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On SD-Prime Labelings of Some One Point Unions of Gears

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Abstract: Let $G = (V(G), E(G))$ be a simple, finite and undirected graph of order n. Given a bijection $f : V(G) \to \{1, \ldots, n\}$, and every edge uv in $E(G)$, let $S = f(u) + f(v)$ and $D = |f(u) - f(v)|$. The labeling f induces an edge labeling $f' : E(G) \to \{0, 1\}$ such that for an edge uv in $E(G)$, $f'(uv) = 1$ if $gcd(S, D) = 1$, and $f'(uv) = 0$ otherwise. Such a labeling is called an SD-prime labeling if $f'(uv) = 1$ for all $uv \in E(G)$. We provide SD-prime labelings for some one point unions of gear graphs.

Keywords: SD-prime labeling; prime labeling; one point union; gear graph

MSC: 05C78; 05C25

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1. Introduction

Let $G = (V(G), E(G))$ (or simply $G = (V, E)$ for short, if there is no ambiguity) be a simple, finite and undirected graph of order $|V| = n$ and size $|E| = m$. For convenience, we shall use $[a, b]$ to denote the set of integers from a to b, where $a \leq b$. All notation not defined in this paper can be found in [1].

Since the introduction of the concept of graph labeling by Rosa in 1967 [2], thousands of papers on various graph labeling problems have been published. It is interesting to note that the field of graph labelings is becoming more significant with more applications in other topics of graph theory being found. For example, the friendly index of a graph is related to eigenvalues, bipartition width, max-cut problem and isoperimetric problem of a graph [3].

One of the graph labeling problems that attracts great interest of researchers is the concept of prime graphs introduced in 1982 [4].

Definition 1. *A bijection $f : V \to [1, n]$ induces an edge labeling $f' : E \to \{0, 1\}$ such that for an edge uv in G, $f'(uv) = 1$ if $gcd(f(u), f(v)) = 1$, and $f'(uv) = 0$ otherwise. Such a labeling is called a prime labeling if $f'(uv) = 1$ for all $uv \in E$. We say G is a prime graph if it admits a prime labeling.*

Since then, many papers that attempt to determine the primality of graphs are published (for example, [5–11]). Although a wheel graph of order $2n + 1$ is prime if and only if n is even [12], it was proved that the one point union of cycles and that of wheels at the center vertex are all prime graphs [13]. However, we are unaware of any vertex labelings in which the induced edge labels are associated to the sum and difference of the incident end-vertex labels.

In view of this, Lau and Shiu [14] introduced a variant of prime graph labeling, known as SD-prime labeling, as defined below.

Given a bijection $f : V \to [1, n]$, and every edge uv in E, one can associate two integers $S = f(u) + f(v)$ and $D = |f(u) - f(v)|$.

Definition 2. *A bijection $f : V \to [1, n]$ induces an edge labeling $f' : E \to \{0, 1\}$ such that for an edge uv in G, $f'(uv) = 1$ if $gcd(S, D) = 1$, and $f'(uv) = 0$ otherwise. Such a labeling is called an SD-prime labeling if $f'(uv) = 1$ for all $uv \in E$. We say G is SD-prime if it admits an SD-prime labeling.*

In [15], Lau et al. provided more discussion on SD-prime labelings. This new concept leads to new research direction and possible future applications. In [16], one may find that the one point union of various families of standard graphs have been investigated under various labeling problems (see, for example, [17–19]). However, we are not aware of the study on one point union of gear graphs under any graph labeling problems. Thus, in this paper, we aim to determine the SD-primality of the various families of one point union of gear graphs. The following results follow directly from the definitions.

Corollary 1. *Every spanning subgraph of an SD-prime graph is also an SD-prime graph.*

Corollary 2. *Suppose G admits an SD-prime labeling, then G also admits a prime labeling.*

The following results are obtained in [14].

Theorem 1 ([14], Theorem 1)**.** *Let f be an SD-prime labeling of G, then $f(u)$ and $f(v)$ have different parity for each edge $uv \in E$.*

Theorem 2 ([14], Theorem 3)**.** *Let G be an SD-prime graph of order greater than 1, then G is a spanning subgraph of either $K_{m,m}$ or $K_{m,m+1}$ for some $m \geq 1$.*

Theorem 3 ([14], Theorem 2.2)**.** *A graph G of order n is SD-prime if and only if G is bipartite and that there exists a bijection $f : V \to [1, n]$ such that for each edge uv of G, $f(u)$ and $f(v)$ are of different parity and $gcd(f(u), f(v)) = 1$.*

2. Preliminary

For $n \geq 2$, let G_n be the graph obtained from a wheel graph W_{2n} of order $2n + 1$ by deleting n spokes where no two of the spokes are consecutive. Such a graph is called a gear graph (or gear, for short). The core (center vertex) of the wheel is also called the core of the gear. From ([14], Theorem 2.13) we know that G_n is SD-prime for all $n \geq 2$. In the paper, we want to study the SD-primality of some one point unions of gears.

Let H_i be a graph and $v_i \in V(H_i)$ be fixed, $1 \leq i \leq t$. A one point union of H_i, $1 \leq i \leq t$, is a graph obtained from the disjoint union of H_i by merging all v_i into a single vertex which is called the merged vertex. So, there will be many different one point unions of H_i's.

There are three types of vertices in a gear. We shall say that a noncore vertex of degree 2 is of Type 1; a noncore vertex of degree 3 is of Type 2; a core vertex is of Type 3.

Let G_{n_i} be a gear, $1 \leq i \leq t$ and $t \geq 2$. Suppose $0 \leq k \leq l \leq t$. Let v_i be a Type 1 vertex in G_{n_i} for $1 \leq i \leq k$ (this case does not occur if $k = 0$), a Type 2 vertex for $k + 1 \leq i \leq l$ (this case does not occur if $l = k$) and a Type 3 vertex for $l + 1 \leq i \leq t$ (this case does not occur if $l = t$). The graph $G(n_1, \ldots, n_k; n_{k+1}, \ldots, n_l; n_{l+1}, \ldots, n_t)$ is a one point union of the gear G_{n_i}'s by merging all v_i into a vertex, say v. When one of the sequences $n_1, \ldots, n_k$; $n_{k+1}, \ldots, n_l$ or $n_{l+1}, \ldots, n_t$ is empty, we will denote it by $\varnothing$. We shall also denote $G(n_1, \ldots, n_k; n_{k+1}, \ldots, n_t; \varnothing)$ by $G(n_1, \ldots, n_k; n_{k+1}, \ldots, n_t)$ and $G(n_1, \ldots, n_t; \varnothing; \varnothing)$ by $G(n_1, \ldots, n_t)$, for short.

Theorem 4. *If $G(n_1, \ldots, n_k; n_{k+1}, \ldots, n_l; n_{l+1}, \ldots, n_t)$ is SD-prime, then $l = k+1$ or k.*

Proof. Clearly, G is bipartite. We color each vertex of G by white or by black so that no two adjacent vertices receive the same color. Suppose the merged vertex is white.

Consider the subgraphs G_{n_i} for $1 \le i \le k$. All the Type 1 and Type 3 vertices are colored by white. So, there are $n_i + 1$ white vertices and n_i black vertices.

Consider the subgraphs G_{n_i} for $k+1 \le i \le l$. All the Type 1 and Type 3 vertices are colored by black. So, there are n_i white vertices and $n_i + 1$ black vertices.

Consider the subgraphs G_{n_i} for $l+1 \le i \le t$. All the Type 1 and Type 3 vertices are colored by white. So, there are $n_i + 1$ white vertices and n_i black vertices.

To obtain G, we identify t white vertices as the merged vertex. Thus, there are

$$1 - t + \sum_{i=1}^{k}(n_i+1) + \sum_{i=k+1}^{l} n_i + \sum_{i=l+1}^{t}(n_i+1) = (k-l+1) + \sum_{i=1}^{t} n_i \text{ white vertices and}$$

$$\sum_{i=1}^{k} n_i + \sum_{i=k+1}^{l}(n_i+1) + \sum_{i=l+1}^{t} n_i = l - k + \sum_{i=1}^{t} n_i \text{ black vertices.}$$

By Theorem 2 we have $|2(l-k)-1| \le 1$. It is equivalent to $0 \le 2(l-k) \le 2$. Thus, we have Theorem 4. □

Corollary 3. *If $G(n_1, \ldots, n_k; \varnothing; n_{k+1}, \ldots, n_t)$ is SD-prime, then all the Type 1 and Type 3 vertices are labeled by odd numbers.*

Proof. From the proof of Theorem 4, we know that all the Type 1 and Type 3 vertices are colored white. Since the number of white vertices is one more than that of black vertices, white vertices are labeled by odd numbers. □

Corollary 4. *If $G(n_1, \ldots, n_k; n_{k+1}; n_{k+2}, \ldots, n_t)$ is SD-prime, then the merged vertex is labeled by even numbers.*

Proof. For this case, from the proof of Theorem 4, the number of white vertices is one less than that of black vertices. Hence, the merged vertex is labeled by even number. □

The above theorem implies that at most one Type 2 vertex can be merged to obtain an SD-prime one point union of gears. In the following sections, we shall consider some simple cases. For the gear G_{n_i}, denote its vertex set by

$$\begin{aligned} V(G_{n_i}) &= \{c_i\} \cup \{v_{i,j} \mid 1 \le j \le 2n_i\} \text{ and} \\ E(G_{n_i}) &= \{c_i v_{i,2k} \mid 1 \le k \le n_i\} \cup \{v_{i,j} v_{i,j+1} \mid 1 \le j \le 2n_i\}, \end{aligned}$$

where $v_{i,2n_i+1} = v_{i,1}$.

3. Merging of Type 1 Vertices

In this section, we only consider $G(n_1, \ldots, n_t)$, when $t = 2, 3$. Before the discussion, we define two sequences first.

For $l \ge 2$, we define a sequence s_l of 4 elements by

$$s_l = \begin{cases} (6l+1, 6l+2, 6l+3, 6l+4) & \text{if } l \equiv 1,2,3 \pmod 5; \\ (6l+1, 6l+4, 6l+3, 6l+2) & \text{if } l \equiv 4 \pmod 5; \\ (6l+5, 6l+4, 6l+3, 6l+2) & \text{if } l \equiv 0 \pmod 5). \end{cases}$$

We also define an order pair t_l by

$$t_l = \begin{cases} (6l+5, 6l+6) & \text{if } l \not\equiv 0 \pmod 5; \\ (6l+1, 6l+6) & \text{if } l \equiv 0 \pmod 5). \end{cases}$$

Let $S = (s_l)_{l\geq 2} = (s_2, s_3, \dots,)$ and $T = (t_l)_{l\geq 2}$.

Lemma 1. *Let sequences S and T be defined above. Then,*

(1) *each pair of consecutive terms in S is of different parity and coprime,*
(2) *each pair of consecutive terms in T is of different parity and coprime.*

Proof. (1) Clearly, (1) is true when the two numbers come from the same subsequence s_l for some l.
Suppose $l \equiv 1 \pmod 5$. The last term of s_{l-1} is $6(l-1)+2$ and the first term of s_l is $6l+1$. Now, $(6l-4, 6l+1) = (6l-4, 5) = (l-4, 5) = 1$.
Suppose $l \equiv 2, 3, 4 \pmod 5$. The last term of s_{l-1} is $6(l-1)+4$ and the first term of s_l is $6l+1$. Clearly, $(6l-2, 6l+1) = (6l-2, 3) = (1, 3) = 1$.
Suppose $l \equiv 0 \pmod 5$. The last term of s_{l-1} is $6(l-1)+2$ and the first term of s_l is $6l+5$. Now, $(6l-4, 6l+5) = (6l-4, 9)$. Since $(6l-4, 3) = 1$, $(6l-4, 9) = 1$.

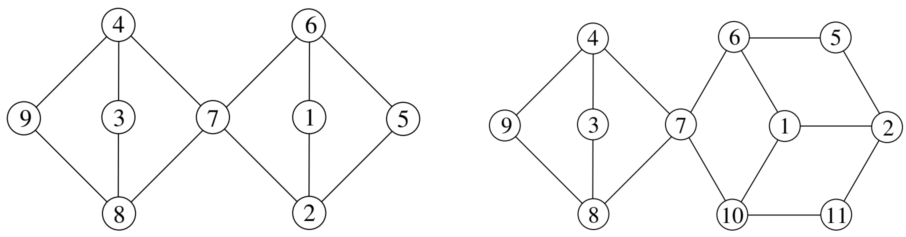
(2) Clearly, (2) is true when the two numbers come from the same subsequence t_l for some l.
Suppose $l \not\equiv 0 \pmod 5$. The last term of t_{l-1} is $6(l-1)+6$ and first term of t_l is $6l+5$. Clearly, $(6l, 6l+5) = (6l, 5) = (l, 5) = 1$.
Suppose $l \equiv 0 \pmod 5$. The last term of t_{l-1} is $6(l-1)+6$ and first term of t_l is $6l+1$. Clearly, $(6l, 6l+1) = (6l, 1) = 1$.

□

Theorem 5. *$G(n_1, n_2)$ is SD-prime for any $n_2 \geq n_1 \geq 2$.*

Proof. For simplicity, we replace $v_{1,x}$ by u_x, and $v_{2,y}$ by v_y, where $1 \leq x \leq n_1$ and $1 \leq y \leq n_2$. Let f be a labeling of G.

When $n_1 = 2$ and $n_2 \geq 2$, we let $f(c_1) = 3$, $f(c_2) = 1$, $f(u_1) = 7 = f(v_1)$, $f(u_2) = 4$, $f(u_3) = 9$, $f(u_4) = 8$, $f(v_{2n_2-2}) = 2$, $f(v_{2n_2-1}) = 5$ and $f(v_{2n_2}) = 6$. When $n_2 = 2$, we have Figure 1a. When $n_2 \geq 3$, labels in $[10, 2n_2+5]$ need to be assigned. For this case, we simply label the vertices from v_2 to v_{2n_2-3} by 10 to $2n_2+5$ consecutively. By Theorem 3, we see that f is an SD-prime labeling. Figure 1b shows the SD-prime labeling for $G(2,3)$ according to the construction above.



(**a**) $G(2,2)$ (**b**) $G(2,3)$

Figure 1. SD-prime labelings.

Now, we assume that $n_1 \geq 3$.

Case 1: Suppose $n_1 = 2k$ for some $k \geq 2$. We let $f(c_1) = 3$, $f(c_2) = 1$, $f(u_1) = 7 = f(v_1)$, $f(u_2) = 4$, $f(u_{2n_1-1}) = 9$, $f(u_{2n_1}) = 8$, $f(v_2) = 10$, $f(v_3) = 11$, $f(v_4) = 12$, $f(v_{2n_2-2}) = 2$, $f(v_{2n_2-1}) = 5$ and $f(v_{2n_2}) = 6$ (see Figure 2a).
Now, there are $4k-4$ and $2n_2-7$ uncolored vertices in G_{n_1} and G_{n_2}, respectively. We label consecutively the vertices from u_3 to u_{2n_1-2} by the first $4k-4$ terms of

S and from v_5 to v_{n_1+2} by the first $2k-2$ terms of T, respectively, where S and T are the sequences in Lemma 1.

Note that $(f(u_{2n_1-2}), f(u_{2n_1-1})) = (6k+4, 9) = 1$ or $(f(u_{2n_1-2}), f(u_{2n_1-1})) = (6k+2, 9) = 1$, and the vertex labeled by $6l+3$ is not adjacent to the core c_1 for every l. So, f satisfies the requirement of SD-prime labeling for $G(n_1)$.

Up to now, we have used labels from 1 to $6k+6 = 3n_1+6$. Finally, we label the unlabeled vertices from v_{n_1+3} to v_{2n_2-3} by $3n_1+7$ to $2n_1+2n_2+1$ consecutively, if $2n_2 \geq n_1+6$. Note that, $(f(v_{2n_2-3}), f(v_{2n_2-2})) = (2n_1+2n_2+1, 2) = 1$. By Theorem 3, we see that f is an SD-prime labeling of G.

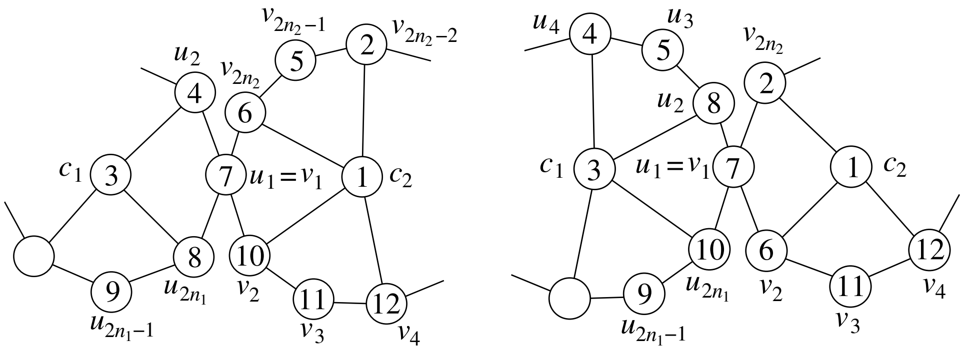
For the remaining case, i.e., $2n_2 \leq n_1+5$. Since $n_2 \geq n_1$, $n_1 \leq 5$. Only $n_2 = n_1 = 4$ is a case. For this case, we have the labeling:

s	1	2	3	4	5	6	7	8
$f(v_{1,s})$	⑦	4	13	14	15	16	9	8

t	1	2	3	4	5	6	7	8
$f(v_{2,t})$	⑦	10	11	12	17	2	5	6

Case 2: Suppose $n_1 = 2k+1$ for some $k \geq 1$. We let $f(c_1) = 3$, $f(c_2) = 1$, $f(u_1) = 7 = f(v_1)$, $f(u_2) = 8$, $f(u_3) = 5$, $f(u_4) = 4$, $f(u_{2n_1-1}) = 9$, $f(u_{2n_1}) = 10$, $f(v_2) = 6$, $f(v_3) = 11$, $f(v_4) = 12$ and $f(v_{2n_2}) = 2$ (see Figure 2b). When $k = 1$, the graph $G(n_1)$ is labeled completely. For this case, we simply label the vertices from v_5 to v_{2n_2-1} by 13 to $2n_2+7$ consecutively.

So, we assume that $k \geq 2$. Similar to Case 1, we may label the vertices from u_5 to u_{2n_1-2} and from v_5 to v_{2n_2-1} using the labels in $[13, 2n_1+2n_2+1]$.



(**a**) n_1 is even (**b**) n_1 is odd

Figure 2. Initial labeling for $G(n_1, n_2)$.

□

Following that, we consider $G = G(n_1, n_2, n_3)$, where $n_3 \geq n_2 \geq n_1 \geq 2$. We always merge vertices $v_{1,1}$, $v_{2,1}$ and $v_{3,1}$ as v and let $f(c_1) = 3$, $f(c_2) = 9$, $f(c_3) = 1$ and $f(v) = 7$, where f is an expected SD-prime labeling of G.

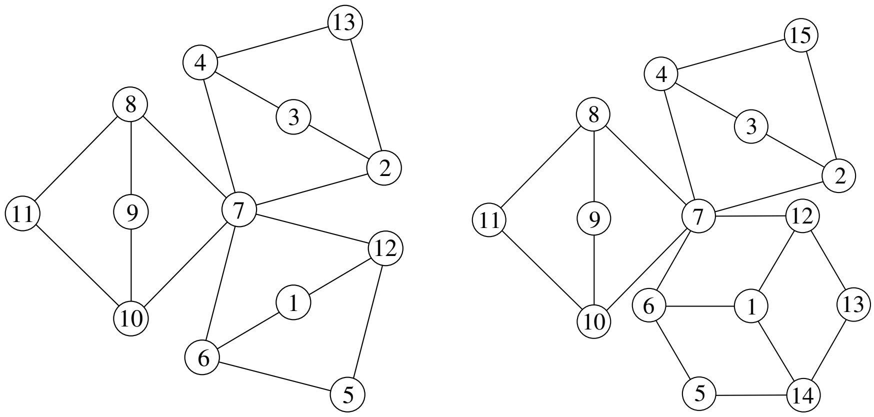
Theorem 6. *$G(2,2,n_3)$ is SD-prime for any $n_3 \geq 2$.*

Proof. The following are SD-prime labelings for $G(2,2,2)$ and $G(2,2,3)$.

For $n_3 \geq 4$, the labeling f for G_{n_3} is listed in the following table.

r	1	2	3	4	5	6	7	$\cdots$	r	$\cdots$	$2n_3-1$	$2n_3$
$f(v_{3,r})$	⑦	6	5	14	13	12	17	$\cdots$	$r+10$	$\cdots$	$2n_3+9$	16

The labeling f for the two G_2's are same as the two G_2's in $G(2,2,3)$ (see Figure 3b).



(**a**) $G(2,2,2)$ (**b**) $G(2,2,3)$

Figure 3. SD-prime labelings.

It is easy to check that f is an SD-prime labeling. □

Theorem 7. *$G(n_1, n_2, n_3)$ is SD-prime for any $n_3 \geq n_2 \geq 3$, $n_1 \geq 2$ and $n_1 + n_2 + n_3 \geq 41$.*

Proof. Similar to the proof of Theorem 5, we let S_1 be the sequence obtained from S by removing subsequences s_4 and s_{13}, where S is defined in Lemma 1. Observe that the subsequences $(s_3, s_5) = (19, 20, 21, 22, 35, 32, 33, 34)$ and $(s_{12}, s_{14}) = (73, 74, 75, 76, 85, 88, 87, 86)$ have adjacent terms that are coprime. Hence, each pair of consecutive terms in S_1 is of different parity and coprime.

Case 1: Suppose $n_1 = 2k_1$ and $n_2 = 2k_2$ for some $k_1 \geq 1$ and $k_2 \geq 2$. The initial labeling for some vertices are listed as follows:

s	1	2	$\cdots$	$2n_1 - 1$	$2n_1$
$f(v_{1,s})$	⑦	2		27	26

t	1	2	3	4	5	6	$\cdots$	$2n_2 - 1$	$2n_2$
$f(v_{2,t})$	⑦	10	11	28	25	8		81	80

r	1	2	3	4	5	$\cdots$	$2n_3 - 3$	$2n_3 - 2$	$2n_3 - 1$	$2n_3$
$f(v_{3,r})$	⑦	6	5	12				4	79	82

When $k_1 + k_2 = 3$, then G_{n_1} and G_{n_2} are already labeled. For this case, the set of unassigned labels is $[13, 24] \cup [29, 78] \cup [83, 2n_3 + 13]$. Here, we may simply label the remaining vertices of G_{n_3} consecutively.

Suppose $k_1 + k_2 \geq 4$. We label consecutively the vertices from $v_{1,3}$ to $v_{1,2n_1-2}$ and then $v_{2,7}$ to $v_{2,2n_2-2}$ by the first $4(k_1 + k_2 - 3)$ terms of S_1 and from $v_{3,5}$ to v_{3,n_1+n_2-4} by the first $2(k_1 + k_2 - 4)$ terms of T, respectively, where T is the sequence defined in Lemma 1. Note that, when $k_1 + k_2 = 4$, no vertices of G_{n_3} are labeled at this step.

After this step, let R be the set of remaining labels arranged in natural order. Namely,

$$R = \begin{cases} [17, 2(n_1 + n_2 + n_3) + 1] \setminus s_{13} & \text{if } k_1 + k_2 = 4; \\ t_{k_1+k_2-2} \cup t_{k_1+k_2-1} \cup \\ \qquad \cup [6k_1 + 6k_2 + 1, 2(n_1 + n_2 + n_3) + 1] \setminus s_{13} & \text{if } 5 \leq k_1 + k_2 \leq 12; \\ t_{k_1+k_2-2} \cup t_{k_1+k_2-1} \cup t_{k_1+k_2} \cup \\ \qquad \cup [6k_1 + 6k_2 + 7, 2(n_1 + n_2 + n_3) + 1] & \text{if } k_1 + k_2 \geq 13. \end{cases}$$

By part (2) of Lemma 1, we see that each pair of consecutive terms in R is of different parity and coprime.

So, we may label the vertices from v_{3,n_1+n_2-5} to $v_{3,2n_3-3}$.

Hence, we obtain an SD-prime labeling for this case.

Case 2: Suppose $n_1 = 2k_1$ and $n_2 = 2k_2 + 1$ for some $k_1 \geq 1$ and $k_2 \geq 1$.

If $k_2 = 1$, then $n_1 = 2$ and $n_2 = 3$. The labeling is defined as follows:

s	1	2	3	4
$f(v_{1,s})$	⑦	2	15	4

t	1	2	3	4	5	6
$f(v_{2,t})$	⑦	10	11	14	13	8

r	1	2	3	4	5	$\cdots$	r	$\cdots$	$2n_3-1$	$2n_3$
$f(v_{3,r})$	⑦	6	5	12	17	$\cdots$	$r+12$	$\cdots$	$2n_3+11$	16

Suppose $k_2 \geq 2$. The initial labeling for some vertices are listed as follows:

s	1	2	$\cdots$	$2n_1-1$	$2n_1$
$f(v_{1,s})$	⑦	2		27	26

t	1	2	3	4	5	$\cdots$	$2n_2-2$	$2n_2-1$	$2n_2$
$f(v_{2,t})$	⑦	10	11	8				81	80

r	1	2	3	4	5	6	7	$\cdots$	$2n_3-3$	$2n_3-2$	$2n_3-1$	$2n_3$
$f(v_{3,r})$	⑦	6	5	12	25	28				4	79	82

Similar to Case 1, we label consecutively the vertices from $v_{1,3}$ to $v_{1,2n_1-2}$ and then $v_{2,5}$ to $v_{2,2n_2-2}$ by the first $4(k_1 + k_2 - 2)$ terms of S_1 and from $v_{3,7}$ to v_{3,n_1+n_2-1} by the first $2(k_1 + k_2 - 3)$ terms of T, respectively. The rest is similar to Case 1.

Case 3: Suppose $n_1 = 2k_1 + 1$ and $n_2 = 2k_2$ for some $k_1 \geq 1$ and $k_2 \geq 2$. The initial labeling for some vertices are listed as follows:

s	1	2	3	4	$\cdots$	$2n_1-1$	$2n_1$
$f(v_{1,s})$	⑦	10	11	8		81	80

t	1	2	3	$\cdots$	$2n_2-2$	$2n_2-1$	$2n_2$
$f(v_{2,t})$	⑦	2				27	26

r	1	2	3	4	5	6	7	$\cdots$	$2n_3-3$	$2n_3-2$	$2n_3-1$	$2n_3$
$f(v_{3,r})$	⑦	6	5	12	25	28				4	79	82

This case is similar to Case 2.

Case 4: Suppose $n_1 = 2k_1 + 1$ and $n_2 = 2k_2 + 1$ for some $k_1 \geq 1$ and $k_2 \geq 1$. The initial labeling for some vertices are listed as follows:

s	1	2	3	4	$\cdots$	$2n_1-1$	$2n_1$
$f(v_{1,s})$	⑦	2	25	28		27	26

t	1	2	3	4	$\cdots$	$2n_2-1$	$2n_2$
$f(v_{2,t})$	⑦	10	11	8		81	80

r	1	2	3	4	5	$\cdots$	$2n_3-3$	$2n_3-2$	$2n_3-1$	$2n_3$
$f(v_{3,r})$	⑦	6	5	12				4	79	82

Similar to Case 1, we label consecutively the vertices from $v_{1,5}$ to $v_{1,2n_1-2}$ and then $v_{2,5}$ to $v_{2,2n_2-2}$ by the first $4(k_1 + k_2 - 2)$ terms of S_1 and from $v_{3,5}$ to v_{3,n_1+n_2-4} by the first $2(k_1 + k_2 - 3)$ terms of T, respectively. The rest is similar to Case 1.

□

Example 1. *Below is the SD-prime labeling of $G(12, 14, 15)$ following the construction in Case 1 of the proof of Theorem 7.*

s	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$f(v_{1,s})$	⑦	2	13	14	15	16	19	20	21	22	35	34	33	32	37	38	39	40	43

s	20	21	22	23	24
$f(v_{1,s})$	44	45	46	27	28

t	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$f(v_{2,t})$	⑦	10	11	28	25	8	49	50	51	52	55	58	57	56	65	64	63	62	67

t	20	21	22	23	24	25	26	27	28
$f(v_{2,t})$	68	69	70	73	74	75	76	81	80

r	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$f(v_{3,r})$	⑦	6	5	12	17	18	23	24	29	30	31	36	41	42	47	48	53	54	59

r	20	21	22	23	24	25	26	27	28	29	30
$f(v_{3,r})$	60	61	66	71	72	77	78	83	4	79	82

Example 2. *Below is the SD-prime labeling of $G(11, 15, 15)$ following the construction in Case 4 of the proof of Theorem 7.*

s	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$f(v_{1,s})$	⑦	2	25	28	13	14	15	16	19	20	21	22	35	34	33	32	37	38

s	19	20	21	22
$f(v_{1,s})$	39	40	27	28

t	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$f(v_{2,t})$	⑦	10	11	8	43	44	45	46	49	50	51	52	55	58	57	56	65	64

t	19	20	21	22	23	24	25	26	27	28	29	30
$f(v_{2,t})$	63	62	67	68	69	70	73	74	75	76	81	80

The labeling for $G_{n_3} = G_{15}$ is the same as in Example 1.

Corollary 5. *$G(n_1, n_2, n_3)$ is SD-prime for any $n_3 \geq n_2 \geq 3$, $n_1 \geq 2$ and $n_1 + n_2 + n_3 = 40$.*

Proof. Basically, the proof is the same as that of Theorem 7. The difference is that 82 is not an available label. So, we have to change the initial labeling of each case in the proof of Theorem 7.

Here, we only show the first case below. Other cases are similar.

Suppose $n_1 = 2k_1$ and $n_2 = 2k_2$ for some $k_1 \geq 1$ and $k_2 \geq 2$. The initial labeling for some vertices are listed as follows:

s	1	2	$\cdots$	$2n_1 - 1$	$2n_1$
$f(v_{1,s})$	⑦	2		27	26

t	1	2	3	4	5	6	$\cdots$	$2n_2 - 1$	$2n_2$
$f(v_{2,t})$	⑦	10	11	28	25	8		81	80

r	1	2	3	4	5	$\cdots$	$2n_3 - 1$	$2n_3$
$f(v_{3,r})$	⑦	6	5	12				4

The rest assignment is same as that in the proof of Theorem 7.

Note that, in all cases, 79 is the largest unassigned label. Follow the labeling shown in the proof of Theorem 7, 79 is labeled at $v_{3,2n_3-1}$. □

From Case 2 of the proof of Theorem 7 we have

Corollary 6. *$G(2, 3, n_3)$ is SD-prime for any $n_3 \geq 3$.*

For the remaining cases, we may assume $n_1 + n_2 \geq 6$. Hence, $9 \leq n_1 + n_2 + n_3 \leq 39$.

Theorem 8. *Suppose $6 \leq n_1 + n_2 \leq 8$, then $G(n_1, n_2, n_3)$ is SD-prime for $n_3 \geq n_2 \geq n_1 \geq 2$.*

Proof. We label G_{n_1} and G_{n_2} as follows:

Case 1: $n_1 + n_2 = 6$. We have $3 \leq n_3 \leq 33$.

When $n_3 = 3$. This implies that $n_1 = n_2 = 3$. For this case, following is an SD-prime labeling for $G(3,3,3)$:

s	1	2	3	4	5	6
$f(v_{1,s})$	⑦	2	5	14	15	16

t	1	2	3	4	5	6
$f(v_{2,t})$	⑦	8	11	10	13	4

r	1	2	3	4	5	6
$f(v_{3,r})$	⑦	6	17	12	19	18

Now, we assume that $n_3 \geq 4$.

When $(n_1, n_2) = (3, 3)$. We label G_{n_1} and G_{n_2} as follows:

s	1	2	3	4	5	6
$f(v_{1,s})$	⑦	2	5	14	15	16

t	1	2	3	4	5	6
$f(v_{2,t})$	⑦	20	21	10	13	4

When $(n_1, n_2) = (2, 4)$. We label G_{n_1} and G_{n_2} as follows:

s	1	2	3	4
$f(v_{1,s})$	⑦	2	5	4

t	1	2	3	4	5	6	7	8
$f(v_{2,t})$	⑦	20	21	10	13	14	15	16

The initial labeling for some vertices of G_{n_3} are as follows:

r	1	2	3	4	5	6	7	$\cdots$	$2n_3$
$f(v_{3,r})$	⑦	6	11	12	17	18	19		8

The unassigned labels in $[22, 2n_3 + 13]$, if any, can be labeled from $v_{3,8}$ to $v_{3,2n_3-1}$ consecutively.

Case 2: $n_1 + n_2 = 7$.

When $(n_1, n_2) = (3, 4)$.

s	1	2	3	4	5	6
$f(v_{1,s})$	⑦	2	5	22	23	4

The labeling for G_{n_2} is as in Case 1.

When $(n_1, n_2) = (2, 5)$. We label G_{n_1} as in Case 1 and label G_{n_2} as

t	1	2	3	4	5	6	7	8	9	10
$f(v_{2,t})$	⑦	20	21	22	23	10	13	14	15	16

The initial labeling for G_{n_3} is as in Case 1. The unassigned labels in $[24, 2n_3 + 15]$ can be labeled from $v_{3,8}$ to $v_{3,2n_3-1}$ consecutively.

Case 3: $n_1 + n_2 = 8$.

Suppose $(n_1, n_2) = (4, 4)$. If $n_3 = 4$, we label G_{n_2} as in Case 1.

The labeling for G_{n_1} and G_{n_3} are as follows:

s	1	2	3	4	5	6	7	8
$f(v_{1,s})$	⑦	2	5	4	23	8	25	22

r	1	2	3	4	5	6	7	8
$f(v_{3,r})$	⑦	6	11	12	17	18	19	24

If $n_3 \geq 5$, the labeling for G_{n_1} is as follows:

s	1	2	3	4	5	6	7	8
$f(v_{1,s})$	⑦	2	5	22	23	4	25	26

The labeling for G_{n_2} is as in Case 1.

When $(n_1, n_2) = (3, 5)$. We label G_{n_1} as

s	1	2	3	4	5	6
$f(v_{1,s})$	⑦	2	5	4	25	26

The labeling for G_{n_2} is as in Case 2.

When $(n_1, n_2) = (2, 6)$. We label G_{n_1} as in Case 1 and label G_{n_2} as

t	1	2	3	4	5	6	7	8	9	10	11	12
$f(v_{2,t})$	⑦	20	21	22	23	10	13	14	15	16	25	26

The initial labeling for G_{n_3} is as follows:

r	1	2	3	4	5	6	7	8	9	$\cdots$	$2n_3$
$f(v_{3,r})$	⑦	6	11	12	17	18	19	24	27		8

The unassigned labels in $[28, 2n_3 + 17]$, if any, can be labeled from $v_{3,10}$ to $v_{3,2n_3-1}$ consecutively.

It is easy to see that all above labelings are SD-prime labelings. □

Now, the remaining case is $n_1 + n_2 \geq 9$ and $n_1 + n_2 + n_3 \leq 39$. This implies that $n_1 + n_2 \leq 26$ and $n_1 + n_2 + n_3 \geq 14$.

Theorem 9. *Suppose* $9 \leq n_1 + n_2 \leq 26$ *and* $14 \leq n_1 + n_2 + n_3 \leq 39$. $G(n_1, n_2, n_3)$ *is SD-prime for* $n_3 \geq n_2 \geq n_1 \geq 2$.

Proof. Let S_2 be the subsequence of S by removing s_2 and s_4, and T_2 be the subsequence of T by removing t_2, where S and T are defined in Lemma 1. That is, $S_2 = 19, 20, 21, 22; 35, 34, 33, 32; s_6, s_7, \cdots$ and $T_2 = 23, 24; t_4, \cdots$. Clearly, each pair of consecutive terms in S_2 or T_2 is of different parity and coprime.

Note that $n_1 \leq 13$ and $n_2 \geq 5$.

Case 1: Suppose $n_1 = 2k_1$ and $n_2 = 2k_2$ for some $k_1 \geq 1$ and $k_2 \geq 3$. In this case, $5 \leq k_1 + k_2 \leq 13$. The initial labeling for some vertices in G_{n_1} and G_{n_2} are listed as follows:

t	1	2	3	4	5	6	7	8	9	10	$\cdots$	$2n_2 - 1$	$2n_2$
$f(v_{2,t})$	⑦	26	25	28	5	14	15	16	13	2		17	10

s	1	2	$\cdots$	$2n_1 - 1$	$2n_1$
$f(v_{1,s})$	⑦	8		27	4

We label consecutively the vertices from $v_{2,11}$ to $v_{2,2n_2-2}$ and then $v_{1,3}$ to $v_{2,2n_1-2}$, if any, by the first $4(k_1 + k_2 - 4)$ terms of S_2. It is easy to see that the last term of s_l is not a multiple of 17 for $3 \leq l \leq 12$. The maximum used label is at most $6(k_1 + k_2 - 1) + 5 \leq 2(n_1 + n_2 + n_3) - 1$. So, we may label G_{n_3} initially as follows:

r	1	2	3	4	5	$\cdots$	$2n_3 - 2$	$2n_3 - 1$	$2n_3$
$f(v_{3,r})$	⑦	12	11	18				$2(n_1 + n_2 + n_3) + 1$	6

We label consecutively the vertices from $v_{3,5}$ to v_{3,n_1+n_2-2} by the first $2(k_1 + k_2 - 3)$ terms of T_2. The rest is similar to Case 1 of the proof of Theorem 7. Since $n_1 + n_2 - 2 \leq 2n_3 - 2$, $f(v_{3,2n_3-2}) = 2(n_1 + n_2 + n_3)$. The above labeling is SD-prime if $n_1 + n_2 + n_3 \not\equiv 1 \pmod 3$.

If $n_1 + n_2 + n_3 \equiv 1 \pmod 3$, then $n_1 + n_2 + n_3 \in \{16, 19, 22, 25, 28, 31, 34, 37\}$. We swap the labels of $v_{3,2n_3-1}$ and $v_{2,9}$. One may check that the resulting labeling is SD-prime.

Case 2: Suppose $n_1 = 2k_1$ and $n_2 = 2k_2 + 1$ for some $k_1 \geq 1$ and $k_2 \geq 2$. In this case, $4 \leq k_1 + k_2 \leq 12$. The initial labeling for some vertices in G_{n_1} and G_{n_2} are listed as follows:

t	1	2	3	4	5	6	7	8	$\cdots$	$2n_2 - 1$	$2n_2$
$f(v_{2,t})$	⑦	26	25	14	15	16	13	2		17	10

s	1	2	$\cdots$	$2n_1-1$	$2n_1$
$f(v_{1,s})$	⑦	8		27	4

We label consecutively the vertices from $v_{2,9}$ to $v_{2,2n_2-2}$ and then $v_{1,3}$ to $v_{1,2n_1-2}$, if any, by the first $4(k_1+k_2-3)$ terms of S_2. Note that $n_1+1 \le n_2 \le n_3$. The maximum used label is at most $6(k_1+k_2)+4 = 3(n_1+n_2)+1 \le 2(n_1+n_2+n_3)$ when $k_1+k_2 \not\equiv 0 \pmod 5$. So, we may label G_{n_3} initially as follows:

r	1	2	3	4	5	6	7	$\cdots$	$2n_3-2$	$2n_3-1$	$2n_3$
$f(v_{3,r})$	⑦	12	11	28	5	18				$2(n_1+n_2+n_3)+1$	6

We label consecutively the vertices from $v_{3,7}$ to v_{3,n_1+n_2-1} by the first $2(k_1+k_2-3)$ terms of T_2. The rest is similar to Case 1. Since $n_1+n_2-1 \le 2n_3-2$, $f(v_{3,2n_3-2}) = 2(n_1+n_2+n_3)$, the above labeling is SD-prime if $n_1+n_2+n_3 \not\equiv 1 \pmod 3$.

When $n_1+n_2+n_3 \equiv 1 \pmod 3$. We swap the labels of $v_{3,2n_3-1}$ and $v_{2,7}$. Similar to Case 1, one may check that the resulting labeling is SD-prime.

When $k_1+k_2 \equiv 0 \pmod 5$, there are only two cases: $k_1+k_2 = 5, 10$.

Suppose $k_1+k_2 = 5$. After labeling G_{n_1} and G_{n_2}, the maximum used labels is 35. Since $2k_2+1 \ge 2k_1$, $k_2 \ge 3$, we have $n_1+3 \le n_2 \le n_3$ so that $2(n_1+n_2+n_3)+1 \ge 37$. Hence, the labeling method is as the case when $k_1+k_2 \not\equiv 0 \pmod 5$.

Suppose $k_1+k_2 = 10$. After labeling G_{n_1} and G_{n_2}, the maximum used labels is 65. In this case, we have $2(n_1+n_2+n_3)+1 \ge 65$. When $2(n_1+n_2+n_3)+1 > 65$, the labeling method is as the case when $k_1+k_2 \not\equiv 0 \pmod 5$. We now deal with $2(n_1+n_2+n_3)+1 = 65$. It implies that $(n_1,n_2,n_3) = (10,11,11)$. Actually, it is not a case according to the previous construction so we list a required labeling as follows:

t	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$f(v_{2,t})$	⑦	26	25	14	15	16	13	2	19	20	21	22	35	34	33

t	16	17	18	19	20	21	22
$f(v_{2,t})$	32	37	38	39	40	17	10

s	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$f(v_{1,s})$	⑦	8	43	44	45	46	49	50	51	52	55	58	57	56	65

s	16	17	18	19	20
$f(v_{1,s})$	64	63	62	27	4

r	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$f(v_{3,r})$	⑦	12	11	28	5	18	23	24	29	30	31	36	41	42	47

r	16	17	18	19	20	21	22
$f(v_{3,r})$	48	53	54	59	60	61	6

Case 3: Suppose $n_1 = 2k_1+1$ and $n_2 = 2k_2$ for some $k_1 \ge 1$ and $k_2 \ge 3$. In this case, $4 \le k_1+k_2 \le 12$. The initial labeling for some vertices in G_{n_1} and G_{n_2} are listed as follows:

t	1	2	3	4	5	6	7	$\cdots$	$2n_2-2$	$2n_2-1$	$2n_2$
$f(v_{2,t})$	⑦	16	13	2	15	14				17	10

s	1	2	3	4	$\cdots$	$2n_1-1$	$2n_1$
$f(v_{1,s})$	⑦	26	25	8		27	4

We label consecutively the vertices from $v_{2,7}$ to $v_{2,2n_2-2}$ and then $v_{1,5}$ to $v_{1,2n_1-2}$, if any, by the first $4(k_1+k_2-3)$ terms of S_2. Note that $n_1+1 \le n_2 \le n_3$. The rest is as Case 2.

Case 4: Suppose $n_1 = 2k_1+1$ and $n_2 = 2k_2+1$ for some $k_1 \ge 1$ and $k_2 \ge 2$. In this case, $4 \le k_1+k_2 \le 12$. The initial labeling for some vertices in G_{n_1} and G_{n_2} are listed as follows:

t	1	2	3	4	5	6	7	8	$\cdots$	$2n_2-1$	$2n_2$
$f(v_{2,t})$	⑦	25	28	16	13	2	15	14		17	4

s	1	2	3	4	$\cdots$	$2n_1-1$	$2n_1$
$f(v_{1,s})$	⑦	10	11	8		27	26

We label consecutively the vertices from $v_{2,9}$ to $v_{2,2n_2-2}$ and then $v_{1,5}$ to $v_{1,2n_1-2}$, if any, by the first $4(k_1+k_2-3)$ terms of S_2. The maximum used label is at most $6(k_1+k_2)+4 = 3(n_1+n_2)-2 \le 2(n_1+n_2+n_3)-2$. So, we may label G_{n_3} initially as follows:

r	1	2	3	4	5	$\cdots$	$2n_3-2$	$2n_3-1$	$2n_3$
$f(v_{3,r})$	⑦	18	5	12				$2(n_1+n_2+n_3)+1$	6

The rest is similar to Case 1.

□

Example 3. *Following is an SD-prime labeling of $G(4,5,7)$ according to the construction in the proof of Theorem 9:*

t	1	2	3	4	5	6	7	8	9	10
$f(v_{2,t})$	⑦	26	25	14	15	16	$\boxed{33}$	2	17	10

s	1	2	3	4	5	6	7	8
$f(v_{1,s})$	⑦	8	19	20	21	22	27	4

r	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$f(v_{3,r})$	⑦	12	11	28	5	18	23	24	29	30	31	32	$\boxed{13}$	6

Conjecture 1. *$G(n_1,\ldots,n_t)$ is SD-prime for $t \ge 4$.*

4. Merging of Type 1 and Exactly One Type 2 Vertices

In this section, we assume that $k = t-1$. For convenience, we use $a^{[n]}$ to denote a sequence of length n whose terms are a.

Theorem 10. *For $t \ge 2$, $G = G(3^{[t-1]};3)$ is SD-prime.*

Proof. Without loss of generality, we assume that $v_{t,2}$ is merged with $v_{i,1}$, $1 \le i \le t-1$. Now, we define a labeling f (see Figure 4) for G by

$$f(c_t) = 7, f(v_{t,j}) = j \text{ for } 1 \le j \le 6; \text{ and}$$
$$f(c_i) = 6i+4, f(v_{i,1}) = 2, f(v_{i,2}) = 6i+3, f(v_{i,3}) = 6i+2,$$
$$f(v_{i,j}) = 6i+j+1 \text{ for } 1 \le i \le t-1, j = 4,5,6.$$

It is easy to check that f is an SD-prime labeling. □

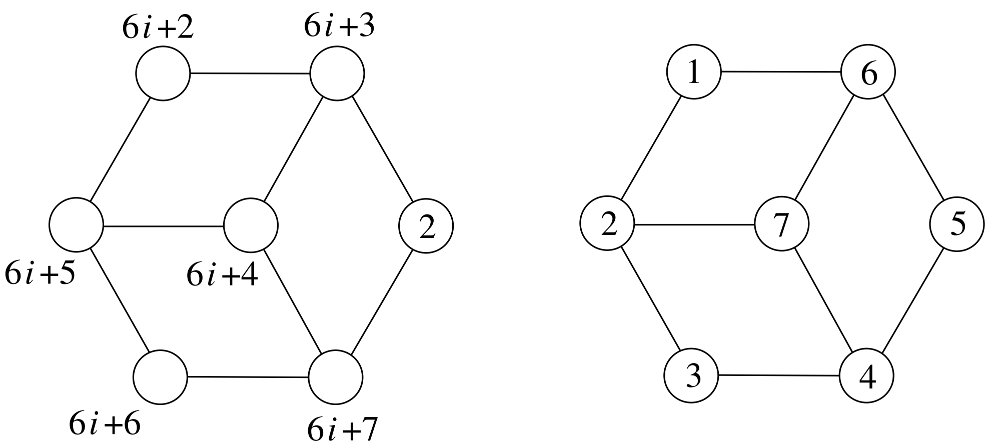


Figure 4. Labels assignment for the i-th G_3 and the t-th G_3.

Theorem 11. *For $a, b \geq 2$, $G = G(a; b)$ is SD-prime.*

Proof. Same as the proof of Theorem 5, we use u_x to replace $v_{1,x}$ and v_y to replace $v_{2,y}$, where $1 \leq x \leq a$ and $1 \leq y \leq b$. Suppose u_1 is merged with v_2. Let f be a labeling of G. We always set $f(c_1) = 2$, $f(c_2) = 1$ and $f(u_1) = 4 = f(v_2)$. We assign the labels to other vertices as follows.

Case 1: Suppose $b \not\equiv 2 \pmod 3$. Let

$$f(u_i) = 2b + i + 1 \text{ for } 2 \leq i \leq 2a,$$
$$f(v_1) = 3, f(v_j) = j + 2 \text{ for } 3 \leq j \leq 2b.$$

Here, $(f(v_1), f(v_{2b})) = (3, 2b + 2) = 1$ since $b \not\equiv 2 \pmod 3$.

Case 2: Suppose $a = 2 = b$. Let

$$f(u_2) = 9, f(u_3) = 8, f(u_4) = 3,$$
$$f(v_1) = 5, f(v_3) = 7, f(v_4) = 6.$$

Case 3: Suppose $a = 2$ and $b \equiv 2 \pmod 3$, $b \geq 5$. Let

$$f(u_2) = 3, f(u_3) = 8, f(u_4) = 9,$$
$$f(v_3) = 5, f(v_4) = 6, f(v_5) = 7,$$
$$f(v_1) = 2b + 5, f(v_j) = j + 4 \text{ for } 6 \leq j \leq 2b.$$

Case 4: Suppose $a = 3$ and $b \equiv 2 \pmod 3$. Let

$$f(u_2) = 3, f(u_3) = 8, f(u_4) = 5, f(u_5) = 6, f(u_6) = 7,$$
$$f(v_1) = 2b + 7, f(v_j) = j + 6 \text{ for } 3 \leq j \leq 2b.$$

Case 5: Suppose $a \geq 4$ and $b \equiv 2 \pmod 3$. Let

$$f(u_2) = 3, f(u_{2a}) = 9, f(u_{2a-1}) = 10, f(u_{2a-2}) = 7,$$
$$f(u_{2a-3}) = 6, f(u_{2a-4}) = 5, f(u_{2a-5}) = 8,$$
$$f(u_i) = 2b + i + 7 \text{ for } 3 \leq i \leq 2a - 6 \text{ (this line can be ignored when } a = 4\text{)},$$
$$f(v_1) = 9 + 2b, f(v_j) = j + 8 \text{ for } 3 \leq j \leq 2b.$$

Now, $(f(u_2), f(u_3)) = (3, 2b + 10) = (3, 2) = 1$.

It is easy to check the coprimality of other adjacent vertices. By Theorem 3, f is an SD-prime labeling of G. □

Theorem 12. *For $b \geq 2$, $G = G(2, 2; b)$ is SD-prime.*

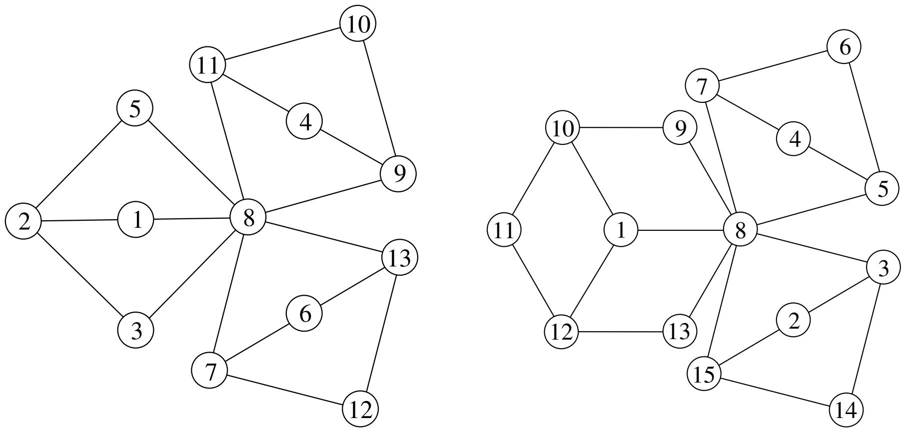
Proof. When $b = 2, 3$, we have the following SD-prime labelings:

For $b \geq 4$, we merge vertices $v_{1,1}$, $v_{2,1}$ and $v_{3,2}$ and let $f(v_{1,1}) = f(v_{2,1}) = f(v_{3,2}) = 8$, $f(c_1) = 4$, $f(c_2) = 2$ and $f(c_3) = 1$.

We define the labeling f by the following table.

r	2	3	4	5	6	7	8	$\cdots$	r	$\cdots$	$2b$	1
$f(v_{3,r})$	⑧	9	10	11	12	13	16	$\cdots$	$r+8$	$\cdots$	$2b+8$	$2b+9$

The labeling of the two G_2's are same as shown in Figure 5b.



(a) $G(2,2;2)$ **(b)** $G(2,2;3)$

Figure 5. SD-prime labelings for $G(2,2;2)$ and $G(2,2;3)$.

Clearly, f is an SD-prime labeling of $G(2,2;b)$. □

Theorem 13. *For $n_2 \geq n_1$, $G = G(n_1, n_2; 2)$ is SD-prime.*

Proof. When $n_2 = 2$. We have proved in Theorem 12. So, we assume that $n_2 \geq 3$. We merge vertices $v_{1,1}$, $v_{2,1}$ and $v_{3,2}$ and let $f(v_{1,1}) = f(v_{2,1}) = f(v_{3,2}) = 8$, $f(c_1) = 4$, $f(c_2) = 2$ and $f(c_3) = 1$.

Case 1: Suppose $n_1 \not\equiv 0 \pmod 3$. We define the labeling f by the following table.

s	1	2	3	$\cdots$	s	$\cdots$	$2n_1$
$f(v_{1,s})$	⑧	3	14	$\cdots$	$s+11$	$\cdots$	$2n_1+11$

t	1	2	3	4	5	6	7	$\cdots$	t	$\cdots$	$2n_2$
$f(v_{2,t})$	⑧	13	12	11	10	9	$2n_1+12$	$\cdots$	$2n_1+t+5$	$\cdots$	$2n_1+2n_2+5$

r	2	3	4	1
$f(v_{3,r})$	⑧	5	6	7

Since $n_1 \not\equiv 0 \pmod 3$, we have $2n_1 + 12 \equiv 1, 2 \pmod 3$. So, $(3, 2n_1 + 12) = 1$ and $(f(v_{2,6}), f(v_{2,7})) = 1$ when $n_2 \geq 4$.

Case 2: Suppose $n_1 \equiv 0 \pmod 3$. We list the labeling as in the following table.

s	1	2	3	$\cdots$	s	$\cdots$	$2n_1$
$f(v_{1,s})$	⑧	11	12	$\cdots$	$s+9$	$\cdots$	$2n_1+9$

t	1	2	3	4	5	$\cdots$	t	$\cdots$	$2n_2$
$f(v_{2,t})$	⑧	9	10	3	$2n_1+10$	$\cdots$	$2n_1+t+5$	$\cdots$	$2n_1+2n_2+5$

r	2	3	4	1
$f(v_{3,r})$	⑧	5	6	7

Clearly, $(f(v_{2,4}), f(v_{2,5})) = (3, 2n_1 + 10) = (3, 10) = 1$.

By Theorem 3, f is an SD-prime labeling of G for each case. □

Example 4. *By the construction from the proof of Case 1 above, we have an SD-prime labeling of $G(2,3;2)$ (Figure 6):*

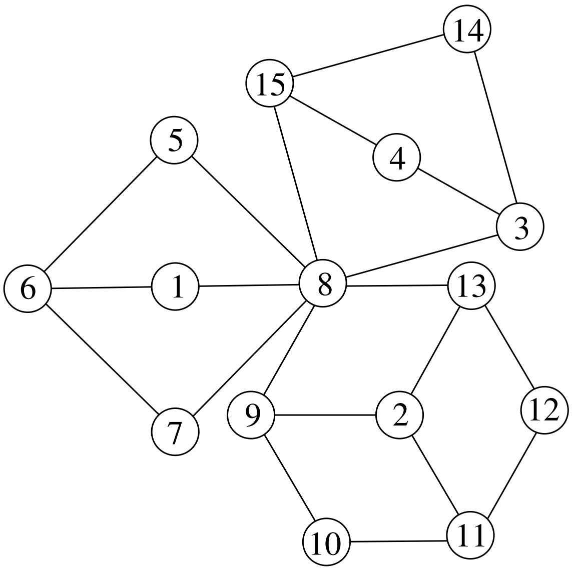


Figure 6. SD-prime labeling of $G(2,3;2)$.

Theorem 14. *For $n_2 \geq n_1$, $n_2 \geq 3$ and $b \geq 3$, $G = G(n_1, n_2; b)$ is SD-prime.*

Proof. We always merge vertices $v_{1,1}$, $v_{2,1}$ and $v_{3,2}$ and let $f(v_{1,1}) = f(v_{2,1}) = f(v_{3,2}) = 8$, $f(c_1) = 4$, $f(c_2) = 2$ and $f(c_3) = 1$.

Case 1: Suppose $b \not\equiv 0 \pmod 3$ and $b + n_1 + 1 \not\equiv 0 \pmod 3$. This implies that $b \geq 4$. We define the labeling f by the following table.

s	1	2	3	4	5	$\cdots$	s	$\cdots$	$2n_1$
$f(v_{1,s})$	⑧	11	10	9	$2b+12$	$\cdots$	$2b+s+7$	$\cdots$	$2b+2n_1+7$

t	1	2	3	4	5	6
$f(v_{2,t})$	⑧	15	14	13	16	3

t	7	$\cdots$	t	$\cdots$	$2n_2$
$f(v_{2,t})$	$2b+2n_1+8$	$\cdots$	$2b+2n_1+t+1$	$\cdots$	$2b+2n_1+2n_2+1$

r	2	3	4	5	6	7	$\cdots$	r	$\cdots$	$2b$	1
$f(v_{3,r})$	⑧	5	6	7	12	17	$\cdots$	$r+10$	$\cdots$	$2b+10$	$2b+11$

Clearly, f satisfies the last two conditions of Theorem 3 for all vertices in G_b. Since $b \not\equiv 0 \pmod 3$, $(3, 2b+12) = (3, 2b) = 1$. Thus, $(f(v_{1,4}), f(v_{1,5})) = (9, 2b+12) = 1$. Since $b + n_1 + 1 \not\equiv 0 \pmod 3$, $(f(v_{2,6}), f(v_{2,7})) = (3, 2b+2n_1+8) = (3, 2(b+n_1+1)) = 1$.

- Suppose $b \not\equiv 0 \pmod 3$ and $b + n_1 + 1 \equiv 0 \pmod 3$. This implies that $b \geq 4$ and $n_1 \geq 3$. We define the labeling f by the following table.

s	1	2	3	4	5	6	7	$\cdots$	s	$\cdots$	$2n_1$
$f(v_{1,s})$	⑧	11	10	13	16	3	$2b+12$	$\cdots$	$2b+s+5$	$\cdots$	$2b+2n_1+5$

t	1	2	3	4	5	$\cdots$	t	$\cdots$	$2n_2$
$f(v_{2,t})$	⑧	15	14	9	$2b+2n_1+6$	$\cdots$	$2b+2n_1+t+1$	$\cdots$	$2b+2n_1+2n_2+1$

r	2	3	4	5	6	7	$\cdots$	r	$\cdots$	$2b$	1
$f(v_{3,r})$	⑧	5	6	7	12	17	$\cdots$	$r+10$	$\cdots$	$2b+10$	$2b+11$

Same as the Case 1, f satisfies the last two conditions of Theorem 3 for all vertices in G_b and G_{n_1}. Since $b + n_1 \equiv 2 \pmod 3$, $(f(v_{2,4}), f(v_{2,5})) = (3, 2b+2n_1+6) = 1$.

- Suppose $b \equiv 0 \pmod 3$ and $n_1 \not\equiv 2 \pmod 3$. This implies that $n_1 \geq 3$. We define the labeling f by the following table.

s	1	2	3	4	5	6	7	$\cdots$	s	$\cdots$	$2n_1$
$f(v_{1,s})$	⑧	13	12	11	10	9	$2b+14$	$\cdots$	$2b+s+7$	$\cdots$	$2b+2n_1+7$

t	1	2	3	4	5	6
$f(v_{2,t})$	⑧	7	6	5	16	3

t	7	$\cdots$	t	$\cdots$	$2n_2$
$f(v_{2,t})$	$2b+2n_1+8$	$\cdots$	$2b+2n_1+t+1$	$\cdots$	$2b+2n_1+2n_2+1$

r	2	3	4	5	$\cdots$	r	$\cdots$	$2b$	1
$f(v_{3,r})$	⑧	15	14	17	$\cdots$	$r+12$	$\cdots$	$2b+12$	$2b+13$

Since $b \equiv 0 \pmod 3$, $(3, 2b+14) = (3, 14) = 1$. Thus, $(f(v_{1,6}), f(v_{1,7})) = (9, 2b+14) = 1$. Since $n_1 \not\equiv 2 \pmod 3$, $(f(v_{2,6}), f(v_{2,7})) = (3, 2b+2n_1+8) = (3, 2n_1-1)) = 1$. Others are similar to the previous cases, we omit the arguments.

Case 2: Suppose $b \equiv 0 \pmod 3$ and $n_1 \equiv 2 \pmod 3$. If $n_2 = 3$, then $n_1 = 2$. We have the following labeling.

s	1	2	3	4
$f(v_{1,s})$	⑧	9	10	3

t	1	2	3	4	5	6
$f(v_{2,t})$	⑧	11	12	13	14	15

r	2	3	4	5	6	$\cdots$	r	$\cdots$	$2b$	1
$f(v_{3,r})$	⑧	5	6	7	16	$\cdots$	$r+10$	$\cdots$	$2b+10$	$2b+11$

We define the labeling f by the following table for $n_2 \geq 4$.

s	1	2	3	4	5	$\cdots$	s	$\cdots$	$2n_1$
$f(v_{1,s})$	⑧	11	10	9	$2b+14$	$\cdots$	$2b+s+9$	$\cdots$	$2b+2n_1+9$

t	1	2	3	4	5	6	7	8
$f(v_{2,t})$	⑧	13	12	7	6	5	16	3

t	9	$\cdots$	t	$\cdots$	$2n_2$
$f(v_{2,t})$	$2b+2n_1+10$	$\cdots$	$2b+2n_1+t+1$	$\cdots$	$2b+2n_1+2n_2+1$

r	2	3	4	5	$\cdots$	r	$\cdots$	$2b$	1
$f(v_{3,r})$	⑧	15	14	17	$\cdots$	$r+12$	$\cdots$	$2b+12$	$2b+13$

Since $n_1 \equiv 2 \pmod 3$, $(f(v_{2,8}), f(v_{2,9})) = (3, 2b+2n_1+10) = (3, 5) = 1$. If others are similar to the previous cases, we omit the arguments.

By Theorem 3, f is an SD-prime labeling of G for each case. □

Conjecture 2. *$G(n_1, \ldots, n_k; b)$ is SD-prime, where $k \geq 3$ and $b \geq 2$.*

5. Merging of Type 3 Vertices

We study the case $k = l = 0$ first. We shall obtain some ad hoc results.

Theorem 15. *For $n \geq 2$, $G = G(\varnothing; \varnothing; 3^{[n]})$ is SD-prime.*

Proof. Let f be a labeling of G defined by $f(c_i) = 1$ and

j	1	2	3	4	5	6
$f(v_{i,j})$	$6i-4$	$6i-3$	$6i-2$	$6i+1$	$6i$	$6i-1$

for $1 \leq i \leq n$. Now, $(f(v_{i,1}), f(v_{i,6})) = (6i-4, 6i-1) = (6i-4, 3) = (-4, 3) = 1$ and $(f(v_{i,4}), f(v_{i,3})) = (6i+1, 6i-2) = (3, 6i-2) = (3, -2) = 1$. Hence, by Theorem 3, f is an SD-prime labeling of G. □

Theorem 16. *For $2 \leq m \leq 5$, $G = G(\varnothing; \varnothing; 4^{[m]})$ is SD-prime.*

Proof. Let f be a labeling of G defined by
$f(c_i) = 1$ and $f(v_{i,j}) = 1 + j + 8(i-1)$ for $1 \le i \le m, 1 \le j \le 8$. In this case,

$$(f(v_{i,1}), f(v_{i,8})) = (8i-6, 8i+1) = (8i-6, 7) = (i-6, 7) = 1.$$

So, we have the theorem. □

Remark 1. *If we extend the labeling f in the proof of Theorem 16 to $m = 6$, then $f(v_{6,8}) = 49$. Here, $(f(v_{6,8}), f(v_{6,1})) = (49, 42) = 7$. For this case, we may swap the labels of $v_{6,8}$ and $v_{1,2}$. Then, f is an SD-prime labeling of $G(\varnothing;\varnothing;4^{[6]})$. Now, by the same way, we may extend f up to $m = 12$. For the 13-th G_4, we may use the same f and swap the labels of $v_{13,8}$ and a suitable labeled vertex. So, that we believe that f can be extended to $G(\varnothing;\varnothing;4^{[13]})$ and then to $G(\varnothing;\varnothing;4^{[19]})$, and so on. So, we make the following conjecture.*

Conjecture 3. *$G(\varnothing;\varnothing;4^{[m]})$ is SD-prime for $m \ge 13$.*

Theorem 17. *Let v be the merged vertex of $H = G(n_1, \dots, n_k; n_{k+1})$ with the following conditions, where $k \ge 1$,*

(1) $\sum_{i=1}^{k+1} n_i \equiv 0 \pmod 3$,

(2) *there is an SD-prime labeling h of H such that $h(v) = 2^a$ for some $a \ge 1$.*

Then, $G = G(n_1, \dots, n_k; n_{k+1}; 3^{[n]})$ is SD-prime, for $n \ge 1$.

Proof. Let $N = 1 + 2\sum_{i=1}^{k+1} n_i$. By the assumption, all cores of G_3's are merged to v. Now, we want to extend h to G by assigning labels in $[N+1, N+6n]$. Let the vertices of the i-th G_3 be $x_{i,j}$, $1 \le i \le n$, $1 \le j \le 6$. For $1 \le i \le n$, let

j	1	2	3	4	5	6
$h(x_{i,j})$	$6i+N-4$	$6i+N-3$	$6i+N$	$6i+N-1$	$6i+N-2$	$6i+N-5$

Then, $(h(x_{i,5}), h(x_{i,6})) = (6i+N-2, 6i+N-5) = (3, 6i+N-5) = (3,2) = 1$ and $(h(x_{i,3}), h(x_{i,2})) = (6i+N, 6i+N-3) = (3, 6i+N-3) = (3,1) = 1$. Hence, h is extended to be an SD-prime labeling of $G(n_1, \dots, n_k; n_{k+1}; 3^{[n]})$ for $n \ge 1$. □

Theorem 18. *For $n \ge 1$ and $1 \le k \le 4$, $G = G(\varnothing;\varnothing;3^{[n]}, 4^{[3k]})$ is SD-prime.*

Proof. From Remark 1, we label $G(\varnothing;\varnothing;4^{[3k]})$ first, $1 \le k \le 4$. Note that the merged vertex is labeled by 1. Then, shift the original labeling f defined in the proof of Theorem 15 by $24k$. By the same argument in the proof of Theorem 15, we have the theorem. □

To illustrate the proof of the above theorem, we provide the following example.

Example 5. *Consider $G = G(\varnothing;\varnothing;3^{[n]}, 4^{[3]})$, where $n \ge 1$. We use the labeling defined in the proof of Theorem 16 to label the three G_4's. Here, labels in $[1, 25]$ are occupied.*

Let $v_{i,j}$ be vertex of the i-th G_3, $1 \le i \le n$. We let

j	1	2	3	4	5	6
$f(v_{i,j})$	$6i+20$	$6i+21$	$6i+22$	$6i+25$	$6i+24$	$6i+23$

Now, $(f(v_{i,1}), f(v_{i,6})) = (6i+20, 6i+23) = (6i+20, 3) = (20, 3) = 1$ and $(f(v_{i,4}), f(v_{i,3})) = (6i+25, 6i+22) = (3, 6i+22) = (3, 22) = 1$. Hence, by Theorem 3, f is an SD-prime labeling of G.

6. Conclusions

In this paper, to consider SD-prime labelings for one point union of gear graphs is an initial work. We have found SD-prime labelings for a few one point union of gear graphs. Besides the two conjectures proposed in Sections 3 and 4, there are many other cases that we can consider. For example, one may consider $G(n_1, \dots, n_k; \emptyset; n_{k+1})$ when $n_{k+1} \geq 5$; or $G(n_1, \dots, n_k; n_{k+1}; n_{k+2})$, where $n_i \geq 2$.

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