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Applications of Certain Conic Domains to a Subclass of q -Starlike Functions Associated with the Janowski Functions

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Abstract: In our present investigation, with the help of the basic (or q -) calculus, we first define a new domain which involves the Janowski function. We also define a new subclass of the class of q -starlike functions, which maps the open unit disk \mathbb{U} , given by $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$, onto this generalized conic type domain. We study here some such potentially useful results as, for example, the sufficient conditions, closure results, the Fekete-Szegő type inequalities and distortion theorems. We also obtain the lower bounds for the ratio of some functions which belong to this newly-defined function class and for the sequences of the partial sums. Our results are shown to be connected with several earlier works related to the field of our present investigation. Finally, in the concluding section, we have chosen to reiterate the well-demonstrated fact that any attempt to produce the rather straightforward (p, q) -variations of the results, which we have presented in this article, will be a rather trivial and inconsequential exercise, simply because the additional parameter p is obviously redundant.

Keywords: analytic functions; conic domains; starlike functions; k -uniformly starlike functions; q -starlike functions; sufficient conditions; partial sums; distortion theorems; Janowski functions; principle of subordination; Carathéodory functions; q -derivative operator; q -hypergeometric functions

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1. Introduction, Motivation and Definitions

Let $\mathcal{H}(\mathbb{U})$ denote the class of analytic functions in the open unit disk:

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

A function f , which is analytic in \mathbb{U} and normalized by

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1,$$

is placed in the class \mathcal{A} . Thus, clearly, each function $f \in \mathcal{A}$ has the following series representation:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (\forall z \in \mathbb{U}). \quad (1)$$

The familiar class of normalized starlike functions in \mathbb{U} is denoted by \mathcal{S}^* , which consists of functions $f \in \mathcal{A}$ that satisfy the following condition:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad (\forall z \in \mathbb{U}).$$

Definition 1. For two analytic functions f_j ($j = 1, 2$) in \mathbb{U} , the function f_1 is said to be subordinate to the function f_2 , which is written as follows:

$$f_1 \prec f_2 \quad \text{or} \quad f_1(z) \prec f_2(z) \quad (z \in \mathbb{U}),$$

if there exists a Schwarz function w , which is analytic in \mathbb{U} , with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1,$$

such that

$$f_1(z) = f_2(w(z)).$$

Furthermore, the following equivalence relation is satisfied whenever the function f_2 is univalent in \mathbb{U} :

$$f_1(z) \prec f_2(z) \quad (z \in \mathbb{U}) \iff f_1(0) = f_2(0) \quad \text{and} \quad f_1(\mathbb{U}) \subset f_2(\mathbb{U}).$$

We next denote by \mathcal{P} the Carathéodory class of functions p , which are analytic in \mathbb{U} and have a series representation of the following form (see, for example, [1]):

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad (2)$$

such that

$$\Re\{p(z)\} > 0 \quad (\forall z \in \mathbb{U}).$$

We next recall that the class \mathcal{S}^* of starlike functions was generalized by Janowski [2] as follows.

Definition 2. A function h such that $h(0) = 1$ is said to belong to the Janowski class $\mathcal{P}[A, B]$ if and only if

$$h(z) \prec \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1).$$

Janowski [2] also proved that, for a function $p \in \mathcal{P}$, a function $h(z)$ belongs to the class $\mathcal{P}[A, B]$ if the following relation holds true:

$$h(z) = \frac{(A+1)p(z) - (A-1)}{(B+1)p(z) - (B-1)} \quad (-1 \leq B < A \leq 1).$$

Definition 3. A normalized analytic function f is placed in the class $\mathcal{S}^*[A, B]$ if

$$\frac{zf'(z)}{f(z)} = \frac{(A+1)p(z) - (A-1)}{(B+1)p(z) - (B-1)} \quad (-1 \leq B < A \leq 1). \quad (3)$$

Historically speaking, Kanas et al. (see [3–5]) were the first to define the conic domain Ω_k ($k \geq 0$) as follows:

$$\Omega_k = \left\{ u + iv : u > k\sqrt{(u-1)^2 + v^2} \right\} \quad (4)$$

and, subjected to this domain, the corresponding class $k\text{-}\mathcal{ST}$ of k -starlike functions is defined (see Definition 4 below). Furthermore, on specifying the parameter k , it is worth mentioning that Ω_k denotes certain important domain regions. For instance, the case $k = 0$ represents the conic region bounded by the imaginary axis. Moreover, if we let $k = 1$, this domain is seen to be a parabola. If k is constrained by $0 < k < 1$, then this domain is the right-hand branch of the hyperbola. Moreover, if $k > 1$, this domain represent an ellipse.

We note that, for the conic regions Ω_k , the following functions act as extremal functions:

$$p_k(z) = \begin{cases} \frac{1+z}{1-z} = 1 + 2z + 2z^2 + \dots & (k = 0) \\ 1 + \frac{2}{\pi^2} \left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right)^2 & (k = 1) \\ 1 + \frac{2}{1-k^2} \sinh^2 \left\{ \left(\frac{2}{\pi} \arccos k \right) \operatorname{arctanh}(\sqrt{z}) \right\} & (0 \leq k < 1) \\ 1 + \frac{1}{k^2-1} \sin \left(\frac{\pi}{2K(\kappa)} \int_0^{\frac{u(z)}{\sqrt{\kappa}}} \frac{dt}{\sqrt{1-t^2}\sqrt{1-\kappa^2 t^2}} \right) + \frac{1}{k^2-1} & (k > 1), \end{cases} \quad (5)$$

where

$$u(z) = \frac{z - \sqrt{\kappa}}{1 - \sqrt{\kappa}z} \quad (\forall z \in \mathbb{U})$$

and we choose $\kappa \in (0, 1)$ such that

$$k = \cosh \left(\frac{\pi K'(\kappa)}{4K(\kappa)} \right).$$

Here $K(\kappa)$ is Legendre's complete elliptic integral of the first kind and $K'(\kappa)$, given by

$$K'(\kappa) = K(\sqrt{1-\kappa^2}),$$

is the complementary integral of $K(\kappa)$.

We assume that

$$p_k(z) = 1 + P_1 z + P_2 z^2 + \dots \quad (\forall z \in \mathbb{U}).$$

Then, in [6], it has been shown that, for (5), one can have

$$P_1 = \begin{cases} \frac{2N^2}{1-k^2} & (0 \leq k < 1) \\ \frac{8}{\pi^2} & (k = 1) \\ \frac{\pi^2}{4k^2(\kappa)^2(1+\kappa)\sqrt{\kappa}} & (k > 1) \end{cases} \quad (6)$$

and

$$P_2 = D(k)P_1, \quad (7)$$

where

$$D(k) = \begin{cases} \frac{N^2+2}{3} & (0 \leq k < 1) \\ \frac{2}{3} & (k = 1) \\ \frac{[4K(\kappa)]^2(\kappa^2+6\kappa+1)-\pi^2}{24[K(\kappa)]^2(1+\kappa)\sqrt{\kappa}} & (k > 1) \end{cases} \quad (8)$$

with

$$N = \frac{2}{\pi} \arccos k.$$

The above-mentioned conic regions have been studied vastly by many authors and researcher (see, for example, [7–9]). The corresponding class k -ST of k -uniformly starlike functions associated with the conic domain is given as follows.

Definition 4. A normalized analytic function f having the form (1) is said to be in the class k -ST if and only if

$$\frac{zf'(z)}{f(z)} \prec p_k(z) \quad (\forall z \in \mathbb{U}; k \geq 0).$$

Definition 5 below was given by Noor et al. [10] by combining the concepts of the Janowski functions and the conic regions.

Definition 5. A function $h \in \mathcal{P}$ is said to be in the function class k - $\mathcal{P}[A, B]$ if and only if

$$h(z) \prec \frac{(A+1)p_k(z) - (A-1)}{(B+1)p_k(z) - (B-1)} \quad (-1 \leq B < A \leq 1; k \geq 0), \quad (9)$$

where $p_k(z)$ is defined by (5).

Geometrically, each function $h \in k$ - $\mathcal{P}[A, B]$ takes all values in the domain $\Omega_k[A, B]$ ($-1 \leq B < A \leq 1; k \geq 0$), which is defined as follows:

$$\Omega_k[A, B] = \left\{ w : \Re \left(\frac{(B-1)w - (A-1)}{(B+1)w - (A+1)} \right) > k \left| \frac{(B-1)w - (A-1)}{(B+1)w - (A+1)} - 1 \right| \right\}.$$

Equivalently, $\Omega_k[A, B]$ is a set of numbers $w = u + iv$ such that

$$\begin{aligned} & \left[(B^2 - 1)(u^2 + v^2) - 2(AB - 1)u + (A^2 - 1) \right]^2 \\ & > k \left[(-2(B+1)(u^2 + v^2) + 2(A+B+2)u - 2(A+1))^2 + 4(A-B)^2v^2 \right]. \end{aligned}$$

The domain $\Omega_k[A, B]$ represents certain conic type regions, which were studied by Noor and Malik [10].

Definition 6. (see [10]) A function $f \in \mathcal{A}$ is said to be in the class k -ST $[A, B]$ if and only if

$$\frac{zf'(z)}{f(z)} \in k$$
- $\mathcal{P}[A, B] \quad (\forall z \in \mathbb{U}; k \geq 0).$

In order to present some of the noteworthy and useful details of the definitions and principles of the basic (or q -) calculus, we assume throughout this article that

$$0 < q < 1 \quad \text{and} \quad k \in \mathbb{N} \cup \{0\},$$

where

$$\mathbb{N} = \{1, 2, 3, \dots\} = \mathbb{N}_0 \setminus \{0\} \quad (\mathbb{N}_0 := \{0, 1, 2, \dots\}).$$

Definition 7. For $0 < q < 1$, we define the q -number $[\lambda]_q$ by:

$$[\lambda]_q = \begin{cases} \frac{1 - q^\lambda}{1 - q} & (\lambda \in \mathbb{C} \setminus \{0\}) \\ \sum_{k=0}^{j-1} q^k = 1 + q + q^2 + \dots + q^{j-1} & (\lambda = j \in \mathbb{N}) \\ 0 & (\lambda = 0). \end{cases}$$

Definition 8. For $f \in \mathcal{A}$, the q -difference (or the q -derivative) operator D_q is defined, in a given subset of the set \mathbb{C} of complex numbers, by (see [11,12]):

$$(D_q f)(z) = \begin{cases} \frac{f(z) - f(qz)}{(1 - q)z} & (z \neq 0) \\ f'(0) & (z = 0), \end{cases} \tag{10}$$

provided that $f'(0)$ exists.

We can easily see from (10) that:

$$\lim_{q \rightarrow 1^-} (D_q f)(z) = \lim_{q \rightarrow 1^-} \frac{f(z) - f(qz)}{(1 - q)z} = f'(z)$$

for a differentiable function f in a given subset of \mathbb{C} . Furthermore, from (1) and (10), we obtain

$$(D_q f)(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}. \tag{11}$$

The intensive applications of the q -calculus in exploring new directions in various diverse areas of mathematics and physics have fascinated a number of researchers to work in several distinctive areas of the mathematical and physical sciences. The versatile applications of the q -derivative operator D_q makes it remarkably significant. Initially, in the year 1990, Ismail et al. [13] presented the idea of a q -extension of the class \mathcal{S}^* of starlike functions. However, historically speaking, in the article [14] published in 1989, Srivastava gave a firm footing on the usages of the q -calculus and the basic (or q -) hypergeometric functions:

$${}_t\Phi_s \left(\tau, s \in \mathbb{N}_0 = \{0, 1, 2, \dots\} = \mathbb{N} \cup \{0\} \right)$$

in the study of Geometric Function Theory (GFT) (see, for details, [14], pp. 347 *et seq.*; see also [15–19]).

We find it to be worthwhile to mention here that, more recently, the state-of-the-art survey and applications of the operators of the q -calculus and the fractional q -calculus such as the q -derivative operator and the fractional q -derivative operators in Geometric Function Theory of Complex Analysis were systematically presented in a survey-cum-expository review article by Srivastava [20]. In this same survey-cum-expository review article by Srivastava [20], the triviality and inconsequential nature of the so-called (p, q) -calculus, associated with an obviously redundant parameter p , was clearly revealed (see, for details, [20], p. 340).

In the advancement of Geometric Function Theory of Complex Analysis, the aforementioned works [13,20] have inspired a number of researchers to contribute significantly toward this subject. Several convolution and fractional q -operators that have been already studied were surveyed in the above-cited work [20]. For example, Kanas and Răducanu [7] introduced the q -analogue of Ruschewyh’s derivative operator, while the ideas of conic

domains and q -calculus, which also involved the Janowski functions, were combined systematically in [21]. We also briefly describe some of the recent developments based on the operators of the q -calculus. For instance, for some subclasses of q -starlike functions, various inclusion properties, coefficient inequalities, and sufficient conditions were studied by Srivastava et al. [22]. Subsequently, Srivastava et al. [23] systematically generalized their work [22]. In fact, Srivastava et al. (see [22,23]) used the q -calculus and the Janowski functions in order to define three new subclasses of q -starlike functions. Moreover, several authors (see, for example, [22–28]) have concentrated upon the classes of q -starlike functions related with the Janowski and other functions from several different viewpoints. For some more recent investigations about q -calculus, one may refer to such works as those in [29–37].

Definition 9. (see [13]) A function $f \in \mathcal{A}$ is said to be in the function class \mathcal{S}_q^* if

$$f(0) = f'(0) - 1 = 0 \quad (12)$$

and

$$\left| \frac{z}{f(z)} (D_q f)(z) - \frac{1}{1-q} \right| \leq \frac{1}{1-q}. \quad (13)$$

We find it to be worthwhile to mention that the above inequality in the limit as $q \rightarrow 1-$ yields

$$\left| w - \frac{1}{1-q} \right| \leq \frac{1}{1-q}.$$

The last inequality represents a closed disk which geometrically depicts the right-half plane. Furthermore, the class \mathcal{S}_q^* of q -starlike functions naturally yields, in the limit when $q \rightarrow 1-$, the familiar class \mathcal{S}^* of starlike function in \mathbb{U} . Furthermore, in an article published by Uçar [38], the equivalent form of the conditions in (12) and (13) is given as follows:

$$\frac{z}{f(z)} (D_q f)(z) \prec \hat{p}(z) \quad \left(\hat{p}(z) = \frac{1+z}{1-qz} \right).$$

We recall that the notation \mathcal{S}_q^* for q -starlike functions was used earlier by Sahoo and Sharma [39].

On the account of the principle of subordination in conjunction with the aforementioned q -calculus, the following function class $k\text{-}\mathcal{P}_q$ is presented next.

Definition 10. (see [26,28,40]) A function p of the class \mathcal{A} is said to be in the class $k\text{-}\mathcal{P}_q$ if and only if

$$p(z) \prec \hat{p}_k(z) \quad \left(\hat{p}_k(z) = \frac{2p_k(z)}{(1+q) + (1-q)p_k(z)} \right), \quad (14)$$

where $p_k(z)$ is defined by (5).

Geometrically, the function $p(z) \in k\text{-}\mathcal{P}_q$ takes on all values from the domain $\Omega_{k,q}$, which is defined as follows (see [26,28,40]):

$$\Omega_{k,q} = \left\{ w : \Re \left(\frac{(1+q)w}{(q-1)w+2} \right) > k \left| \frac{(1+q)w}{(q-1)w+2} - 1 \right| \right\}.$$

We now give the generalization of the class $k\text{-}\mathcal{P}[A, B]$ by replacing the function $p_k(z)$ in (9) by the function $\hat{p}_k(z)$ which is involved in (14).

The replacement of the function $p_k(z)$ in (9) by the function $\hat{p}_k(z)$, which is also involved in (14), gives rise to another way to generalize the class $k\text{-}\mathcal{P}[A, B]$ in Definition 6. The appropriate definition of the corresponding q -extension of the class $k\text{-}\mathcal{P}[A, B]$ is given below.

Definition 11. A function $h \in \mathcal{P}$ is said to belong to the class $\mathcal{P}(q, k, A, B)$ if and only if

$$h(z) \prec \frac{(A + 1)\widehat{p}_k(z) - (A - 1)}{(B + 1)\widehat{p}_k(z) - (B - 1)} \quad (-1 \leq B < A \leq 1; k \geq 0),$$

where

$$\widehat{p}_k(z) = \frac{2p_k(z)}{(1 + q) + (1 - q)p_k(z)}$$

and $p_k(z)$ is defined by (5).

Geometrically, the function $p \in \mathcal{P}(q, k, A, B)$ takes on all values from the domain $\Omega_k[q, A, B]$ which is defined as follows:

$$\Omega_k[q, A, B] = \left\{ w : \Re \left(\frac{(1 + q)\{(B - 1)w - (A - 1)\}}{\{(B + 3) + q(B - 1)\}w - \{(A + 3) + q(A - 1)\}} \right) > k \left| \frac{(1 + q)\{(B - 1)w - (A - 1)\}}{\{(B + 3) + q(B - 1)\}w - \{(A + 3) + q(A - 1)\}} - 1 \right| \right\}.$$

The domain $\Omega_k[q, A, B]$ represents certain conic type regions which involve the q -calculus.

In our application based upon the above definition (see Definition 11), we introduce and study the corresponding q -extension of the function class $k\text{-}\mathcal{S}^*[A, B]$ as follows.

Definition 12. A normalized analytic function f of the form (1) is said to belong to the class $\mathcal{S}^*(q, k, A, B)$ if and only if

$$\Re(\mathcal{F}(q, k, A, B)) > k|\mathcal{F}(q, k, A, B) - 1|,$$

where

$$\mathcal{F}(q, k, A, B) = \frac{(1 + q) \left\{ (B - 1) \frac{z(D_q f)(z)}{f(z)} - (A - 1) \right\}}{\{(B + 3) + q(B - 1)\} \frac{z(D_q f)(z)}{f(z)} - \{(A + 3) + q(A - 1)\}}. \tag{15}$$

Equivalently, we have

$$\frac{z(D_q f)(z)}{f(z)} \in \mathcal{P}(q, k, A, B). \tag{16}$$

Each of the following special cases of the above-defined function class $\mathcal{S}^*(q, k, A, B)$ is worthy of note.

I. Upon setting

$$k = 0, \quad A = 1 - 2\alpha \quad \text{and} \quad B = -1 \quad (0 \leq \alpha < 1),$$

if we let $q \rightarrow 1-$ in Definition 12, we are led to the class $\mathcal{S}^*(\alpha)$ which was introduced and studied by Silverman (see [41]).

II. If, after putting

$$A = 1 \quad \text{and} \quad B = -1,$$

we let $q \rightarrow 1-$ in Definition 12, we get the function class $k\text{-}\mathcal{ST}$. This class was studied by Kanas and Wiśniowska [4].

III. If we first put

$$A = 1 - 2\alpha \quad (0 \leq \alpha < 1) \quad \text{and} \quad B = -1,$$

and then let $q \rightarrow 1-$ in Definition 12, we have the class $\mathcal{SD}(k, \alpha)$ due to Shams et al. [9].

IV. By virtue of (16), in its special case when

$$k = 0, \quad A = 1 \quad \text{and} \quad B = -1,$$

if we let $q \rightarrow 1-$ in Definition 12, we deduce the class \mathcal{S}_q^* which was studied by Ismail et al. [13]; see also [14].

- V. If, in Definition 12, we let $q \rightarrow 1-$, we are led to the class $k\text{-}\mathcal{S}^*[A, B]$, which was introduced and studied by Noor and Sarfaraz [10].
- VI. If, in Definition 12, we put $k = 0$, we are led to the class $\mathcal{S}_q^*[A, B]$, which was introduced and studied by Srivastava et al. [27].

2. Sufficient Conditions

This section is devoted to the study of sufficient conditions for a function f to be in the class $\mathcal{S}^*(q, k, A, B)$.

Theorem 1. *A normalized analytic function f having the series expansion given in (1) is placed in the class $\mathcal{S}^*(q, k, A, B)$ if the following condition holds true:*

$$\sum_{n=2}^{\infty} \Lambda(n, k, A, B, q) |a_n| < |B - A|(1 + q), \tag{17}$$

where

$$\Lambda(n, k, A, B, q) = 4(k + 1)q[n - 1]_q + |\mathcal{L}(n, k, A, B, q)| \tag{18}$$

and

$$\mathcal{L}(n, k, A, B, q) = \{(B + 3) + q(B - 1)\}[n]_q - \{(A + 3) + q(A - 1)\}. \tag{19}$$

Proof. Assuming that the inequality (17) holds true, it suffices to show that

$$k|\mathcal{F}(q, k, A, B) - 1| - \Re(\mathcal{F}(q, k, A, B) - 1) < 1,$$

where $\mathcal{F}(q, k, A, B)$ is given by (15).

We now have

$$\begin{aligned} & (k + 1)|\mathcal{F}(q, k, A, B) - 1| \\ & \leq (k + 1) \left| \frac{(1 + q)\{(B - 1)z(D_q f)(z) - (A - 1)f(z)\}}{\mathcal{M}(A, B, k, q)} - 1 \right| \\ & = 4(k + 1) \left| \frac{f(z) - z(D_q f)(z)}{\mathcal{M}(A, B, k, q)} \right| \\ & = 4(k + 1) \left| \frac{\sum_{n=2}^{\infty} (1 - [n]_q) a_n z^n}{(B - A)(1 + q)z + \sum_{n=2}^{\infty} \mathcal{L}(n, k, A, B, q) a_n z^n} \right| \\ & \leq \frac{4(k + 1) \sum_{n=2}^{\infty} |1 - [n]_q| |a_n|}{|(B - A)(1 + q)| - \sum_{n=2}^{\infty} |\mathcal{L}(n, k, A, B, q)| |a_n|}, \tag{20} \end{aligned}$$

where

$$\mathcal{M}(A, B, k, q) = [(B + 3) + q(B - 1)]z(D_q f)(z) - [(A + 3) + q(A - 1)]f(z)$$

and $\mathcal{L}(n, k, A, B, q)$ is given by (19).

The last expression in (20) is bounded above by 1 if

$$\sum_{n=2}^{\infty} \Lambda(n, k, A, B, q) |a_n| < |B - A|(1 + q).$$

Hence the proof of Theorem 1 is completed. \square

Each of the following (known or new) corollaries and consequences of Theorem 1 is worthy of note.

1. Upon letting $q \rightarrow 1-$, Theorem 1 yields the following known result.

Corollary 1. (see [10]) *A normalized analytic function f having series expansion given in (1) is in the class $k\text{-}\mathcal{S}^*[A, B]$ if the following condition holds true:*

$$\sum_{n=2}^{\infty} \{2(k+1)(n-1) + |n(B+1) - (A+1)|\} |a_n| < |B - A|.$$

2. If we first set

$$k = 0, \quad A = 1 - 2\alpha \quad (0 \leq \alpha < 1) \quad \text{and} \quad B = -1$$

and then let $q \rightarrow 1-$, then Theorem 1 leads to the following known result.

Corollary 2. (see [41]) *A normalized analytic function f having series expansion given in (1) is in the class $\mathcal{S}^*(\alpha)$ if the following condition holds true:*

$$\sum_{n=2}^{\infty} (n - \alpha) |a_n| < 1 - \alpha \quad (0 \leq \alpha < 1).$$

3. If we first put

$$A = 1 \quad \text{and} \quad B = -1$$

and then let $q \rightarrow 1-$ in Theorem 1, we get the following Corollary.

Corollary 3. (see [4]) *A normalized analytic function f having series expansion given in (1) is in the class $k\text{-}\mathcal{ST}$ if the following condition holds true:*

$$\sum_{n=2}^{\infty} \{n + k(n-1)\} |a_n| < 1.$$

4. If we first put

$$A = 1 - 2\alpha \quad (0 \leq \alpha < 1) \quad \text{and} \quad B = -1$$

and then let $q \rightarrow 1-$ in Theorem 1, we get the following known result.

Corollary 4. (see [9]) *A normalized analytic function f having series expansion given in (1) is in the class $\mathcal{SD}(k, \alpha)$ if it satisfies the following condition:*

$$\sum_{n=2}^{\infty} \{n(k+1) - (k+\alpha)\} |a_n| < (1 - \alpha).$$

3. Closure Theorems

Let the functions $f_{\varkappa}(z)$ ($\varkappa = 1, 2, 3, \dots, l$) be defined by

$$f_{\varkappa}(z) = z + \sum_{n=2}^{\infty} a_{n,\varkappa} z^n \quad (z \in \mathbb{U}). \quad (21)$$

Now we present and prove the following result.

Theorem 2. Let the functions $f_{\varkappa}(z)$ ($\varkappa = 1, 2, 3, \dots, l$) defined by (21) be in the class $\mathcal{S}^*(q, k, A, B)$. Then the function $T \in \mathcal{S}^*(q, k, A, B)$, where

$$T(z) = \sum_{\varkappa=1}^l \Gamma_{\varkappa} f_{\varkappa}(z) \quad \left(\Gamma_{\varkappa} \geq 0 \quad \sum_{\varkappa=1}^l \Gamma_{\varkappa} = 1 \right).$$

Proof. From (21), we have

$$T(z) = z + \sum_{n=2}^{\infty} \left(\sum_{\varkappa=1}^l \Gamma_{\varkappa} a_{n,\varkappa} \right) z^n.$$

Now, making use of Theorem 1, we find that

$$\begin{aligned} & \sum_{n=2}^{\infty} \Lambda(n, k, A, B, q) \left| \sum_{\varkappa=1}^l \Gamma_{\varkappa} a_{n,\varkappa} \right| \\ &= \sum_{\varkappa=1}^l \Gamma_{\varkappa} \left(\sum_{n=2}^{\infty} \Lambda(n, k, A, B, q) |a_{n,\varkappa}| \right) \\ &\leq \sum_{\varkappa=1}^l \Gamma_{\varkappa} |B - A| (1 + q) = |B - A| (1 + q) \quad \left(\sum_{\varkappa=1}^l \Gamma_{\varkappa} = 1 \right), \end{aligned}$$

where $\Lambda(n, k, A, B, q)$ is given by (18).

Finally, by applying Theorem 1, the proof of Theorem 2 is completed. \square

Theorem 3. The class $\mathcal{S}^*(q, k, A, B)$ is closed under convex combination.

Proof. Let the functions $f_{\varkappa}(z)$ ($\varkappa = 1, 2$) defined by (21) be in the class $\mathcal{S}^*(q, k, A, B)$. It is enough to show that

$$g(z) = \varrho f_1(z) + (1 - \varrho) f_2(z) \quad (0 \leq \varrho \leq 1)$$

is in the class $\mathcal{S}^*(q, k, A, B)$. Since

$$g(z) = z + \sum_{n=2}^{\infty} (\varrho a_{n,1} + (1 - \varrho) a_{n,2}) z^n \quad (0 \leq \varrho \leq 1).$$

By Theorem 1, we have

$$\begin{aligned} & \sum_{n=2}^{\infty} \Lambda(n, k, A, B, q) |(\varrho a_{n,1} + (1 - \varrho) a_{n,2})| \\ &\leq \sum_{n=2}^{\infty} \Lambda(n, k, A, B, q) |\varrho a_{n,1}| + \sum_{n=2}^{\infty} \Lambda(n, k, A, B, q) |(1 - \varrho) a_{n,2}| \\ &\leq \varrho |A - B| (q + 1) + (1 - \varrho) |A - B| (q + 1) = |A - B| (q + 1), \end{aligned}$$

where $\Lambda(n, k, A, B, q)$ is given by (18). This evidently completes the proof of Theorem 3. \square

4. The Fekete-Szegő Functional

The problem to evaluate the maximum values for the functional $|a_3 - \mu a_2^2|$ is what we call the Fekete-Szegő problem. For μ , a real or complex number, this functional has been extensively studied from different viewpoints and perspectives. While studying this functional, some interesting geometric characteristics of the image domains were obtained by many authors (see, for example, [25,27,37,42]). In this section, we aim to investigate the

Fekete-Szegö functional $|a_3 - \mu a_2^2|$ for the class $\mathcal{S}^*(q, k, A, B)$ of Janowski type q -starlike functions which is associated with a certain conic domain.

In order to prove the result of this section, we need the following Lemma 1.

Lemma 1. (see [43,44]) *Let $p \in \mathcal{P}$ be in the Carathéodory class of functions with positive real part in \mathbb{U} and have the following form:*

$$p(z) = 1 + c_1z + c_2z^2 + \dots$$

Then, for any number $v \in \mathbb{C}$,

$$|c_2 - vc_1^2| \leq 2 \max\{1, |1 - 2v|\}$$

and, for the case when $v \in \mathbb{R}$,

$$|c_2 - vc_1^2| \leq \begin{cases} -4v + 2 & (v \leq 0) \\ 2 & (0 \leq v \leq 1) \\ 4v - 2 & (v \geq 1). \end{cases} \tag{22}$$

For $v < 0$ or $v > 1$, the equality in (22) holds true if and only if

$$p(z) = \frac{1+z}{1-z}$$

for one of its rotations. When $0 < v < 1$, the equality in (22) holds true whenever

$$p(z) = \frac{1+z^2}{1-z^2}$$

for one of its rotations. For $v = 0$, the equality in (22) is satisfied if and only if

$$p(z) = \left(\frac{1+\rho}{2}\right) \frac{1+z}{1-z} + \left(\frac{1-\rho}{2}\right) \frac{1-z}{1+z} \quad (0 \leq \rho \leq 1)$$

for one of its rotations. Furthermore, if we set $v = 1$, then the equality in (22) holds true if $p(z)$ is a reciprocal of one of the functions such that the equality holds true in the case when $v = 0$.

Theorem 4. *Let the function $f(z)$ having the form (1) be in the class $\mathcal{S}^*(q, k, A, B)$ with $0 \leq k \leq 1$. Then, for $\mu \in \mathbb{C}$,*

$$|a_3 - \mu a_2^2| \leq \left(\frac{A-B}{4q}\right) P_1 \max\left\{1, \left| \frac{P_2}{P_1} + \frac{Y(q)}{4q} P_1 - \frac{\mu(A-B)(1+q)^2}{4q} P_1 \right| \right\}. \tag{23}$$

Furthermore, for a real parameter μ , it is asserted that

$$|a_3 - \mu a_2^2| \leq \begin{cases} \left(\frac{A-B}{4q}\right) \left(P_2 + \frac{Y(q)}{4q} P_1^2 - \frac{\mu(1+q)^2}{4q} P_1^2\right) & (\mu < \sigma_1) \\ \left(\frac{A-B}{4q}\right) P_1 & (\sigma_1 \leq \mu \leq \sigma_2) \\ \left(\frac{B-A}{4q}\right) \left(P_2 + \frac{Y(q)}{4q} P_1^2 - \frac{\mu(1+q)^2}{4q} P_1^2\right) & (\mu > \sigma_2), \end{cases} \tag{24}$$

where

$$Y(q) = [(A - B) + (A - 2B - 3)q + (1 - B)q^2], \tag{25}$$

$$\sigma_1 = \frac{4q}{(A-B)(1+q)^2 P_1^2} \left(\frac{Y(q)}{4q} P_1^2 - P_1 + P_2 \right),$$

$$\sigma_2 = \frac{4q}{(A-B)(1+q)^2 P_1^2} \left(P_1 + P_2 + \frac{Y(q)}{4q} P_1^2 \right)$$

and P_1 and P_2 are defined by (6) and (7), respectively.

Proof. We start by proving that, for $f \in \mathcal{S}^*(q, k, A, B)$, the inequalities stated in (23) and (24) hold true. Let us consider a function $m(z)$ given by

$$m(z) = \frac{z(D_q f)(z)}{f(z)} \quad (\forall z \in \mathbb{U}).$$

Then, since $f \in \mathcal{S}^*(q, k, A, B)$, we have the following subordination relation:

$$m(z) \prec \phi(z), \quad (26)$$

where

$$\phi(z) = \frac{(1+q)(A+1)(p_k(z)-1) + 2(p_k(z)+1 - q(p_k(z)-1))}{(1+q)(B+1)(p_k(z)-1) + 2(p_k(z)+1 - q(p_k(z)-1))}.$$

Thus, if

$$p_k(z) = 1 + P_1 z + P_2 z^2 + \dots,$$

then we find after some simplification that

$$\phi(z) = 1 + \frac{1}{4}(A-B)(q+1)P_1 z + \frac{1}{16}(A-B)(q+1) \cdot [4P_2 - (3-q+(q+1)B)P_1^2] z^2 + \dots.$$

Now, in light of (26), it is obvious that the function $h(z)$ given by

$$h(z) = \frac{1 + \phi^{-1}(m(z))}{1 - \phi^{-1}(m(z))} = 1 + c_1 z + c_2 z^2 + \dots \quad (\forall z \in \mathbb{U})$$

is analytic and has a positive real part in the open unit disk \mathbb{U} . We also have

$$m(z) = \phi \left(\frac{h(z)-1}{h(z)+1} \right), \quad (27)$$

where

$$m(z) = \frac{z(D_q f)(z)}{f(z)} = 1 + qa_2 z + [(q+q^2)a_3 - qa_2^2] z^2 + \dots \quad (28)$$

and

$$\phi \left(\frac{h(z)-1}{h(z)+1} \right) = 1 + \frac{1}{8}(A-B)(q+1)P_1 c_1 z + \frac{1}{8}(A-B)(q+1) \cdot \left[P_1 c_2 + \left(\frac{P_2}{2} - \frac{(3-q+(q+1)B)}{8} P_1^2 - \frac{P_1}{2} \right) c_1^2 \right] z^2 + \dots \quad (29)$$

Next, from the equations (28) and (29), we find that

$$a_2 = \frac{(A-B)(q+1)}{8q} P_1 c_1 \quad (30)$$

and

$$a_3 = \frac{(A-B)}{8q} \left[P_1 c_2 + \left(\frac{P_2}{2} - \frac{P_1}{2} + \frac{Y(q)P_1^2}{8q} \right) c_1^2 \right], \quad (31)$$

where $Y(q)$ is given by (25). Thus, clearly, we get

$$|a_3 - \mu a_2^2| = \left(\frac{A - B}{8q}\right) P_1 |c_2 - \zeta c_1^2|, \tag{32}$$

where

$$\zeta = \frac{1}{2} \left(1 - \frac{P_2}{P_1} - \frac{Y(q)P_1}{4q} + \frac{\mu(A - B)(1 + q)^2 P_1}{4q} \right).$$

Finally, by applying the above Lemma in conjunction with (32), we obtain the result asserted by Theorem 4. \square

5. Partial Sums for the Function Class $\mathcal{S}^*(q, k, A, B)$

In this section, we are propose to consider the ratio of the partial sums for a function having the form (1) to the following sequence of its partia sums:

$$f_j(z) = z + \sum_{n=2}^j a_n z^n$$

whenever the coefficients of f are sufficiently small in order to satisfy the condition (17). We also find sharp lower bounds for each of the following expressions:

$$\Re\left(\frac{f(z)}{f_j(z)}\right), \quad \Re\left(\frac{f_j(z)}{f(z)}\right), \quad \Re\left(\frac{D_q f(z)}{D_q f_j(z)}\right) \quad \text{and} \quad \Re\left(\frac{(D_q f_j)(z)}{(D_q f)(z)}\right).$$

Theorem 5. *If the function f of the form (1) satisfies condition (17), then*

$$\Re\left(\frac{f(z)}{f_j(z)}\right) \geq 1 - \frac{1}{\rho_{j+1}} \quad (\forall z \in \mathbb{U}) \tag{33}$$

and

$$\Re\left(\frac{f_j(z)}{f(z)}\right) \geq \frac{\rho_{j+1}}{1 + \rho_{j+1}} \quad (\forall z \in \mathbb{U}), \tag{34}$$

where

$$\rho_j = \frac{\Lambda(j, k, A, B, q)}{(1 + q)|A - B|} \tag{35}$$

and $\Lambda(j, k, A, B, q)$ is given by (18).

Proof. It is easy to verify that

$$\rho_{n+1} \geq \rho_n \geq 1 \quad \text{for} \quad n \geq 2.$$

Thus, in order to prove the inequality (33), we set

$$\begin{aligned} \rho_{j+1} \left[\frac{f(z)}{f_j(z)} - \left(1 - \frac{1}{\rho_{j+1}} \right) \right] &= \frac{1 + \sum_{n=2}^j a_n z^{n-1} + \rho_{j+1} \sum_{n=j+1}^{\infty} a_n z^{n-1}}{1 + \sum_{n=2}^j a_n z^{n-1}} \\ &= \frac{1 + h_1(z)}{1 + h_2(z)}. \end{aligned}$$

We now consider

$$\frac{1 + h_1(z)}{1 + h_2(z)} = \frac{1 + w(z)}{1 - w(z)}.$$

We then find after some suitable simplification that

$$w(z) = \frac{h_1(z) - h_2(z)}{2 + h_1(z) + h_2(z)}.$$

Thus, clearly, we have

$$w(z) = \frac{\rho_{j+1} \sum_{n=j+1}^{\infty} a_n z^{n-1}}{2 + 2 \sum_{n=2}^j a_n z^{n-1} + \rho_{j+1} \sum_{n=j+1}^{\infty} a_n z^{n-1}}.$$

By applying the trigonometric inequalities together with $|z| < 1$, we arrive at the following inequality:

$$|w(z)| \leq \frac{\rho_{j+1} \sum_{n=j+1}^{\infty} |a_n|}{2 - 2 \sum_{n=2}^j |a_n| - \rho_{j+1} \sum_{n=j+1}^{\infty} |a_n|}.$$

We can now see that

$$|w(z)| \leq 1$$

if and only if

$$2\rho_{j+1} \sum_{n=j+1}^{\infty} |a_n| \leq 2 - 2 \sum_{n=2}^j |a_n|,$$

which implies that

$$\sum_{n=2}^j |a_n| + \rho_{j+1} \sum_{n=j+1}^{\infty} |a_n| \leq 1. \tag{36}$$

Finally, in order to prove the inequality in (33), it suffices to show that the left-hand side of (36) is bounded above by the following sum:

$$\sum_{n=2}^{\infty} \rho_n |a_n|,$$

which is equivalent to

$$\sum_{n=2}^j (\rho_n - 1) |a_n| + \sum_{n=j+1}^{\infty} (\rho_n - \rho_{j+1}) |a_n| \geq 0. \tag{37}$$

Thus, by virtue of (37), the proof of the inequality in (33) is now complete.

Next, in order to prove the inequality (34), we set

$$\begin{aligned} (1 + \rho_{j+1}) \left(\frac{f_j(z)}{f(z)} - \frac{\rho_{j+1}}{1 + \rho_{j+1}} \right) &= \frac{1 + \sum_{n=2}^j a_n z^{n-1} - \rho_{j+1} \sum_{n=j+1}^{\infty} a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} a_n z^{n-1}} \\ &= \frac{1 + w(z)}{1 - w(z)}, \end{aligned}$$

where

$$|w(z)| \leq \frac{(1 + \rho_{j+1}) \sum_{n=j+1}^{\infty} |a_n|}{2 - 2 \sum_{n=2}^j |a_n| - (\rho_{j+1} - 1) \sum_{n=j+1}^{\infty} |a_n|} \leq 1. \tag{38}$$

This last inequality in (38) is equivalent to the following inequality:

$$\sum_{n=2}^j |a_n| + \rho_{j+1} \sum_{n=j+1}^{\infty} |a_n| \leq 1. \tag{39}$$

Finally, it is easy to check that the left-hand side of the inequality in (39) is bounded above by the following sum:

$$\sum_{n=2}^{\infty} \rho_n |a_n|,$$

so we have completed the proof of the assertion (34). The proof of Theorem 5 is thus completed. □

We next turn to the ratios involving derivatives.

Theorem 6. *If a function f of the form (1) satisfies the condition (17), then*

$$\Re \left(\frac{(D_q f)(z)}{(D_q f_j)(z)} \right) \geq 1 - \frac{[j + 1]_q}{\rho_{j+1}} \quad (\forall z \in \mathbb{U}) \tag{40}$$

and

$$\Re \left(\frac{(D_q f_j)(z)}{(D_q f)(z)} \right) \geq \frac{\rho_{j+1}}{\rho_{j+1} + [j + 1]_q} \quad (\forall z \in \mathbb{U}), \tag{41}$$

where ρ_j is given by (35).

Proof. Theorem 6 can be proved by using arguments similar to those of Theorem 5. □

6. Analytic Functions with Negative Coefficients

In this section, we consider certain new subclasses of q -starlike functions associated with the generalized conic type domain, but with negative coefficients. Let \mathcal{T} be a subset of the normalized analytic function class \mathcal{A} consisting of functions with negative Taylor-Maclaurin coefficients, that is,

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n. \tag{42}$$

We also let $\mathcal{TS}^*(A, B, q, k)$ be the subclass of the analytic function class \mathcal{T} . We see that the function class $\mathcal{TS}^*(A, B, q, k)$ is a subclass of $\mathcal{S}^*(A, B, q, k)$. We now state the following distortion theorems for the function class $\mathcal{TS}^*(A, B, q, k)$.

Theorem 7. *If $f \in \mathcal{TS}^*(A, B, q, k)$, then*

$$r - \frac{|B - A|(1 + q)}{\Lambda(2, k, A, B, q)} r^2 \leq |f(z)| \leq r + \frac{|B - A|(1 + q)}{\Lambda(2, k, A, B, q)} r^2 \quad (|z| = r; 0 < r < 1),$$

where $\Lambda(2, k, A, B, q)$ is given by (18).

Proof. By making use of Theorem 1, we can deduce the following inequality:

$$\Lambda(2, k, A, B, q) \sum_{n=2}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} \Lambda(n, k, A, B, q) |a_n| < |B - A|(1 + q),$$

which implies that

$$|f(z)| \leq r + \sum_{n=2}^{\infty} |a_n| r^n \leq r + r^2 \sum_{n=2}^{\infty} |a_n| \leq r + \frac{|B - A|(1 + q)}{\Lambda(2, k, A, B, q)} r^2.$$

On the other hand, we can see that

$$|f(z)| \geq r - \sum_{n=2}^{\infty} |a_n| r^n \geq r - r^2 \sum_{n=2}^{\infty} |a_n| \geq r - \frac{|B - A|(1 + q)}{\Lambda(2, k, A, B, q)} r^2.$$

This completes the proof of Theorem 7. \square

As a special case of Theorem 7, if first we set

$$k = 0, A = 1 - 2\alpha \quad (0 \leq \alpha < 1) \quad \text{and} \quad B = -1,$$

and then let $q \rightarrow 1-$, we arrive at the following known result.

Corollary 5. (see [41]) *If $f \in \mathcal{TS}^*(\alpha)$, then*

$$r - \frac{1 - \alpha}{2 - \alpha} r^2 \leq |f(z)| \leq r + \frac{1 - \alpha}{2 - \alpha} r^2 \quad (|z| = r; 0 < r < 1).$$

The proof of the following result is similar to the proof of Theorem 7. We, therefore, only present the statement here.

Theorem 8. *If $f \in \mathcal{TS}^*(A, B, q, k)$, then*

$$1 - \frac{2|B - A|(1 + q)}{\Lambda(2, k, A, B, q)} r \leq |f'(z)| \leq 1 + \frac{2|B - A|(1 + q)}{\Lambda(2, k, A, B, q)} r \quad (|z| = r; 0 < r < 1)$$

where $\Lambda(2, k, A, B, q)$ is given by (18).

7. Concluding Remarks and Observations

In our present work, we are motivated by the well-established usage of the basic (or q -) calculus and the fractional basic (or q -) calculus in Geometric Function Theory of Complex Analysis as described in the survey-cum-expository review article by Srivastava [20]. Here, in our present investigation, we successfully studied the q -extension of conic domains with the Janowski functions. We derived coefficient estimates and the sufficient conditions and obtained the lower bounds for the ratios of some functions belonging to this newly-defined function class and the sequences of their partial sums. We also derived several properties of a corresponding class of q -starlike functions with negative Taylor-Maclaurin coefficients including (for example) distortion theorems. The importance of the results demonstrated in this paper is obvious from the fact that these results would generalize and extend various previously known results derived in many earlier works. Moreover, with a view to motivating and encouraging further researches on the subject of our investigation, we have chosen to cite several recently-published articles (see, for example, [45–48]) on a wide variety of developments in Geometric Function Theory of Complex Analysis.

As mentioned in the introduction, the basic (or q -) polynomials and the basic (or q -) series, especially the basic (or q -) hypergeometric functions and basic (or q -) hypergeometric polynomials, are relevant and potentially useful in many areas. Moreover, as we remarked above and in Section 1, in the recently-published survey-cum-expository review article

by Srivastava [20], the so-called (p, q) -calculus was clearly demonstrated to be a relatively insignificant and inconsequential variation of the traditional q -calculus, the extra parameter p being redundant or superfluous (see, for details, [20], p. 340). This observation by Srivastava [20] will indeed apply also to any attempt to produce the rather straightforward (p, q) -variations of the results which we have presented in this paper.

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