Second-Order Impulsive Delay Differential Systems: Necessary and Sufficient Conditions for Oscillatory or Asymptotic Behavior

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Abstract: In this work, we aimed to obtain sufficient and necessary conditions for the oscillatory or asymptotic behavior of an impulsive differential system. It is easy to notice that most works that study the oscillation are concerned only with sufficient conditions and without impulses, so our results extend and complement previous results in the literature. Further, we provide two examples to illustrate the main results.

Keywords: impulsive differential systems; delay; Lebesgue's dominated convergence theorem; oscillation; asymptotic behavior

1. Introduction

Nowadays, impulsive differential equations are attracting a lot of attention. They appear in the study of several real world problems (see, for instance, [1–3]). In general, it is well-known that several natural phenomena are driven by differential equations, but the descriptions of some real world problems subjected to sudden changes in their states have become very interesting from a mathematical point of view because they should be described while considering systems of differential equations with impulses. Examples of the aforementioned phenomena are related to mechanical systems, biological systems, population dynamics, pharmacokinetics, theoretical physics, biotechnology processes, chemistry, engineering and control theory.

The literature related to impulsive differential equations is very vast. Here we mention some recent developments in this field.

In [4], Shen and Wang considered impulsive differential equations with the following form:

$$\begin{cases} u'(\xi) + r(\xi)u(\xi - \nu) = 0, \ \xi \neq \chi_j, \ \xi \geq \xi_0 \\ u(\chi_j^+) - u(\chi_j^-) = I_j(u(\chi_j)), \ j \in \mathbb{N} \end{cases} \tag{1}$$

where $r, I_j \in C(\mathbb{R}, \mathbb{R})$ (that is, r, I_j are continuous in $(\mathbb{R}, \mathbb{R})$) for $j \in \mathbb{N}$, and established some sufficient conditions for the oscillatory and the asymptotic behavior of the solutions of the problem (1).

In [5], the authors considered the problem

$$\begin{cases} (u(\xi) - q(\xi)u(\xi - \sigma))' + r(\xi)|u(\xi - \nu)|^{\lambda}\mathrm{sgn}\, u(\xi - \nu) = 0, \ \xi \geq \xi_0 \\ u(\chi_j^+) = b_j u(\chi_j), \ j \in \mathbb{N} \end{cases} \tag{2}$$

assuming that $q(\xi) \in PC([\xi_0, \infty), \mathbb{R}_+)$ (that is, $q(\xi)$ is piecewise continuous in $[\xi_0, \infty)$), and established sufficient conditions for the oscillation of (2).

In [6], the authors studied the first order impulsive systems of the form:

$$\begin{cases} (u(\xi) - q(\xi)u(\xi - \sigma))' + r(\xi)u(\xi - \nu_1) - v(\xi)u(\xi - \nu_2) = 0, \ \nu_1 \geq \nu_2 > 0 \\ u(\chi_j^+) = I_j(u(\chi_j)), \ j \in \mathbb{N} \end{cases} \tag{3}$$

and obtained sufficient conditions for the oscillation of (3) when $q(\xi) \in PC([\xi_0, \infty), \mathbb{R}_+)$ and $b_j \leq \frac{I_j(u)}{u} \leq 1$.

Karpuz et al. in [7] extended the results contained in [6] by taking the non-homogeneous counterpart of the system (3) with variable delays.

In [8], Tripathy and Santra considered the the impulsive system

$$\begin{cases} (u(\xi) - qu(\xi - \sigma))'' + ru(\xi - \nu) = 0, \ \xi \neq \chi_j, \ j \in \mathbb{N} \\ \Delta(u(\chi_j) - qu(\chi_j - \sigma))' + \tilde{r}u(\chi_j - \nu) = 0, \ j \in \mathbb{N} \end{cases} \tag{4}$$

where

$$\Delta u(a) = \lim_{s \to a^+} u(s) - \lim_{s \to a^-} u(s),$$

and studied oscillation and non-oscillation properties for (4). In an another paper, Tripathy and Santra studied the following impulsive systems:

$$\begin{cases} (p(\xi)(u(\xi) + q(\xi)u(\xi - \sigma))')' + r(\xi)g(u(\xi - \nu)), \ \xi \neq \chi_j, \ j \in \mathbb{N} \\ \Delta(p(\chi_j)(u(\chi_j) + q(\chi_j)u(\chi_j - \sigma))') + r(\chi_j)g(u(\chi_j - \nu)) = 0, \ j \in \mathbb{N} \end{cases}. \tag{5}$$

In [9], in particular, the authors are interested in oscillating systems that, after a perturbation by instantaneous change of state, remain oscillating.

In [10], Santra and Tripathy studied a special type first-order impulsive systems of the form

$$\begin{cases} (u(\xi) - q(\xi)u(\xi - \sigma))' + r(\xi)g(u(\xi - \nu)) = 0, \ \xi \neq \chi_j, \ \xi \geq \xi_0 \\ u(\chi_j^+) = I_j(u(\chi_j)), \ j \in \mathbb{N} \\ u(\chi_j^+ - \tau) = I_j(u(\chi_j - \tau)), \ j \in \mathbb{N} \end{cases} \tag{6}$$

for different values of the neutral coefficient q.

We also mention the paper [11] in which Santra and Dix, using the Lebesgue's dominated convergence theorem, studied the following impulsive system:

$$\begin{cases} (p(\xi)(w'(\xi))^{\mu})' + \sum_{j=1}^{m} r_j(\xi)g_j(u(\nu_j(\xi))) = 0, \ \xi \geq \xi_0, \ \xi \neq \chi_j, \ j \in \mathbb{N} \\ \Delta(p(\chi_j)(w'(\chi_j))^{\mu}) + \sum_{j=1}^{m} \tilde{r}_j(\chi_j)g_j(u(\nu_j(\chi_j))) = 0, \end{cases} \tag{7}$$

where

$$w(\xi) = u(\xi) + q(\xi)u(\sigma(\xi)), \quad \Delta u(a) = \lim_{s \to a^+} u(s) - \lim_{s \to a^-} u(s).$$

In line with the contents of [11], Tripathy and Santra in [12] examined oscillation and non-oscillation behavior of the following impulsive system:

$$\begin{cases} (p(\xi)(u(\xi) + q(\xi)u(\xi - \sigma))')' + r(\xi)g(u(\xi - \nu)) = f(\xi), \ \xi \neq \chi_j, \ j \in \mathbb{N} \\ \Delta(p(\chi_j)(u(\chi_j) + q(\chi_j)u(\chi_j - \sigma))') + \tilde{r}(\chi_j)g(u(\chi_j - \nu)) = \tilde{f}(\chi_j), \ j \in \mathbb{N} \end{cases} \tag{8}$$

for different values of $q(\xi)$.

Finally, we mention the recent work [13] in which Tripathy and Santra established some characterizations for the oscillation and asymptotic behavior of the impulsive differential system of the form

$$
\begin{cases}
(p(\xi)(w'(\xi))^\mu)' + \sum_{j=1}^m r_j(\xi)x^{\alpha_j}(v_j(\xi)) = 0, \ \xi \geq \xi_0, \ \xi \neq \chi_j \\
\Delta(p(\chi_j)(w'(\chi_j))^\mu) + \sum_{j=1}^m h_j(\chi_j)x^{\alpha_j}(v_j(\chi_j)) = 0, \ j \in \mathbb{N}
\end{cases}
\tag{9}
$$

where $w(\xi) = u(\xi) + q(\xi)u(\sigma(\xi))$ and $-1 < q(\xi) \leq 0$.

For further details on neutral impulsive differential equation and for recent results related to the oscillation theory for ordinary differential equations, we refer the reader to the papers [14–28] and to the references therein.

Motivated by the aforementioned findings, in this paper we prove necessary and sufficient conditions for the oscillatory or asymptotic behavior of solutions to a second-order non-linear impulsive differential system of the form

$$
(E1) \begin{cases}
\left(a(\xi)(w'(\xi))^\mu\right)' + c(\xi)g(u(\vartheta(\xi))) = 0, \quad \xi \geq \xi_0, \ \xi \neq \chi_j, \\
\Delta\left(a(\chi_j)(w'(\chi_j))^\mu\right) + \tilde{c}(\chi_j)g(u(\vartheta(\chi_j))) = 0, \ j \in \mathbb{N},
\end{cases}
$$

where

$$
w(\xi) = u(\xi) + b(\xi)u(\sigma(\xi)), \quad \Delta u(a) = \lim_{s \to a^+} u(s) - \lim_{s \to a^-} u(s),
$$

and the functions $g, b, c, \tilde{c}, a, \vartheta, \sigma$ are continuous such that

(A1) $\vartheta \in C([0, \infty), \mathbb{R})$, $\sigma \in C^2([0, \infty), \mathbb{R})$, (in general C^k means the function has k derivatives and they are all continuous functions) $\vartheta(\xi) < \xi$, $\sigma(\xi) < \xi$, $\lim_{\xi \to \infty} \vartheta(\xi) = \infty$, $\lim_{\xi \to \infty} \sigma(\xi) = \infty$;

(A2) $a \in C^1([0, \infty), \mathbb{R})$, $c, \tilde{c} \in C([0, \infty), \mathbb{R})$; $0 < a(\xi)$, $0 \leq c(\xi)$, $0 \leq \tilde{c}(\xi)$, $\xi \geq 0$;

(A3) $g \in C(\mathbb{R}, \mathbb{R})$ is non-decreasing and $g(u)u > 0$ for $u \neq 0$;

(A4) $\lim_{\xi \to \infty} A(\xi) = \infty$ where $A(\xi) = \int_{\xi_1}^{\xi} a^{-1/\mu}(s) \, ds$;

(A5) The sequence $\{\chi_j\}$ satisfying $0 < \chi_1 < \chi_2 < \cdots < \chi_j < \cdots \to \infty$ as $j \to \infty$ are fixed moments of impulsive effects;

(A6) ν is the quotient of two positive odd integers. In particular, the assumption of ν can be replaced by $\nu > 0$, by using $|u|^\nu \text{sgn}(u)$ instead of u^ν, but the notation will be much longer.

2. Main Results

Lemma 1. *Assume (A1)–(A6), $-1 < -b_0 \leq b(\xi) \leq 0$ for $\xi \geq \xi_0$, and that u is an eventually positive solution of (E1). Then only one of the following two cases happens:*

(1) $\lim_{\xi \to \infty} u(\xi) = 0$;

(2) *There exist $\xi_1 \geq \xi_0$ and $\delta > 0$, such that*

$$
0 < w(\xi) \leq \delta A(\xi), \tag{10}
$$

$$
A(\xi)\Lambda^{1/\mu} \leq w(\xi) \leq u(\xi), \tag{11}
$$

for $\xi \geq \xi_1$ and where

$$
\Lambda = \int_\xi^\infty c(\zeta)g(u(\vartheta(\zeta))) \, d\zeta + \sum_{\chi_j \geq \xi} \tilde{c}(\chi_j)g(u(\vartheta(\chi_j))).
$$

Proof. Let u be an eventually positive solution. Then we can find a ξ^* such that $u(\xi) > 0$, $u(\sigma(\xi)) > 0$ and $u(\vartheta(\xi)) > 0$ for all $\xi \geq \xi^*$. Note that z is continuous and $w(\xi) \leq u(\xi)$. From (E1), we obtain

$$\left(a(\xi)\left(w'(\xi)\right)^{\mu}\right)' = -c(\xi)g\left(u(\vartheta(\xi))\right) \leq 0 \quad \text{for } \xi \neq \chi_j,$$

$$\Delta\left(a(\chi_j)\left(w'(\chi_j)\right)^{\mu}\right) = -\tilde{c}(\chi_j)g\left(u(\vartheta(\chi_j))\right) \leq 0 \quad \text{for } j \in \mathbb{N}. \tag{12}$$

From (12), we have $a(\xi)\left(w'(\xi)\right)^{\mu}$ is non-increasing, including impulses for $\xi \geq \xi^*$. By contradiction we assume that $a(\xi)\left(w'(\xi)\right)^{\mu} \leq 0$ at a certain time $\xi \geq \xi^*$. Using that c is not identically zero on any interval $[b, \infty)$, and that $g(\xi) > 0$ for $\xi > 0$, by (12), there exist $\xi_1 \geq \xi^*$ such that

$$a(\xi)\left(w'(\xi)\right)^{\mu} \leq a(\xi_1)\left(w'(\xi_1)\right)^{\mu} < 0 \quad \text{for all } \xi \geq \xi_1.$$

Then

$$w'(\xi) \leq \left(\frac{a(\xi_1)}{a(\xi)}\right)^{1/\mu} w'(\xi_1) \quad \text{for } \xi \geq \xi_1.$$

Integrating from ξ_1 to ξ, we have

$$w(\xi) \leq w(\xi_1) + \left(a(\xi_1)\right)^{1/\mu} w'(\xi_1) A(\xi). \tag{13}$$

Using (A4), we arrive at $\lim_{\xi \to \infty} w(\xi) = -\infty$. Since b is bounded and w is unbounded, u can not be bounded. This allows the existence of a sequence $\{s_j\} \to \infty$ such that $u(s_j) = \sup\{u(s) : s \leq s_j\}$. Then $u(\sigma(s_j)) \leq u(s_j)$ and

$$w(s_j) = u(s_j) + b(s_j)u(\sigma(s_j)) \geq (1 + b(s_j))u(s_j) \geq (1 - b_0)u(s_j) \geq 0,$$

which contradicts $\lim_{j \to \infty} w(s_j) = -\infty$. Therefore, $a(\xi)\left(w'(\xi)\right)^{\mu} > 0$ for all $\xi \geq \xi^*$. Since $a(\xi) > 0$, ultimately $w'(\xi) > 0$. Then, there is $\xi_1 \geq \xi^*$ such that only one of the following two cases happens.

Case 1: $w(\xi) < 0$ for all $\xi \geq \xi_1$. Note that by (A1), $\limsup_{\xi \to \infty} u(\xi) = \limsup_{\xi \to \infty} u(\sigma(\xi))$. Then $0 > w(\xi) \geq u(\xi) - b_0 u(\sigma(\xi))$ implies

$$0 \geq (1 - b_0) \limsup_{\xi \to \infty} u(\xi).$$

Since $(1 - b_0) > 0$, it follows that $\limsup_{\xi \to \infty} u(\xi) = 0$; hence, $\lim_{\xi \to \infty} u(\xi) = 0$.

Case 2: $w(\xi) > 0$ for all $\xi \geq \xi_1$. Note that $u(\xi) \geq w(\xi)$ and w is positive and increasing. From $a(\xi)\left(w'(\xi)\right)^{\mu}$ being non-increasing, we have

$$w'(\xi) \leq \left(\frac{a(\xi_1)}{a(\xi)}\right)^{1/\mu} w'(\xi_1) \quad \text{for } \xi \geq \xi_1.$$

Integrating this inequality from ξ_1 to ξ, we get

$$w(\xi) \leq w(\xi_1) + \left(a(\xi_1)\right)^{1/\mu} w'(\xi_1) A(\xi).$$

Since $\lim_{\xi \to \infty} A(\xi) = \infty$, there is a constant $\delta > 0$ such that (10) holds.

Since $a(\xi)\left(w'(\xi)\right)^{\mu} > 0$ and is non-increasing, $\lim_{\xi \to \infty} a(\xi)\left(w'(\xi)\right)^{\mu}$ exists and is non-negative. Integrating (E1) from ξ to a, we have

$$a(a)\left(w'(a)\right)^{\mu} - a(\xi)\left(w'(\xi)\right)^{\mu} = \int_{\xi}^{a} \left(a(s)\left(w'(s)\right)^{\mu}\right)' ds + \sum_{\xi \leq \chi_j < a} \Delta\left(a(\chi_j)w'(\chi_j)\right)^{\mu}.$$

Computing the limit as $a \to \infty$,

$$a(\xi)(w'(\xi))^\mu \geq \int_\xi^\infty c(s)g\big(u(\vartheta(s))\big)\, ds + \sum_{\chi_j \geq \xi} \tilde{c}(\chi_j)g\big(u(\vartheta(\chi_j))\big). \tag{14}$$

Then

$$w'(\xi) \geq \left[\frac{1}{a(\xi)}\left[\int_\xi^\infty c(s)g\big(u(\vartheta(s))\big)\, ds + \sum_{\xi \leq \chi_j} \tilde{c}(\chi_j)g\big(u(\vartheta(\chi_j))\big)\right]\right]^{1/\mu}.$$

Since $w(\xi_1) > 0$, we integrate the above inequality from ξ_1 to ξ, and so

$$w(\xi) \geq \int_{\xi_1}^\xi \left[\frac{1}{a(s)}\left[\int_s^\infty c(\zeta)g\big(u(\vartheta(\zeta))\big)\, d\zeta + \sum_{s \leq \chi_j} \tilde{c}(\chi_j)g\big(u(\vartheta(\chi_j))\big)\right]\right]^{1/\mu} ds$$

$$\geq A(\xi)\left[\int_\xi^\infty c(\zeta)g\big(u(\vartheta(\zeta))\big)\, d\zeta + \sum_{\xi \leq \chi_j} \tilde{c}(\chi_j)g\big(u(\vartheta(\chi_j))\big)\right]^{1/\mu},$$

which yields (11). □

For the next theorem, we suppose that there is a constant α, which is a ratio of two positive odd integers, with $\alpha < \mu$, such that

$$\frac{g(u)}{u^\alpha} \text{ is non-increasing for } 0 < u. \tag{15}$$

For example $g(u) = |u|^\beta \, \mathrm{sgn}(u)$, with $0 < \beta < \alpha$ satisfies this condition.

Theorem 1. *Assume (A1)–(A6), (15), and that $-1 < -b_0 \leq b(\xi) \leq 0$ holds for all $\xi \geq \xi_0$. Then, each solution of (E1) is oscillatory or tends to zero, if and only if*

$$\int_{\xi_2}^\infty c(s)g\big(\delta A(\vartheta(s))\big)\, ds + \sum_{k=1}^\infty \tilde{c}(\chi_j)g\big(\delta A(\vartheta(\chi_j))\big) = \infty \quad \forall \delta \neq 0. \tag{16}$$

Proof. Assume the contrary and suppose that (E1) has a non-oscillatory solution u which is positive and does not converge to zero. Then, case 1 in Lemma 1 leads to $\lim_{\xi\to\infty} u(\xi) = 0$, which a contradiction.

Case 2 of Lemma 1 also leads to a contradiction. In case 2 there exists ξ_1 such that

$$u(\xi) \geq w(\xi) \geq A(\xi)\Lambda^{1/\mu}(\xi) \geq 0 \quad \forall \xi \geq \xi_1. \tag{17}$$

Now, we see that w is left continuous at χ_j,

$$\Lambda'(\xi) = -c(\xi)g\big(u(\vartheta(\xi))\big) \quad \text{for } \xi \neq \chi_j,$$
$$\Delta\Lambda(\chi_j) = -\tilde{c}(\chi_j)g\big(u(\vartheta(\chi_j))\big) \leq 0.$$

It is clear that $\Lambda(\xi) > 0$ for $\xi \geq \xi_1$. Computing the derivative,

$$\left(\Lambda^{1-\alpha/\mu}(\xi)\right)' = \left(1 - \frac{\alpha}{\mu}\right)\Lambda^{-\alpha/\mu}(\xi)\Lambda'(\xi) \quad \text{for } \xi \neq \chi_j. \tag{18}$$

To estimate the discontinuities of $\Lambda^{1-\alpha/\mu}$ we use a first order Taylor polynomial for the function $h(u) = u^{1-\alpha/\mu}$, with $0 < \alpha < \mu$, about $u = e$:

$$d^{1-\alpha/\mu} - e^{1-\alpha/\mu} \leq \left(1 - \frac{\alpha}{\mu}\right)d^{-\alpha/\mu}(e - d).$$

Then $\Delta \Lambda^{1-\alpha/\mu}(\chi_j) \leq \left(1 - \frac{\alpha}{\mu}\right) \Lambda^{-\alpha/\mu}(\chi_j) \Delta\Lambda(\chi_j)$. Integrating (18) from ξ_2 to ξ, and using that $\Lambda > 0$, we have

$$
\begin{aligned}
\Lambda^{1-\alpha/\mu}(\xi_2) &\geq \left(1 - \frac{\alpha}{\mu}\right)\left[-\int_{\xi_2}^{\xi} \Lambda^{-\alpha/\mu}(s)\Lambda'(s)\,ds - \sum_{\xi_2 \leq \chi_j < \xi} \Lambda^{-\alpha/\mu}(\chi_j)\Delta\Lambda(\chi_j) \right] \\
&= \left(1 - \frac{\alpha}{\mu}\right)\left[\int_{\xi_2}^{\xi} \Lambda^{-\alpha/\mu}(s)\Big(c(s)g(u(\vartheta(s)))\Big)\,ds + \sum_{\xi_2 \leq \chi_j < \xi} \Lambda^{-\alpha/\mu}(\chi_j)\tilde{c}(\chi_j)g(u(\vartheta(\chi_j))) \right]. \quad (19)
\end{aligned}
$$

Since $w \leq u$, by (A3), (15), (10), and (17), we have

$$
\begin{aligned}
g(u(\xi)) &\geq g(w(\xi))\frac{w^{\alpha}(\xi)}{w^{\alpha}(\xi)} \geq \frac{g(\delta A(\xi))}{(\delta A(\xi))^{\alpha}} w^{\alpha}(\xi) \\
&\geq \frac{g(\delta A(\xi))}{(\delta A(\xi))^{\alpha}}\left(A(\xi)\Lambda^{1/\mu}(\xi)\right)^{\alpha} = \frac{g(\delta A(\xi))}{\delta^{\alpha}}\Lambda^{\alpha/\mu}(\xi) \quad \text{for } \xi \geq \xi_2.
\end{aligned}
$$

Since $\alpha/\mu > 0$ and $\vartheta(s) < s$, we have

$$
g(u(\vartheta(s))) \geq \frac{g(\delta A(\vartheta(s)))}{\delta^{\alpha}}\Lambda^{\alpha/\mu}(\vartheta(s)) \geq \frac{g(\delta A(\vartheta(s)))}{\delta^{\alpha}}\Lambda^{\alpha/\mu}(s). \quad (20)
$$

Going back to (19), we have

$$
\Lambda^{1-\alpha/\mu}(\xi_2) \geq \frac{\left(1 - \frac{\alpha}{\mu}\right)}{\delta^{\alpha}}\left[\int_{\xi_2}^{\xi} c(s)g(\delta A(\vartheta(s)))\,ds + \sum_{\xi_2 \leq \chi_j < \xi} \tilde{c}(\chi_j)g(\delta A(\vartheta(\chi_j))) \right], \quad (21)
$$

which contradicts (16). This completes the proof of sufficient part of the theorem when the solution is a eventually positive.

For an eventually negative solution u, we will define a new variable $v = -u$ and $g(\xi) = -g(\xi)$. Then v is an eventually positive solution of (E1) with f instead of g. We find that f satisfies (A3) and (15). Therefore, the above method can be applied to the v solution.

Next, by a contrapositive argument, we show the necessity part—that is, if (16) is not true then there is a non-oscillatory solution. Let (16) be untrue for some $\delta > 0$; then for each $\epsilon > 0$ there exists $\xi_1 \geq \xi_0$ such that

$$
\int_s^{\infty} c(\zeta)g(\delta A(\vartheta(\zeta)))\,d\zeta + \sum_{\chi_j \geq s} \tilde{c}(\chi_j)g(\delta A(\vartheta(\chi_j))) \leq \epsilon, \quad (22)
$$

for all $s \geq \xi_1$. In particular we use a positive ϵ such that

$$
(2\epsilon)^{1/\mu} = (1 - b_0)\delta, \quad (23)
$$

so that $0 < \epsilon^{1/\mu} \leq (1 - b_0)\delta/2^{1/\mu} < \delta$. Note that ξ_1 depends on δ. We define

$$
S = \{u \in C([0,\infty)) : \epsilon^{1/\mu}A(\xi) \leq u(\xi) \leq \delta A(\xi),\ \xi \geq \xi_1\}.
$$

Then we can define an operator Φ on S as follows:

$$
(\Phi u)(\xi) = \begin{cases}
0 & \text{if } \xi \leq \xi_1 \\
-b(\xi)u(\sigma(\xi)) + \int_{\xi_1}^{\xi}\left[\frac{1}{a(s)}\left[\epsilon + \int_s^{\infty} c(\zeta)g(u(\vartheta(\zeta)))\,d\zeta \right.\right. & \\
\left.\left. + \sum_{\chi_j \geq s}\tilde{c}(\chi_j)g(u(\vartheta(\chi_j)))\right]\right]^{1/\mu}ds & \text{if } \xi > \xi_1.
\end{cases}
$$

Now we are going to show that u is a fixed point of Φ in S, that is, $\Phi u = u$; u is an eventually positive solution: of (E1).

For $u \in S$, we have $0 \leq \epsilon^{1/\mu} A(\xi) \leq u(\xi)$. By (A3), we have $0 \leq g\big(u(\vartheta(s))\big)$ and by (A2) we have

$$(\Phi u)(\xi) \geq 0 + \int_{\xi_1}^{\xi} \Big[\frac{1}{a(s)}[\epsilon + 0 + 0]\Big]^{1/\mu} ds = \epsilon^{1/\mu} A(\xi).$$

For u in S, by (A2) and (A3), we have $g\big(u(\vartheta(\zeta))\big) \leq g\big(\delta a(\vartheta(\zeta))\big)$. Then by (22) and (23),

$$(\Phi x)(\xi) \leq b_0 \delta\big(A(\sigma(\xi))\big) + \int_{\xi_1}^{\xi} \Big[\frac{1}{a(s)}\Big[\epsilon + \int_{s}^{\infty} c(\zeta) g(\delta a(\vartheta(\zeta))) \, d\zeta + \sum_{\chi_j \geq s} \tilde{c}(\chi_j) g(\delta a(\vartheta(\chi_j)))\Big]\Big]^{1/\mu} ds$$

$$\leq b_0 \delta A(\xi) + (2\epsilon)^{1/\mu} A(\xi) = \delta A(\xi).$$

Therefore, Φ maps S to S. In the next section, we search a fixed point for Φ in S. Let us define a recurrence relation

$$v_0(\xi) = 0 \quad \text{for } \xi \geq \xi_0,$$

$$v_1(\xi) = (\Phi v_0)(\xi) = \begin{cases} 0 & \text{if } \xi < \xi_1 \\ \epsilon^{1/\mu} A(\xi)) & \text{if } \xi \geq \xi_1 \end{cases},$$

$$v_{n+1}(\xi) = (\Phi v_n)(\xi) \quad \text{for } n \geq 1, \ \xi \geq \xi_1.$$

Note that for each fixed v, we have $v_1(\xi) \geq v_0(\xi)$. Using the mathematical induction and the fact that g is non-decreasing, one can prove $v_{n+1}(\xi) \geq v_n(\xi)$. Therefore, u is a fixed point of Φ in S; that is, $\Phi u = u$ by using the lebesgue dominated convergence theorem. Thus, we have a eventually positive solution. This completes the proof. $\quad\square$

Remark 1. *If all the conditions stated in Theorem 1 hold, then every unbounded solution of (E1) is oscillatory if and only if* (16) *holds.*

In the next theorem, we assume that ϑ_0 is a differentiable function, such that

$$0 < \vartheta_0(\xi) \leq \vartheta(\xi), \quad \exists \beta > 0 : \beta \leq \vartheta_0'(\xi) \quad \text{for } \xi \geq \xi_0. \tag{24}$$

Additionally, we assume that there exists a constant α, with $\mu < \alpha$, such that

$$\frac{g(u)}{u^\alpha} \text{ is non-decreasing for } 0 < u \tag{25}$$

where α is a ratio of two positive odd integers.
For example, $g(u) = |u|^\beta \, \mathrm{sgn}(u)$, with $\alpha < \beta$, satisfies this condition.

Theorem 2. *Assume (A1)–(A6), (24), (25), $a(\xi)$ is non-decreasing and $-1 < -b_0 \leq b(\xi) \leq 0$ for all $\xi \geq \xi_0$. Every solution of (E1) is oscillatory or tends to zero, if and only if*

$$\int_{\xi_1}^{\infty} \Big[\frac{1}{a(s)}\Big[\int_{s}^{\infty} c(\zeta) \, d\zeta + \sum_{\chi_j \geq s} \tilde{c}(\chi_j)\Big]\Big]^{1/\mu} ds = \infty. \tag{26}$$

Proof. We prove sufficiency part by contradiction. Suppose that u is an eventually positive solution that does not tends to zero. Using the same argument as in Lemma 1, there exists $\xi_1 \geq \xi_0$ such that: $u(\vartheta(\xi)) > 0$, $u(\sigma(\xi)) > 0$, and $a(\xi)\big(w'(\xi)\big)^\mu$ is positive and non-increasing. Case 1 of Lemma 1 leads to $\lim_{\xi \to \infty} u(\xi) = 0$, which contradicts the assumption that u does not tend to zero.

Case 2 of Lemma 1 also leads to a contradiction. In case 2, $w(\xi)$ is positive and increasing for $\xi \leq \xi_1$. Since $-1 < -b_0 \leq b(\xi) \leq 0$, it follows that $w(\xi) = u(\xi) + b(\xi)u(\sigma(\xi)) \leq u(\xi)$. From (A3), $w(\xi) \geq w(\xi_1)$ and (25), we have

$$g(u(\xi)) \geq \frac{g(w(\xi))}{z^{\alpha}(\xi)}z^{\alpha}(\xi) \geq \frac{g(w(\xi_1))}{z^{\alpha}(\xi_1)}z^{\alpha}(\xi).$$

By (A1) there exists a $\xi_2 \geq \xi_1$ such that $\vartheta(\xi) \geq \xi_1$ for $\xi \geq \xi_2$. Then

$$g\big(u(\vartheta(\xi))\big) \geq \frac{g(w(\xi_1))}{z^{\alpha}(\xi_1)}z^{\alpha}(\vartheta(\xi)) \quad \forall \xi \geq \xi_2. \tag{27}$$

Using this inequality, (14), that $\vartheta(\xi) \geq \vartheta_0(\xi)$ which is an increasing function and that z is increasing, we have

$$a(\xi)\big(w'(\xi)\big)^{\mu} \geq \frac{z^{\alpha}(\vartheta_0(\xi))}{z^{\alpha}(\xi_1)}\Big[\int_{\xi}^{\infty}c(s)g\big(w(\xi_1)\big)\,ds + \sum_{\chi_j \geq \xi}\tilde{c}(\chi_j)g\big(w(\xi_1)\big)\Big]$$

for $\xi \geq \xi_2$. From $a(\xi)\big(w'(\xi)\big)^{\mu}$ being non-increasing and $\vartheta_0(\xi) \leq \xi$, we have

$$a(\vartheta_0(\xi))\big(w'(\vartheta_0(\xi))\big)^{\mu} \geq a(\xi)\big(w'(\xi)\big)^{\mu}.$$

By then dividing by $a(\vartheta_0(\xi)) > 0$, taking $1/\mu$ on the power of both sides and dividing by $z^{\alpha/\mu}(\vartheta_0(\xi)) > 0$, we have

$$\frac{w'(\vartheta_0((\xi)))}{z^{\alpha/\mu}(\vartheta_0(\xi))} \geq \Big[\frac{1}{a(\vartheta_0(\xi))z^{\alpha}(\xi_1)}\Big[\int_{\xi}^{\infty}c(s)g(w(\xi_1))\,ds + \sum_{\chi_j \geq \xi}\tilde{c}(\chi_j)g(w(\xi_1))\Big]\Big]^{1/\mu}$$

for $\xi \geq \xi_2$. On the left-hand side we multiply by $\vartheta_0'(\xi)/\beta \geq 1$, and then integrate over ξ_1 to ξ:

$$\frac{1}{\beta}\int_{\xi_1}^{\xi}\frac{w'(\vartheta_0(s))\vartheta_0'(s)}{z^{\alpha/\mu}(\vartheta_0(s))}\,ds \geq \frac{1}{z^{\alpha/\mu}(\xi_1)}\int_{\xi_1}^{\xi}\Big[\frac{1}{a(\vartheta_0(s))}\Big[\int_{s}^{\infty}c(\zeta)g(w(\xi_1))\,d\zeta$$
$$+ \sum_{s \leq \chi_j}\tilde{c}(\chi_j)g\big(w(\xi_1)\big)\Big]\Big]^{1/\mu}\,ds \quad \forall \xi \geq \xi_2. \tag{28}$$

We know $\mu < \alpha$, and by integrating the left-hand side of (28) from ξ_2 to ξ, we obtain

$$\frac{1}{\beta(1-\alpha/\mu)}\Big[z^{1-\alpha/\mu}(\vartheta_0(s))\Big]_{s=\xi_2}^{\xi} \leq \frac{1}{\beta(\alpha/\mu-1)}z^{1-\alpha/\mu}(\vartheta_0(\xi_2)). \tag{29}$$

Using $g(w(\xi_1)) > 0$ and $a(\vartheta_0(s)) \leq a(s)$, and combining (28) and (29), we are getting a contradiction to (26). This completes the proof of sufficient part for the eventually positive solution.

For eventually negative solutions, we will use the same variables that were defined in Theorem 1, and follow the same method used in Theorem 1.

For the necessary part, we suppose that (26) does not suffice, and obtain an eventually positive solution that does not converge to zero. Let (26) not hold; then for each $\epsilon > 0$ there exists $\xi_1 \geq \xi_0$ such that

$$\int_{\xi_1}^{\infty}\Big[\frac{1}{a(s)}\Big[\int_{s}^{\infty}c(\zeta)\,d\zeta + \sum_{\chi_j \geq s}\tilde{c}(\chi_j)\Big]\Big]^{1/\mu}\,ds < \epsilon \quad \forall \xi \geq \xi_1. \tag{30}$$

In particular we use $\epsilon = \left(g(2/(1-b_0))\right)^{-1/\mu} > 0$. Let us consider

$$S = \left\{ u \in C([0,\infty)) : 1 \le u(\xi) \le \frac{2}{1-b_0} \text{ for } \xi \ge \xi_1 \right\}$$

the set of continuous functions. Then we define the operator

$$(\Phi u)(\xi) = \begin{cases} \frac{1}{1+b(\xi_1)} & \text{if } \sigma(\xi_1) = \xi_1, \ \xi \le \xi_1, \\ \frac{u - \sigma(\xi_1)}{\xi_1 - \sigma(\xi_1)} & \text{if } \sigma(\xi_1) < \xi_1, \ \xi \le \xi_1, \\ 1 - b(\xi)u(\sigma(\xi)) + \int_{\xi_1}^{\xi} \left[\frac{1}{a(s)} \left[\int_s^\infty c(\zeta) g\big(u(\vartheta(\zeta))\big) \, d\zeta \right. \right. \\ \left. \left. + \sum_{\chi_j \ge s} \tilde{c}(\chi_j) g\big(u(\vartheta(\chi_j))\big) \right]\right]^{1/\mu} ds & \text{if } \xi > \xi_1. \end{cases}$$

Note that if u is continuous, Φu is also continuous at $\xi = \xi_1$. This follows by taking the right and left limits in the three possible cases in the the definition of Φ. Additionally, if $\Phi u = u$, then u is solution of (E1).

Let $u \in S$. Then $1 \le u$, and by (A3), we have $(\Phi u)(\xi) \ge 1 + 0 + 0$, on $[\xi_1, \infty)$.

Let $u \in S$. Then $u \le 2/(1-b_0)$ and

$$(\Phi u)(\xi) \le 1 - b(\xi) \frac{2}{1-b_0} + \int_{\xi_1}^{\xi} \left[\frac{1}{a(s)} \left[\int_s^\infty c(\zeta) g\left(\frac{2}{1-b_0}\right) d\zeta \ + \sum_{\chi_j \ge s} \tilde{c}(\chi_j) g\left(\frac{2}{1-b_0}\right) \right] \right]^{1/\mu} ds.$$

$$\le 1 + \frac{2b_0}{1-b_0} + \left(g(2/(1-b_0))\right)^{1/\mu} \epsilon = 1 + \frac{2b_0}{1-b_0} + 1 = \frac{2}{1-b_0}.$$

Therefore, Φ maps S to S.

We need to prove there is a fixed point of Φ in S so we are going to define a sequence as follows

$$v_0(\xi) = 0 \quad \text{for } \xi \ge \xi_0,$$
$$v_1(\xi) = (\Phi v_0)(\xi) = 1 \quad \text{for } \xi \ge \xi_1,$$
$$v_{n+1}(\xi) = (\Phi v_n)(\xi) \quad \text{for } n \ge 1, \ \xi \ge \xi_1.$$

The rest of necessary part follows from Theorem 1. □

The next theorem does not require neither (15) nor (25), but considers only bounded solutions.

Theorem 3. *Assume (A1)–(A6) and $-1 < -b_0 \le b(\xi) \le 0$ for all $\xi \ge \xi_0$. Then every bounded solution of (E1) is oscillatory or converges to zero, if and only if (26) holds.*

Proof. We prove sufficiency by contradiction. Suppose u is an eventually positive solution that does not converge to zero. Then we proceed as in Lemma 1 up to Equation (13). u and b are bounded, so w is bounded. Then the left-hand side of (13) is bounded, and the right-hand side approaches $-\infty$ as $\xi \to \infty$. This contradiction implies that $w'(\xi) > 0$ for $\xi \ge \xi_1$. As in Lemma 1, we find two possible cases.

Case 1: $w(\xi) < 0$ for all $\xi \ge \xi_1$. This leads to a contradiction. As in case 1 of Lemma 1, we have $\lim_{\xi \to \infty} u(\xi) = 0$, which contradicts the assumption that u does not converge to zero.

Case 2: $w(\xi) > 0$ for all $\xi \ge \xi_1$. This also leads to a contradiction. Since z is positive and increasing, $w(\xi) \ge w(\xi_1)$ for $\xi \ge \xi_1$. Recall that $u(\xi) \ge w(\xi)$, so u cannot converge to zero. By (A2), there is a $\xi_2 \ge \xi_1$ such that $\vartheta(\xi) \ge \xi_1$ and $u(\vartheta(\xi)) \ge w(\xi_1)$ for $\xi \ge \xi_2$. From (A4), $g\big(u(\vartheta(\xi))\big) \ge g(w(\xi_1)) > 0$. Then, integrating as we did for (14), we have

$$\lim_{\xi \to \infty} w(\xi) - w(\xi_2) \ge \int_{\xi_2}^\infty \left[\frac{1}{a(s)} \left[\int_s^\infty c(\zeta) g(w(\xi_1)) \, d\zeta + \sum_{s \le \chi_j} \tilde{c}(\chi_j) g(w(\xi_1)) \right] \right]^{1/\mu} ds.$$

By (26), the right-hand side approaches $+\infty$, which contradicts w being bounded.

For eventually negative solutions, we proceed as above to obtain also a contradiction. Therefore, every bounded and solution must be oscillatory or converge to zero.

The proof of the necessity part is the same as that in Theorem 2, taking into account that if $u \in S$, then $w(\xi_1) \leq u(\xi_1) \leq 2/(1-p)$. $\square$

3. Example

In this section, we are giving one example to show the effectiveness and feasibility of our main results.

Example 1. *Consider the delay differential equation*

$$
\begin{cases}
\dfrac{\mathrm{d}}{\mathrm{d}\xi}\left(\left(\dfrac{\mathrm{d}}{\mathrm{d}\xi}\left(u(\xi)-\beta u(\xi-3)\right)\right)\right)+3\left(\dfrac{\beta\mathrm{e}^3-\alpha^3}{\alpha\mathrm{e}^2}\right)^3\left(u(\xi-2)\right)^3=0, & \xi\neq j \\[2mm]
\Delta\left(\left((u(j)-\beta u(j-3))'\right)\right)+(\alpha^3-1)\left(\dfrac{\alpha^3-\beta\mathrm{e}^3}{\alpha^4\mathrm{e}^2}\right)^3\left(u(j-2)\right)^3=0, & \xi=j.
\end{cases}
$$

where we have $a(\xi) \equiv 1$, $\chi_j = j$, $\mu = 1$, $b(\xi) \equiv \beta$, $c(\xi) \equiv 3\left(\dfrac{\beta\alpha^3\mathrm{e}^3-8}{2\alpha^2\mathrm{e}^2}\right)^3$, $\tilde{c}_k := (\alpha^3 - 1)\left(\dfrac{\alpha^3-\beta\mathrm{e}^3}{\alpha^4\mathrm{e}^2}\right)^3$, $\sigma(\xi) = \xi - 3$, $\vartheta(\xi) = \xi - 2$, $g(u) = u^3$. *With* $0 < \alpha < 1$ *and* $\left(\dfrac{\alpha}{\mathrm{e}}\right)^3 < \beta < 1$ *all the conditions of Theorem 2 are satisfied. Further, a solution of the equation is* $u(\xi) := \left(\prod_{-3\leq j<\xi}\alpha\right)\mathrm{e}^{-\xi} = \alpha^{\lfloor\xi\rfloor+3}\mathrm{e}^{-\xi} > 0$, *which tends to zero as* $\xi \to \infty$. *Indeed, we have*

$$
\begin{cases}
\dfrac{\mathrm{d}}{\mathrm{d}\xi}\left(\left(\dfrac{\mathrm{d}}{\mathrm{d}\xi}\left(\underbrace{\alpha^{\lfloor\xi\rfloor+3}\mathrm{e}^{-\xi}-\beta\alpha^{\lfloor\xi\rfloor}\mathrm{e}^{-(\xi-3)}}_{(\alpha^3-\beta\mathrm{e}^3)\alpha^{\lfloor\xi\rfloor}\mathrm{e}^{-\xi}}\right)\right)\right)+3\left(\dfrac{\beta\mathrm{e}^3-\alpha^3}{\alpha\mathrm{e}^2}\right)^3\left(\alpha^{\lfloor\xi\rfloor+1}\mathrm{e}^{-(\xi-2)}\right)^3=0, & \xi\neq j \\[4mm]
\Delta\left(\left(\left((\alpha^3-\beta\mathrm{e}^3)\alpha^{j-3}\mathrm{e}^{-j}\right)'\right)\right)+(\alpha^3-1)\left(\dfrac{\alpha^3-\beta\mathrm{e}^3}{\alpha^4\mathrm{e}^2}\right)^3\left(\alpha^{j+1}\mathrm{e}^{-(j-2)}\right)^3=0, & \xi=j.
\end{cases}
$$

4. Conclusions and an Open Problem

In this work, we established sufficient and necessary conditions for the oscillatory or asymptotic behavior of a second-order neutral delay impulsive system of the form (E1) when the neutral coefficient $b \in (-1, 0]$. Based on this work and [9–13,27], we can frame the following open problem.

Q. Can we find the necessary and sufficient conditions for the oscillation of solutions to the impulsive differential system (E1) for other ranges of the neutral coefficient, i.e., for $b > 1$ and $-\infty < b(\xi) < -1$?

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