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Small Solutions of the Perturbed Nonlinear Partial Discrete Dirichlet Boundary Value Problems with (p,q)-Laplacian Operator

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Abstract: In this paper, we consider a perturbed partial discrete Dirichlet problem with the (p,q)-Laplacian operator. Using critical point theory, we study the existence of infinitely many small solutions of boundary value problems. Without imposing the symmetry at the origin on the nonlinear term f, we obtain the sufficient conditions for the existence of infinitely many small solutions. As far as we know, this is the study of perturbed partial discrete boundary value problems. Finally, the results are exemplified by an example.

Keywords: boundary value problem; partial difference equation; infinitely many small solutions; (p,q)-Laplacian; critical point theory

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Citation: Xiong, F.; Zhou, Z. Small Solutions of the Perturbed Nonlinear Partial Discrete Dirichlet Boundary Value Problems with (p,q)-Laplacian Operator. *Symmetry* **2021**, *13*, 1207. https://doi.org/10.3390/sym13071207

Academic Editor: Jan Awrejcewicz

Received: 25 May 2021
Accepted: 3 July 2021
Published: 5 July 2021

1. Introduction

Let $\mathbf{Z}$ and $\mathbf{R}$ denote the sets of integers and real numbers, respectively. Denote $\mathbf{Z}(a,b) = \{a, a+1, \cdots, b\}$ when $a \le b$.

We consider the following partial discrete problem, namely $(D^{\lambda,\mu})$
$-[\Delta_1(\phi_p(\Delta_1 y(s-1,t))) + \Delta_2(\phi_p(\Delta_2 y(s,t-1)))] + l(s,t)\phi_q(y(s,t)) = \lambda f((s,t), y(s,t))$
$+\mu g((s,t), y(s,t)), (s,t) \in \mathbf{Z}(1,a) \times \mathbf{Z}(1,b)$, with Dirichlet boundary conditions as follows:

$$\begin{aligned} y(s,0) = y(s,b+1) = 0, &\quad s \in \mathbf{Z}(0,a+1), \\ y(0,t) = y(a+1,t) = 0, &\quad t \in \mathbf{Z}(0,b+1), \end{aligned} \tag{1}$$

where a and b are the given positive integers, λ and μ are the positive real parameters, Δ_1 and Δ_2 are the forward difference operators defined by $\Delta_1 y(s,t) = y(s+1,t) - y(s,t)$ and $\Delta_2 y(s,t) = y(s,t+1) - y(s,t)$, $\Delta_1^2 y(s,t) = \Delta_1(\Delta_1 y(s,t))$ and $\Delta_2^2 y(s,t) = \Delta_2(\Delta_2 y(s,t))$, $\phi_r(y) = |y|^{r-2}y$ with $y \in \mathbf{R}$, $1 < q \le p < +\infty$, $l(s,t) \ge 0$ for all $(s,t) \in \mathbf{Z}(1,a) \times \mathbf{Z}(1,b)$, and $f((s,t),\cdot), g((s,t),\cdot) \in C(\mathbf{R},\mathbf{R})$ for each $(s,t) \in \mathbf{Z}(1,a) \times \mathbf{Z}(1,b)$.

Difference equations are widely applied in diverse domains, including natural science, and biological neural networks, as shown in [1–4]. For the existence and multiplicity of solutions to boundary value problems, some authors derived a number of conclusions using nonlinear analysis methods, such as fixed point methods as well as the Brouwer degree [5–9]. In 2003, Yu and Guo [10] used firstly the critical point theory to study a class of difference equations. Since then, many mathematical researchers have explored difference equations and made great achievements, which include the results of periodic solutions [10,11], homoclinic solutions [12–18], boundary value problems [19–25] and so on.

Bonanno et al. [20] in 2016 considered the following discrete Dirichlet problem:

$$\begin{cases} \Delta^2 y_{h-1} + \lambda f(h, y_h) = 0, & h \in \mathbf{Z}(1,N), \\ y_0 = y_{N+1} = 0, \end{cases} \tag{2}$$

and acquired at least two positive solutions of (2).

Mawhin et al. [21] in 2017 studied the following boundary value problem:

$$\begin{cases} -\Delta(\phi_p(\Delta y_{h-1})) + q_h\phi_p(y_h) = \lambda f(h, y_h), \quad h \in \mathbf{Z}(1, N), \\ y_0 = y_{N+1} = 0, \end{cases} \tag{3}$$

extending the results in [20] with $p = 2$.

Nastasi et al. [22] in 2017 studied the discrete Dirichlet problem involving the (p, q)-Laplacian operator as follows:

$$\begin{cases} -\Delta(\phi_p(\Delta y(h-1))) - \Delta(\phi_q(\Delta y(h-1))) + \alpha(h)\phi_p(y(h)) + \beta(h)\phi_q(y(h)) = \lambda g(h, y(h)), \ h \in \mathbf{Z}(1, N), \\ y(0) = y(N+1) = 0, \end{cases} \tag{4}$$

and obtained at least two positive solutions of (4).

In 2019, Ling and Zhou [24] considered the Dirichlet problem involving ϕ_c-Laplacian as follows:

$$\begin{cases} -\Delta(\phi_c(\Delta y_{h-1})) + q_h\phi_c(y_h) = \lambda f(h, y_h), \quad h \in \mathbf{Z}(1, N), \\ y_0 = y_{N+1} = 0. \end{cases} \tag{5}$$

They studied the existence of positive solutions of (5) when $q_h \equiv 0$ in [23].

In 2020, Wang and Zhou [25] considered discrete Dirichlet boundary value problem as follows:

$$\begin{cases} \Delta(\phi_{p,c}(\Delta y_{h-1})) + \lambda f(h, y_h) = 0, \quad h \in \mathbf{Z}(1, N), \\ y_0 = y_{N+1} = 0. \end{cases} \tag{6}$$

The difference equations studied above involve only one variable. However, the difference equations containing two or more variables are less studied, and such difference equations are called partial difference equations. Recently, partial difference equations were widely used in many fields. Boundary value problems of partial difference equations seem to be challenging problem that has attracted many mathematical researchers [26,27].

In 2015, Heidarkhani and Imbesi [26] adopted two critical points theorems to establish multiple solutions of the partial discrete problem as shown below:

$$\Delta_1^2 y(s-1, t) + \Delta_2^2 y(s, t-1) + \lambda f((s, t), y(s, t)) = 0, \quad (s, t) \in \mathbf{Z}(1, a) \times \mathbf{Z}(1, b), \tag{7}$$

with Dirichlet boundary conditions (1).

Recently, in 2020, Du and Zhou [27] studied a partial discrete Dirichlet problem as follows:

$$\Delta_1(\phi_p(\Delta_1 y(s-1, t))) + \Delta_2(\phi_p(\Delta_2 y(s, t-1))) + \lambda f((s, t), y(s, t)) = 0, \quad (s, t) \in \mathbf{Z}(1, a) \times \mathbf{Z}(1, b), \tag{8}$$

with Dirichlet boundary conditions (1).

Inspired by the above research, we found that the perturbed partial difference equations had rarely been studied, so this paper aims at studying small solutions of the perturbed partial discrete Dirichlet problems with the (p, q)-Laplacian operator. Here, the perturbed partial difference equations mean that the term with the parameter μ in the right hand of the equation for the problem $(D^{\lambda,\mu})$ is very small. A solution $y(s, t)$ of $(D^{\lambda,\mu})$ is called a small solution if the norm $\|y(s, t)\|$ is small. In fact, without the symmetric assumption on the origin for the nonlinear term f, we can still verify that problem $(D^{\lambda,\mu})$ possesses a sequence of solutions which converges to zero by using the Lemma 2. Moreover, by Lemma 1, we can show that all of these solutions are positive. Furthermore, by truncation techniques, we obtain two sequences of constant-sign solutions, which converge to zero (with one being positive and the other being negative). As far as we know, our study takes the lead in addressing small solutions of the perturbed partial discrete Dirichlet problems with the (p, q)-Laplacian operator.

The rest of this paper is organized as follows. In Section 2, we establish the variational framework linked to $(D^{\lambda,\mu})$ and recall the abstract critical point theorem. In Section 3, we give the main results. In Section 4, we provide an example to demonstrate our results. We make a conclusion in the last section.

2. Preliminaries

The current section is the first one to establish the variational framework linked to $(D^{\lambda,\mu})$. We consider the ab-dimensional Banach space

$Y = \{y : \mathbf{Z}(0,a+1) \times \mathbf{Z}(0,b+1) \to \mathbf{R} : y(s,0) = y(s,b+1) = 0, s \in \mathbf{Z}(0,a+1)$ and $y(0,t) = y(a+1,t) = 0, t \in \mathbf{Z}(0,b+1)\}$, endowed with the norm

$$\|y\| = \left(\sum_{t=1}^{b} \sum_{s=1}^{a+1} |\Delta_1 y(s-1,t)|^p + \sum_{s=1}^{a} \sum_{t=1}^{b+1} |\Delta_2 y(s,t-1)|^p \right)^{\frac{1}{p}}, \quad y \in Y.$$

and $\|y\|_\infty = \max\{|y(s,t)| : (s,t) \in \mathbf{Z}(1,a) \times \mathbf{Z}(1,b)\}$ is another norm in Y.

Let $l_* = \min\{l(s,t) : (s,t) \in \mathbf{Z}(1,a) \times \mathbf{Z}(1,b)\}$.

Proposition 1. *The following inequality holds:*

$$\|y\|_\infty \le \max\left\{ \left(\frac{p(a+b+2)^{p-1}}{4^p + l_*(a+b+2)^{p-1}} \right)^{1/q} \left(\frac{\|y\|^p}{p} + \frac{\sum\limits_{t=1}^{b} \sum\limits_{s=1}^{a} l(s,t)|y(s,t)|^q}{q} \right)^{1/q}, \right.$$
$$\left. \left(\frac{p(a+b+2)^{p-1}}{4^p + l_*(a+b+2)^{p-1}} \right)^{1/p} \left(\frac{\|y\|^p}{p} + \frac{\sum\limits_{t=1}^{b} \sum\limits_{s=1}^{a} l(s,t)|y(s,t)|^q}{q} \right)^{1/p} \right\}.$$

Proof. According to the result of ([27], Proposition 1), we have the following:

$$\|y\|_\infty^p \le \frac{(a+b+2)^{p-1}}{4^p} \|y\|^p. \tag{9}$$

When $\|y\|_\infty > 1$, according to (9), we have the following:

$$\begin{aligned} \frac{(1 + l_* \frac{(a+b+2)^{p-1}}{4^p})\|y\|_\infty^q}{p} &\le \frac{\|y\|_\infty^p + \frac{(a+b+2)^{p-1}}{4^p} \sum\limits_{t=1}^{b} \sum\limits_{s=1}^{a} l(s,t)|y(s,t)|^q}{p} \\ &\le \frac{(a+b+2)^{p-1}}{p4^p} \left(\|y\|^p + \sum_{t=1}^{b} \sum_{s=1}^{a} l(s,t)|y(s,t)|^q \right) \\ &\le \frac{(a+b+2)^{p-1}}{p4^p} \|y\|^p + \frac{(a+b+2)^{p-1}}{q4^p} \sum_{t=1}^{b} \sum_{s=1}^{a} l(s,t)|y(s,t)|^q, \end{aligned} \tag{10}$$

that is,

$$\|y\|_\infty \le \left(\frac{p(a+b+2)^{p-1}}{4^p + l_*(a+b+2)^{p-1}} \right)^{1/q} \left(\frac{\|y\|^p}{p} + \frac{\sum\limits_{t=1}^{b} \sum\limits_{s=1}^{a} l(s,t)|y(s,t)|^q}{q} \right)^{1/q}. \tag{11}$$

When $\|y\|_\infty \le 1$, according to (9), we have the following:

$$\frac{(1+l_*\frac{(a+b+2)^{p-1}}{4^p})\|y\|_\infty^p}{p} \le \frac{\|y\|_\infty^p + \frac{(a+b+2)^{p-1}}{4^p}\sum\limits_{t=1}^{b}\sum\limits_{s=1}^{a} l(s,t)|y(s,t)|^q}{p} \le \frac{(a+b+2)^{p-1}}{p4^p}\|y\|^p + \frac{(a+b+2)^{p-1}}{q4^p}\sum_{t=1}^{b}\sum_{s=1}^{a} l(s,t)|y(s,t)|^q, \tag{12}$$

that is,

$$\|y\|_\infty \le \left(\frac{p(a+b+2)^{p-1}}{4^p + l_*(a+b+2)^{p-1}}\right)^{1/p}\left(\frac{\|y\|^p}{p} + \frac{\sum\limits_{t=1}^{b}\sum\limits_{s=1}^{a} l(s,t)|y(s,t)|^q}{q}\right)^{1/p}. \tag{13}$$

In summary, we have the following:

$$\|y\|_\infty \le \max\left\{\left(\frac{p(a+b+2)^{p-1}}{4^p + l_*(a+b+2)^{p-1}}\right)^{1/q}\left(\frac{\|y\|^p}{p} + \frac{\sum\limits_{t=1}^{b}\sum\limits_{s=1}^{a} l(s,t)|y(s,t)|^q}{q}\right)^{1/q}, \left(\frac{p(a+b+2)^{p-1}}{4^p + l_*(a+b+2)^{p-1}}\right)^{1/p}\left(\frac{\|y\|^p}{p} + \frac{\sum\limits_{t=1}^{b}\sum\limits_{s=1}^{a} l(s,t)|y(s,t)|^q}{q}\right)^{1/p}\right\}.$$

Define

$$\Phi(y) = \Phi_1(y) + \Phi_2(y),$$

$$\Psi(y) = \sum_{t=1}^{b}\sum_{s=1}^{a}\Big(F((s,t),y(s,t)) + \frac{\mu}{\lambda}G((s,t),y(s,t))\Big),$$

for every $y \in Y$, where $\Phi_1(y) = \frac{\|y\|^p}{p}$, $\Phi_2(y) = \frac{\sum\limits_{t=1}^{b}\sum\limits_{s=1}^{a} l(s,t)|y(s,t)|^q}{q}$, $F((s,t),y) = \int_0^y f((s,t),\tau)\,d\tau$, $G((s,t),y) = \int_0^y g((s,t),\tau)d\tau$ for each $((s,t),y) \in \mathbf{Z}(1,a)\times\mathbf{Z}(1,b)\times\mathbf{R}$.

Let

$$I_\lambda(y) = \Phi(y) - \lambda\Psi(y),$$

for any $y \in Y$. Obviously, $\Phi, \Psi \in C^1(Y,R)$, that is, Φ_1, Φ_2 and Ψ are continuously Fréchet differentiable in Y.

$$\begin{aligned}\Phi_1'(y)(v) &= \lim_{t\to 0}\frac{\Phi_1(y+tv)-\Phi_1(y)}{t} \\ &= \sum_{t=1}^{b}\sum_{s=1}^{a+1}\phi_p(\Delta_1 y(s-1,t))\Delta_1 v(s-1,t) + \sum_{s=1}^{a}\sum_{t=1}^{b+1}\phi_p(\Delta_2 y(s,t-1))\Delta_2 v(s,t-1) \\ &= -\sum_{t=1}^{b}\sum_{s=1}^{a}\Delta_1\phi_p(\Delta_1 y(s-1,t))v(s,t) - \sum_{s=1}^{a}\sum_{t=1}^{b}\Delta_2\phi_p(\Delta_2 y(s,t-1))v(s,t),\end{aligned}$$

$$\Phi_2'(y)(v) = \sum_{t=1}^{b}\sum_{s=1}^{a} l(s,t)\phi_q(y(s,t))v(s,t),$$

and

$$\Psi'(y)(v) = \lim_{t\to 0}\frac{\Psi(y+tv)-\Psi(y)}{t} = \sum_{t=1}^{b}\sum_{s=1}^{a}\Big(f((s,t),y(s,t)) + \frac{\mu}{\lambda}g((s,t),y(s,t))\Big)v(s,t),$$

for $y, v \in Y$.

Thus

$$\begin{aligned}[\Phi'(y)-\lambda\Psi'(y)](v) = -\sum_{t=1}^{b}\sum_{s=1}^{a}\{&\Delta_1\phi_p(\Delta_1 y(s-1,t)) + \Delta_2\phi_p(\Delta_2 y(s,t-1)) \\ &- l(s,t)\phi_q(y(s,t)) + \lambda f((s,t),y(s,t)) + \mu g((s,t),y(s,t))\}v(s,t) = 0, \quad \forall v(s,t)\in Y,\end{aligned} \tag{14}$$

is equivalent to

$$-\Big[\Delta_1(\phi_p(\Delta_1 y(s-1,t))) + \Delta_2(\phi_p(\Delta_2 y(s,t-1)))\Big] + l(s,t)\phi_q(y(s,t)) = \lambda f((s,t),y(s,t)) +$$

$\mu g((s,t),y(s,t))$, for any $(s,t) \in \mathbf{Z}(1,a)\times\mathbf{Z}(1,b)$ with $y(s,0) = y(s,b+1) = 0$, $s \in \mathbf{Z}(0,a+1)$, $y(0,t) = y(a+1,t) = 0, t\in\mathbf{Z}(0,b+1)$. Thus, we reduce the existence of the solutions of $(D^{\lambda,\mu})$ to the existence of the critical points of $\Phi - \lambda\Psi$ on Y. □

Lemma 1. *Suppose that there exists y:* $\mathbf{Z}(0,a+1)\times\mathbf{Z}(0,b+1)\to\boldsymbol{R}$ *such that the following is true:*

$$y(s,t) > 0 \ \ or \ \ -\Delta_1(\phi_p(\Delta_1 y(s-1,t))) - \Delta_2(\phi_p(\Delta_2 y(s,t-1))) + l(s,t)\phi_q(y(s,t)) \ge 0, \tag{15}$$

for all $(s,t)\in\mathbf{Z}(1,a)\times\mathbf{Z}(1,b)$ *and* $y(s,0) = y(s,b+1) = 0, s\in\mathbf{Z}(0,a+1)$, $y(0,t) = y(a+1,t) = 0$, $t\in\mathbf{Z}(0,b+1)$.

Then, either $y(s,t) > 0$ *for all* $(s,t)\in\mathbf{Z}(1,a)\times\mathbf{Z}(1,b)$ *or* $y\equiv 0$.

Proof. Let $h\in\mathbf{Z}(1,a), k\in\mathbf{Z}(1,b)$ and

$$y(h,k) = \min\{y(s,t) : s\in\mathbf{Z}(1,a), t\in\mathbf{Z}(1,b)\}.$$

If $y(h,k) > 0$, then it is clear that $y(s,t) > 0$ for all $s\in\mathbf{Z}(1,a), t\in\mathbf{Z}(1,b)$. If $y(h,k)\le 0$, then $y(h,k) = \min\{y(s,t) : s\in\mathbf{Z}(0,a+1), t\in\mathbf{Z}(0,b+1)\}$, since $\Delta_1 y(h-1,k) = y(h,k) - y(h-1,k)\le 0$, $\Delta_2 y(h,k-1) = y(h,k) - y(h,k-1)\le 0$, and $\Delta_1 y(h,k) = y(h+1,k) - y(h,k)\ge 0$, $\Delta_2 y(h,k) = y(h,k+1) - y(h,k)\ge 0$, $\phi_p(s)$ is increasing in s, and $\phi_p(0) = 0$, we have

$$\phi_p(\Delta_1 y(h,k)) \ge 0 \ge \phi_p(\Delta_1 y(h-1,k)),$$

and

$$\phi_p(\Delta_2 y(h,k)) \ge 0 \ge \phi_p(\Delta_2 y(h,k-1)).$$

Owing to $\Delta_1(\phi_p(\Delta_1 y(h-1,k))) = \phi_p(\Delta_1 y(h,k)) - \phi_p(\Delta_1 y(h-1,k)) \ge 0$, $\Delta_2(\phi_p(\Delta_2 y(h,k-1))) = \phi_p(\Delta_2 y(h,k)) - \phi_p(\Delta_2 y(h,k-1)) \ge 0$. Thus, we have

$$\Delta_1(\phi_p(\Delta_1 y(h-1,k))) + \Delta_2(\phi_p(\Delta_2 y(h,k-1))) \ge 0. \tag{16}$$

By (15), we have the following:

$$-\Delta_1(\phi_p(\Delta_1 y(h-1,k))) - \Delta_2(\phi_p(\Delta_2 y(h,k-1))) \ge -l(h,k)\phi_q(y(h,k)) \ge 0,$$

that is

$$\Delta_1(\phi_p(\Delta_1 y(h-1,k))) + \Delta_2(\phi_p(\Delta_2 y(h,k-1))) \le 0. \tag{17}$$

By combining (16) with (17), we have the following:

$$\Delta_1(\phi_p(\Delta_1 y(h-1,k))) + \Delta_2(\phi_p(\Delta_2 y(h,k-1))) = 0,$$

namely,

$$\begin{cases}\phi_p(\Delta_1 y(h,k)) = \phi_p(\Delta_1 y(h-1,k)) = 0,\\ \phi_p(\Delta_2 y(h,k)) = \phi_p(\Delta_2 y(h,k-1)) = 0.\end{cases}$$

Therefore,

$$\begin{cases}y(h+1,k) = y(h,k) = y(h-1,k),\\ y(h,k+1) = y(h,k) = y(h,k-1).\end{cases}$$

If $h+1 = a+1$, we have $y(h,k) = 0$. Otherwise, $(h+1) \in \mathbf{Z}(1,a)$. Replacing h by $h+1$, we get $y(h+2,k) = y(h+1,k)$. Continuing this process $(a+1-h)$ times, we have $y(h,k) = y(h+1,k) = y(h+2,k) = \cdots = y(a+1,k) = 0$. Similarly, we have $y(h,k) = y(h-1,k) = y(h-2,k) = \cdots = y(0,k) = 0$. Hence, $y(s,k) = 0$ for each $s \in \mathbf{Z}(1,a)$. In the same way, we can prove that $y \equiv 0$, and the proof is completed. □

From Lemma 1, we have the following:

Corollary 1. *Suppose that there exists y: $\mathbf{Z}(0,a+1) \times \mathbf{Z}(0,b+1) \to \boldsymbol{R}$ such that*

$$y(s,t) < 0 \;\; or \;\; -\Delta_1(\phi_p(\Delta_1 y(s-1,t))) - \Delta_2(\phi_p(\Delta_2 y(s,t-1))) + l(s,t)\phi_q(y(s,t)) \le 0, \tag{18}$$

for all $(s,t) \in \mathbf{Z}(1,a) \times \mathbf{Z}(1,b)$ and $y(s,0) = y(s,b+1) = 0, s \in \mathbf{Z}(0,a+1)$, $y(0,t) = y(a+1,t) = 0, t \in \mathbf{Z}(0,b+1)$.

Then, either $y(s,t) < 0$ for all $(s,t) \in \mathbf{Z}(1,a) \times \mathbf{Z}(1,b)$ or $y \equiv 0$.

The existence of constant-sign solutions is discussed by truncation techniques. So, we introduce the following truncations of the functions $f((s,t),\xi)$ and $g((s,t),\xi)$ for every $(s,t) \in \mathbf{Z}(1,a) \times \mathbf{Z}(1,b)$.

If $f((s,t),0) \ge 0$ and $g((s,t),0) \ge 0$ for every $(s,t) \in \mathbf{Z}(1,a) \times \mathbf{Z}(1,b)$, let

$$f^+((s,t),\xi) := \begin{cases} f((s,t),\xi), & if\ \xi \ge 0,\\ f((s,t),0), & if\ \xi < 0,\end{cases} \quad g^+((s,t),\xi) := \begin{cases} g((s,t),\xi), & if\ \xi \ge 0,\\ g((s,t),0), & if\ \xi < 0.\end{cases}$$

Define problem (D^{λ,μ^+}) as follows:

$-\Big[\Delta_1(\phi_p(\Delta_1 y(s-1,t))) + \Delta_2(\phi_p(\Delta_2 y(s,t-1)))\Big] + l(s,t)\phi_q(y(s,t)) = \lambda f^+((s,t), y(s,t)) + \mu g^+((s,t),y(s,t)), (s,t) \in \mathbf{Z}(1,a) \times \mathbf{Z}(1,b)$, *with Dirichlet boundary conditions (1).*

Obviously, $f^+((s,t),\cdot)$ and $g^+((s,t),\cdot)$ are also continuous for every $(s,t) \in \mathbf{Z}(1,a) \times \mathbf{Z}(1,b)$. By Lemma 1, the solutions of problem (D^{λ,μ^+}) are also those of problem $(D^{\lambda,\mu})$. Therefore, when problem (D^{λ,μ^+}) has non-zero solutions, then problem $(D^{\lambda,\mu})$ possesses positive solutions.

If $f((s,t),0) \le 0$ and $g((s,t),0) \le 0$ for every $(s,t) \in \mathbf{Z}(1,a) \times \mathbf{Z}(1,b)$, let

$$f^-((s,t),\xi) := \begin{cases} f((s,t),0), & if\ \xi > 0,\\ f((s,t),\xi), & if\ \xi \le 0,\end{cases} \quad g^-((s,t),\xi) := \begin{cases} g((s,t),0), & if\ \xi > 0,\\ g((s,t),\xi), & if\ \xi \le 0.\end{cases}$$

Define problem (D^{λ,μ^-}) as follows:

$-\Big[\Delta_1(\phi_p(\Delta_1 y(s-1,t))) + \Delta_2(\phi_p(\Delta_2 y(s,t-1)))\Big] + l(s,t)\phi_q(y(s,t)) = \lambda f^-((s,t), y(s,t)) + \mu g^-((s,t),y(s,t)), (s,t) \in \mathbf{Z}(1,a) \times \mathbf{Z}(1,b)$, *with Dirichlet boundary conditions (1).*

By Corollary 1, the solutions of problem (D^{λ,μ^-}) are also those of problem $(D^{\lambda,\mu})$. Therefore, when problem (D^{λ,μ^-}) has non-zero solutions, then problem $(D^{\lambda,\mu})$ possesses negative solutions.

Here, we present the main tools used in this paper.

Lemma 2 (Theorem 4.3 of [28])**.** *Let X be a finite dimensional Banach space and let $I_\lambda : X \to \boldsymbol{R}$ be a function satisfying the following structure hypothesis:*

(H) $I_\lambda(u) := \Phi(u) - \lambda\Psi(u)$ *for all* $u \in X$*, where* $\Phi, \Psi : X \to \boldsymbol{R}$ *be two continuously Gâteux differentiable functions with* Φ *coercive, i.e.,* $\lim_{\|u\|\to+\infty} \Phi(u) = +\infty$*, and such that* $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$.

For all $r > 0$*, put the following:*

$$\varphi(r) := \frac{\sup_{v\in\Phi^{-1}[0,r]} \Psi(v)}{r}, \quad \varphi_0 := \liminf_{r\to 0^+} \varphi(r).$$

Assume that $\varphi_0 < +\infty$ *and for every* $\lambda \in (0, \frac{1}{\varphi_0})$*, 0 is not a local minima of functional* I_λ*. Then, there is a sequence* $\{u_n\}$ *of pairwise distinct critical points (local minima) of* I_λ *such that* $\lim_{n\to+\infty} u_n = 0$.

3. Main Results

In this section, the existence of constant-sign solutions of problem $(D^{\lambda,\mu})$ is discussed. Our aim is to use Lemma 2 for the function $I_\lambda^\pm : X \to R$,

$$I_\lambda^\pm(y) := \Phi(y) - \lambda\Psi^\pm(y),$$

where

$$\Psi^\pm(y) = \sum_{t=1}^{b}\sum_{s=1}^{a}(F^\pm((s,t),y(s,t)) + \frac{\mu}{\lambda}G^\pm((s,t),y(s,t))),$$

$$F^\pm((s,t),y) := \int_0^y f^\pm((s,t),\tau)d\tau, \;\; G^\pm((s,t),y) := \int_0^y g^\pm((s,t),\tau)d\tau,$$

for each $(s,t) \in \mathbf{Z}(1,a) \times \mathbf{Z}(1,b)$. Then, we apply Lemma 1 or Corollary 1 to obtain our results.

Let

$$A_{0*} = \liminf_{c\to 0^+} \frac{\sum_{t=1}^{b}\sum_{s=1}^{a}\max_{0\le m\le c} F((s,t),\pm m)}{c^p}, \quad B^{0*} = \limsup_{c\to 0^+} \frac{\sum_{t=1}^{b}\sum_{s=1}^{a} F((s,t),c)}{c^p}.$$

$$C^{0*} = \limsup_{c\to 0^+} \frac{\sum_{t=1}^{b}\sum_{s=1}^{a}\max_{0\le m\le c} G((s,t),\pm m)}{c^p}, \quad \tilde{l} = \sum_{t=1}^{b}\sum_{s=1}^{a} l(s,t).$$

In addition, put the following:

$$\bar{\mu}_\lambda^* := \frac{1}{C^{0*}}\left(\frac{4^p + l_*(a+b+2)^{p-1}}{p(a+b+2)^{p-1}} - \lambda A_{0*}\right).$$

It should be pointed out that if the denominator is 0, we regard $\frac{1}{0}$ as $+\infty$.

Theorem 1. *Let* $f((s,t),y)$ *be a continuous function of* y*, and* $f((s,t),0) \ge 0$*,* $g((s,t),\cdot) \in C(\boldsymbol{R},\boldsymbol{R})$ *for every* $(s,t) \in \mathbf{Z}(1,a) \times \mathbf{Z}(1,b)$*. Suppose the following:*

(i_1) $\frac{p(a+b+2)^{p-1}A_{0+}}{4^p+l_*(a+b+2)^{p-1}} < \frac{pqB^{0^+}}{2bq+2aq+p\tilde{l}}$,

(g_1) *there exsits* $\delta > 0$ *such that at* $[0,\delta]$*,* $G((s,t),y) \ge 0$ *and* $C^{0^+} < +\infty$.

Then, for each $\lambda \in \Lambda = \left(\frac{2bq+2aq+p\tilde{l}}{pqB^{0^+}}, \frac{4^p+l_*(a+b+2)^{p-1}}{p(a+b+2)^{p-1}A_{0+}}\right)$ *and* $\mu \in [0, \bar{\mu}_\lambda^+)$*, problem* $(D^{\lambda,\mu})$ *has a sequence of positive solutions, which converges to zero.*

Proof. We take $X = Y$, Φ_1, Φ_2 and Ψ as in Section 2. Obviously, for each $(s,t) \in \mathbf{Z}(1,a) \times \mathbf{Z}(1,b)$, $g((s,t),0) \ge 0$.

Now, we consider the auxiliary problem (D^{λ,μ^+}).
Clearly Φ and Ψ^+ satisfy the hypothesis required in Lemma 2.
Let

$$r = \frac{4^p + l_*(a+b+2)^{p-1}}{p(a+b+2)^{p-1}} \min\{c^q, c^p\}, \quad \text{for } c > 0.$$

Assume $y \in Y$, and the following:

$$\Phi(y) = \frac{\|y\|^p}{p} + \frac{\sum_{t=1}^{b}\sum_{s=1}^{a} l(s,t)|y(s,t)|^q}{q} \leq r.$$

If $r = \frac{4^p + l_*(a+b+2)^{p-1}}{p(a+b+2)^{p-1}} c^q$, it means that $c \geq 1$. According to Proposition 1, we have the following:

$$\begin{aligned}\|y\|_\infty &\leq \max\left\{\left(\frac{p(a+b+2)^{p-1}}{4^p + l_*(a+b+2)^{p-1}}\right)^{1/q} r^{1/q},\right.\\ &\qquad \left.\left(\frac{p(a+b+2)^{p-1}}{4^p + l_*(a+b+2)^{p-1}}\right)^{1/p} r^{1/p}\right\}\\ &= \max\{c, c^{q/p}\} = c.\end{aligned}$$

If $r = \frac{4^p + l_*(a+b+2)^{p-1}}{p(a+b+2)^{p-1}} c^p$, we know $0 < c < 1$, then

$$\begin{aligned}\|y\|_\infty &\leq \max\left\{\left(\frac{p(a+b+2)^{p-1}}{4^p + l_*(a+b+2)^{p-1}}\right)^{1/q} r^{1/q},\right.\\ &\qquad \left.\left(\frac{p(a+b+2)^{p-1}}{4^p + l_*(a+b+2)^{p-1}}\right)^{1/p} r^{1/p}\right\}\\ &= \max\{c^{p/q}, c\} = c,\end{aligned}$$

and

$$\frac{c^p}{r} = \frac{p(a+b+2)^{p-1}}{4^p + l_*(a+b+2)^{p-1}}. \tag{19}$$

Therefore, we have $\Phi^{-1}[0,r] \subseteq \{y \in Y : \|y\|_\infty \leq c\}$.
By the definition of φ, we have the following:

$$\begin{aligned}\varphi(r) &= \frac{\sup_{v \in \Phi^{-1}[0,r]} \Psi^+(v)}{r}\\ &\leq \frac{1}{r} \sup_{\|y\|_\infty \leq c} \sum_{t=1}^{b}\sum_{s=1}^{a}\left(F^+((s,t),y(s,t)) + \frac{\mu}{\lambda} G^+((s,t),y(s,t))\right)\\ &\leq \frac{c^p}{r}\left(\frac{\sum_{t=1}^{b}\sum_{s=1}^{a} \max_{0\leq m\leq c} F((s,t),m)}{c^p} + \frac{\mu}{\lambda}\frac{\sum_{t=1}^{b}\sum_{s=1}^{a} \max_{0\leq m\leq c} G((s,t),m)}{c^p}\right).\end{aligned}$$

According to condition (i_1), (g_1) and (19), we have the following:

$$\varphi_0 \leq \frac{p(a+b+2)^{p-1}}{4^p + l_*(a+b+2)^{p-1}} (A_{0^+} + \frac{\mu}{\lambda} C^{0^+}) < +\infty.$$

We assert that if $\lambda \in \left(\frac{2bq+2aq+p\tilde{l}}{pqB^{0^+}}, \frac{4^p + l_*(a+b+2)^{p-1}}{p(a+b+2)^{p-1}A_{0^+}}\right)$ and $\mu \in [0, \bar{\mu}^+_\lambda)$, then $\lambda \in (0, \frac{1}{\varphi_0})$.

In fact, for $\lambda \in \Lambda$, we have $\lambda > 0$.
When $C^{0^+} = 0$, then

$$\varphi_0 \leq \frac{p(a+b+2)^{p-1}}{4^p + l_*(a+b+2)^{p-1}} A_{0^+} < \frac{1}{\lambda}.$$

When $C^{0^+} > 0$, then

$$\begin{aligned}\varphi_0 &< \frac{p(a+b+2)^{p-1}}{4^p + l_*(a+b+2)^{p-1}} \left(A_{0^+} + \frac{\bar{\mu}_\lambda^+}{\lambda} C^{0^+} \right) \\ &= \frac{p(a+b+2)^{p-1}}{4^p + l_*(a+b+2)^{p-1}} \left(A_{0^+} + \frac{1}{\lambda} \frac{1}{C^{0^+}} \left(\frac{4^p + l_*(a+b+2)^{p-1}}{p(a+b+2)^{p-1}} - \lambda A_{0^+} \right) C^{0^+} \right) \\ &= \frac{1}{\lambda}.\end{aligned}$$

Clearly, $(0, 0, \cdots, 0) \in Y$ is a global minima of Φ.

Next, we need to prove that $(0, 0, \cdots, 0)$ is not a local minima of I_λ^+. Let us prove this in two cases: $B^{0^+} = +\infty$ and $B^{0^+} < +\infty$.

Firstly, when $B^{0^+} = +\infty$, fix M such that $M > \frac{2aq+2bq+\tilde{l}p}{pq}$ and there exists a sequence of positive numbers $\{c_n\}$ such that $\lim\limits_{n\to+\infty} c_n = 0$, and

$$\sum_{t=1}^{b}\sum_{s=1}^{a} F^+((s,t), c_n) = \sum_{t=1}^{b}\sum_{s=1}^{a} F((s,t), c_n) \geq \frac{Mc_n^q}{\lambda}, \quad \text{for } n \in \mathbf{Z}(1).$$

Define a sequence $\{\eta_n\}$ in Y with the following:

$$\eta_n(s,t) = \begin{cases} c_n, \text{ if } \ (s,t) \in \mathbf{Z}(1,a) \times \mathbf{Z}(1,b), \\ 0, \text{ if } \ s = 0, t \in \mathbf{Z}(0, b+1), \text{ or, } s = a+1, t \in \mathbf{Z}(0, b+1), \\ 0, \text{ if } \ t = 0, s \in \mathbf{Z}(0, a+1), \text{ or, } t = b+1, s \in \mathbf{Z}(0, a+1). \end{cases}$$

According to $G^+((s,t), \eta_n(s,t)) = G((s,t), \eta_n(s,t)) \geq 0$, $(s,t) \in \mathbf{Z}(1,a) \times \mathbf{Z}(1,b)$, we acquire the following:

$$\begin{aligned}I_\lambda^+(\eta_n) &\leq \left(\frac{2a+2b}{p}\right) c_n^p + \frac{\tilde{l}}{q} c_n^q - \lambda \left(\sum_{t=1}^{b}\sum_{s=1}^{a} F((s,t), c_n) \right) \\ &\leq \left(\frac{2a+2b}{p}\right) c_n^q + \frac{\tilde{l}}{q} c_n^q - M c_n^q \\ &= \left(\frac{2a+2b}{p} + \frac{\tilde{l}}{q} - M \right) c_n^q \\ &< 0.\end{aligned}$$

Secondly, when $B^{0^+} < +\infty$, let $\lambda \in \left(\frac{2bq+2aq+p\tilde{l}}{pqB^{0^+}}, \frac{4^p + l_*(a+b+2)^{p-1}}{p(a+b+2)^{p-1} A_{0^+}} \right)$, choose $\varepsilon_0 > 0$ such that

$$\frac{2a+2b}{p} + \frac{\tilde{l}}{q} - \lambda(B^{0^+} - \varepsilon_0) < 0.$$

Then, there is a positive sequence $\{c_n\} \subset (0, \delta)$ such that $\lim\limits_{n\to+\infty} c_n = 0$ and

$$(B^{0^+} - \varepsilon_0) c_n^q \leq \sum_{t=1}^{b}\sum_{s=1}^{a} F^+((s,t), c_n) = \sum_{t=1}^{b}\sum_{s=1}^{a} F((s,t), c_n) \leq (B^{0^+} + \varepsilon_0) c_n^q.$$

By the definition of the sequence $\{\eta_n\}$ in Y being the same as the case where $B^{0^+} = +\infty$, we have the following:

$$\begin{aligned} I_\lambda^+(\eta_n) &\leq \left(\frac{2a+2b}{p}\right)c_n^p + \frac{\tilde{l}}{q}c_n^q - \lambda\left(\sum_{t=1}^{b}\sum_{s=1}^{a} F((s,t),c_n)\right) \\ &\leq \left(\frac{2a+2b}{p}\right)c_n^p + \frac{\tilde{l}}{q}c_n^q - \lambda(B^{0^+} - \varepsilon_0)c_n^q \\ &\leq \left(\frac{2a+2b}{p} + \frac{\tilde{l}}{q} - \lambda(B^{0^+} - \varepsilon_0)\right)c_n^q \\ &< 0. \end{aligned}$$

According to the above discussion, we have $I_\lambda^+(\eta_n) < 0$.

Since $I_\lambda^+(0,0,\cdots,0) = 0$ and $\lim_{n\to\infty} \eta_n = (0,0,\cdots,0)$. By combining the above two cases, we obtain that $(0,0,\cdots,0) \in Y$ is a global minima of Φ but $(0,0,\cdots,0)$ is not a local minima of I_λ^+.

Through the above discussion, I_λ^+ satisfies every condition of Lemma 2. According to Lemma 2, there exists a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ^+ such that $\lim_{n\to+\infty} u_n = 0$. So, $(s,t) \in \mathbf{Z}(1,a) \times \mathbf{Z}(1,b)$, $y(s,t)$ is a non-zero solution of problem (D^{λ,μ^+}), by Lemma 1, $y(s,t)$ is a positive solution of problem $(D^{\lambda,\mu})$. Therefore, the proof of Theorem 1 is completed. □

Remark 1. *When the nonlinear terms f and g are symmetric on the origin, i.e., $f(\cdot,-y) = -f(\cdot,y)$, $g(\cdot,-y) = -g(\cdot,y)$, it is easy to obtain infinitely many small solutions to problem $(D^{\lambda,\mu})$ by using the critical point theory with symmetries. However, in this paper, we obtain infinitely many small solutions to problem $(D^{\lambda,\mu})$ without the symmetry on f.*

When $\lambda = 1$, according to Theorem 1, we obtain the following.

Corollary 2. *Let $f((s,t),y)$ is a continuous function of y, and $f((s,t),0) \geq 0$, $g((s,t),\cdot) \in C(\boldsymbol{R},\boldsymbol{R})$ for every $(s,t) \in \mathbf{Z}(1,a) \times \mathbf{Z}(1,b)$. Suppose the following:*

(i_2) $\frac{p(a+b+2)^{p-1}A_{0^+}}{4^p+l_*(a+b+2)^{p-1}} < 1 < \frac{pqB^{0^+}}{2bq+2aq+p\tilde{l}}$,

(g_1) *there exsits $\delta > 0$ such that at $[0,\delta]$, $G((s,t),y) \geq 0$ and $C^{0^+} < +\infty$.*

Then, for each $\mu \in [0,\bar{\mu}_1^+)$, the following problem (D^μ)

$-\left[\Delta_1(\phi_p(\Delta_1 y(s-1,t))) + \Delta_2(\phi_p(\Delta_2 y(s,t-1)))\right] + l(s,t)\phi_q(y(s,t)) = f((s,t),$ *$y(s,t)) + \mu g((s,t),y(s,t))$, $(s,t) \in \mathbf{Z}(1,a) \times \mathbf{Z}(1,b)$, with Dirichlet boundary conditions (1), has a sequence of positive solutions which converges to zero.*

Similarly, we obtain the following results.

Theorem 2. *Let $f((s,t),y)$ is a continuous function of y, and $f((s,t),0) \leq 0$, $g((s,t),\cdot) \in C(\boldsymbol{R},\boldsymbol{R})$ for every $(s,t) \in \mathbf{Z}(1,a) \times \mathbf{Z}(1,b)$. Suppose the following:*

(i_3) $\frac{p(a+b+2)^{p-1}A_{0_-}}{4^p+l_*(a+b+2)^{p-1}} < \frac{pqB^{0-}}{2bq+2aq+p\tilde{l}}$,

(g_2) *there exsits $\delta > 0$ such that at $[-\delta,0]$, $G((s,t),y) \geq 0$ and $C^{0-} < +\infty$.*

Then, for every $\lambda \in \left(\frac{2bq+2aq+p\tilde{l}}{pqB^{0-}}, \frac{4^p+l_*(a+b+2)^{p-1}}{p(a+b+2)^{p-1}A_{0_-}}\right)$ and $\mu \in [0,\bar{\mu}_\lambda^-)$, problem $(D^{\lambda,\mu})$ has a sequence of negative solutions which converges to zero.

When $\lambda = 1$, according to Theorem 2, we obtain the following.

Corollary 3. *Let $f((s,t),y)$ is a continuous function of y, and $f((s,t),0) \leq 0$, $g((s,t),\cdot) \in C(\boldsymbol{R},\boldsymbol{R})$ for every $(s,t) \in \mathbf{Z}(1,a) \times \mathbf{Z}(1,b)$. Suppose the following:*

(i_4) $\frac{p(a+b+2)^{p-1}A_{0_-}}{4^p+l_*(a+b+2)^{p-1}} < 1 < \frac{pqB^{0-}}{2bq+2aq+p\tilde{l}}$,

(g_2) *there exsits* $\delta > 0$ *such that at* $[-\delta, 0]$, $G((s,t),y) \geq 0$ *and* $C^{0-} < +\infty$.

Then, for each $\mu \in [0, \bar{\mu}_1^-)$, *problem* (D^μ) *has a sequence of negative solutions, which converges to zero.*

Combining Theorem 1 with Theorem 2, we have the following.

Theorem 3. *Let* $f((s,t),y)$ *is a continuous function of* y, *and* $f((s,t),0) = 0$, $g((s,t),\cdot) \in C(\boldsymbol{R},\boldsymbol{R})$ *for every* $(s,t) \in \boldsymbol{Z}(1,a) \times \boldsymbol{Z}(1,b)$. *Suppose that*

(i_5) $\frac{p(a+b+2)^{p-1}\max\{A_{0+},A_{0_-}\}}{4^p+l_*(a+b+2)^{p-1}} < \frac{pq\min\{B^{0^+},B^{0-}\}}{2bq+2aq+p\tilde{l}}$,

(g_3) *there exists* $\delta > 0$ *such that at* $[-\delta,\delta]$, $G((s,t),y) \geq 0$ *and* $C^{0*} < +\infty$.

Then, for every $\lambda \in \left(\frac{2bq+2aq+p\tilde{l}}{pq\min\{B^{0^+},B^{0-}\}}, \frac{4^p+l_*(a+b+2)^{p-1}}{p(a+b+2)^{p-1}\max\{A_{0+},A_{0_-}\}}\right)$ *and* $\mu \in [0, \min\{\bar{\mu}_\lambda^+, \bar{\mu}_\lambda^-\})$, *problem* $(D^{\lambda,\mu})$ *has two sequences of constant-sign solutions, which converge to zero (with one being positive and the other being negative).*

When $\lambda = 1$, according to Theorem 3, we acquire the following.

Corollary 4. *Let* $f((s,t),y)$ *is a continuous function of* y, *and* $f((s,t),0) = 0$, $g((s,t),\cdot) \in C(\boldsymbol{R},\boldsymbol{R})$ *for every* $(s,t) \in \boldsymbol{Z}(1,a) \times \boldsymbol{Z}(1,b)$. *Suppose the following:*

(i_6) $\frac{p(a+b+2)^{p-1}A_*}{4^p+l_*(a+b+2)^{p-1}} < 1 < \frac{pqB^{0*}}{2bq+2aq+p\tilde{l}}$,

(g_3) *there exsits* $\delta > 0$ *such that at* $[-\delta,\delta]$, $G((s,t),y) \geq 0$ *and* $C^{0*} < +\infty$.

Then, for each $\mu \in [0, \min\{\bar{\mu}_1^+, \bar{\mu}_1^-\})$, *problem* (D^μ) *has two sequences of constant-sign solutions which converge to zero (with one being positive and the other being negative).*

Remark 2. *As a special case of Theorem 1, when* $\mu = 0$.

Considering the following problem, namely (D^λ)

$-[\Delta_1(\phi_p(\Delta_1 y(s-1,t))) + \Delta_2(\phi_p(\Delta_2 y(s,t-1)))] + l(s,t)\phi_q(y(s,t)) = \lambda f((s,t),y(s,t))$, $(s,t) \in \boldsymbol{Z}(1,a) \times \boldsymbol{Z}(1,b)$, *with Dirichlet boundary conditions (1).*

Theorem 4. *Let* $f((s,t),y)$ *be a continuous function of* y, *and* $f((s,t),0) \geq 0$ *for every* $(s,t) \in \boldsymbol{Z}(1,a) \times \boldsymbol{Z}(1,b)$. *Suppose the following:*

(i_7) $\frac{p(a+b+2)^{p-1}A_{0+}}{4^p+l_*(a+b+2)^{p-1}} < \frac{pqB^{0^+}}{2bq+2aq+p\tilde{l}}$.

Then, for each $\lambda \in \left(\frac{2bq+2aq+p\tilde{l}}{pqB^{0^+}}, \frac{4^p+l_*(a+b+2)^{p-1}}{p(a+b+2)^{p-1}A_{0+}}\right)$, *problem* (D^λ) *has a sequence of positive solutions, which converges to zero.*

4. Example

We provide an example to illustrate our Theorem 3.

Example 1. *Suppose that* $l(s,t) = s+t$. *Let* $a = 2$, $b = 2$, $p = 3$, $q = 2$, *and* $\tilde{l} = \sum_{t=1}^{2}\sum_{s=1}^{2} l(s,t) = 12 < \frac{16+18l_*}{3} = \frac{52}{3}$, f *and* g *are two functions defined as follows:*

$$f((s,t),c) = f(c) = \begin{cases} \frac{5}{4}pc^{p-1} + pc^{p-1}\sin(\frac{1}{5}\ln c^p) + \frac{1}{5}pc^{p-1}\cos(\frac{1}{5}\ln c^p), & c > 0, \\ 0, & c \leq 0, \end{cases} \tag{20}$$

and

$$g((s,t),c) = g(c) = 2pc^{p-1}. \tag{21}$$

Then, for each $\lambda_1 \in (\frac{49}{54}, \frac{34}{27})$ *and* $\mu_1 \in [0, (\frac{34}{216} - \frac{1}{8}\lambda_1))$, *the following problem, namely* (D^{λ_1,μ_1}).

$-[\Delta_1(\phi_p(\Delta_1 y(s-1,t))) + \Delta_2(\phi_p(\Delta_2 y(s,t-1)))] + l(s,t)\phi_q(y(s,t)) = \lambda_1 f((s,t), y(s,t)) + \mu_1 g((s,t), y(s,t)), (s,t) \in \mathbf{Z}(1,2)\times\mathbf{Z}(1,2)$, with the following Dirichlet boundary conditions:

$$y(s,0) = y(s,2) = 0, \quad s \in \mathbf{Z}(0,3),$$
$$y(0,t) = y(2,t) = 0, \quad t \in \mathbf{Z}(0,3),$$

possesses two sequences of constant-sign solutions, which converge to zero (with one being positive and the other being negative).

In fact,

$$F((s,t),c) = \int_0^c f((s,t),\tau)d\tau = \begin{cases} \frac{5}{4}c^p + c^p\sin(\frac{1}{5}\ln c^p), & c > 0, \\ 0, & c \le 0, \end{cases} \tag{22}$$

$$G((s,t),c) = \int_0^c g((s,t),\tau)d\tau = 2c^p. \tag{23}$$

Since $f((s,t),c) > 0, g((s,t),c) > 0$ for $c > 0$, we know that $F((s,t),c)$ and $G((s,t),c)$ are increasing at $c \in (0,+\infty)$. Thus, $\max\limits_{0\le m\le c} F((s,t),m) = F((s,t),c)$ and $\max\limits_{0\le m\le c} G((s,t),m) = G((s,t),c)$, for every $c > 0$. Obviously,

$$A_{0*} = \liminf_{c\to 0^+} \frac{abF((s,t),c)}{c^p} = \liminf_{c\to 0^+} \frac{4(\frac{5}{4}c^p + c^p\sin(\frac{1}{5}\ln c^p))}{c^p} = 1,$$

$$B^{0*} = \limsup_{c\to 0^+} \frac{abF((s,t),c)}{c^p} = \limsup_{c\to 0^+} \frac{4(\frac{5}{4}c^p + c^p\sin(\frac{1}{5}\ln c^p))}{c^p} = 9.$$

We can verify condition (i_5) of Theorem 3 since

$$\frac{p(a+b+2)^{p-1}\max\{A_{0^+}, A_{0_-}\}}{4^p + l_*(a+b+2)^{p-1}} = \frac{27}{34} < \frac{pq\min\{B^{0^+}, B^{0-}\}}{2bq + 2aq + p\tilde{l}} = \frac{54}{49}.$$

Next, we can further verify condition (g_3) of Theorem 3 since

$$C^{0*} = \limsup_{c\to 0^+} \frac{\sum\limits_{t=1}^{b}\sum\limits_{s=1}^{a} G((s,t),c)}{c^p} = \limsup_{c\to 0^+} \frac{ab2c^p}{c^p} = 8 < +\infty.$$

In summary, every condition of Theorem 3 is met.

Therefore, for each $\lambda_1 \in (\frac{49}{54}, \frac{34}{27})$ and $\mu_1 \in [0, (\frac{34}{216} - \frac{1}{8}\lambda_1))$, problem (D^{λ_1,μ_1}) possesses two sequences of constant-sign solutions, which converge to zero (with one being positive and the other being negative).

5. Conclusions

In this paper, we studied the existence of small solutions of perturbed partial discrete Dirichlet problems with the (p,q)-Laplacian operator. Unlike the results in [25], we obtained some sufficient conditions of the existence of infinitely many small solutions, as shown in Theorems 1–3. Firstly, according to Theorem 4.3 of [28] and Lemma 1 of this paper, we obtained a sequence of positive solutions, which converges to zero in Theorem 1. Furthermore, by truncation techniques, we acquired two sequences of constant-sign solutions, which converge to zero (with one being positive and the other being negative). Secondly, the Corollaries 2–4 was acquired when $\lambda = 1$. Finally, as a special case of Theorem 1, we obtained a sequence of positive solutions, which converges to zero in Theorem 4. The existence of large constant-sign solutions of partial difference equations with the (p,q)-Laplacian operator will be discussed by the method used in this paper as our future research direction.

Author Contributions: All authors contributed equally to this paper. All authors have read and agreed to the published version of the manuscript.

Funding: This work is supported by the National Natural Science Foundation of China (grant No. 11971126), the Program for Changjiang Scholars and Innovative Research Team in University (grant No. IRT_16R16) and the Innovation Research for the Postgraduates of Guangzhou University (grant No. 2020GDJC-D07).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interests.

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