in $[1+\sum_{j=1}^{s-1}\delta_j, p-m_s-\sum_{j=1}^{s-1}(m_j+1)]_{\mathbb{Z}}$, respectively. Then, the largest new induced edge labels related to the neighbors of u_{s-1} are

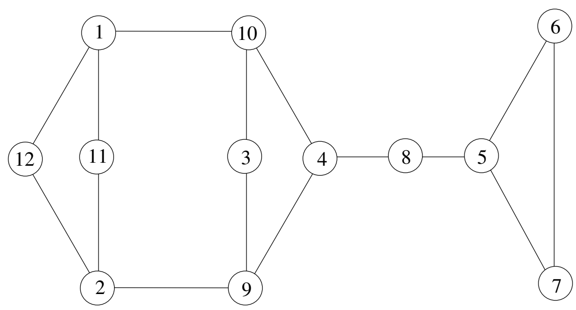
$$p-\sum_{j=1}^{s-1}(m_j+1)+\sum_{j=1}^{s-1}\delta_j = p-(m_1+1-\delta_1)-\widetilde{z}_{s-1} \le p-1+\delta_1 < p+\delta_1.$$

The largest new induced edge labels in K_r are

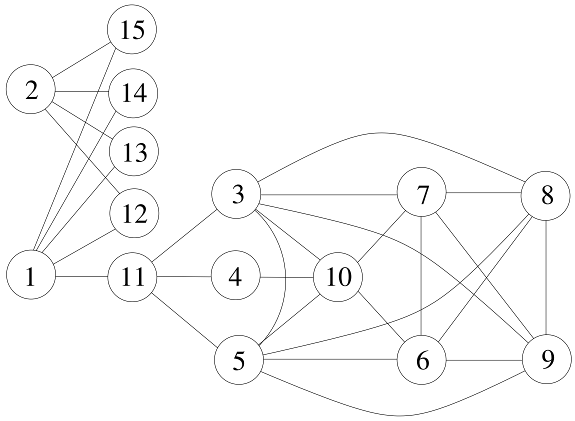
$$[p-m_s-\sum_{j=1}^{s-1}(m_j+1)]+[p-1-m_s-\sum_{j=1}^{s-1}(m_j+1)]$$
$$= [p-m_s-\sum_{j=1}^{s-1}(m_j+1)]+[\sum_{j=1}^{s}\delta_j] = p+1-\sum_{j=1}^{s}(m_j+1-\delta_j)$$
$$\le p+1-m_1-1+\delta_1 = p+\delta_1.$$

Hence, we have a numbering f such that $str_f(G) = p+\delta_1$. Therefore, $str(G) \le p+\delta_1$. By Lemma 2, $str(G) = p+\delta_1$. □

Example 3. *Consider the graph G_1 described in Example 1. Using the second δ-sequence of G_1 and following the construction in the proof of Theorem 1, we have the following strength labeling f of G_1 such that $str_f(G_1) = 14$.*



From Example 2, G does not satisfy the hypothesis of Theorem 1, but there is a strength labeling f for it with $str_f(G) = p + \delta(G) = 17$ as follows.



So, the converse of Theorem 1 is not true.

Thus, Theorem 1 provides a solution to Problem 1. Note that every tree T has the property that $\delta_i = 1$ and $m_i \geq 0$ for each $i \geq 2$. We immediately have $str(T) = |V(T)| + 1$ and the following corollary that answers more than what Problem 3 asks.

Corollary 2. *If G is a forest without an isolated vertex, then* $str(G) = |V(G)| + 1$.

Corollary 3. *The one-point union of cycles G of order p has* $str(G) = p + 2$.

Proof. Remove a degree 2 vertex that is adjacent to the maximum degree vertex of G and its neighbors to obtain a subgraph $\mathcal{G}_2$, which is a disjoint union of path(s). So, G admits a δ-sequence that satisfies (1). □

Corollary 4. *If G is a wheel or fan graph of order p, then* $str(G) = p + \delta(G)$.

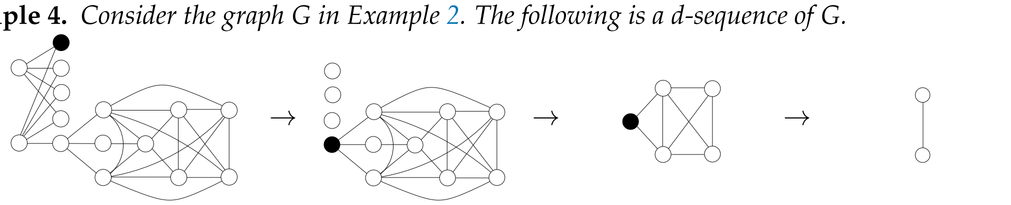
In constructing a δ-sequence of G, if we change the choice of choosing a vertex of degree δ_i to a vertex of degree d_i, then we get another sequence of subgraphs of G. This sequence is called a d-sequence of G. Let $y_j(G) = m_j + 1 - d_j$, $z_i(G) = \sum_{j=2}^{i} y_j(G)$ and denote d_1 as d_G. By the same argument as that when proving Theorem 1, we have:

Theorem 5. *For a graph G of order p with* $p - 2 \geq \delta(G) \geq 1$, *if there is a d-sequence* $\{\mathcal{G}_i\}_{i=1}^{s}$ *of G such that*

$$z_i(G) \geq 0 \text{ for } 2 \leq i \leq s, \tag{2}$$

then $str(G) \leq p + d_G$. *The equality holds if* $d_G = \delta(G)$.

Example 4. *Consider the graph G in Example 2. The following is a d-sequence of G.*



$\mathcal{G}_1 = G$ $\quad$ $\mathcal{G}_2 = 3K_1 + G_2$ $\quad$ $\mathcal{G}_3 = G_3$ $\quad$ $\mathcal{G}_4 = K_2$

$d_G = d_1 = \delta(G) = 2$ $\quad$ $d_2 = 3, m_2 = 3$ $\quad$ $d_3 = 2, m_3 = 0$ $\quad$ $d_4 = 1, m_4 = 0$

Now, $z_2 = 1$, $z_3 = 0$, *and* $z_4 = 0$. *By Theorem 5,* $str(G) = 17$, *as shown in Example 3.*

Lemma 4. *Let T be a forest without an isolated vertex of an order at of least 3, and let* P_T *be the set of pendant vertices that are adjacent to a vertex of a degree of at least 2. There is a* δ*-sequence of T* *of length s such that $\widetilde{z}_s(T) \geq |P_T| - |N_T(P_T)|$, where $\widetilde{z}_i(T)$ is defined in (1). Moreover, all $\widetilde{z}_i(T)$ satisfy (1).*

Proof. Obviously, the lemma holds when T is of order 3. Suppose that the lemma holds when the order of T is k or less, where $k \geq 3$.

Now, consider a forest T of order $k+1$. Choose a vertex $u \in P_T$. Let v be the vertex adjacent to u with degree d. We shall consider the forest $T - u - v$.

Suppose that $T = K_{1,k}$, which is a star; then, $T - u - v = (k-1)K_1$ and $\{T, (k-1)K_1\}$ is a δ-sequence of T. Note that $|P_T| = k$ and $|N_T(P_T)| = 1$. Clearly, $\widetilde{z}_2(T) = (k-1) + 1 - 0 = k > |P_T| - |N_T(P_T)|$.

Now, we assume that T is not a star. Let $T - u - v = mK_1 + T'$, where $m \geq 0$ and T' is a forest without an isolated vertex.

Suppose that the order of T' is 2; then, $T' = K_2$ and $\{T, mK_1 + K_2\}$ is a δ-sequence of T, where $m = k-3$. Now, $\widetilde{z}_2(T) = m = k-3$. If $T = K_{1,k-2} + K_2$, then $|P_T| = k-2$ and $|N_T(P_T)| = 1$. We get $\widetilde{z}_2(T) = |P_T| - |N_T(P_T)|$. If T is a tree, then $|P_T| = k-1$ and $|N_T(P_T)| = 2$. So, we still get $\widetilde{z}_2(T) = |P_T| - |N_T(P_T)|$.

Suppose that the order of T' is greater than 2. By the induction assumption, there is a δ-sequence of T', say $\{T' = \mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_s\}$, such that $\widetilde{z}_s(T') \geq |P_{T'}| - |N_{T'}(P_{T'})|$ for some $s \geq 2$.

Now, consider $\{T, \mathcal{T}_1, \ldots, \mathcal{T}_s\}$ of T. Note that $\widetilde{z}_2(T) = m + 1 - \delta(T') = m$. Let a be the number of vertices of degree 2 in T, but degree 1 in T'; then, $|P_{T'}| = |P_T| - (m+1) + a$, where $a \geq 0$. Let $b = |N_{T'}(P_{T'}) \setminus N_T(P_T)|$. Therefore, $|N_{T'}(P_{T'})| \leq |N_T(P_T)| - 1 + b$ (some vertices in $N_T(P_T)$ may not be in $N_{T'}(P_{T'})$).

If a vertex w in T of degree 2 becomes of degree 1 in T', then w may be in $P_{T'}$ and may have at most one neighbor in $N_{T'}(P_{T'})$ that is not in $N_T(P_T)$. So, $a \geq b$.

Now, we have

$$\begin{aligned}
\widetilde{z}_{s+1}(T) &= \widetilde{z}_2(T) + \widetilde{z}_s(T') = m + \widetilde{z}_s(T') \geq m + |P_{T'}| - |N_{T'}(P_{T'})| \\
&\geq m + [|P_T| - (m+1) + a] - [|N_T(P_T)| - 1 + b] \\
&= |P_T| - |N_T(P_T)| + a - b \geq |P_T| - |N_T(P_T)|.
\end{aligned}$$

By induction, the lemma holds for any forest T of an order greater than 2. □

Remark 1. *In the proof of Lemma 4, we can see that $\widetilde{z}_2(T) = |P_T| - |N_T(P_T)| + 1$ if T is a star.*

Theorem 6. *Keep all notations defined in Theorem 1 and Lemma 4. Let H be a graph with $\delta(H) \geq 1$. Suppose that $\{H = \mathcal{H}_1, \ldots, \mathcal{H}_s\}$ is a d-sequence of H. Suppose that*

$$Z = \min\{z_i(H) \mid 2 \leq i \leq s\} \tag{3}$$

is not positive. Suppose that T is a graph with a d-sequence $\{T = \mathcal{T}_1, \ldots, \mathcal{T}_t\}$ satisfying (2). Let $G = H + T$. If $z_t(T) \geq d_H - Z$; then, $|V(G)| + \delta(G) \leq str(G) \leq |V(G)| + d_T$.

Proof. Since $\mathcal{T}_t$ is either mK_1 for some $m \geq 1$ or $mK_1 + K_r$ for some $m \geq 0$, we have the following two cases.

For the first case, $y_t(T) = m+1$ and $\{\mathcal{T}_1, \ldots, \mathcal{T}_{t-1}, mK_1 + H, \mathcal{H}_2, \ldots, \mathcal{H}_s\}$ is a d-sequence of G. Then,
$z_j(G) = z_j(T) \geq 0, 2 \leq j \leq t-1$;
$z_t(G) = z_{t-1}(T) + y_t(H+T) = z_{t-1}(T) + [m+1-d_H] = z_t(T) - d_H \geq z_t(T) - d_H + Z \geq 0$;
$z_{t+1}(G) = z_{t-1}(T) + (m+1-d_H) + z_2(H) = z_t(T) - d_H + y_2(H) = z_t(T) - d_H + z_2(H)$.

In general,

$$z_{t+j}(G) = z_{t-1}(T) + (m+1-d_H) + z_{j+1}(H) = z_t(T) - d_H + z_{j+1}(H)$$
$$\geq z_t(T) - d_H + Z \geq 0, \quad 1 \leq j \leq s-1.$$

For the last case, $\{\mathcal{T}_1, \ldots, \mathcal{T}_t, \mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_s\}$ is a d-sequence of G. We will get that
$z_j(G) = z_j(T) \geq 0, 2 \leq j \leq t$;
$z_{t+1}(G) = z_t(G) + [1 - d_H] = z_t(T) + 1 - d_H > 0$;
$z_{t+j}(G) = z_t(T) + 1 - d_H + z_j(H) > 0, 2 \leq j \leq s$.
By Theorem 5, $str(G) \leq |V(G)| + d_T$. The lower bound follows from Lemma 2. □

Remark 2. *In Theorem 6, suppose that H and T are connected. Let $x \in V(\mathcal{T}_{t-1})$ be chosen to construct $\mathcal{T}_t$, let $v \in V(\mathcal{T}_{t-1})$ be a neighbor of x, and let $u \in V(H)$, which is not chosen to construct $\mathcal{H}_2$. We add an edge vu to the graph $H+T$. All of the y_j values of this connected graph are the same as those of $H+T$.*

Theorem 7. *Keep all notations defined in Theorem 1 and Lemma 4. Let H be a graph with $\delta(H) \geq 1$. Suppose that $\{H = \mathcal{H}_1, \ldots, \mathcal{H}_s\}$ is a d-sequence of H. Suppose*

$$Z = \min\{z_i(H) \mid 2 \leq i \leq s\} \leq 0.$$

Let $G = H + T$, where T is a forest without an isolated vertex of an order of at least 3. If $|P_T| - |N_T(P_T)| \geq d_H - Z$; then, $str(G) = |V(G)| + 1$.

Proof. By Lemma 4, there is a δ-sequence $\{\mathcal{T}_1, \ldots, \mathcal{T}_t\}$ of T such that $\tilde{z}_t(T) \geq |P_T| - |N_T(P_T)|$ and all $\tilde{z}_i(T)$ satisfy (1). Since this δ-sequence is a particular d-sequence of T, it satisfies the condition of Theorem 6. Since $d_T = 1$ now, by Theorem 6, we have $str(G) \leq |V(G)| + 1$. Hence, we have the theorem, since $\delta(G) = 1$. □

Corollary 8. *Let H be a graph with $\delta(H) \geq 1$. Suppose that $\{H = \mathcal{H}_1, \ldots, \mathcal{H}_s\}$ is a d-sequence of H.*

Suppose

$$Z = \min\{z_i(H) \mid 2 \leq i \leq s\} \leq 0.$$

Let $G = H + K_{1,k}$. If $k \geq d_H - Z$, then $str(G) = |V(G)| + 1$.

Let $G = K_{m,n}$ for $n \geq m \geq 1$. It is proven in [7] (Theorem 3.5) that $str(G) = |V(G)| + m$. So, we have the following.

Corollary 9. *There exists a graph G with $str(G) = |V(G)| + \delta(G)$ for each $\delta(G) \geq 1$.*

Suppose that H is a graph with $str(H) > |V(H)| + \delta(H)$. So, there is a d-sequence of H with $d_H = \delta(H)$ such that Z is non-positive, where Z is defined in (3). Let $T = K_{m,n}$. Since $\{T, (n-1)K_1\}$ is a d-sequence of T, $z_2(T) = n$. Suppose that $n \geq d_H - Z$ and $m = \delta(H)$. Since $n \geq d_H - Z \geq \delta(H)$, $\delta(H+T) = \delta(H)$. By Theorem 6, we have $str(H+T) = |V(H+T)| + \delta(H)$. So, together with Remark 2, we have the following theorem.

Theorem 10. *For every graph H, either $str(H) = |V(H)| + \delta(H)$ or H is a proper subgraph of a graph G such that $str(G) = |V(G)| + \delta(G)$ with $\delta(G) = \delta(H)$.*

Example 5. *It is easy to obtain a δ-sequence of Q_4 with*

$$\begin{aligned}
&\delta_1 = 4;\\
&\delta_2 = 2, m_2 = 0, \text{ so } \tilde{z}_2 = -1;\\
&\delta_3 = 1, m_3 = 0, \text{ so } \tilde{z}_3 = -1;\\
&\delta_4 = 1, m_4 = 1, \text{ so } \tilde{z}_4 = 0;\\
&\delta_5 = 0, m_5 = 3, \text{ so } \tilde{z}_5 = 4.
\end{aligned}$$

Thus, $Z = -1$. Let $n = 4 + 1 = 5$. By the construction before Theorem 10, we have $str(Q_4 + K_{4,5}) = 25 + 4 = 29$. A required labeling can be obtained by similarly following the proof of Theorem 1. Moreover, adding an edge joining a vertex of degree 5 of $K_{4,5}$ and a vertex of Q_4 gives a connected graph G that contains Q_4 as a proper subgraph with $str(G) = str(Q_4 + K_{4,5})$, as required.

Example 6. *For 2-regular graphs $\mathcal{C}_k$ with exactly $k \geq 1$ odd cycles, we have $Z = -k + 1$. Let $n = k + 1$. By the construction before Theorem 10, we have $str(\mathcal{C}_k + K_{2,k+1}) = |V(\mathcal{C}_k)| + (k + 3) + 2$.*

3. New Lower Bounds

Theorem 11. *Suppose that G is a graph of order p with an independent number α; then, $str(G) \geq 2p - 2\alpha + 1$.*

Proof. For any numbering of G, by the pigeonhole principle, at least two integers in $[p - \alpha, p]_{\mathbb{Z}}$ are assigned to two adjacent vertices. So, the induced edge label is at least $2p - 2\alpha + 1$. This completes the proof. □

Corollary 12. *Suppose that G is a graph of order p with minimum degree δ. Suppose that $str(G) = p + \delta$; then, $\alpha \geq \left\lceil \frac{p-\delta+1}{2} \right\rceil$, where α is the independence number of G.*

Proof. From Theorem 11, we have $\alpha \geq \frac{p-\delta+1}{2}$. □

Let G be a graph of order p. Let

$$\begin{aligned}
x_i &= \min\{|N_G(S) \setminus S| \ : \ |S| = i\};\\
\xi = \xi(G) &= \max\{x_i - i + 1 \mid 1 \leq i \leq p - 1\}.
\end{aligned}$$

Theorem 13. *Let G be a graph of order p; then, $str(G) \geq p + \xi$.*

Proof. Let $\xi = x_i - i + 1$ for some i. Let f be a strength labeling of G. Consider the labels in $[p - i + 1, p]_{\mathbb{Z}}$. Let $T = f^{-1}([p - i + 1, p])$; then, $|T| = i$. Now, $|f(N_G(T) \setminus T)| = |N_G(T) \setminus T| \geq x_i$. Let a be the largest label in $f(N_G(T) \setminus T)$. There is a vertex $u \in N_G(T) \setminus T$ such that $f(u) = a$. Moreover, u is adjacent to $v \in T$. Thus,

$$str_f(G) \geq f(v) + f(u) \geq p - i + 1 + a \geq p - i + 1 + x_i = p + \xi.$$

□

Thus, we provided two good bounds for the strength of a graph, as raised in Problem 2. Note that Lemma 2 is a corollary of Theorem 13 when $\xi = \delta = x_1 \geq x_i - i + 1$ for $i \geq 2$.

Theorem 14. *If $G = \sum_{i=1}^{h} C_{2m_i} + \sum_{j=1}^{k} C_{2n_j+1}$, where $m_i \geq 2$, $n_j \geq 1$, and $h + k \geq 1$, then $str(G) = \max\{p + 2, p + 1 + k\}$.*

Proof. Note that if $h = 0$, then the first summand does not appear, similarly for the second summand. Now, $\alpha(G) = \sum_{i=1}^{h} m_i + \sum_{j=1}^{k} n_j$. By Theorem 11, we have $str(G) \geq p + 1 + k$.

Let $H = \sum_{i=1}^{h} C_{2m_i}$ and $K = \sum_{j=1}^{k} C_{2n_j+1}$. Let $M = \sum_{i=1}^{h} m_i$ such that $M = 0$ when $h = 0$.

We shall construct a numbering f on G. If $h \geq 1$, we first label H with integers in $[1, M]_{\mathbb{Z}} \cup [p - M + 1, p]_{\mathbb{Z}}$ as follows.

Label C_{2m_1} with $1, p, 2, \ldots, m_1, p - m_1 + 1$ in the natural order. In general, for $i \geq 2$, we label the vertices of even cycle C_{2m_i} with $1 + \sum_{l=1}^{i-1} m_l$, $p - \sum_{l=1}^{j-1} m_l$, $2 + \sum_{l=1}^{j-1} m_l$, $\ldots$, $\sum_{l=1}^{i} m_l$, $p - \sum_{l=1}^{j} m_l + 1$ in the natural order. Continue this process until $i = h$. Hence, the maximum induced edge label is $p + 2$.

If $k \geq 1$, then we label the vertices of odd cycle C_{2n_1+1} by $M + 1, p - M, M + 2, \ldots, p - M - n_1 + 1, M + n_1 + 1$ in the natural order. Up to now, the maximum induced edge label is still $p + 2$.

Now, we label the vertices of odd cycle C_{2n_2+1} with $M + n_1 + 2, p - M - n_1, M + n_1 + 3, \ldots, p - M - n_1 - n_2 + 1, M + n_1 + n_2 + 2$ in the natural order. Note that $(M + n_1 + 2) + (M + n_1 + n_2 + 2) = 2M + (2n_1 + 1) + n_2 + 2 \leq p$. So, the current maximum induced edge label is $p + 3$.

In general, for $j \geq 2$, we label the vertices of odd cycle C_{2n_j+1} with $M + 1 + \sum_{l=1}^{j-1}(n_l + 1)$, $p - M - \sum_{l=1}^{j-1} n_l$, $M + 2 + \sum_{l=1}^{j-1}(n_l + 1)$, $\ldots$, $p - M - \sum_{l=1}^{j} n_l + 1$, $M + \sum_{l=1}^{j}(n_l + 1)$ in the natural order. Note that $(M + 1 + \sum_{l=1}^{j-1}(n_l + 1)) + (M + \sum_{l=1}^{j}(n_l + 1)) = 2M + \sum_{l=1}^{j-1}(2n_l + 1) + n_j + j \leq p - 2 + j$. So, the current maximum induced edge label is $p + j + 1$.

Continue this process until $j = k$. Hence, we have $str_f(G) = p + k + 1$. $\square$

Example 7. *Consider $G = C_4 + C_6 + C_5 + C_5 + C_7$. Now, $p = 27$ and $k = 3$.*

We label the vertices of
C_4 with integers in $[1, 27, 2, 26]$ (max. induced edge label is 29);
C_6 with integers in $[3, 25, 4, 24, 5, 23]$ (max. induced edge label is 29);
C_5 with integers in $[6, 22, 7, 21, 8]$; (max. induced edge label is 29);
C_5 with integers in $[9, 20, 10, 19, 11]$ (max. induced edge label is 30);
C_7 with integers in $[12, 18, 13, 17, 14, 16, 15]$ (max. induced edge label is 31).
So, $str(G) = 31$.

Let $G \times H$ be the Cartesian product of graphs G and H.

Lemma 5. *Let G be a bipartite graph with bipartition (X, Y) such that $|X| = |Y| = m$. Suppose that there is a numbering f of G such that $f(X) = [1, m]_{\mathbb{Z}}$; then, there is a numbering F of $G \times K_2$ such that $F([1, 2m])_{\mathbb{Z}} = \widetilde{X}$, where $(\widetilde{X}, \widetilde{Y})$ is a bipartition of $G \times K_2$. Moreover, $str_F(G \times K_2) = 5m + 1$.*

Note that the following proof is modified from the proof of Theorem 3.10 in [7].

Proof. Note that, from the hypotheses, $f(Y) = [m + 1, 2m]_{\mathbb{Z}}$. Let u and v be vertices of K_2. Then,
$\widetilde{X} = \{(x, u) \mid x \in X\} \cup \{(y, v) \mid y \in Y\}$ and $\widetilde{Y} = \{(y, u) \mid y \in Y\} \cup \{(x, v) \mid x \in X\}$.

Define $F : V(G \times K_2) \to [1, 4m]_{\mathbb{Z}}$ by

$$\begin{aligned} F(x, u) &= f(x) \in [1, m]_{\mathbb{Z}} \\ F(y, v) &= (3m + 1) - f(y) \in [m + 1, 2m]_{\mathbb{Z}} \\ F(x, v) &= (3m + 1) - f(x) \in [2m + 1, 3m]_{\mathbb{Z}} \\ F(y, u) &= 2m + f(y) \in [3m + 1, 4m]_{\mathbb{Z}}. \end{aligned}$$

Clearly, $F(\widetilde{X}) = [1, 2m]_{\mathbb{Z}}$.

Now, $F(x, u) + F(x, v) = f(x) + (3m + 1) - f(x) = 3m + 1$ and $F(y, u) + F(y, v) = 2m + f(y) + (3m + 1) - f(y) = 5m + 1$. Suppose that (x_1, u) and (x_2, u) are adjacent in

$G \times K_2$. By definition, $F(x_1,u) + F(x_2,u) \le 2m$. Similarly, if (y_1,v) and (y_2,v) are adjacent in $G \times K_2$, then $F(y_1,v) + F(y_2,v) \le 4m$. Thus, $str_F(G \times K_2) = 5m + 1$. $\square$

Let Q_n be a hypercube of dimension n, $n \ge 2$. Since there is a strength numbering f of Q_2 satisfying the hypotheses of Lemma 5, by applying this lemma repeatedly, we get the following.

Theorem 15. *For $n \ge 2$, $str(Q_n) \le 2^n + 2^{n-2} + 1$.*

This is a known result in [7] (Theorem 3.10).

We shall improve the lower bound of the strength of Q_n. The vertices of Q_n often used the elements of the vector space $\mathbb{Z}_2^n$ over $\mathbb{Z}_2$. Two vertices u and v are adjacent if and only if $u + v = e_i$, where e_i is the standard basis of $\mathbb{Z}_2^n$. Note that $v = -v$ for any vector $v \in \mathbb{Z}_2^n$. In the proofs of the following lemmas, all algebra involving vectors is over $\mathbb{Z}_2$.

For any vertex v, we let $N_G[v] = N_G(v) \cup \{v\}$, the closed neighborhood of v. Hence, for any subset of vertices S, $N_G(S) \setminus S = \left(\bigcup\limits_{v \in S} N_G[v] \right) \setminus S$. We shall omit the subscript G when there is no ambiguity.

Lemma 6. *If u and v are two distinct vertices of Q_n, $n \ge 3$, then $|N[u] \cap N[v]|$ is either 0 or 2.*

Proof. Suppose that u and v are adjacent. Clearly, $|N[u] \cap N[v]| = 2$. Suppose that u and v are not adjacent. If $z \in N[u] \cap N[v]$, then the distance between u and v is two. Hence, $u \notin N(v)$, $v \notin N(u)$ and $u + v = e_i + e_j$ (equivalently, $u + e_i = v + e_j$), where $i \ne j$. Since $z \in N(u) \cap N(v)$, $z = u + e_k = v + e_l$ for some k, l. So, $u + e_k + v + e_l = \mathbf{0}$ or $u + v = e_k + e_l$. Thus, $\{i, j\} = \{k, l\}$. Hence, $|N[u] \cap N[v]| = |\{u + e_i, u + e_j\}| = 2$. $\square$

Lemma 7. *For any three distinct vertices u, v, and w of Q_n, $n \ge 3$,*

$$|N[u] \cap N[v] \cap N[w]| = \begin{cases} 0 & \text{if at least one of } |N[u] \cap N[v]|,\ |N[u] \cap N[w]| \text{ and } |N[v] \cap N[w]| \text{ is } 0; \\ 1 & \text{if all of } |N[u] \cap N[v]|,\ |N[u] \cap N[w]| \text{ and } |N[v] \cap N[w]| \text{ are } 2. \end{cases}$$

Proof. If one of $|N[u] \cap N[v]|$, $|N[u] \cap N[w]|$ and $|N[v] \cap N[w]|$ is 0, then $|N[u] \cap N[v] \cap N[w]| = 0$. Otherwise, Lemma 6 implies that $|N[u] \cap N[v]| = |N[u] \cap N[w]| = |N[v] \cap N[w]| = 2$.

(1). Suppose that only one pair of u, v, and w are adjacent—say, uv is an edge—then, the distances from w to u and to v are 2. This creates a five-cycle, which is impossible.

(2). Suppose that two pairs of u, v, and w are adjacent—say, uv and uw are edges. Note that v and w cannot be adjacent. Then, $N[u] \cap N[v] = \{u, v\}$ and $N[u] \cap N[w] = \{u, w\}$. Hence, $u \in N[v] \cap N[w]$. This implies that $N[u] \cap N[v] \cap N[w] = \{u\}$.

(3). Suppose that none of u, v, and w are adjacent. By the proof of Lemma 6, we have $u + v = e_{i_1} + e_{j_1}$ and $v + w = e_{i_2} + e_{j_2}$ for some i_1, i_2, j_1, j_2, $i_1 \ne j_1$, and $i_2 \ne j_2$. This implies that $u + w = e_{i_1} + e_{j_1} + e_{i_2} + e_{j_2}$. Since the distance of u and w is 2, $|\{i_1, j_1\} \cap \{i_2, j_2\}| = 1$. Without loss of generality, we may assume that $i_1 = i_2$. Now, $N[u] \cap N[v] = \{u + e_{i_1}, u + e_{j_1}\}$, $N[u] \cap N[w] = \{u + e_{j_1}, u + e_{j_2}\}$, and $N[v] \cap N[w] = \{v + e_{i_2}, v + e_{j_2}\}$. Here, $v + e_{i_2} = v + e_{i_1} = u + e_{j_1}$. Hence, $u + e_{j_1} \in N[u] \cap N[v] \cap N[w]$. Since $u + e_{j_2} \notin N[u] \cap N[v]$, $N[u] \cap N[v] \cap N[w] = \{u + e_{j_1}\}$.

This completes the proof. $\square$

Theorem 16. *For the hypercube Q_n, $n \ge 2$, we have $str(Q_2) \ge 6$; $str(Q_3) \ge 11$; $str(Q_4) \ge 21$; and $str(Q_n) \ge 2^n + 4n - 12$ for $n \ge 5$.*

Proof. Keeping the notations defined in Theorem 13, we want to compute x_i and ξ. Clearly, $x_1 = \delta = n$.

Suppose that $S = \{u, v\}$ with $u \neq v$.

$$|N[u] \cup N[v]| = |N[u]| + |N[v]| - |N[u] \cap N[v]| \geq 2(n+1) - 2 = 2n.$$

So, $|N(S) \setminus S| = |N[u] \cup N[v]| - |S| \geq 2n - 2$. Actually, when $S = \{\mathbf{0}, e_1 + e_2\}$, $|N(S) \setminus S| = 2n - 2$. Thus, $x_2 = 2n - 2$.

Suppose that $S = \{u, v, w\}$, where u, v, w are distinct.

(1). If all $|N[u] \cap N[v]|$, $|N[u] \cap N[w]|$ and $|N[v] \cap N[w]|$ are not zero, then by Lemma 7, $|N[u] \cap N[v] \cap N[w]| = 1$. Thus,

$$\begin{aligned} |N[u] \cup N[v] \cup N[w]| = {} & |N[u]| + |N[v]| + |N[w]| \\ & - |N[u] \cap N[v]| - |N[u] \cap N[w]| - |N[v] \cap N[w]| + |N[u] \cap N[v] \cap N[w]| \\ = {} & 3(n+1) - 3 \times 2 + 1 = 3n - 2. \end{aligned}$$

Actually, $S = \{\mathbf{0}, e_1 + e_2, e_1 + e_3\}$.

(2). If all $|N[u] \cap N[v]|$, $|N[u] \cap N[w]|$ and $|N[v] \cap N[w]|$ are zero, then $|N[u] \cup N[v] \cup N[w]| = 3(n+1)$.

(3). If at least one of $|N[u] \cap N[v]|$, $|N[u] \cap N[w]|$, and $|N[v] \cap N[w]|$ is not zero and at least one of them is zero, then

$$\begin{aligned} |N[u] \cup N[v] \cup N[w]| = {} & |N[u]| + |N[v]| + |N[w]| \\ & - |N[u] \cap N[v]| - |N[u] \cap N[w]| - |N[v] \cap N[w]| + |N[u] \cap N[v] \cap N[w]| \\ \geq {} & 3(n+1) - 2 \times 2 = 3n - 1. \end{aligned}$$

Thus, $x_3 = 3n - 5$.

Let us consider $S = \{u_1, u_2, u_3, u_4\}$, where u_1, u_2, u_3, u_4 are distinct. Then,

$$\begin{aligned} \left| \bigcup_{l=1}^{4} N[u_l] \right| = {} & \sum_{l=1}^{4} |N[u_l])| - \sum_{1 \leq j < l \leq 4} |N[u_j] \cap N[u_l]| + \sum_{1 \leq h < j < l \leq 4} |N[u_h] \cap N[u_j] \cap N[u_l]| \\ & - |N[u_1] \cap N[u_2] \cap N[u_3] \cap N[u_4]|. \end{aligned}$$

(1). If only one of $N[u_j] \cap N[u_l] = \varnothing$, then by Lemma 7, the third summand is 1 and the fourth summand is 0. Then, $\left| \bigcup_{l=1}^{4} N[u_l] \right| \geq 4n + 4 - 5 \times 2 + 1 = 4n - 5$.

Actually, $S = \{\mathbf{0}, e_1 + e_2, e_1 + e_3, e_1 + e_2 + e_3 + e_4\}$.

(2). If more than one of $N[u_j] \cap N[u_l] = \varnothing$, then the third and fourth summands are 0. Thus, $\left| \bigcup_{l=1}^{4} N[u_l] \right| \geq 4n + 4 - 4 \times 2 = 4n - 4$.

(3). If all of $N[u_j] \cap N[u_l] \neq \varnothing$, then

$$\begin{aligned} \left| \bigcup_{l=1}^{4} N[u_l] \right| & = 4n + 4 - 6 \times 2 + 4 - |N[u_1] \cap N[u_2] \cap N[u_3] \cap N[u_4]| \\ & \geq 4n - 4 - 1 = 4n - 5. \end{aligned}$$

Therefore, $x_4 = 4n - 9$.

Hence, by Theorem 13, we have $\xi \geq 2$ when $n = 2$; $\xi \geq 3$ when $n = 3$; $\xi \geq 5$ when $n = 4$; and $\xi \geq 4n - 12$ when $n \geq 5$. Thus, we have $str(Q_2) \geq 6$; $str(Q_3) \geq 11$; $str(Q_4) \geq 21$; and $str(Q_n) \geq 2^n + 4n - 12$ when $n \geq 5$. $\square$

From the proof of Theorem 16, we have $x_1 = n$, $x_2 = 2n - 2$, $x_3 = 3n - 5$, and $x_4 = 4n - 9$ for Q_n.

Suppose that $x_{i+1} = |N(S) \setminus S|$ for some subset of vertices S with $|S| = i + 1$. Let $S = \{u_1, \ldots, u_{i+1}\}$ with $|S| = i + 1$.

$$x_{i+1} + (i+1) = \left|\bigcup_{l=1}^{i+1} N[u_l]\right| = |N[u_1]| + \left|\bigcup_{l=2}^{i+1} N[u_l]\right| - \left|N[u_1] \cap \left(\bigcup_{l=2}^{i+1} N[u_l]\right)\right|$$
$$= |N[u_1]| + \left|\bigcup_{l=2}^{i+1} N[u_l]\right| - \left|\bigcup_{l=2}^{i+1} (N[u_1] \cap N[u_l])\right|$$
$$\geq (n+1) + (x_i + i) - 2i \qquad (\text{since } |N[u_1] \cap N[u_l]| \leq 2)$$

So, $x_{i+1} \geq n + x_i - 2i$. Since $x_4 = 4n - 9$, by induction, we will get $x_i \geq in + 3 - (i-1)i$, where $i \geq 4$.

Let $\eta_i = x_i - i + 1$; then, $\eta_{i+1} \geq \eta_i + n - 2i - 1$. So, η_i is increasing when $i \leq (n-1)/2$.

Suppose that $n = 2m$, $m \geq 2$; then, $\eta_m \geq \eta_{m-1} + 2m - 2(m-1) - 1 = \eta_{m-1} + 1$. So, $\xi \geq \eta_m = x_m - m + 1 \geq m(2m) + 3 - (m-1)m - m + 1 = m^2 + 4$. We have $str(Q_{2m}) \geq 2^{2m} + m^2 + 4$.

Suppose that $n = 2m - 1$, $m \geq 2$; then, $\eta_m \geq \eta_{m-1} + (2m-1) - 2(m-1) - 1 = \eta_{m-1}$. So, $\xi \geq \eta_m = x_m - m + 1 \geq m^2 - m - 4$. We have $str(Q_{2m-1}) \geq 2^{2m-1} + m^2 - m + 4$.

Combining with Theorem 16, we have the following.

Theorem 17. *For $n \geq 2$,*

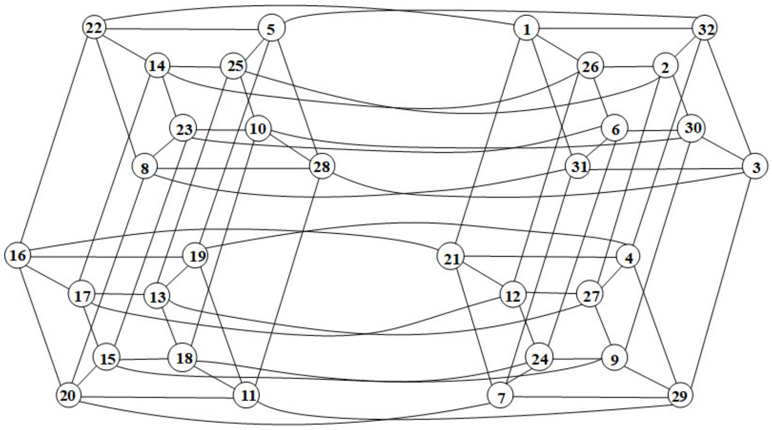
1. *$str(Q_2) \geq 6$; $str(Q_3) \geq 11$; $str(Q_4) \geq 21$;*
2. *$str(Q_n) \geq 2^n + 4n - 12$ for $5 \leq n \leq 9$;*
3. *$str(Q_{2m}) \geq 2^{2m} + m^2 + 4$ for $m \geq 5$.*
4. *$str(Q_{2m-1}) \geq 2^{2m-1} + m^2 - m + 4$ for $m \geq 6$.*

Corollary 18. *$str(Q_2) = 6$, $str(Q_3) = 11$, $str(Q_4) = 21$, and $str(Q_5) = 40$.*

Proof. Combining with Theorems 15 and 16, we have $str(Q_2) = 6$, $str(Q_3) = 11$, and $str(Q_4) = 21$. By considering the following labeling of Q_5, we have $str(Q_5) = 40$. □

Q_2 \ Q_3	**000**	**100**	**110**	**010**	**001**	**101**	**111**	**011**	
00	00000 1	10000 32	11000 3	01000 31	00100 26	10100 2	11100 30	01100 6	37
10	00010 21	10010 4	11010 29	01010 7	00110 12	10110 27	11110 9	01110 24	39
11	00011 16	10011 19	11011 11	01011 20	00111 17	10111 13	11111 18	01111 15	36
01	00001 22	10001 5	11001 28	01001 8	00101 14	10101 25	11101 10	01101 23	39
	38	37	40	39	40	40	40	39	max. induced edge label

Note that $Q_5 \cong Q_3 \times Q_2$. The first row and the first column are vertices of Q_3 and Q_2, respectively. The following is the corresponding figure.



The following is a numbering for $Q_6 \cong Q_3 \times Q_3$.

Q_3	**000**	**100**	**110**	**010**	**001**	**101**	**111**	**011**	
000	000000 1	100000 64	110000 3	010000 63	001000 62	101000 2	111000 52	011000 7	70
100	000100 61	100100 4	110100 45	010100 8	001100 11	101100 57	111100 17	011100 58	75
110	000110 15	100110 49	110110 28	010110 38	001110 56	101110 21	111110 33	011110 19	75
010	000010 60	100010 6	110010 50	010010 10	001010 13	101010 54	111010 25	011010 46	79
001	000001 59	100001 5	110001 37	010001 9	001001 12	101001 53	111001 23	011001 48	76
101	000101 14	100101 44	110101 30	010101 43	001101 55	101101 20	111101 42	011101 18	75
111	000111 39	100111 26	110111 35	010111 31	001111 22	101111 40	111111 32	011111 34	75
011	000011 16	100011 51	110011 27	010011 41	001011 47	101011 24	111011 36	011011 29	78
	76	77	78	73	78	78	77	77	max. induced edge label

Thus, $76 \leq str(Q_6) \leq 79$.

4. Conclusions and Open Problems

We obtained a sufficient condition for $str(G) = |V(G)| + \delta(G)$. Many open problems were solved and new results were obtained immediately. A new lower bound of $str(G)$ in terms of $\alpha(G)$ was obtained. Consequently, the strengths of all 2-regular graphs were determined. An approach for obtaining the lower bound of $str(G)$ in terms of the neighborhood size of all possible subsets of $V(G)$ was also obtained. This gave us a sharp lower bound of $str(Q_n)$ and partially answered Problem 4. The following problems naturally arise.

Problem 5. *Find sufficient and/or necessary conditions such that $str(G) = 2p - 2\alpha(G) + 1$ or $str(G) = p + \xi(G)$.*

Problem 6. *Determine the exact strength of all r-regular graphs for $r \geq 3$.*

Note that for $G = C_{2n+1}, n \geq 1$, $str(G) = 2n + 3 = |V(G)| + \delta(G) = 2|V(G)| - 2\alpha(G) + 1$.

Problem 7. *Characterize all graphs G of order p with (i)* $str(G) = p + \delta(G) = 2p - 2\alpha(G) + 1$*, (ii)* $str(G) = p + \xi(G) = 2p - 2\alpha(G) + 1$*, or (iii)* $str(G) = p + \delta(G) = p + \xi(G)$*.*

Note that every 2-regular graph $\mathcal{C}_k$ that has $k \geq 2$ odd cycles has $str(\mathcal{C}_k) = 2|V(\mathcal{C}_k)| - 2\alpha(\mathcal{C}_k) + 1 > |V(\mathcal{C}_k)| + \delta(\mathcal{C}_k)$. Observe that if $\mathcal{C}_k$ contains an even cycle C, then it contains a component C with $str(C) = |V(C)| + \delta(C)$.

Problem 8. *Prove that if* $str(G) = 2|V(G)| - 2\alpha(G) + 1 > |V(G)| + \delta(G)$*, then G contains a proper subgraph H with* $str(H) = |V(H)| + \delta(H)$*.*

Problem 9. *Prove that for each graph G, if* $str(G) > |V(G)| + \delta(G)$*, then* $str(G) = 2|V(G)| - 2\alpha(G) + 1$*. Otherwise, either G is a proper subgraph of a graph H with* $str(H) = 2|V(H)| - 2\alpha(H) + 1$ *with* $\alpha(H) \geq \alpha(G)$*, or else G contains a proper subgraph H with* $str(H) = |V(H)| + \delta(H)$*.*

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