

Article

General Fractional Integrals and Derivatives of Arbitrary Order

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Abstract: In this paper, we introduce the general fractional integrals and derivatives of arbitrary order and study some of their basic properties and particular cases. First, a suitable generalization of the Sonine condition is presented, and some important classes of the kernels that satisfy this condition are introduced. Whereas the kernels of the general fractional derivatives of arbitrary order possess integrable singularities at the point zero, the kernels of the general fractional integrals can—depending on their order—be both singular and continuous at the origin. For the general fractional integrals and derivatives of arbitrary order with the kernels introduced in this paper, two fundamental theorems of fractional calculus are formulated and proved.

Keywords: Sonine kernel; general fractional derivative of arbitrary order; general fractional integral of arbitrary order; first fundamental theorem of fractional calculus; second fundamental theorem of fractional calculus

MSC: 26A33; 26B30; 44A10; 45E10



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1. Introduction

In his papers [1,2], Abel derived and studied a mathematical model for the tautochrone problem in the form of the following integral equation (with slightly different notations):

$$f(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{\phi'(\tau) d\tau}{\sqrt{t-\tau}}. \quad (1)$$

In fact, he considered the even more general integral equation

$$f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\phi'(\tau) d\tau}{(t-\tau)^\alpha} \quad (2)$$

under an implicit restriction $0 < \alpha < 1$. It is easy to see that the right-hand side of (2) is the operator that is currently referred to as the Caputo fractional derivative ${}^*D_{0+}^\alpha$ of the order α , $0 < \alpha < 1$. Abel's solution formula to Equation (2) is nothing else than the operator now called the Riemann–Liouville fractional integral I_{0+}^α of the order $\alpha > 0$:

$$\phi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau =: (I_{0+}^\alpha f)(t), \quad t > 0. \quad (3)$$

In modern notation, Formulas (2) and (3) correspond to the second fundamental theorem of FC for the Caputo fractional derivative of a function that takes the value zero at the point zero:

$$(I_{0+}^\alpha f)(t) = (I_{0+}^\alpha {}^*D_{0+}^\alpha \phi)(t) = \phi(t) - \phi(0) = \phi(t), \quad (4)$$

where the validity of the condition $\phi(0) = 0$ follows from the construction of Abel's mathematical model for the tautochrone problem. For more details regarding Abel's results and derivations presented in [1,2], see the recent paper presented in [3].

To solve the integral Equation (2), in [2], published in 1826, Abel employed the relation

$$(h_\alpha * h_{1-\alpha})(t) = \{1\}, t > 0, h_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \alpha > 0, \quad (5)$$

where the operation $*$ stands for the Laplace convolution,

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau) d\tau \quad (6)$$

and $\{1\}$ is the function that is identically equal to 1 for $t \geq 0$.

In [4], published in 1884, Sonine recognized that the relation (5) is the most crucial ingredient of Abel's solution method that can be generalized and applied to the analytical treatment of a larger class of integral equations. In place of (5), Sonine considered a pair of functions κ, k (Sonine kernels) that satisfy the relation

$$(\kappa * k)(t) = \{1\}, t > 0. \quad (7)$$

In what follows, we denote the set of the Sonine kernels by \mathcal{S} . For a given Sonine kernel κ , the kernel k that satisfies the Sonine condition (7) is called its associate Sonine kernel. Following Abel's solution method, Sonine showed that the integral equation

$$f(t) = \int_0^t \kappa(t-\tau)\phi(\tau) d\tau = (\kappa * \phi)(t) \quad (8)$$

has a solution in the form

$$\phi(t) = \frac{d}{dt} \int_0^t k(t-\tau)f(\tau) d\tau = \frac{d}{dt}(k * f)(t), \quad (9)$$

provided that the kernels κ, k satisfy the Sonine condition (7). Indeed, we obtain

$$(k * f)(t) = (k * \kappa * \phi)(t) = (\{1\} * \phi)(t) = \int_0^t \phi(\tau) d\tau$$

which immediately leads to the Formula (9). Of course, any concrete realization of the Sonine schema requires a precise characterization of the Sonine kernels and the spaces of functions where the operators from the right-hand sides of (8) and (9) are well defined. In [4], Sonine introduced a large class of the Sonine kernels in the form

$$\kappa(t) = h_\alpha(t) \cdot \kappa_1(t), \kappa_1(t) = \sum_{k=0}^{+\infty} a_k t^k, a_0 \neq 0, 0 < \alpha < 1, \quad (10)$$

$$k(t) = h_{1-\alpha}(t) \cdot k_1(t), k_1(t) = \sum_{k=0}^{+\infty} b_k t^k, \quad (11)$$

where the functions $\kappa_1 = \kappa_1(t)$, $k_1 = k_1(t)$ are analytical on \mathbb{R} and their coefficients are connected by the relations

$$a_0 b_0 = 1, \sum_{k=0}^n \Gamma(k+1-\alpha)\Gamma(\alpha+n-k)a_{n-k}b_k = 0, n \geq 1. \quad (12)$$

The most prominent pair of the Sonine kernels from this class is given by the formulas

$$\kappa(t) = (\sqrt{t})^{\alpha-1} J_{\alpha-1}(2\sqrt{t}), k(t) = (\sqrt{t})^{-\alpha} I_{-\alpha}(2\sqrt{t}), 0 < \alpha < 1, \quad (13)$$

where

$$J_\nu(t) = \sum_{k=0}^{+\infty} \frac{(-1)^k (t/2)^{2k+\nu}}{k! \Gamma(k+\nu+1)}, \quad I_\nu(t) = \sum_{k=0}^{+\infty} \frac{(t/2)^{2k+\nu}}{k! \Gamma(k+\nu+1)}, \quad \Re(\nu) > -1, \quad t \in \mathbb{C} \quad (14)$$

are the Bessel and the modified Bessel functions, respectively.

Later, the evolution equations with the integro-differential operators of the convolution type (compare them with the solution by Sonine in Formula (9)),

$$(\mathbb{D}_{(k)} f)(t) = \frac{d}{dt} \int_0^t k(t-\tau) f(\tau) d\tau, \quad t > 0, \quad (15)$$

were actively studied in the framework of the abstract Volterra integral equations on the Banach spaces (see [5] and references therein). For example, in [6], the case of operators with the completely positive kernels $k \in L^1(0, +\infty)$ was considered. The kernels from this class satisfy the condition (compare it with the Sonine condition (7))

$$a k(t) + \int_0^t k(t-\tau) l(\tau) d\tau = \{1\}, \quad t > 0, \quad (16)$$

where $a \geq 0$ and $l \in L^1(0, +\infty)$ is a non-negative and non-increasing function.

However, until recently, no interpretation of these general results in the framework of fractional calculus (FC) had been suggested. The situation changed with the publication of the paper presented in [7] (see also [8–10]). In [7], Kochubei introduced a class \mathcal{K} of kernels that satisfy the following conditions:

(K1) The Laplace transform \tilde{k} of k ,

$$\tilde{k}(p) = (\mathcal{L}k)(p) = \int_0^{+\infty} k(t) e^{-pt} dt, \quad (17)$$

exists for all $p > 0$;

(K2) $\tilde{k}(p)$ is a Stieltjes function (see [11] for details regarding the Stieltjes functions);

(K3) $\tilde{k}(p) \rightarrow 0$ and $p\tilde{k}(p) \rightarrow +\infty$ as $p \rightarrow +\infty$;

(K4) $\tilde{k}(p) \rightarrow +\infty$ and $p\tilde{k}(p) \rightarrow 0$ as $p \rightarrow 0$.

Using the technique of the complete Bernstein functions, Kochubei investigated the integro-differential operators in the form of (15) and their Caputo type modifications

$$(*\mathbb{D}_{(k)} f)(t) = (\mathbb{D}_{(k)} f)(t) - f(0)k(t) \quad (18)$$

with the kernels from \mathcal{K} . In [7], Kochubei showed the inclusion $\mathcal{K} \subset \mathcal{S}$, introduced the corresponding integral operator

$$(\mathbb{I}_{(\kappa)} f)(t) = (\kappa * f)(t) = \int_0^t \kappa(t-\tau) f(\tau) d\tau, \quad (19)$$

and proved the validity of the first fundamental theorem of FC; i.e., that the operators (15) and (18) are left-inverse to the integral operator (19) on the suitable spaces of functions.

Moreover, Kochubei treated some basic ordinary and partial fractional differential equations with the time-derivative in the form of (18) and proved that the solution to the Cauchy problem for the relaxation equation with the operator (18) and a positive initial condition is completely monotonic and that the fundamental solution to the Cauchy problem for the fractional diffusion equation with the time-derivative in the form of (18) can be interpreted as a probability density function. These results justified calling the operators (15) and (18) the general fractional derivatives (GFDs) in the Riemann–Liouville and Caputo sense, respectively. The integral operator (19) was called the general fractional integral (GFI).

The GFDs (15) and (18) with the kernels $k \in \mathcal{K} \subset \mathcal{S}$ possess a series of important properties. However, the conditions (K1)–(K4) are very strong (especially the condition (K2)), and thus in subsequent publications, the operators (15) and (18) with the Sonine kernels from some larger classes were considered from the viewpoint of FC and its applications. In [12], a class of the kernels was introduced that ensures the validity of a maximum principle for the general time-fractional diffusion equations with the operators of type (18). Another important class of the Sonine kernels was described in [8] in terms of the completely monotone functions. As shown in [8], any singular (unbounded in a neighborhood of the point zero) locally integrable completely monotone function κ is a Sonine kernel, and its associate kernel k is also a locally integrable completely monotone function.

In the recent publications presented in [9,13], the operators (15) and (18) with the Sonine kernels from the class $\mathcal{S}_{-1} \subset \mathcal{S}$ that satisfy only some minimal restrictions were studied from the viewpoint of FC. The Sonine kernels κ , $k \in \mathcal{S}_{-1}$ are continuous on \mathbb{R}_+ and possess the integrable singularities of the power function type at the point zero. In particular, in [9], the first and second fundamental theorems of FC for the operators (15) and (18) with the kernels $k \in \mathcal{S}_{-1}$ were formulated and proved. In [13], an operational calculus of the Mikusiński type for the operators (18) with the Sonine kernels $k \in \mathcal{S}_{-1}$ was constructed and applied for the analytical treatment of some initial value problems for the fractional differential equations with these operators.

It is clear that weakening the Kochubei conditions (K1)–(K4) on the Sonine kernels from \mathcal{K} leads to the abandonment of some properties that were derived in [7] for the GFDs (15) and (18). However, it was shown in [9,13] that the operators (15) and (18) with the Sonine kernels $k \in \mathcal{S}_{-1}$ and the corresponding integral operator (19) still satisfy the main properties that the fractional derivatives and integrals should fulfill (see [14] and the references therein). Thus, these operators can also be interpreted as the GFDs and GFIs.

Another important point concerns the “generalized order” of the GFDs (15) and (18) with the Sonine kernels from the classes mentioned above. While projecting these operators to the conventional Riemann–Liouville and Caputo fractional derivatives (the case of the kernel $k(t) = h_{1-\alpha}(t)$), the derivatives’ order is restricted only to the case of $\alpha \in (0, 1)$. The reason is that the Sonine condition (5) for the power functions h_α and $h_{1-\alpha}$ holds true only in the case $0 < \alpha < 1$. Moreover, even in the definition of the Caputo type general fractional derivative (18), only one initial condition is contained, which again indicates that the “generalized order” of this operator does not exceed one.

Because the Riemann–Liouville fractional integral and the Riemann–Liouville and Caputo fractional derivatives are defined for arbitrary order $\alpha \geq 0$, an extension of the GFDs (15) and (18) to the case of arbitrary order is worthy of investigation.

In a recent paper [9], the n -fold GFIs and GFDs were introduced as an attempt to extend their order behind the interval $(0, 1)$. For example, the two-fold general fractional derivative constructed for the operator (15) with the kernel $\kappa(t) = h_{1-\alpha}(t)$, $0 < \alpha < 1$ is the Riemann–Liouville fractional derivative of the order 2α :

$$(D_{0+}^{2\alpha} f)(t) = \begin{cases} \frac{d^2}{dt^2} (I_{0+}^{2-2\alpha} f)(t), & \frac{1}{2} < \alpha < 1, t > 0, \\ \frac{d}{dt} (I_{0+}^{1-2\alpha} f)(t), & 0 < \alpha \leq \frac{1}{2}, t > 0. \end{cases} \quad (20)$$

Thus, we cannot ensure that the order of this two-fold GFD is always greater than one. Depending on the values of α and n , the “generalized order” of the n -fold GFD can be any number in the interval $(0, n)$.

The main objective of this paper is to introduce the GFIs and GFDs of an arbitrary order in analogy to the Riemann–Liouville fractional integral and the Riemann–Liouville and Caputo fractional derivatives. This is done by a suitable generalization of the Sonine condition (7) and by the corresponding adjustment of Formulas (15) and (18), which define the GFDs in the Riemann–Liouville and Caputo senses.

The rest of the paper is organized as follows. In Section 2, following [9,13], we provide some basic definitions and properties of the GFDs (15) and (18) with the Sonine kernels $k \in \mathcal{S}_{-1}$. Section 3 presents our main results. First, a suitable generalization of the Sonine

condition (7) is introduced and some examples of the kernels that satisfy this condition are discussed. Then, the GFDs of an arbitrary order with these kernels are defined and their properties are studied. The conventional Riemann–Liouville and Caputo fractional derivatives of arbitrary order are particular cases of these GFDs. Another important example is the integro-differential operators of convolution type with the Bessel and the modified Bessel functions in the kernels. The constructions introduced in this section allow the formulation of the fractional differential equations with the GFDs with a generalized order greater than one with several initial conditions.

2. General Fractional Integrals and Derivatives with the Sonine Kernels

In this section, we provide some basic definitions and results regarding the GFIs and GFDs with the Sonine kernels from the class \mathcal{S}_{-1} introduced in [9]. For more details, other relevant results and the proofs, see [9,13].

In what follows, we employ the space of functions $C_{-1}(0, +\infty)$ and its sub-spaces. A family of the spaces $C_\alpha(0, +\infty)$, $\alpha \geq -1$ was first introduced in [15] as follows:

$$C_\alpha(0, +\infty) := \{f : f(t) = t^p f_1(t), t > 0, p > \alpha, f_1 \in C[0, +\infty)\}. \tag{21}$$

Evidently, the spaces $C_\alpha(0, +\infty)$ are ordered by the inclusion $\alpha_1 \geq \alpha_2 \Rightarrow C_{\alpha_1}(0, +\infty) \subseteq C_{\alpha_2}(0, +\infty)$, and thus the inclusion $C_\alpha(0, +\infty) \subseteq C_{-1}(0, +\infty)$, $\alpha \geq -1$ holds true.

In the further discussions, we also use the sub-spaces $C_{-1}^m(0, +\infty)$, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ of the space $C_{-1}(0, +\infty)$, which are defined as follows:

$$C_{-1}^m(0, +\infty) := \{f : f^{(m)} \in C_{-1}(0, +\infty)\}. \tag{22}$$

The spaces $C_{-1}^m(0, +\infty)$ were first introduced and studied in [16]. In particular, we have the following properties:

- (1) $C_{-1}^0(0, +\infty) \equiv C_{-1}(0, +\infty)$;
- (2) $C_{-1}^m(0, +\infty)$, $m \in \mathbb{N}_0$ is a vector space over the field \mathbb{R} (or \mathbb{C});
- (3) If $f \in C_{-1}^m(0, +\infty)$ with $m \geq 1$, then $f^{(k)}(0+) := \lim_{t \rightarrow 0+} f^{(k)}(t) < +\infty$, $0 \leq k \leq m - 1$, and the function

$$\tilde{f}(t) = \begin{cases} f(t), & t > 0, \\ f(0+), & t = 0 \end{cases}$$

belongs to the space $C^{m-1}[0, +\infty)$;

- (4) If $f \in C_{-1}^m(0, +\infty)$ with $m \geq 1$, then $f \in C^m(0, +\infty) \cap C^{m-1}[0, +\infty)$.
- (5) For $m \geq 1$, the following representation holds true:

$$f \in C_{-1}^m(0, +\infty) \Leftrightarrow f(t) = (I_{0+}^m \phi)(t) + \sum_{k=0}^{m-1} f^{(k)}(0) \frac{t^k}{k!}, t \geq 0, \phi \in C_{-1}(0, +\infty);$$

- (6) Let $f \in C_{-1}^m(0, +\infty)$, $m \in \mathbb{N}_0$, $f(0) = \dots = f^{(m-1)}(0) = 0$ and $g \in C_{-1}^1(0, +\infty)$. Then, the Laplace convolution $h(t) = (f * g)(t)$ belongs to the space $C_{-1}^{m+1}(0, +\infty)$ and $h(0) = \dots = h^{(m)}(0) = 0$.

For our aims, we also need another two-parameter family of sub-spaces of $C_\alpha(0, +\infty)$ that allows us to better control the behavior of the functions at the origin:

$$C_{\alpha,\beta}(0, +\infty) = \{f : f(t) = t^p f_1(t), t > 0, \alpha < p < \beta, f_1 \in C[0, +\infty)\}. \tag{23}$$

In particular, the sub-space $C_{-1,0}(0, +\infty)$ contains the functions that are continuous on \mathbb{R}_+ and possess the integrable singularities of the power function type at the origin.

As mentioned in [17] (see also [8]), any Sonine kernel has an integrable singularity at the point zero. On the other hand, the kernels of the fractional integrals and derivatives should be singular [18]. Thus, the fractional integrals and derivatives with the Sonine

kernels are worthy of investigation. In what follows, we consider the GFI (19) and the GFDs (15) and (18) of the Riemann–Liouville and Caputo types, respectively, with the Sonine kernels κ and k that belong to the sub-space $C_{-1,0}(0, +\infty)$ of the space $C_{-1}(0, +\infty)$.

Definition 1. Let $\kappa, k \in C_{-1,0}(0, +\infty)$ be a pair of the Sonine kernels; i.e., let the Sonine condition (7) be fulfilled. The set of such Sonine kernels is denoted by \mathcal{S}_{-1} :

$$(\kappa, k \in \mathcal{S}_{-1}) \Leftrightarrow (\kappa, k \in C_{-1,0}(0, +\infty)) \wedge ((\kappa * k)(t) = \{1\}). \tag{24}$$

Several important features of the GFI (19) on the space $C_{-1}(0, +\infty)$ follow from the well-known properties of the Laplace convolution. In particular, we mention the mapping property

$$\mathbb{I}_{(\kappa)} : C_{-1}(0, +\infty) \rightarrow C_{-1}(0, +\infty), \tag{25}$$

the commutativity law

$$\mathbb{I}_{(\kappa_1)} \mathbb{I}_{(\kappa_2)} = \mathbb{I}_{(\kappa_2)} \mathbb{I}_{(\kappa_1)}, \kappa_1, \kappa_2 \in \mathcal{S}_{-1}, \tag{26}$$

and the index law

$$\mathbb{I}_{(\kappa_1)} \mathbb{I}_{(\kappa_2)} = \mathbb{I}_{(\kappa_1 * \kappa_2)}, \kappa_1, \kappa_2 \in \mathcal{S}_{-1} \tag{27}$$

that are valid on the space $C_{-1}(0, +\infty)$.

Let $\kappa \in \mathcal{S}_{-1}$ and k be its associate Sonine kernel. The GFDs of the Riemann–Liouville and the Caputo types associated to the GFI (19) are given by the Formulas (15) and (18), respectively. It is easy to see that the GFD (18) in the Caputo sense can be rewritten as a regularized GFD (15) in the Riemann–Liouville sense:

$$(*\mathbb{D}_{(k)} f)(t) = (\mathbb{D}_{(k)} [f(\cdot) - f(0)])(t), t > 0. \tag{28}$$

For the functions from $C_{-1}^1(0, +\infty)$, the Riemann–Liouville GFD (15) can be represented as

$$(\mathbb{D}_{(k)} f)(t) = (k * f')(t) + f(0)k(t), t > 0, \tag{29}$$

which immediately leads to the useful representation

$$(*\mathbb{D}_{(k)} f)(t) = (k * f')(t), t > 0 \tag{30}$$

of the Caputo type GFD (18) that is valid on the space $C_{-1}^1(0, +\infty)$.

In the rest of this section, we formulate the first and second fundamental theorems of FC for the GFDs in the Riemann–Liouville and Caputo senses.

Theorem 1 (First Fundamental Theorem for the GFD). Let $\kappa \in \mathcal{S}_{-1}$ and k be its associate Sonine kernel.

Then, the GFD (15) is a left-inverse operator to the GFI (19) on the space $C_{-1}(0, +\infty)$,

$$(\mathbb{D}_{(k)} \mathbb{I}_{(\kappa)} f)(t) = f(t), f \in C_{-1}(0, +\infty), t > 0, \tag{31}$$

and the GFD (18) is a left inverse operator to the GFI (19) on the space $C_{-1,(k)}(0, +\infty)$:

$$(*\mathbb{D}_{(k)} \mathbb{I}_{(\kappa)} f)(t) = f(t), f \in C_{-1,(k)}(0, +\infty), t > 0, \tag{32}$$

where $C_{-1,(k)}(0, +\infty) := \{f : f(t) = (\mathbb{I}_{(k)} \phi)(t), \phi \in C_{-1}(0, +\infty)\}$.

As shown in [9], the space $C_{-1,(k)}(0, +\infty)$ can be also characterized as follows:

$$C_{-1,(k)}(0, +\infty) = \{f : \mathbb{I}_{(\kappa)} f \in C_{-1}^1(0, +\infty) \wedge (\mathbb{I}_{(\kappa)} f)(0) = 0\}.$$

Now, we proceed with the second fundamental theorem of FC for the GFDs in the Riemann–Liouville and Caputo senses.

Theorem 2 (Second Fundamental Theorem for the GFD). *Let $\kappa \in \mathcal{S}_{-1}$ and k be its associate Sonine kernel.*

Then, the relations

$$(\mathbb{I}_{(\kappa)} * \mathbb{D}_{(k)} f)(t) = f(t) - f(0), \quad t > 0, \tag{33}$$

$$(\mathbb{I}_{(\kappa)} \mathbb{D}_{(k)} f)(t) = f(t), \quad t > 0 \tag{34}$$

hold valid for the functions $f \in C_{-1}^1(0, +\infty)$.

In [9,13], the n -fold GFIs and GFDs with the Sonine kernels from \mathcal{S}_{-1} were introduced and studied. For more details, we refer interested readers to these publications.

3. General Fractional Integrals and Derivatives of Arbitrary Order

As already mentioned in the Introduction, the “generalized order” of the GFIs and GFDs introduced so far is restricted to the interval $(0, 1)$. The order of the n -fold GFIs and GFDs recently introduced in [9] belongs to the interval $(0, n)$. However, it is hardly possible to fix their order between two neighboring natural numbers as in the case of the conventional Riemann–Liouville and Caputo fractional derivatives and thus to study, for example, the fractional oscillator equations or the time-fractional diffusion-wave equations with the GFDs of the order from the interval $(1, 2)$.

In this section, we define the GFIs and GFDs of arbitrary order and study their basic properties. As in the case of the conventional Riemann–Liouville and Caputo fractional derivatives, for the GFDs, we also have to distinguish between two completely different cases; namely, between the case of the integer order and the case of non-integer order. In the first case, the conventional Riemann–Liouville and Caputo fractional derivatives are defined as the integer-order derivatives, while in the second case, they are non-local integro-differential operators. Because the conventional Riemann–Liouville and Caputo fractional derivatives are important particular cases of the GFDs, we have no other choice but to follow the same strategy; namely, to separately define the GFDs of integer order as the integer-order derivatives and the GFDs of non-integer order as some integro-differential operators. In what follows, we focus on the case of the GFDs of non-integer order (the integer-order GFDs are simply the integer-order derivatives).

To introduce the GFIs and the GFDs of arbitrary non-integer order, we first formulate a condition on their kernels that generalizes the Sonine condition (7):

$$(\kappa * k)(t) = \{1\}^n(t), \quad n \in \mathbb{N}, \quad t > 0, \tag{35}$$

where

$$\{1\}^n(t) := \underbrace{(\{1\} * \dots * \{1\})}_{n \text{ times}}(t) = h_n(t) = \frac{t^{n-1}}{(n-1)!}.$$

Evidently, the Sonine condition corresponds to the case $n = 1$ of the more general condition (35).

Another important ingredient of our definitions is a set of the kernels that satisfy the condition (35) and belong to the suitable spaces of functions.

Definition 2. *Let the functions κ and k satisfy the condition (35) and the inclusions $\kappa \in C_{-1}(0, +\infty)$ and $k \in C_{-1,0}(0, +\infty)$ hold true.*

The set of pairs (κ, k) of such kernels is denoted by \mathcal{L}_n .

Remark 1. *The set \mathcal{L}_1 coincides with the set of the Sonine kernels \mathcal{S}_{-1} discussed in the previous section (see Definition 1). Indeed, in this case, the kernel $\kappa \in C_{-1}(0, +\infty)$ is a Sonine kernel,*

and therefore it has an integrable singularity at the point zero. Thus, it belongs to the subspace $C_{-1,0}(0, +\infty)$ as required in Definition 1.

Remark 2. For $n > 1$, Definition 2 is not symmetrical with respect to the kernels κ and k because of the non-symmetrical inclusions $\kappa \in C_{-1}(0, +\infty)$ and $k \in C_{-1,0}(0, +\infty)$ (in the case $n = 1$, Definition 1 is symmetrical and one can interchange the kernels κ and k).

However, the same statement is valid for the kernel $\kappa(t) = h_\alpha(t)$, $\alpha > 0$ of the Riemann–Liouville integral I_{0+}^α and the kernel $k(t) = h_{n-\alpha}(t)$ of the Riemann–Liouville and Caputo fractional derivatives of order α , $n - 1 < \alpha < n$, $n \in \mathbb{N}$, defined as follows:

$$(D_{0+}^\alpha f)(t) := \frac{d^n}{dt^n} (I_{0+}^{n-\alpha} f)(t), \quad t > 0, \tag{36}$$

$$(*D_{0+}^\alpha f)(t) := \left(D_{0+}^\alpha \left(f(\cdot) - \sum_{j=0}^{n-1} f^{(j)}(0)h_{j+1}(\cdot) \right) \right)(t), \quad t > 0, \tag{37}$$

with I_{0+}^α being the Riemann–Liouville fractional integral of order α :

$$(I_{0+}^\alpha f)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau, \quad t > 0, \quad \alpha > 0. \tag{38}$$

The solution to defining the integer-order Riemann–Liouville and Caputo fractional derivatives consists of a separate definition of the Riemann–Liouville fractional integral of the order $\alpha = 0$:

$$(I_{0+}^0 f)(t) := f(t). \tag{39}$$

Of course, the definition (39) is not arbitrary and is justified inter alia by the formula

$$\lim_{\alpha \rightarrow 0+} \|I_{0+}^\alpha f(t) - f(t)\|_{L_1(0,T)} = 0 \tag{40}$$

that is valid for $f \in L^1(0, T)$ in every Lebesgue point of f ; i.e., almost everywhere on the interval $(0, T)$, $T > 0$ (see, e.g., [19]).

Example 1. The kernels $\kappa(t) = h_\alpha(t)$, $\alpha > 0$ and $k(t) = h_{n-\alpha}(t)$, $n - 1 < \alpha < n$, $n \in \mathbb{N}$ provide a first example of the kernels from \mathcal{L}_n . Please note that the power functions h_α and $h_{n-\alpha}$ build a pair of the Sonine kernels only in the case $n = 1$; i.e., only in the case when the fractional derivatives' order is less than one.

Because both the Sonine condition (7) and its generalization (35) contain the Laplace convolution of two kernels, it is very natural to transform them into the Laplace domain. Providing that the Laplace transforms $\tilde{\kappa}$, \tilde{k} of the functions κ and k exist, the convolution theorem for the Laplace transform leads to the relation

$$\tilde{\kappa}(p) \cdot \tilde{k}(p) = \frac{1}{p}, \quad \Re(p) > p_{\kappa,k} \in \mathbb{R} \tag{41}$$

for the Laplace transforms of the Sonine kernels and to a more general relation

$$\tilde{\kappa}(p) \cdot \tilde{k}(p) = \frac{1}{p^n}, \quad \Re(p) > p_{\kappa,k} \in \mathbb{R}, \quad n \in \mathbb{N} \tag{42}$$

for the kernels from the set \mathcal{L}_n introduced in Definition 2.

Example 2. Formula (42) along with the works in [20,21] for the direct and inverse Laplace transforms, respectively, can be used to deduce other nontrivial examples of the kernels from \mathcal{L}_n . For instance, we employ the Laplace transform formulas (see [20])

$$\left(\mathcal{L} t^{\nu/2} J_{\nu}(2\sqrt{t})\right)(p) = p^{-\nu-1} \exp(-1/p), \Re(\nu) > -1, \Re(p) > 0,$$

$$\left(\mathcal{L} t^{\nu/2} I_{\nu}(2\sqrt{t})\right)(p) = p^{-\nu-1} \exp(1/p), \Re(\nu) > -1, \Re(p) > 0$$

for the Bessel function J_{ν} and the modified Bessel function I_{ν} defined by the power series (14) to introduce the kernels

$$\kappa(t) = t^{\nu/2} J_{\nu}(2\sqrt{t}), \quad k(t) = t^{n/2-\nu/2-1} I_{n-\nu-2}(2\sqrt{t}), \quad n-2 < \nu < n-1, \quad n \in \mathbb{N}. \quad (43)$$

These kernels satisfy the condition (42). Moreover, for $n-2 < \nu < n-1$, $n \in \mathbb{N}$, the inclusions $\kappa \in C_{-1}(0, +\infty)$ and $k \in C_{-1,0}(0, +\infty)$ hold true, and thus the pair of the kernels (κ, k) given by (43) is from \mathcal{L}_n .

Now let us consider a pair of the Sonine kernels (κ, k) from \mathcal{L}_1 (in [4,8,9,13,17] and other related publications, many pairs of such kernels were presented). There are at least two reasonable possibilities to construct a pair (κ_n, k_n) of the kernels from \mathcal{L}_n , $n > 1$ based on the Sonine kernels κ, k from \mathcal{L}_1 .

The first strategy consists of building the kernels $\kappa_n = \kappa^n$ and $k_n = k^n$. Evidently, the kernels κ_n and k_n satisfy the relation (35) because κ and k are the Sonine kernels:

$$(\kappa_n * k_n)(t) = (\kappa^n * k^n)(t) = (\kappa * k)^n(t) = \{1\}^n(t). \quad (44)$$

However, the pair (κ_n, k_n) does not always belong to the set \mathcal{L}_n . This is the case only under an additional condition; namely, only when the inclusion $k^n \in C_{-1,0}(0, +\infty)$ holds true (of course, $\kappa^n \in C_{-1}(0, +\infty)$ for any $n \in \mathbb{N}$). This is a very strong and restrictive condition. For example, in the case of the Riemann–Liouville fractional integral I_{0+}^{α} with the kernel $\kappa(t) = h_{\alpha}(t)$, $0 < \alpha < 1$ and the Riemann–Liouville fractional derivative D_{0+}^{α} with the kernel $k(t) = h_{1-\alpha}(t)$, the kernel k^n takes the form $k^n(t) = h_{n(1-\alpha)}(t)$. It belongs to the space $C_{-1,0}(0, +\infty)$ only under the condition $0 < n(1-\alpha) < 1$; i.e., if $1 - \frac{1}{n} < \alpha < 1$, which is very restrictive. Moreover, the example of the kernels (43) shows that not every pair of the kernels from \mathcal{L}_n can be represented in the form (κ^n, k^n) with the kernels $(\kappa, k) \in \mathcal{L}_1$.

Another and even more general and important possibility for the construction of a pair (κ_n, k_n) of the kernels from \mathcal{L}_n , $n > 1$ based on the Sonine kernels κ, k from \mathcal{L}_1 is presented in the following theorem:

Theorem 3. Let (κ, k) be a pair of the Sonine kernels from \mathcal{L}_1 .

Then, the pair (κ_n, k_n) of the kernels given by the formula

$$\kappa_n(t) = (\{1\}^{n-1} * \kappa)(t), \quad k_n(t) = k(t) \quad (45)$$

belongs to the set \mathcal{L}_n .

Proof. First, we check that the kernels (45) satisfy the condition (35):

$$(\kappa_n * k_n)(t) = (\{1\}^{n-1} * \kappa * k)(t) = (\{1\}^{n-1} * \{1\})(t) = \{1\}^n(t). \quad (46)$$

Moreover, because of the inclusions $\kappa, k \in \mathcal{L}_1$, the inclusions $\kappa_n \in C_{-1}(0, +\infty)$ and $k_n = k \in C_{-1,0}(0, +\infty)$ are satisfied, and thus the kernels κ_n and k_n defined by (45) belong to the set \mathcal{L}_n . \square

In the rest of this section, we introduce the general fractional integrals and derivatives of an arbitrary (non-integer) order and discuss their basic properties and examples.

Definition 3. Let (κ, k) be a pair of the kernels from \mathcal{L}_n . The GFI with the kernel κ is specified by the standard formula

$$(\mathbb{I}_{(\kappa)} f)(t) := \int_0^t \kappa(t-\tau) f(\tau) d\tau, \quad t > 0, \quad (47)$$

whereas the GFDs of the Riemann–Liouville and Caputo types with the kernel k are defined as follows:

$$(\mathbb{D}_{(k)} f)(t) := \frac{d^n}{dt^n} \int_0^t k(t-\tau) f(\tau) d\tau, \quad t > 0, \quad (48)$$

$$(*\mathbb{D}_{(k)} f)(t) := \left(\mathbb{D}_{(k)} \left(f(\cdot) - \sum_{j=0}^{n-1} f^{(j)}(0) h_{j+1}(\cdot) \right) \right)(t), \quad t > 0. \quad (49)$$

Example 3. Evidently, the GFI (47) with the kernel $\kappa(t) = h_\alpha(t)$, $\alpha > 0$ is reduced to the Riemann–Liouville fractional integral (38), and the Riemann–Liouville and Caputo fractional derivatives of the order α , $n-1 < \alpha < n$, $n \in \mathbb{N}$ defined by (36) and (37), respectively, are particular cases of the GFDs (48) and (49) with the kernel $k(t) = h_{n-\alpha}(t)$. As mentioned in Example 1, the inclusion $(h_\alpha, h_{n-\alpha}) \in \mathcal{L}_n$ holds valid if and only if $n-1 < \alpha < n$, $n \in \mathbb{N}$.

It is worth mentioning that the Riemann–Liouville fractional integral (38) and the Riemann–Liouville and Caputo fractional derivatives of an arbitrary order α , $n-1 < \alpha < n$, $n \in \mathbb{N}$ can be introduced based on the Sonine pair $\kappa = h_\beta, k = h_{1-\beta}$, $0 < \beta < 1$ and using the construction (45) presented in Theorem 3. Indeed, in this case, we have the relations

$$\kappa_n(t) = (\{1\}^{n-1} * \kappa)(t) = (\{1\}^{n-1} * h_\beta)(t) = h_{n-1+\beta}(t), \quad k_n(t) = k(t) = h_{1-\beta}(t). \quad (50)$$

Thus, the GFI (47) and the GFDs (48) and (49) with the kernels $(\kappa_n, k_n) \in \mathcal{L}_n$ take the form

$$(\mathbb{I}_{(\kappa)} f)(t) = (h_{n-1+\beta} * f)(t) = (I_{0+}^{n-1+\beta} f)(t), \quad t > 0, \quad (51)$$

$$(\mathbb{D}_{(k)} f)(t) = \frac{d^n}{dt^n} (h_{1-\beta} * f)(t) = \frac{d^n}{dt^n} (I_{0+}^{1-\beta} f)(t), \quad t > 0, \quad (52)$$

$$(*\mathbb{D}_{(k)} f)(t) = \frac{d^n}{dt^n} \left(I_{0+}^{1-\beta} \left(f(\cdot) - \sum_{j=0}^{n-1} f^{(j)}(0) h_{j+1}(\cdot) \right) \right)(t), \quad t > 0. \quad (53)$$

Now we introduce a new variable $\alpha := n-1+\beta$. Then, $1-\beta = n-\alpha$ and the inequalities $n-1 < \alpha < n$ are fulfilled because of the condition $0 < \beta < 1$. Thus, the operator (51) is the Riemann–Liouville fractional integral (38) of the order α , and the operators (52) and (53) coincide with the Riemann–Liouville and Caputo fractional derivatives of the order α , $n-1 < \alpha < n$, $n \in \mathbb{N}$.

Example 4. Another interesting and nontrivial particular case of the GFI (47) and the GFDs (48) and (49) is constructed for the pair $(\kappa, k) \in \mathcal{L}_n$ of the kernels defined by Formula (43) with $n-2 < \nu < n-1$, $n \in \mathbb{N}$:

$$(\mathbb{I}_{(\kappa)} f)(t) = \int_0^t (t-\tau)^{\nu/2} J_\nu(2\sqrt{t-\tau}) f(\tau) d\tau, \quad t > 0, \quad (54)$$

$$(\mathbb{D}_{(k)} f)(t) = \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n/2-\nu/2-1} I_{n-\nu-2}(2\sqrt{t-\tau}) f(\tau) d\tau, \quad t > 0, \quad (55)$$

$$(*\mathbb{D}_{(k)} f)(t) := \left(\mathbb{D}_{(k)} \left(f(\cdot) - \sum_{j=0}^{n-1} f^{(j)}(0) h_{j+1}(\cdot) \right) \right)(t), \quad t > 0. \quad (56)$$

It is worth mentioning that the Caputo type GFD (49) can be represented in a slightly different form:

$$\begin{aligned}
 (*\mathbb{D}_{(k)} f)(t) &= \left(\mathbb{D}_{(k)} \left(f(\cdot) - \sum_{j=0}^{n-1} f^{(j)}(0) h_{j+1}(\cdot) \right) \right)(t) = \\
 &= \mathbb{D}_{(k)} f(t) - \sum_{j=0}^{n-1} f^{(j)}(0) \mathbb{D}_{(k)} h_{j+1}(t) = \mathbb{D}_{(k)} f(t) - \sum_{j=0}^{n-1} f^{(j)}(0) \frac{d^n}{dt^n} (k * h_{j+1})(t) = \\
 &= \mathbb{D}_{(k)} f(t) - \sum_{j=0}^{n-1} f^{(j)}(0) \frac{d^n}{dt^n} (I_{0+}^{j+1} k)(t) = \mathbb{D}_{(k)} f(t) - \sum_{j=0}^{n-1} f^{(j)}(0) \frac{d^{n-j-1}}{dt^{n-j-1}} k(t), \quad t > 0. \tag{57}
 \end{aligned}$$

As regards the basic properties of the GFI (47) of an arbitrary order on $C_{-1}(0, +\infty)$, they follow from the well-known properties of the Laplace convolution (compare these to the properties of the GFI (19) of the order less than one):

$$\mathbb{I}_{(\kappa)} : C_{-1}(0, +\infty) \rightarrow C_{-1}(0, +\infty) \text{ (mapping property),} \tag{58}$$

$$\mathbb{I}_{(\kappa_1)} \mathbb{I}_{(\kappa_2)} = \mathbb{I}_{(\kappa_2)} \mathbb{I}_{(\kappa_1)} \text{ (commutativity law),} \tag{59}$$

$$\mathbb{I}_{(\kappa_1)} \mathbb{I}_{(\kappa_2)} = \mathbb{I}_{(\kappa_1 * \kappa_2)} \text{ (index law).} \tag{60}$$

To justify this denotation of GFIs and GFDs, in the rest of this section, we formulate and prove the two fundamental theorems of FC for the GFDs (48) and (49) of the Riemann–Liouville and Caputo types.

Theorem 4 (First Fundamental Theorem for the GFD of an Arbitrary Order). *Let (κ, k) be a pair of the kernels from \mathcal{L}_n .*

Then, the GFD (48) is a left-inverse operator to the GFI (47) on the space $C_{-1}(0, +\infty)$,

$$(\mathbb{D}_{(k)} \mathbb{I}_{(\kappa)} f)(t) = f(t), \quad f \in C_{-1}(0, +\infty), \quad t > 0, \tag{61}$$

and the GFD (49) is a left-inverse operator to the GFI (47) on the space $C_{-1,(k)}(0, +\infty)$:

$$(*\mathbb{D}_{(k)} \mathbb{I}_{(\kappa)} f)(t) = f(t), \quad f \in C_{-1,(k)}(0, +\infty), \quad t > 0, \tag{62}$$

where the space $C_{-1,(k)}(0, +\infty)$ is defined as in Theorem 1.

Proof. We start with a proof of the Formula (61):

$$\begin{aligned}
 (\mathbb{D}_{(k)} \mathbb{I}_{(\kappa)} f)(t) &= \frac{d^n}{dt^n} (k * (\kappa * f))(t) = \frac{d^n}{dt^n} ((k * \kappa) * f)(t) = \\
 &= \frac{d^n}{dt^n} (\{1\}^n * f)(t) = \frac{d^n}{dt^n} (I_{0+}^n f)(t) = f(t).
 \end{aligned}$$

A function $f \in C_{-1,(k)}(0, +\infty)$ can be represented in the form $f(t) = (\mathbb{I}_{(k)} \phi)(t)$, $\phi \in C_{-1}(0, +\infty)$, and thus the following chain of equations is valid:

$$(\mathbb{I}_{(\kappa)} f)(t) = (\mathbb{I}_{(\kappa)} \mathbb{I}_{(k)} \phi)(t) = ((\kappa * k) * f)(t) = (\{1\}^n \phi)(t) = (I_{0+}^n \phi)(t).$$

The last relation implicates the inclusion $\mathbb{I}_{(\kappa)} f \in C_{-1}^n(0, +\infty)$ and the relations

$$\left. \frac{d^j}{dt^j} (\mathbb{I}_{(\kappa)} f)(t) \right|_{t=0} = \left. (I_{0+}^{n-j} \phi)(t) \right|_{t=0} = 0, \quad j = 0, \dots, n - 1. \tag{63}$$

To derive Formula (62), we employ the representation (57) of the GFD of the Caputo type, Formula (63) and the relation (61) that we already proved:

$$({}_*\mathbb{D}_{(k)}\mathbb{I}_{(\kappa)}f)(t) = (\mathbb{D}_{(k)}\mathbb{I}_{(\kappa)}f)(t) - \sum_{j=0}^{n-1} \frac{d^j}{dt^j} (\mathbb{I}_{(\kappa)}f)(t) \Big|_{t=0} \frac{d^{n-j-1}}{dt^{n-j-1}} k(t) = f(t).$$

□

Theorem 5 (Second Fundamental Theorem for the GFD of an Arbitrary Order). *Let (κ, k) be a pair of the kernels from \mathcal{L}_n .*

Then, the relation

$$(\mathbb{I}_{(\kappa)}{}_*\mathbb{D}_{(k)}f)(t) = f(t) - \sum_{j=0}^{n-1} f^{(j)}(0) h_{j+1}(t) \tag{64}$$

holds true on the space $C_{-1}^n(0, +\infty)$ and the formula

$$(\mathbb{I}_{(\kappa)}\mathbb{D}_{(k)}f)(t) = f(t), \quad t > 0 \tag{65}$$

is valid for the functions $f \in C_{-1,(\kappa)}^n(0, +\infty)$.

Proof. As already mentioned in Section 2, any function f from $C_{-1}^n(0, +\infty)$ can be represented as follows (see [16]):

$$f(t) = (I_{0+}^n \phi)(t) + \sum_{j=0}^{n-1} f^{(j)}(0) h_{j+1}(t), \quad t \geq 0, \quad \phi \in C_{-1}(0, +\infty). \tag{66}$$

Then, we employ this representation and Formula (49) and arrive at the following chain of relations:

$$\begin{aligned} ({}_*\mathbb{D}_{(k)}f)(t) &= (\mathbb{D}_{(k)}\left(f(\cdot) - \sum_{j=0}^{n-1} f^{(j)}(0) h_{j+1}(\cdot)\right))(t) = (\mathbb{D}_{(k)}I_{0+}^n \phi)(t) = \\ &= \frac{d^n}{dt^n} (k * \{1\}^n * \phi)(t) = \frac{d^n}{dt^n} (\{1\}^n * (k * \phi))(t) = (k * \phi)(t). \end{aligned}$$

Finally, we take into account the representation (66) and obtain Formula (64):

$$\begin{aligned} (\mathbb{I}_{(\kappa)}{}_*\mathbb{D}_{(k)}f)(t) &= (\mathbb{I}_{(\kappa)}(k * \phi))(t) = ((\kappa * k) * \phi)(t) = \\ &= (\{1\}^n * \phi)(t) = (I_{0+}^n \phi)(t) = f(t) - \sum_{j=0}^{n-1} f^{(j)}(0) h_{j+1}(t). \end{aligned}$$

To prove Formula (65), we first mention that a function $f \in C_{-1,(\kappa)}^n(0, +\infty)$ can be represented in the form $f(t) = (\mathbb{I}_{(\kappa)}\phi)(t)$, $\phi \in C_{-1}(0, +\infty)$, and thus the following chain of equations is valid:

$$\begin{aligned} (\mathbb{I}_{(\kappa)}\mathbb{D}_{(k)}f)(t) &= (\mathbb{I}_{(\kappa)}\frac{d^n}{dt^n}(k * f)(t)) = (\mathbb{I}_{(\kappa)}\frac{d^n}{dt^n}(k * (\kappa * \phi)))(t) = \\ &= (\mathbb{I}_{(\kappa)}\frac{d^n}{dt^n}(\{1\}^n * \phi))(t) = (\mathbb{I}_{(\kappa)}\phi)(t) = f(t). \end{aligned}$$

□

In conclusion, we emphasize once again the result of Theorem 3 and its implications on the definitions of the GFIs and the GFDs of an arbitrary order. If (κ, k) is a pair of the

Sonine kernels from \mathcal{L}_1 , the pair (κ_n, k_n) of the kernels given by the Formula (45) belongs to the set \mathcal{L}_n , $n > 1$. The GFI (47) with the kernel $\kappa_n = (\{1\}^{n-1} * \kappa)(t)$ takes the form

$$(\mathbb{I}_{(\kappa_n)} f)(t) = (I_{0+}^{n-1} \mathbb{I}_{(\kappa)} f)(t), \quad t > 0, \quad (67)$$

whereas the GFDs of the Riemann–Liouville and Caputo types with the kernel $k_n = k$ can be represented as follows:

$$(\mathbb{D}_{(k_n)} f)(t) = \frac{d^n}{dt^n} (\mathbb{I}_{(k)} f)(t), \quad t > 0, \quad (68)$$

$$(*\mathbb{D}_{(k_n)} f)(t) = \frac{d^n}{dt^n} \left(\mathbb{I}_{(k)} \left(f(\cdot) - \sum_{j=0}^{n-1} f^{(j)}(0) h_{j+1}(\cdot) \right) \right) (t), \quad t > 0. \quad (69)$$

As we see, these constructions are completely analogical to the definitions of the Riemann–Liouville fractional integral and the Riemann–Liouville and Caputo fractional derivatives of an arbitrary order.

Another point that is worth mentioning is that the kernel $\kappa_n = (\{1\}^{n-1} * \kappa)(t)$ of the GFI (67) possesses an integrable singularity of the power function type at the origin in the case $n = 1$; i.e., in the case that its order is less than one ($\kappa_1 = \kappa \in C_{-1,0}(0, +\infty)$). If the order of the GFI (67) is greater than one ($n = 2, 3, \dots$), κ_n is continuous at the origin and $\kappa_n(0) = 0$ as in the case of the Riemann–Liouville fractional integral of the order $\alpha > 1$. Indeed, as mentioned in [16], the inclusion $g * f \in C_{\alpha_1 + \alpha_2 + 1}(0, +\infty)$ holds true for the Laplace convolution of the functions $f \in C_{\alpha_1}(0, +\infty)$, $g \in C_{\alpha_2}(0, +\infty)$, $\alpha_1, \alpha_2 \geq -1$. Thus, the function $\kappa_n = (\{1\}^{n-1} * \kappa)(t)$ with $\kappa \in C_{-1,0}(0, +\infty)$ belongs to the space $C_{n-2}(0, +\infty)$ and thus can be represented in the form $\kappa_n(t) = t^p f(t)$, $p > n - 2 \geq 0$, $f \in C[0, +\infty)$.

4. Conclusions

Starting from the work presented in [7], the so-called GFDs of the Riemann–Liouville and Caputo types have become a topic of active research in FC. In particular, both the ordinary and the partial fractional differential equations with these derivatives have been considered (see [10] for a survey of some recent results). However, the GFDs introduced to date have been based on the classical Sonine condition, and thus their “generalized order” was restricted to the interval $(0, 1)$. In particular, the initial value problems for the fractional differential equations with these derivatives permitted only one initial condition, and thus no models for the intermediate processes between diffusion and wave propagation could be formulated in terms of these GFDs.

The main contribution of this paper is an extension of the definitions of the GFIs and GFDs to the case of arbitrary order. To achieve this aim, a suitable generalization of the Sonine condition was introduced, and some important classes of the kernels that satisfy this generalized condition were described. The kernels of the GFDs of an arbitrary order possess integrable singularities at the point zero. However, the kernels of the GFIs can be both singular (in the case of an order less than one) and continuous (in the case of an order greater or equal to one) at the origin. The conventional Riemann–Liouville and Caputo fractional derivatives of arbitrary order are particular cases of these GFDs. Another important example is the integro-differential operators of the convolution type with the Bessel and the modified Bessel functions in the kernels.

To justify the denotation of GFIs and GFDs of arbitrary order, in this paper, two fundamental theorems of fractional calculus for these operators were formulated and proved. The constructions introduced in this paper allow the formulation of the initial-value problems for the fractional differential equations with GFDs of a generalized order greater than one with several initial conditions. Thus, further research regarding the properties of the GFIs and GFDs of an arbitrary order introduced in this paper as well as applications of the fractional differential equations with the GFDs of arbitrary order to

model, for instance, the processes intermediate between diffusion and wave propagation is needed.

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