

Article

The Irregularity and Modular Irregularity Strength of Fan Graphs

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Abstract: For a simple graph G with no isolated edges and at most, one isolated vertex, a labeling $\varphi : E(G) \rightarrow \{1, 2, \dots, k\}$ of positive integers to the edges of G is called irregular if the weights of the vertices, defined as $wt_\varphi(v) = \sum_{u \in N(v)} \varphi(uv)$, are all different. The irregularity strength of a graph G is known as the maximal integer k , minimized over all irregular labelings, and is set to ∞ if no such labeling exists. In this paper, we determine the exact value of the irregularity strength and the modular irregularity strength of fan graphs.

Keywords: irregular labeling; modular irregular labeling; irregularity strength; modular irregularity strength; fan graph

MSC: 05C78, 05C70



Citation: Bača, M.; Kimáková, Z.; Lascsóková, M.; Semaničová-Feňovčíková, A. The Irregularity and Modular Irregularity Strength of Fan Graphs. *Symmetry* **2021**, *13*, 605. <https://doi.org/10.3390/sym13040605>

Academic Editor: Alice Miller

Received: 5 March 2021

Accepted: 4 April 2021

Published: 6 April 2021

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1. Introduction

It is well-known that a simple graph of an order of at least two must contain a pair of vertices with the same degree. However, a multigraph can be irregular, that is, each vertex can have a different degree. By Frieze et al. in [1], a natural question would be: What is the least number of edges we would need to add to a graph in order to convert a simple graph into an irregular multigraph?

Motivated by this question, Chartrand et al. in [2] introduced an *edge k -labeling* $\varphi : E(G) \rightarrow \{1, 2, \dots, k\}$ of a graph G (i.e., a mapping that assigns numbers $1, 2, \dots, k$ to the edges of G) with the property that the weights of the vertices are all different. The *weight of a vertex* $v \in V(G)$ is defined as $wt_\varphi(v) = \sum_{u \in N(v)} \varphi(uv)$, where $N(v)$ denotes the set of neighbors of v in G . Such labelings were called *irregular assignments*. Note that as the induced vertex weights are all distinct and the only edges of a graph are labeled with the numbers $1, 2, \dots, k$ this assignment can also be called *vertex irregular edge k -labeling*. The *irregularity strength* $s(G)$ of a graph G is known as the maximal integer k , minimized over all irregular assignments. This means that the irregularity strength of a graph G is the minimum k for which a graph admits an irregular assignment using the number k as the largest edge label. If no such labeling of G exists, then $s(G) = \infty$. Clearly, the irregularity strength is finite only for graphs that contain, at most, one isolated vertex and no isolated edges. To view the irregularity strength via the degree-based problem, this graph invariant is connected to the maximal number of edges joining any pair of vertices in an irregular multigraph corresponding to the given graph G .

The lower bound of the irregularity strength is given in [2] in the form

$$s(G) \geq \max \left\{ \frac{n_i + i - 1}{i} : 1 \leq i \leq \Delta \right\}, \quad (1)$$

where n_i denotes the number of vertices of degree i and Δ is the maximum degree of the graph G . For d -regular graphs of order n , the lower bound (1) reduces to

$$s(G) \geq \frac{n+d-1}{d}. \quad (2)$$

Faudree and Lehel in [3] showed that if G is a d -regular graph of order n , $d \geq 2$, then $s(G) \leq \lceil \frac{n}{2} \rceil + 9$, and they conjectured that there exists an absolute constant C such that $s(G) \leq \frac{n}{d} + C$. For general graphs with no component of order at most 2, it is known that $s(G) \leq |V(G)| - 1$, see [4,5]. This upper bound was gradually improved by Cuckler and Lazebnik in [6], Przybyło in [7], Kalkowski, Karonski, and Pfender in [8], and recently by Majerski and Przybyło in [9]. Other interesting results on the irregularity strength can be found in [1,10].

The exact value of the irregularity strength of particular families of graphs are known, where among them are paths, complete graphs [2], cycles, most of the complete bipartite graphs, Turan graphs [11], generalized Petersen graphs [12], circulant graphs [13], and trees [14]. For more results, see [15].

A natural modification of an irregular assignment is a modular irregular assignment introduced in [16]. Edge k -labeling $\varphi : E(G) \rightarrow \{1, 2, \dots, k\}$ of positive integers to the edges of a graph G of order n is called a *modular irregular assignment* of G if the weight function $\vartheta : V(G) \rightarrow \mathbb{Z}_n$ defined by

$$\vartheta(v) = wt_\varphi(v) = \sum_{u \in N(v)} \varphi(uv) \quad (3)$$

is bijective and is called as the *modular weight* of the vertex v , where \mathbb{Z}_n is the group of integers modulo n . The *modular irregularity strength*, $ms(G)$, is defined as the minimum k for which G has a modular irregular assignment. If there is no such labeling for the graph G , then the value of $ms(G)$ is defined as ∞ .

In [16], a lower bound of the modular irregularity strength is established, and the exact values of this parameter for certain families of graphs, namely paths, cycles, stars, triangular graphs, and gear graphs are determined.

A *fan graph* F_n , $n \geq 2$ is a graph obtained by joining all vertices of path P_n on n vertices to a further vertex, called the centre. Thus, F_n contains $n + 1$ vertices, say, u_1, u_2, \dots, u_n, w , and $2n - 1$ edges, say, $u_i u_{i+1}$, $1 \leq i \leq n - 1$, and $u_i w$, $1 \leq i \leq n$.

In this paper, we determine the exact value of the irregularity strength and the modular irregularity strength of fan graphs F_n of order $n + 1$. The rest of the article is organized as follows. First we deal with the irregularity strength of fan graphs. We describe a desired labeling scheme that proves the exact value of the irregularity strength of fan graphs. We describe a labeling scheme with symmetrical distribution of even weights and odd weights of vertices u_i . We use this symmetrical distribution of the weights to prove that the weight of the centre w is always greater than the weights of u_i . It proves that the labeling scheme is a desired vertex irregular edge labeling that proves the exact value of the irregularity strength of fan graphs. Next, by modifications of this irregular assignment we obtain labelings that imply the results for the modular irregularity strength of fan graphs.

2. Results

2.1. Fan Graphs—The Irregularity Strength

The main result of this subsection is the following theorem.

Theorem 1. *Let F_n , $n \geq 3$, be a fan graph on $n + 1$ vertices. Then,*

$$s(F_n) = \begin{cases} 3, & \text{if } n = 2, \\ \lceil \frac{n+1}{3} \rceil, & \text{if } n \geq 3. \end{cases}$$

To prove the above-mentioned result, we present several lemmas. The first lemma gives a lower bound for the irregularity strength for the fan graphs.

Lemma 1. *Let $F_n, n \geq 3$, be a fan graph on $n + 1$ vertices. Then,*

$$s(F_n) \geq \left\lceil \frac{n+1}{3} \right\rceil.$$

Proof. According to the general lower bound (1), we have that the irregularity strength for the fan graphs $s(F_n) \geq \left\lceil \frac{n}{3} \right\rceil$ for $n \geq 4$. However, we can improve this bound. If we consider only vertices $u_i \in V(F_n), i = 1, 2, \dots, n$, and assume that an edge labeling φ is the irregular assignment of F_n with $s(F_n) = k$, then the smallest weight of each considered vertex is at least 2, and the largest weight admits the value at least $n + 1$, and at most $3k$. Thus, $n + 1 \leq 3k$. This implies

$$k = s(F_n) \geq \left\lceil \frac{n+1}{3} \right\rceil.$$

□

For $n \geq 3$, we define the edge labeling φ in the following way:

$$\begin{aligned} \varphi(u_i u_{i+1}) &= \left\lceil \frac{i-1}{3} \right\rceil + \left\lceil \frac{i}{3} \right\rceil, & \text{for } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor, \\ \varphi(u_{n-i} u_{n-i+1}) &= \left\lceil \frac{i-1}{3} \right\rceil + \left\lceil \frac{i+1}{3} \right\rceil, & \begin{cases} \text{for } 1 \leq i \leq \frac{n}{2} - 1 \text{ if } n \text{ is even,} \\ \text{for } 1 \leq i \leq \frac{n-1}{2} \text{ if } n \text{ is odd,} \end{cases} \\ \varphi(u_i w) &= \begin{cases} 1, & \text{for } i = 1, \\ 2, & \text{for } i = n, \end{cases} \\ \varphi(u_i w) &= \left\lceil \frac{i-2}{3} \right\rceil + \left\lceil \frac{i}{3} \right\rceil, & \begin{cases} \text{for } 2 \leq i \leq \frac{n}{2} + 1 \text{ if } n \text{ is even,} \\ \text{for } 2 \leq i \leq \frac{n-1}{2} \text{ if } n \text{ is odd,} \end{cases} \\ \varphi(u_{n-i} w) &= \left\lceil \frac{i}{3} \right\rceil + \left\lceil \frac{i+1}{3} \right\rceil, & \begin{cases} \text{for } 1 \leq i \leq \frac{n}{2} - 2 \text{ if } n \text{ is even,} \\ \text{for } 1 \leq i \leq \frac{n-1}{2} \text{ if } n \text{ is odd.} \end{cases} \end{aligned} \tag{4}$$

Now we prove that the above-defined labeling φ is an $\left\lceil \frac{n+1}{3} \right\rceil$ -labeling, and that the vertex weights induced by the labeling φ are all distinct.

The following lemma shows that under the edge labeling φ , the edge labels of F_n are bounded from above.

Lemma 2. *The labeling φ is an $\left\lceil \frac{n+1}{3} \right\rceil$ -labeling.*

Proof. Let $n \geq 3$ and let φ be the edge labeling of the fan graph F_n defined above. Let $b = \left\lceil \frac{n-1}{6} \right\rceil + \left\lceil \frac{n+1}{6} \right\rceil$ and $c = \left\lceil \frac{n-2}{6} \right\rceil + \left\lceil \frac{n+2}{6} \right\rceil$.

If n is odd, then

$$\begin{aligned} \max \left\{ \varphi(u_i u_{i+1}) : 1 \leq i \leq \frac{n-1}{2} \right\} &= \varphi(u_{\frac{n-1}{2}} u_{\frac{n+1}{2}}) = \left\lceil \frac{n-3}{6} \right\rceil + \left\lceil \frac{n-1}{6} \right\rceil < b, \\ \max \left\{ \varphi(u_{n-i} u_{n-i+1}) : 1 \leq i \leq \frac{n-1}{2} \right\} &= \varphi(u_{\frac{n+1}{2}} u_{\frac{n+3}{2}}) = \left\lceil \frac{n-3}{6} \right\rceil + \left\lceil \frac{n+1}{6} \right\rceil \leq b, \\ \max \left\{ \varphi(u_i w) : 2 \leq i \leq \frac{n-1}{2} \right\} &= \varphi(u_{\frac{n-1}{2}} w) = \left\lceil \frac{n-5}{6} \right\rceil + \left\lceil \frac{n-1}{6} \right\rceil < b, \\ \max \left\{ \varphi(u_{n-i} w) : 1 \leq i \leq \frac{n-1}{2} \right\} &= \varphi(u_{\frac{n+1}{2}} w) = \left\lceil \frac{n-1}{6} \right\rceil + \left\lceil \frac{n+1}{6} \right\rceil = b. \end{aligned}$$

If n is even, then

$$\max \left\{ \varphi(u_i u_{i+1}) : 1 \leq i \leq \frac{n}{2} \right\} = \varphi(u_{\frac{n}{2}} u_{\frac{n}{2}+1}) = \left\lceil \frac{n-2}{6} \right\rceil + \left\lceil \frac{n}{6} \right\rceil \leq c,$$

$$\begin{aligned}\max\{\varphi(u_{n-i}u_{n-i+1}) : 1 \leq i \leq \frac{n}{2} - 1\} &= \varphi(u_{\frac{n}{2}+1}u_{\frac{n}{2}+2}) = \left\lceil \frac{n-4}{6} \right\rceil + \left\lceil \frac{n}{6} \right\rceil \leq c, \\ \max\{\varphi(u_i w) : 2 \leq i \leq \frac{n}{2} + 1\} &= \varphi(u_{\frac{n}{2}+1}w) = \left\lceil \frac{n-2}{6} \right\rceil + \left\lceil \frac{n+2}{6} \right\rceil = c, \\ \max\{\varphi(u_{n-i}w) : 1 \leq i \leq \frac{n}{2} - 2\} &= \varphi(u_{\frac{n}{2}+2}w) = \left\lceil \frac{n-4}{6} \right\rceil + \left\lceil \frac{n-2}{6} \right\rceil < c.\end{aligned}$$

It is easy to see that if n is odd, then the parameter $b = \left\lceil \frac{n-1}{6} \right\rceil + \left\lceil \frac{n+1}{6} \right\rceil = \left\lceil \frac{n+1}{3} \right\rceil$, and if n is even, then the parameter $c = \left\lceil \frac{n-2}{6} \right\rceil + \left\lceil \frac{n+2}{6} \right\rceil = \left\lceil \frac{n+1}{3} \right\rceil$. Thus, the labeling φ is an edge $\left\lceil \frac{n+1}{3} \right\rceil$ -labeling of F_n . \square

The next two lemmas show the induced weights of the vertices of F_n under the edge labeling φ .

Lemma 3. *The weights of the vertices u_i , $1 \leq i \leq n$ of the fan graph F_n , under the labeling φ , admit the values*

$$wt_{\varphi}(u_i) = \begin{cases} 2i, & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\ 2n - 2i + 3, & \text{for } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n. \end{cases}$$

Proof. One can check that

$$wt_{\varphi}(u_1) = \varphi(u_1u_2) + \varphi(u_1w) = 2$$

and for $i = 2, 3, \dots, \left\lfloor \frac{n}{2} \right\rfloor$ we get

$$\begin{aligned}wt_{\varphi}(u_i) &= \varphi(u_{i-1}u_i) + \varphi(u_iu_{i+1}) + \varphi(u_iw) = \left\lceil \frac{i-2}{3} \right\rceil + \left\lceil \frac{i-1}{3} \right\rceil + \left\lceil \frac{i-1}{3} \right\rceil + \left\lceil \frac{i}{3} \right\rceil + \left\lceil \frac{i-2}{3} \right\rceil \\ &+ \left\lceil \frac{i}{3} \right\rceil = 2i.\end{aligned}$$

If n is even, then

$$\begin{aligned}wt_{\varphi}(u_{\frac{n}{2}+1}) &= \varphi(u_{\frac{n}{2}}u_{\frac{n}{2}+1}) + \varphi(u_{n-\frac{n}{2}+1}u_{n-\frac{n}{2}+2}) + \varphi(u_{\frac{n}{2}+1}w) = \left\lceil \frac{n-2}{6} \right\rceil + \left\lceil \frac{n}{6} \right\rceil + \left\lceil \frac{n-4}{6} \right\rceil \\ &+ \left\lceil \frac{n}{6} \right\rceil + \left\lceil \frac{n-2}{6} \right\rceil + \left\lceil \frac{n+2}{6} \right\rceil = n + 1.\end{aligned}$$

If n is odd, then

$$\begin{aligned}wt_{\varphi}(u_{\frac{n+1}{2}}) &= \varphi(u_{\frac{n-1}{2}}u_{\frac{n+1}{2}}) + \varphi(u_{n-\frac{n-1}{2}}u_{n-\frac{n-3}{2}}) + \varphi(u_{\frac{n-1}{2}}w) = \left\lceil \frac{n-3}{6} \right\rceil + \left\lceil \frac{n-1}{6} \right\rceil \\ &+ \left\lceil \frac{n-3}{6} \right\rceil + \left\lceil \frac{n+1}{6} \right\rceil + \left\lceil \frac{n-1}{6} \right\rceil + \left\lceil \frac{n+1}{6} \right\rceil = n + 1, \\ wt_{\varphi}(u_{\frac{n+3}{2}}) &= \varphi(u_{n-\frac{n-1}{2}}u_{n-\frac{n-3}{2}}) + \varphi(u_{n-\frac{n-3}{2}}u_{n-\frac{n-5}{2}}) + \varphi(u_{n-\frac{n-3}{2}}w) = \left\lceil \frac{n-3}{6} \right\rceil \\ &+ \left\lceil \frac{n+1}{6} \right\rceil + \left\lceil \frac{n-5}{6} \right\rceil + \left\lceil \frac{n-1}{6} \right\rceil + \left\lceil \frac{n-1}{6} \right\rceil + \left\lceil \frac{n-3}{6} \right\rceil = n.\end{aligned}$$

For $i = 1, 2, \dots, \left\lfloor \frac{n}{2} \right\rfloor - 2$ we get

$$\begin{aligned}wt_{\varphi}(u_{n-i}) &= \varphi(u_{n-i}u_{n-i+1}) + \varphi(u_{n-i-1}u_{n-i}) + \varphi(u_{n-i}w) = \left\lceil \frac{i-1}{3} \right\rceil + \left\lceil \frac{i+1}{3} \right\rceil + \left\lceil \frac{i}{3} \right\rceil \\ &+ \left\lceil \frac{i+2}{3} \right\rceil + \left\lceil \frac{i}{3} \right\rceil + \left\lceil \frac{i+1}{3} \right\rceil = 2i + 3, \\ wt_{\varphi}(u_n) &= \varphi(u_{n-1}u_n) + \varphi(u_nw) = 3.\end{aligned}$$

Combining the previous, the result follows. \square

Lemma 4. *The centre w of the fan graph F_n , under the labeling φ , admits the weight*

$$wt_\varphi(w) = \begin{cases} \frac{n(n+3)}{6} + 2, & \text{for } n \equiv 0, 3 \pmod{6}, \\ \frac{(n+4)(n-1)}{6} + 2, & \text{for } n \equiv 1, 2, 5 \pmod{6}, \\ \frac{(n+1)(n+2)}{6} + 1, & \text{for } n \equiv 4 \pmod{6}. \end{cases}$$

Proof. If we consider a triple of edges $(u_{3s+1}w, u_{3s+2}w, u_{3s+3}w)$ for $s = 0, 1, 2, \dots, p$, where $p = \lfloor \frac{n}{6} \rfloor$ when $n \equiv 4 \pmod{6}$ and $p = \lfloor \frac{n}{6} \rfloor - 1$ otherwise, then the sum of the labels of edges for each triple is

$$\sum_{j=1}^3 \varphi(u_{3s+j}w) = \lceil \frac{3s+1-2}{3} \rceil + \lceil \frac{3s+1}{3} \rceil + \lceil \frac{3s+2-2}{3} \rceil + \lceil \frac{3s+2}{3} \rceil + \lceil \frac{3s+3-2}{3} \rceil + \lceil \frac{3s+3}{3} \rceil = 6s + 4.$$

If we consider a triple of edges $(u_{n-3r-1}w, u_{n-3r-2}w, u_{n-3r-3}w)$ for $r = 0, 1, 2, \dots, q$, where $q = \lfloor \frac{n}{6} \rfloor - 2$ when $n \equiv 0, 2 \pmod{6}$ and $q = \lfloor \frac{n}{6} \rfloor - 1$ otherwise, then the sum of labels of edges for each such triple is

$$\sum_{j=1}^3 \varphi(u_{n-3r-j}w) = \lceil \frac{3r+1}{3} \rceil + \lceil \frac{3r+2}{3} \rceil + \lceil \frac{3r+2}{3} \rceil + \lceil \frac{3r+3}{3} \rceil + \lceil \frac{3r+3}{3} \rceil + \lceil \frac{3r+4}{3} \rceil = 6r + 7.$$

Next we consider six cases according to the residue of n modulo 6.

Case 1. $n \equiv 0 \pmod{6}$.

Decompose the edges $u_iw, 1 \leq i \leq \frac{n}{2}$, into $\frac{n}{6}$ triples $(u_{3s+1}w, u_{3s+2}w, u_{3s+3}w)$ for $s = 0, 1, 2, \dots, \frac{n}{6} - 1$, and decompose the edges $u_{n-i}w, 1 \leq i \leq \frac{n}{2} - 3$ into $\frac{n}{6} - 1$ triples $(u_{n-3r-1}w, u_{n-3r-2}w, u_{n-3r-3}w)$ for $r = 0, 1, 2, \dots, \frac{n}{6} - 2$. Then, for the weight of the centre vertex, we get

$$\begin{aligned} wt_\varphi(w) &= \sum_{s=0}^{\frac{n}{6}-1} \sum_{j=1}^3 \varphi(u_{3s+j}w) + \varphi(u_{\frac{n}{2}+1}w) + \varphi(u_{n-\frac{n}{2}+2}w) + \sum_{r=0}^{\frac{n}{6}-2} \sum_{j=1}^3 \varphi(u_{n-3r-j}w) \\ &+ \varphi(u_nw) = \sum_{s=0}^{\frac{n}{6}-1} (6s + 4) + \lceil \frac{n-2}{6} \rceil + \lceil \frac{n+2}{6} \rceil + \lceil \frac{n-4}{6} \rceil + \lceil \frac{n-2}{6} \rceil + \sum_{r=0}^{\frac{n}{6}-2} (6r + 7) \\ &+ 2 = \frac{n(n+3)}{6} + 2. \end{aligned}$$

Case 2. $n \equiv 1 \pmod{6}$.

Decompose the edges $u_iw, 1 \leq i \leq \frac{n-1}{2}$, into $\frac{n-1}{6}$ triples $(u_{3s+1}w, u_{3s+2}w, u_{3s+3}w)$ for $s = 0, 1, 2, \dots, \frac{n-1}{6} - 1$, and decompose the edges $u_{n-i}w, 1 \leq i \leq \frac{n-1}{2}$, into $\frac{n-1}{6}$ triples $(u_{n-3r-1}w, u_{n-3r-2}w, u_{n-3r-3}w)$ for $r = 0, 1, 2, \dots, \frac{n-1}{6} - 1$. Then for the centre vertex weight, we have

$$\begin{aligned} wt_\varphi(w) &= \sum_{s=0}^{\frac{n-1}{6}-1} \sum_{j=1}^3 \varphi(u_{3s+j}w) + \sum_{r=0}^{\frac{n-1}{6}-1} \sum_{j=1}^3 \varphi(u_{n-3r-j}w) + \varphi(u_nw) = \sum_{s=0}^{\frac{n-1}{6}-1} (6s + 4) \\ &+ \sum_{r=0}^{\frac{n-1}{6}-1} (6r + 7) + 2 = \frac{(n+4)(n-1)}{6} + 2. \end{aligned}$$

Case 3. $n \equiv 2 \pmod{6}$.

Decompose the edges $u_iw, 1 \leq i \leq \frac{n}{2} - 1$, into $\frac{n-2}{6}$ triples $(u_{3s+1}w, u_{3s+2}w, u_{3s+3}w)$ for $s = 0, 1, 2, \dots, \frac{n-2}{6} - 1$, and the edges $u_{n-i}w, 1 \leq i \leq \frac{n}{2} - 4$ we decompose into $\frac{n-2}{6} - 1$

triples $(u_{n-3r-1}w, u_{n-3r-2}w, u_{n-3r-3}w)$ for $r = 0, 1, 2, \dots, \frac{n-2}{6} - 2$. Then, for the weight of the centre vertex, we get

$$\begin{aligned} wt_{\varphi}(w) &= \sum_{s=0}^{\frac{n-2}{6}-1} \sum_{j=1}^3 \varphi(u_{3s+j}w) + \varphi(u_{\frac{n}{2}}w) + \varphi(u_{\frac{n}{2}+1}w) + \varphi(u_{n-\frac{n}{2}+3}w) + \varphi(u_{n-\frac{n}{2}+2}w) \\ &+ \sum_{r=0}^{\frac{n-2}{6}-2} \sum_{j=1}^3 \varphi(u_{n-3r-j}w) + \varphi(u_nw) = \sum_{s=0}^{\frac{n-2}{6}-1} (6s + 4) + \lceil \frac{n-4}{6} \rceil + \lceil \frac{n}{6} \rceil + \lceil \frac{n-2}{6} \rceil \\ &+ \lceil \frac{n+2}{6} \rceil + \lceil \frac{n-6}{6} \rceil + \lceil \frac{n-4}{6} \rceil + \lceil \frac{n-4}{6} \rceil + \lceil \frac{n-2}{6} \rceil + \sum_{r=0}^{\frac{n-2}{6}-2} (6r + 7) + 2 \\ &= \frac{(n+4)(n-1)}{6} + 2. \end{aligned}$$

Case 4. $n \equiv 3 \pmod{6}$.

Decompose the edges $u_iw, 1 \leq i \leq \frac{n-3}{2}$, into $\frac{n-3}{6}$ triples $(u_{3s+1}w, u_{3s+2}w, u_{3s+3}w)$ for $s = 0, 1, 2, \dots, \frac{n-3}{6} - 1$, and decompose the edges $u_{n-i}w, 1 \leq i \leq \frac{n-3}{2}$, into $\frac{n-3}{6}$ triples $(u_{n-3r-1}w, u_{n-3r-2}w, u_{n-3r-3}w)$ for $r = 0, 1, 2, \dots, \frac{n-3}{6} - 1$. Then, for the centre vertex weight, we have

$$\begin{aligned} wt_{\varphi}(w) &= \sum_{s=0}^{\frac{n-3}{6}-1} \sum_{j=1}^3 \varphi(u_{3s+j}w) + \varphi(u_{\frac{n-1}{2}}w) + \varphi(u_{n-\frac{n-1}{2}}w) + \sum_{r=0}^{\frac{n-3}{6}-1} \sum_{j=1}^3 \varphi(u_{n-3r-j}w) \\ &+ \varphi(u_nw) = \sum_{s=0}^{\frac{n-3}{6}-1} (6s + 4) + \lceil \frac{n-5}{6} \rceil + \lceil \frac{n-1}{6} \rceil + \lceil \frac{n-1}{6} \rceil + \lceil \frac{n+1}{6} \rceil \\ &+ \sum_{r=0}^{\frac{n-3}{6}-1} (6r + 7) + 2 = \frac{n(n+3)}{6} + 2. \end{aligned}$$

Case 5. $n \equiv 4 \pmod{6}$.

Decompose the edges $u_iw, 1 \leq i \leq \frac{n}{2} + 1$, into $\frac{n-4}{6} + 1$ triples $(u_{3s+1}w, u_{3s+2}w, u_{3s+3}w)$ for $s = 0, 1, 2, \dots, \frac{n-4}{6}$, and the edges $u_{n-i}w, 1 \leq i \leq \frac{n}{2} - 2$ we decompose into $\frac{n-4}{6}$ triples $(u_{n-3r-1}w, u_{n-3r-2}w, u_{n-3r-3}w)$ for $r = 0, 1, 2, \dots, \frac{n-4}{6} - 1$. Then, for the weight of the centre vertex, we get

$$\begin{aligned} wt_{\varphi}(w) &= \sum_{s=0}^{\frac{n-4}{6}} \sum_{j=1}^3 \varphi(u_{3s+j}w) + \sum_{r=0}^{\frac{n-4}{6}-1} \sum_{j=1}^3 \varphi(u_{n-3r-j}w) + \varphi(u_nw) = \sum_{s=0}^{\frac{n-4}{6}} (6s + 4) \\ &+ \sum_{r=0}^{\frac{n-4}{6}-1} (6r + 7) + 2 = \frac{(n+1)(n+2)}{6} + 1. \end{aligned}$$

Case 6. $n \equiv 5 \pmod{6}$.

Decompose the edges $u_iw, 1 \leq i \leq \frac{n-1}{2} - 2$, into $\frac{n-5}{6}$ triples $(u_{3s+1}w, u_{3s+2}w, u_{3s+3}w)$ for $s = 0, 1, 2, \dots, \frac{n-5}{6} - 1$, and decompose the edges $u_{n-i}w, 1 \leq i \leq \frac{n-1}{2} - 2$, into $\frac{n-5}{6}$ triples $(u_{n-3r-1}w, u_{n-3r-2}w, u_{n-3r-3}w)$ for $r = 0, 1, 2, \dots, \frac{n-5}{6} - 1$. Then, for the centre vertex weight, we have

$$wt_{\varphi}(w) = \sum_{s=0}^{\frac{n-5}{6}-1} \sum_{j=1}^3 \varphi(u_{3s+j}w) + \varphi(u_{\frac{n-1}{2}-1}w) + \varphi(u_{\frac{n-1}{2}}w) + \varphi(u_{n-\frac{n-1}{2}+1}w)$$

$$\begin{aligned}
 &+ \varphi(u_{n-\frac{n-1}{2}}w) + \sum_{r=0}^{\frac{n-5}{6}-1} \sum_{j=1}^3 \varphi(u_{n-3r-j}w) + \varphi(u_nw) = \sum_{s=0}^{\frac{n-5}{6}-1} (6s + 4) \\
 &+ \lceil \frac{n-7}{6} \rceil + \lceil \frac{n-3}{6} \rceil + \lceil \frac{n-5}{6} \rceil + \lceil \frac{n-1}{6} \rceil + \lceil \frac{n-3}{6} \rceil + \lceil \frac{n-1}{6} \rceil + \lceil \frac{n-1}{6} \rceil + \lceil \frac{n+1}{6} \rceil \\
 &+ \sum_{r=0}^{\frac{n-5}{6}-1} (6r + 7) + 2 = \frac{(n+4)(n-1)}{6} + 2.
 \end{aligned}$$

□

Combining the previous lemmas, we can prove Theorem 1.

Proof of Theorem 1. The fan graph F_2 is isomorphic to a cycle C_3 . It admits an irregular assignment with edge labels 1, 2, 3 and with the induced vertex weights 3, 4, 5. Thus, $s(F_2) = s(C_3) = 3$.

According to Lemma 1, we have that $s(F_n) \geq \lceil \frac{n+1}{3} \rceil$ for $n \geq 3$. To prove the equality, it suffices to prove the existence of a vertex irregular edge $\lceil \frac{n+1}{3} \rceil$ -labeling of F_n .

For $n \geq 3$, consider the edge labeling φ of F_n defined by (4). From Lemma 2, it follows that φ is an $\lceil \frac{n+1}{3} \rceil$ -labeling.

Lemma 3 proves that weights of the vertices $u_i, i = 1, 2, \dots, n$, under the labeling φ successively attain values $2, 3, \dots, n + 1$. Moreover, with respect to Lemma 4, we get that $w_{\varphi}(w) > n + 1$ for every $n \geq 3$. Thus, the vertex weights are distinct for all pairs of distinct vertices. Therefore, the labeling φ is a suitable vertex irregular edge $\lceil \frac{n+1}{3} \rceil$ -labeling of F_n . This concludes the proof. □

2.2. The Modular Irregularity Strength of the Fan Graphs

Let us recall the following two lemmas.

Lemma 5. [16] *Let G be a graph with no component of order ≤ 2 . Every modular irregular labeling of G is also its irregular assignment.*

In general, the converse of the previous lemma does not hold. For example, the edge labeling of star $K_{1,3}$ with edge labels 1, 2, 3 is an irregular assignment with vertex weights 1, 2, 3, 6. However, this irregular labeling is not modular. If we label the edges of the star $K_{1,3}$ by labels 1, 2, 4, then we get a modular irregular assignment with modular vertex weights 0, 1, 2, 3, and $ms(K_{1,3}) = 4$.

The next statement gives a condition when an irregular assignment of a graph is also its modular irregular labeling.

Lemma 6. [16] *Let G be a graph with no component of order ≤ 2 , and let $s(G) = k$. If there exists an irregular assignment of G with edge values of at most k , where the weights of vertices constitute a set of consecutive integers, then*

$$s(G) = ms(G) = k.$$

The following theorem gives a lower bound of the modular irregularity strength.

Theorem 2. [16] *Let G be a graph with no component of order ≤ 2 . Then,*

$$s(G) \leq ms(G).$$

Now, we give the precise value of the modular irregularity strength for fan graphs F_n , for $n \geq 2$ even.

Theorem 3. Let F_n be a fan graph on $n + 1$ vertices with $n \geq 2$ even. Then,

$$ms(F_n) = \begin{cases} 3, & \text{if } n = 2, \\ 4, & \text{if } n = 8, \\ \lceil \frac{n+1}{3} \rceil, & \text{otherwise.} \end{cases}$$

Proof. Let $n = 2$. We have already mentioned that the fan graph F_2 admits an irregular assignment with edge labels 1, 2, 3 and with vertex weights 3, 4, 5. From Lemma 6, it follows that the irregular assignment of F_2 is modular and $ms(F_2) = 3$.

Let $n = 8$. Suppose that there exists a modular irregular 3-labeling ζ of F_8 . As the vertices $u_i, i = 1, 2, \dots, 8$ are either of degree 2 or 3, the weights of these vertices under any 3-labeling is at least 2 (this can be realizable only on a vertex of degree 2 as the sum of edge labels 1 + 1) and is at most 9 (this can be realizable only on a vertex of degree 3 as the sum of edge labels 3 + 3 + 3). As all the vertices must have distinct modular weights, we get that the weights of the vertices $u_i, i = 1, 2, \dots, 8$, constitute the sequence of consecutive integers from 2 up to 9. Thus, the modular weight 1 can be obtained only by the centre w . Moreover, it is easy to see that the weight of the centre cannot be 10 (must be at least 11 but at most 21). Thus, $wt_\zeta(w) = 19 \equiv 1 \pmod{9}$. Then,

$$\sum_{i=1}^8 \zeta(u_i w) + 2 \sum_{i=1}^7 \zeta(u_i u_{i+1}) = \sum_{i=1}^8 wt_\zeta(u_i). \tag{5}$$

Since $\sum_{i=1}^8 \zeta(u_i w) = wt_\zeta(w) = 19$ and $\sum_{i=1}^8 wt_\zeta(u_i) = 44$, then Equation (5) gives

$$19 + 2 \sum_{i=1}^7 \zeta(u_i u_{i+1}) = 44,$$

which is a contradiction. Thus, there is no modular irregular 3-labeling for F_8 . Figure 1 shows an example of a modular irregular 4-labeling of the fan graph F_8 , where the modular weights are depicted in italic font.

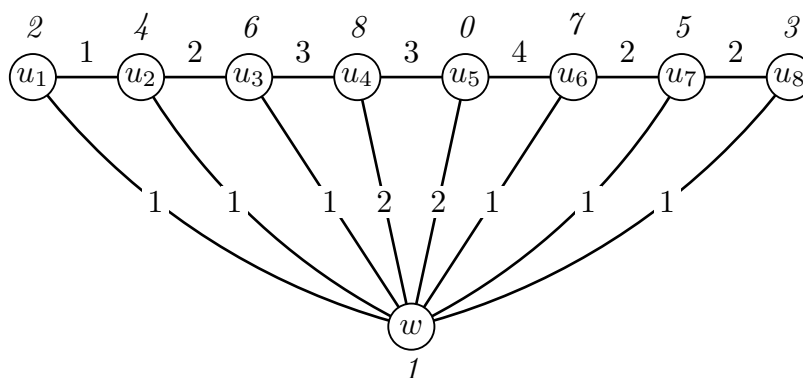


Figure 1. A modular irregular 4-labeling of the fan graph F_8 .

Now, for $n \neq 2, 8$ let us distinguish the following three cases, according to n .

Case 1. $n \equiv 4 \pmod{6}$.

It is sufficient to consider the edge irregular $\lceil \frac{n+1}{3} \rceil$ -labeling φ defined by (4). By Lemma 3, under the labeling φ , the weights of all vertices $u_i \in V(F_n), i = 1, 2, \dots, n$, successively assume values $2, 3, \dots, n, n + 1$, and by Lemma 4, the weight of the centre vertex is $wt_\varphi(w) = \frac{(n+1)(n+2)}{6} + 1$. Since $n \equiv 4 \pmod{6}$, then $\frac{n+2}{6}$ is an integer and

$\frac{(n+1)(n+2)}{6} + 1 \equiv 1 \pmod{n+1}$. This implies that the labeling φ is a suitable modular irregular $\lceil \frac{n+1}{3} \rceil$ -labeling.

Case 2. $n \equiv 0 \pmod{6}$.

Observe that under the labeling φ , by Lemma 4, the centre of F_n admits the weight $wt_\varphi(w) = \frac{n(n+3)}{6} + 2$. Since $n \equiv 0 \pmod{6}$, it follows that $wt_\varphi(w) \not\equiv 1 \pmod{n+1}$ and the labeling φ is not modular irregular. Therefore, we need to modify the labeling φ .

Figure 2 illustrates a modular irregular 3-labeling of the fan graph F_6 , where the modular weights are depicted in italic font.

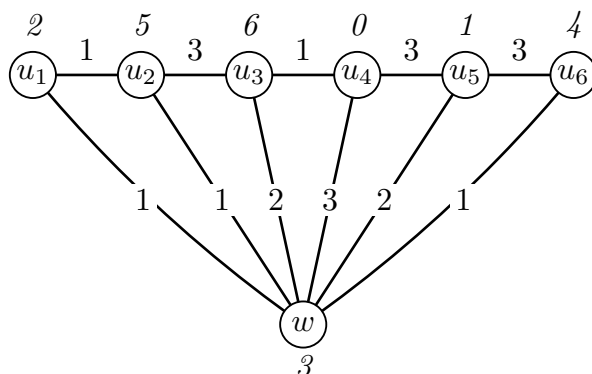


Figure 2. A modular irregular 3-labeling of the fan graph F_6 .

For $n \geq 12$, we define an edge labeling ψ of F_n as follows:

$$\psi(u_i u_{i+1}) = \begin{cases} \varphi(u_i u_{i+1}), & \text{for } 1 \leq i \leq \frac{n}{2}, i \neq 2, 4, \dots, \frac{n}{3} - 2, \frac{n}{3}, \\ \varphi(u_i u_{i+1}) - 1, & \text{for } i = 2, 4, \dots, \frac{n}{3} - 2, \frac{n}{3}, \end{cases}$$

$$\psi(u_{n-i} u_{n-i+1}) = \begin{cases} \varphi(u_{n-i} u_{n-i+1}), & \text{for } 1 \leq i \leq \frac{n}{2} - 1, i \neq 2, 4, \dots, \frac{n}{3} - 2, \frac{n}{3}, \\ \varphi(u_{n-i} u_{n-i+1}) - 1, & \text{for } i = 2, 4, \dots, \frac{n}{3} - 2, \frac{n}{3}, \end{cases}$$

$$\psi(u_i w) = \begin{cases} 1, & \text{for } i = 1, \\ \varphi(u_i w) + 1, & \text{for } 2 \leq i \leq \frac{n}{3} + 1, \\ \varphi(u_i w), & \text{for } \frac{n}{3} + 2 \leq i \leq \frac{n}{2} + 1, \\ 2, & \text{for } i = n, \end{cases}$$

$$\psi(u_{n-i} w) = \begin{cases} \varphi(u_{n-i} w), & \text{for } \frac{n}{3} + 1 \leq i \leq \frac{n}{2} - 2, \\ \varphi(u_{n-i} w) + 1, & \text{for } 1 \leq i \leq \frac{n}{3}. \end{cases}$$

One can see that decreasing the labels of the edges $u_i u_{i+1}$ and $u_{n-i} u_{n-i+1}$, $i = 2, 4, \dots, \frac{n}{3} - 2, \frac{n}{3}$ by one, and increasing the labels of the edges $u_i w$, $2 \leq i \leq \frac{n}{3} + 1$ and $u_{n-i} w$, $1 \leq i \leq \frac{n}{3}$ by one has no effect on the weights of vertices $u_i \in V(F_n)$, as they successively attain the values $2, 3, \dots, n, n + 1$. We note that $\max\{\psi(u_i w) : 2 \leq i \leq \frac{n}{3} + 1\} = \psi(u_{\frac{n}{3}+1} w) = \lceil \frac{n-3}{9} \rceil + \lceil \frac{n+3}{9} \rceil + 1 < \lceil \frac{n+1}{3} \rceil$ and $\max\{\psi(u_{n-i} w) : 1 \leq i \leq \frac{n}{3}\} = \psi(u_{\frac{2n}{3}} w) = \lceil \frac{n}{9} \rceil + \lceil \frac{n+3}{9} \rceil + 1 < \lceil \frac{n+1}{3} \rceil$.

However, by increasing the labels of the edges $u_i w$, $2 \leq i \leq \frac{n}{3} + 1$ and $u_{n-i} w$, $1 \leq i \leq \frac{n}{3}$, the weight of the centre increases, and we have

$$wt_\psi(w) = \frac{n(n+3)}{6} + 2 + \frac{2n}{3} = \frac{(n+1)(n+6)}{6} + 1.$$

Since $\frac{n+6}{6}$ is an integer, then $wt_\psi(w) \equiv 1 \pmod{n+1}$. Thus, the labeling ψ is a required modular irregular $\lceil \frac{n+1}{3} \rceil$ -labeling of F_n .

Case 3. $n \equiv 2 \pmod{6}$, $n \geq 14$.

According to Lemma 4 we have that $wt_\varphi(w) = \frac{(n+4)(n-1)}{6} + 2$ and it is not congruent to 1 (mod $n + 1$). Our next goal is to modify the edge labeling φ such that the weights of vertices $u_i \in V(F_n), i = 1, 2, \dots, n$, will not change but the weight of the centre decreases to a value congruent to one (mod $n + 1$).

Therefore, for $n \geq 14$, we construct an edge labeling ϑ of F_n in the following way:

$$\begin{aligned} \vartheta(u_i u_{i+1}) &= \begin{cases} \varphi(u_i u_{i+1}), & \text{for } 1 \leq i \leq \frac{n}{2}, i \neq 3, 5, \dots, \frac{n+1}{3} - 2, \frac{n+1}{3}, \\ \varphi(u_i u_{i+1}) + 1, & \text{for } i = 3, 5, \dots, \frac{n+1}{3} - 2, \frac{n+1}{3}, \end{cases} \\ \vartheta(u_{n-i} u_{n-i+1}) &= \begin{cases} \varphi(u_{n-i} u_{n-i+1}), & \text{for } 1 \leq i \leq \frac{n}{2} - 1, i \neq 1, 3, \dots, \frac{n+1}{3} - 2, \frac{n+1}{3}, \\ \varphi(u_{n-i} u_{n-i+1}) + 1, & \text{for } i = 1, 3, \dots, \frac{n+1}{3} - 2, \frac{n+1}{3}, \end{cases} \\ \vartheta(u_i w) &= \begin{cases} 1, & \text{for } i = 1, 2, n, \\ \varphi(u_i w) - 1, & \text{for } 3 \leq i \leq \frac{n+1}{3} + 1, \\ \varphi(u_i w), & \text{for } \frac{n+1}{3} + 2 \leq i \leq \frac{n}{2} + 1, \end{cases} \\ \vartheta(u_{n-i} w) &= \begin{cases} \varphi(u_{n-i} w), & \text{for } \frac{n+1}{3} + 1 \leq i \leq \frac{n}{2} - 2, \\ \varphi(u_{n-i} w) - 1, & \text{for } 1 \leq i \leq \frac{n+1}{3}. \end{cases} \end{aligned}$$

By direct computation we can see that increasing the labels of the edges $u_i u_{i+1}, i = 3, 5, \dots, \frac{n+1}{3} - 2, \frac{n+1}{3}$, and $u_{n-i} u_{n-i+1}, i = 1, 3, 5, \dots, \frac{n+1}{3} - 2, \frac{n+1}{3}$ by one and decreasing the labels of the edges $u_i w, 3 \leq i \leq \frac{n+1}{3} + 1$ and $u_{n-i} w, 1 \leq i \leq \frac{n+1}{3}$ by one has no impact to weights of the vertices $u_i \in V(F_n)$, and they preserve the values $2, 3, \dots, n, n + 1$. Since $\max\{\varphi(u_i u_{i+1}) : i = 3, 5, \dots, \frac{n+1}{3}\} = \varphi(u_{\frac{n+1}{3}} u_{\frac{n+4}{3}}) = \lceil \frac{n-2}{9} \rceil + \lceil \frac{n+1}{9} \rceil + 1 < \lceil \frac{n+1}{3} \rceil$ and $\max\{\varphi(u_{n-i} u_{n-i+1}) : i = 1, 3, 5, \dots, \frac{n+1}{3}\} = \varphi(u_{\frac{2n-1}{3}} u_{\frac{2n+2}{3}}) = \lceil \frac{n-2}{9} \rceil + \lceil \frac{n+4}{9} \rceil + 1 < \lceil \frac{n+1}{3} \rceil$, it follows (applying Lemma 2) that all edge labels under the labeling ϑ are at most $\lceil \frac{n+1}{3} \rceil$.

Decreasing the labels of the edges $u_i w, 3 \leq i \leq \frac{n+1}{3} + 1$ and $u_{n-i} w, 1 \leq i \leq \frac{n+1}{3}$, the weight of the centre decreases and we get

$$wt_\vartheta(w) = \frac{(n+4)(n-1)}{6} + 2 - 2 \frac{n+1}{3} = \frac{(n-2)(n+1)}{6} + 1.$$

Because $\frac{n-2}{6}$ is an integer, $wt_\vartheta(w) \equiv 1 \pmod{n + 1}$. It proves that the labeling ϑ is a suitable modular irregular $\lceil \frac{n+1}{3} \rceil$ -labeling. \square

The next theorem, proved in [16], gives a condition when no modular irregular labeling of a graph exists.

Theorem 4. [16] *If G is a graph of order $n, n \equiv 2 \pmod{4}$, then G has no modular irregular labeling, that is, $ms(G) = \infty$.*

An immediate consequence of the above theorem is the following statement.

Corollary 1. *If $n \equiv 1 \pmod{4}$, then the fan graph F_n on $n + 1$ vertices has no modular irregular labeling.*

Theorem 5. *Let F_n be a fan graph on $n + 1$ vertices with $n \geq 3$ odd. Then*

$$ms(F_n) = \begin{cases} \lceil \frac{n+1}{3} \rceil, & \text{if } n \equiv 3 \pmod{4}, \\ \infty, & \text{if } n \equiv 1 \pmod{4}. \end{cases}$$

Proof. It is a matter of routine checking to see that under the edge labeling φ defined by (4), for n odd, the weights of the centre of F_n listed in Lemma 4 are not congruent to 1 (mod $n + 1$). In order to preserve the property of the edge labeling φ that the weights of all vertices $u_i \in V(F_n), i = 1, 2, \dots, n$, form the set $\{2, 3, \dots, n, n + 1\}$, and to attain the weights of the centre congruent to one (mod $n + 1$), we will construct some appropriate modifications of the labeling φ in a similar way as in the proof of Theorem 3.

Since we consider only n odd and moreover, $n \not\equiv 1 \pmod{4}$, then we distinguish the following three cases according to the residue of n modulo 12.

Case 1. $n \equiv 3 \pmod{12}$.

From Theorem 1, it follows that $s(F_3) = 2$ and $wt_\varphi(u_1) = 2, wt_\varphi(u_3) = 3, wt_\varphi(u_2) = 4$ and $wt_\varphi(w) = 5$. According to Lemma 6, we have that $s(F_3) = ms(F_3) = 2$.

For $n \geq 15$, we define an edge labeling ρ of F_n such that:

$$\rho(u_i u_{i+1}) = \begin{cases} \varphi(u_i u_{i+1}), & \text{for } 1 \leq i \leq \frac{n-1}{2}, i \neq 2, 4, \dots, \frac{n-3}{6} - 2, \frac{n-3}{6}, \\ \varphi(u_i u_{i+1}) - 1, & \text{for } i = 2, 4, \dots, \frac{n-3}{6} - 2, \frac{n-3}{6}, \end{cases}$$

$$\rho(u_{n-i} u_{n-i+1}) = \varphi(u_{n-i} u_{n-i+1}), \quad \text{for } 1 \leq i \leq \frac{n-1}{2},$$

$$\rho(u_i w) = \begin{cases} 1, & \text{for } i = 1, \\ 2, & \text{for } i = n, \\ \varphi(u_i w) + 1, & \text{for } 2 \leq i \leq \frac{n+3}{6}, \\ \varphi(u_i w), & \text{for } \frac{n+3}{6} + 1 \leq i \leq \frac{n-1}{2}, \end{cases}$$

$$\rho(u_{n-i} w) = \varphi(u_{n-i} w), \quad \text{for } 1 \leq i \leq \frac{n-1}{2}.$$

By a direct verification, we can detect that under the labeling ρ , all edge labels are at most $\lceil \frac{n+1}{3} \rceil$, the weights of the vertices $u_i, 1 \leq i \leq n$ constitute a sequence of consecutive integers from 2 up to $n + 1$, and the weight of the centre determined by Lemma 4 is increased by $\frac{n-3}{6}$. Consequently,

$$wt_\rho(w) = wt_\varphi(w) + \frac{n-3}{6} = \frac{n(n+3)}{6} + 2 + \frac{n-3}{6} = \frac{(n+3)(n+1)}{6} + 1.$$

As $\frac{n+3}{6}$ is an integer, then $wt_\rho(w) \equiv 1 \pmod{n + 1}$.

Case 2. $n \equiv 7 \pmod{12}$.

Figure 3 depicts a modular irregular 3-labeling of the fan graph F_7 . The modular weights are again illustrated using italic font.

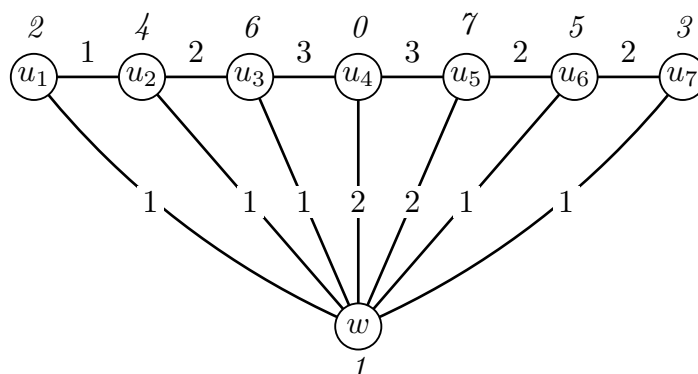


Figure 3. A modular irregular 3-labeling of the fan graph F_7 .

For $n \geq 19$, we define an edge labeling λ of F_n in the following way:

$$\lambda(u_i u_{i+1}) = \begin{cases} \varphi(u_i u_{i+1}), & \text{for } 1 \leq i \leq \frac{n-1}{2}, i \neq 3, \\ 3, & \text{for } i = 3, \end{cases}$$

$$\lambda(u_{n-i}u_{n-i+1}) = \begin{cases} \varphi(u_{n-i}u_{n-i+1}), & \text{for } 1 \leq i \leq \frac{n-1}{2}, i \neq 1, 3, 5, \dots, \frac{n-5}{2} - 2, \frac{n-5}{2}, \\ \varphi(u_{n-i}u_{n-i+1}) + 1, & \text{for } i = 1, 3, 5, \dots, \frac{n-5}{2} - 2, \frac{n-5}{2}, \end{cases}$$

$$\lambda(u_iw) = \begin{cases} 1, & \text{for } i = 1, 2, n, \\ i - 2, & \text{for } i = 3, 4, \\ \varphi(u_iw), & \text{for } 5 \leq i \leq \frac{n-1}{2}, \end{cases}$$

$$\lambda(u_{n-i}w) = \begin{cases} \varphi(u_{n-i}w) - 1, & \text{for } 1 \leq i \leq \frac{n-5}{2}, \\ \varphi(u_{n-i}w), & \text{for } \frac{n-3}{2} \leq i \leq \frac{n-1}{2}. \end{cases}$$

We can see that the labeling λ , as a modification of the labeling φ , did not increase the largest values of the edges and has no effect on the weights of vertices u_i in F_n . The weight of the centre is reduced by $\frac{n+1}{2}$, and we get

$$wt_\lambda(w) = wt_\varphi(w) - \frac{n+1}{2} = \frac{(n+4)(n-1)}{6} + 2 - \frac{n+1}{2} = \frac{(n-1)(n+1)}{6} + 1.$$

Indeed, $\frac{n-1}{6}$ is an integer, and then $wt_\lambda(w) \equiv 1 \pmod{n+1}$.

Case 3. $n \equiv 11 \pmod{12}$.

For $n \geq 11$, we define an edge labeling μ of F_n as follows:

$$\mu(u_iu_{i+1}) = \begin{cases} i, & \text{for } i = 1, 2, \\ \varphi(u_iu_{i+1}), & \text{for } 3 \leq i \leq \frac{n-1}{2}, i \neq 3, 5, \dots, \frac{n+7}{6} - 2, \frac{n+7}{6}, \\ \varphi(u_iu_{i+1}) + 1, & \text{for } i = 3, 5, \dots, \frac{n+7}{6} - 2, \frac{n+7}{6}, \end{cases}$$

$$\mu(u_{n-i}u_{n-i+1}) = \varphi(u_{n-i}u_{n-i+1}), \quad \text{for } 1 \leq i \leq \frac{n-1}{2},$$

$$\mu(u_iw) = \begin{cases} 1, & \text{for } i = 1, 2, \\ \varphi(u_iw) - 1, & \text{for } 3 \leq i \leq \frac{n+7}{6} + 1, \\ \varphi(u_iw), & \text{for } \frac{n+7}{6} + 2 \leq i \leq \frac{n-1}{2}, \\ 2, & \text{for } i = n, \end{cases}$$

$$\mu(u_{n-i}w) = \varphi(u_{n-i}w), \quad \text{for } 1 \leq i \leq \frac{n-1}{2}.$$

Again, it is readily seen that this modification of the labeling φ has no impact on the weights of vertices $u_i \in V(F_n)$ and to the largest values of the edges. Under the labeling μ , the weight of the centre determined by Lemma 4 is decreased by $\frac{n+1}{6}$, and we have

$$wt_\mu(w) = wt_\varphi(w) - \frac{n+1}{6} = \frac{(n+4)(n-1)}{6} + 2 - \frac{n+1}{6} = \frac{(n+1)^2}{6} + 1.$$

Obviously, $\frac{n+1}{6}$ is an integer and $wt_\mu(w) \equiv 1 \pmod{n+1}$. Thus, we arrive at the desired result. \square

3. Conclusions

In this paper, we proved that the exact value of the irregularity strength of the fan graph F_n of order $n + 1$ is

$$s(F_n) = \begin{cases} 3, & \text{if } n = 2, \\ \lceil \frac{n+1}{3} \rceil, & \text{if } n \geq 3. \end{cases}$$

By modifying an irregular assignment of the fan graph, we obtained modular irregular assignments and proved that

$$\text{ms}(F_n) = \begin{cases} 3, & \text{if } n = 2, \\ 4, & \text{if } n = 8, \\ \infty, & \text{if } n \equiv 1 \pmod{4}, \\ \left\lceil \frac{n+1}{3} \right\rceil, & \text{otherwise.} \end{cases}$$

According to the given result, we get that the fan graphs are an example of graphs for which the irregularity strength and the modular irregularity strength are almost the same, up to a small case and one case excluded by the necessary condition for the modular irregularity strength to be finite. Thus, naturally, we conclude our paper with the following open problem.

Problem 1. Find another family of graphs for which the irregularity strength and the modular irregularity strength are the same.

Author Contributions: Conceptualization, M.B. and A.S.-F.; methodology, M.B. and A.S.-F.; validation, M.B., Z.K., M.L. and A.S.-F.; investigation, M.B., Z.K., M.L. and A.S.-F.; resources, M.B., Z.K., M.L. and A.S.-F.; writing—original draft preparation, A.S.-F.; writing—review and editing, M.B. and A.S.-F.; supervision, A.S.-F.; project administration, M.B. and A.S.-F.; funding acquisition, M.B. and A.S.-F. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the Slovak Research and Development Agency under the contract No. APVV-19-0153 and by VEGA 1/0233/18.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: We thank the anonymous reviewers for their careful reading of our manuscript and their many insightful comments and suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

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