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On Forbidden Subgraphs of (K_2, H)-Sim-(Super)Magic Graphs

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Abstract: A graph G admits an H-covering if every edge of G belongs to a subgraph isomorphic to a given graph H. G is said to be H-magic if there exists a bijection $f : V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)| + |E(G)|\}$ such that $w_f(H') = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e)$ is a constant, for every subgraph H' isomorphic to H. In particular, G is said to be H-supermagic if $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$. When H is isomorphic to a complete graph K_2, an H-(super)magic labeling is an edge-(super)magic labeling. Suppose that G admits an F-covering and H-covering for two given graphs F and H. We define G to be (F, H)-sim-(super)magic if there exists a bijection f' that is simultaneously F-(super)magic and H-(super)magic. In this paper, we consider (K_2, H)-sim-(super)magic where H is isomorphic to three classes of graphs with varied symmetry: a cycle which is symmetric (both vertex-transitive and edge-transitive), a star which is edge-transitive but not vertex-transitive, and a path which is neither vertex-transitive nor edge-transitive. We discover forbidden subgraphs for the existence of (K_2, H)-sim-(super)magic graphs and classify classes of (K_2, H)-sim-(super)magic graphs. We also derive sufficient conditions for edge-(super)magic graphs to be (K_2, H)-sim-(super)magic and utilize such conditions to characterize some (K_2, H)-sim-(super)magic graphs.

Keywords: H-supermagic; (K_2, H)-sim-supermagic; edge-magic total; super edge-magic total

Citation: Ashari, Y.F.; Salman, A.N.M.; Simanjuntak, R. On Forbidden Subgraphs of (K_2, H)-Sim-(Super)Magic Graphs. *Symmetry* **2021**, *13*, 1346. https://doi.org/10.3390/sym13081346

Academic Editors: Juan Alberto Rodríguez Velázquez

Received: 17 June 2021
Accepted: 21 July 2021
Published: 26 July 2021

1. Introduction

In this paper, all graphs to be considered are finite, simple, and undirected. We write $[a, b]$ to define the set of consecutive integers $\{a, a+1, a+2, \ldots, b\}$, for any positive integers $a < b$. We denote two isomorphic graphs G and H with $G \cong H$. The *degree of vertex* x of G, denoted by $deg(x)$, is the number of vertices in G adjacent to x.

Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. An *edge-magic total labeling* (or EMT labeling for short) of a graph G is a bijection $\lambda : V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)| + |E(G)|\}$ with the property that there exists a constant k such that $\lambda(x) + \lambda(y) + \lambda(xy) = k$, for any edge $xy \in E(G)$. Then, G is said to be *edge-magic* (EMT) and k is called a *magic sum*. This notion was defined by Kotzig and Rosa [1], who called it *magic valuation*, and later rediscovered by Ringel and Lladó [2]. In [2], Ringel and Lladó conjectured that all trees are EMT. Since then, numerous papers associated with EMT labeling have been published.

In 1998, Enomoto et al. [3] introduced a special case of EMT labeling with the extra property that $\lambda(V(G)) = \{1, 2, \ldots, |V(G)|\}$. It is called a *super edge-magic total labeling* (SEMT labeling). A graph G that admits an SEMT labeling is said to be *super edge-magic* (SEMT). An SEMT labeling has a significant role in graph labeling because it is related to other types of labelings. Figueroa-Centeno et al. [4] found relationships between SEMT and well-known labelings such as harmonious, sequential, and cordial labelings. Bača et al. [5] established the relationship between SEMT and EMT labelings and (a, d)-edge-antimagic vertex labeling. Other relationships and comprehensive surveys about SEMT and EMT graphs can be found in [6–9].

The next Lemma states a necessary and sufficient condition of an SEMT graph. We frequently use this condition to construct SEMT labelings of some graphs.

Lemma 1 ([4])**.** *A graph G is SEMT if and only if there exists a bijective function $f : V(G) \rightarrow [1, |V(G)|]$ such that the set $S = \{f(u) + f(v)|uv \in E(G)\}$ consists of $|E(G)|$ consecutive integers. In such a case, f extends to an SEMT labeling of G with magic sum $k = |V(G)| + |E(G)| + s$, where $s = min(S)$.*

In [3], Enomoto et al. presented a necessary condition for an SEMT graph as stated in the following.

Lemma 2 ([3])**.** *If a graph G with order p and size q is SEMT, then $q \leq 2p - 3$.*

We call an SEMT graph with the maximum number of edges given by Lemma 2 a *maximal SEMT graph*. In [10], Macdougall and Wallis provide some properties of maximal SEMT graphs and construct some particular maximal SEMT graphs such as triangulations of v-cycle, generalized prisms, and graphs with large cliques. Sugeng and Xie [11] presented a construction to extend any non-maximal SEMT graph into a maximal SEMT graph by utilizing the adjacency matrix. Thus, it is interesting to ask the question of which other graphs are maximal SEMT.

Subsequently, Gutiérrez and Lladó [12] generalized the notion of EMT and SEMT into H-(super)magic labelings in 2005. Let G be a graph where each edge belongs to at least one subgraph isomorphic to a given graph H. In this case, G admits an *H-covering*. An *H-magic labeling of G* is a bijection g: $V(G) \cup E(G) \rightarrow \{1, 2, \ldots, |V(G)| + |E(G)|\}$ with the property that there exists a positive integer k such that $w_H(H') = \sum_{v \in V(H')} g(v) + \sum_{e \in E(H')} g(e) = k$, for every subgraph H' of G isomorphic to H. The H-magic labeling g of G with the extra property that $g(V(G)) = \{1, 2, \ldots, |V(G)|\}$ is called *H-supermagic labeling* of G. A graph G is an *H-magic* or *H-supermagic* if it has an H-magic labeling or H-supermagic labeling, respectively.

While working with H-magic graphs, we found labelings of graphs which are simultaneously H-magic and F-magic, for two non isomorphic graphs F and H. For instance, Figure 1 shows an example of a ladder $L_n = P_n \times K_2$ which is C_4-magic and C_{2m}-magic, for any $m \in [3, \lceil \frac{n}{2} \rceil]$, at the same time [13]. This leads us to generalize the concept of H-magic with two or more non-isomorphic covers.

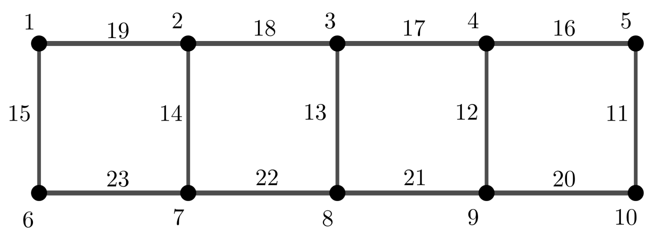

Figure 1. A C_4-supermagic and C_6-supermagic labelings of ladder.

Given two non-isomorphic graphs F and H, let G be a graph admitting an F-covering and H-covering simultaneously. An *(F, H)-simultaneously-magic labeling* of G, denoted by *(F, H)-sim-magic labeling*, is a bijective function f: $V(G) \cup E(G) \rightarrow \{1, 2, \ldots, |V(G)| + |E(G)|\}$ with the property that there exist two positive integers k_F and k_H (not necessarily the same) such that $w_f(F') = \sum_{v \in V(F')} f(v) + \sum_{e \in E(F')} f(e) = k_F$ and $w_f(H') = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e) = k_H$, for each subgraph F' of G isomorphic to F and each subgraph H' of G isomorphic to H. In such a case that $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$, we call f an *(F, H)-simultaneously-supermagic labeling*, denoted by (F, H)-sim-supermagic labeling. The graph G is said to be *(F, H)-sim-magic* or *(F, H)-sim-supermagic* if it has an (F, H)-sim-magic labeling or (F, H)-sim-supermagic labeling, respectively. By the definition of these notions, the construction of (F, H)-sim-(super)magic labelings of graphs can enlarge the collection of graphs that are known to be F-(super)magic and H-(super)magic.

In [13], we established the existence of a $(K_2 + H, 2K_2 + H)$-sim-supermagic labeling of a join product graph $G + H$ and a (C_4, H)-sim-supermagic labeling of a Cartesian product graph $G \times K_2$ where H is isomorphic to a ladder or an even cycle. We also presented the relationship between an α labeling of a tree T not isomorphic to a star and a (C_4, C_6)-sim-supermagic of the Cartesian product $T \times K_2$.

Since SEMT and EMT labelings are known to be related to other well-known graph labelings, in this paper we focus on the study of (K_2, H)-sim-(super)magic labelings; in particular for a graph H that is isomorphic to a path, a star, or a cycle. We denote a *path* on n vertices by P_n and a *cycle* on n vertices by C_n. A *star* S_n is a tree on $n + 1$ vertices with one vertex, called the *center*, having degree n and the remaining vertices having degree one.

An *automorphism* of a graph G is a permutation of $V(G)$ preserving adjacency. A graph G is said to be *vertex-transitive* if, for any two vertices u and w, there is an automorphism of G that maps u to w and it is said to be *edge-transitive* if, for any two edges u and w, there is an automorphism of G that maps u to w. If G is both vertex-transitive and edge-transitive, G is said to be *symmetric*. Recall that a cycle is symmetric; a *star* is edge-transitive but not vertex-transitive; and a path on at least 4 vertices is neither vertex-transitive nor edge-transitive. In other words, in this paper we study (K_2, H)-sim-(super)magic labelings for three classes of graphs H with varied symmetry.

Some of our results enlarge the collection of known (S)EMT and H-(super)magic graphs. To show this, in Section 2 we list some necessary or sufficient conditions for a graph to be H-(super)magic, for H isomorphic to a path, a star, or a cycle.

To recognize whether a graph is not (K_2, H)-sim-(super)magic, we determine *forbidden subgraphs* for (K_2, H)-sim-(super)magic graphs. In Sections 3–5 some forbidden subgraphs for (K_2, H)-sim-(super)magic labelings, where H is isomorphic to a path, a star, or a cycle, are presented. In those sections, we say that G is *H-free* if G does not contain H as a subgraph.

Additionally, in Section 3, we characterize (K_2, P_n)-sim-(super)magic graphs of small order and establish sufficient conditions for (K_2, P_n)-sim-(super)magic graphs. In Section 4, we characterize (K_2, S_n)-(super)magic graphs. In Section 5, we characterize (K_2, C_n)-(super)magic graphs of order $n \geq 3$ by establishing a relation between (S)EMT and C_n-(super)magic labelings and construct some cycles with chords that are (K_2, C_n)-(super)magic. Our constructions subsequently extend known maximal SEMT graphs and cycle-(super)magic graphs. In Section 5, we present sufficient conditions for an SEMT graph with order m to be (K_2, C_n)-sim-(super)magic for $n < m$.

2. Previous Results on *H*-(Super)Magic Labelings

In this paper, we first survey some known necessary conditions of H-(super)magic graphs for H isomorphic to a path and a star. These results are immediately necessary conditions for a (K_2, H)-sim-(super)magic graph. We also list some graphs known to be cycle-(super)magic.

In [12], it is proved that if G is a P_h-magic graph, $h > 2$, then G is C_h-free as stated in the following theorem.

Theorem 1 ([12])**.** *Let G be a P_h-magic graph, $h > 2$. Then G is C_h-free.*

A cycle on n vertices C_n with one pendant edge is denoted by C_n^{+1} (See Figure 2 for C_5^{+1}). Maryati et al. [14] gave the following necessary conditions for path-magic graphs.

Theorem 2 ([14])**.** *Let $n \geq 4$ be a positive integer.*

1. *If G is P_n-magic, then G is C_{n-1}^{+1}-free.*
2. *If G is P_n-magic, then G is C_{n+1}^{+1}-free.*

In [14,15], Maryati et al. provided another forbidden subgraph of path-magic graphs by defining an H_n graph. The H_n graph is a graph with $V(H_n) = \{v_{1,i}, v_{2,i} | i \in [1, 2n + 1]\}$

and $E(H_n) = \{v_{1,i}v_{1,i+1}, v_{2,i}v_{2,i+1} \,|i \in [1,2n]\} \cup \{v_{1,c}v_{2,c}|c = n+1\}$ (Figure 3 illustrates the graph H_3).

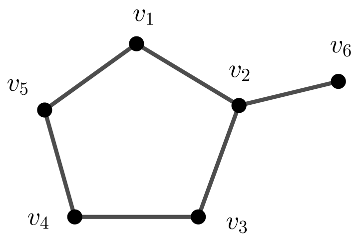

Figure 2. The cycle with one pendant edge C_5^{+1}.

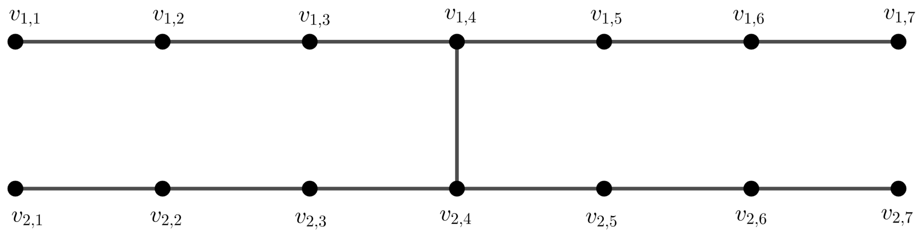

Figure 3. The graph H_3.

Theorem 3 ([14,15])**.** *Let $n \geq 3$ be a positive integer. If G is P_n-magic, then G is H_{n+2}-free.*

In [12], Gutiérrez and A. Lladó also established some necessary conditions of star-magic graphs by considering the degree of vertices.

Theorem 4 ([12])**.** *Let f be a S_h-magic labeling of a graph G with magic constant m_f. If the degree of vertex $x \in V(G)$ verifies $deg(x) > h$, then for every vertex y adjacent to x, we have $f(y) + f(xy) = \frac{1}{h}(m_f - f(x))$.*

Corollary 1 ([12])**.** *Let G be a S_h-magic graph with $h > 1$. Then, for every edge $e = xy$ of G, $min\{deg(x), deg(y)\} \leq h$.*

In the following theorems, we present some known classes of cycle-supermagic graphs, a more complete list can be found in [7]. We recall the definition of the graphs mentioned in the theorems. A *fan* F_n is a graph obtained from connecting a single vertex to all vertices in cycle P_n. A *wheel* W_n is a graph with $n+1$ vertices obtained from connecting a single vertex to all vertices in cycle C_n. For $k \geq 2$, a *windmill* $W(r,k)$ is a graph obtained by identifying one vertex in each of the k disjoint copies of the cycle C_r. For $n \geq 2$, a *ladder* L_n, is defined as $P_n \times K_2$, whose vertex set is $V(L_n) = V(P_n) \times V(K_2) = \{(x_i, y_j)|i \in [1,n]$ and $j \in [1,2]\}$ and edge set is $E(L_n) = \{(x_i, y_j)(x_{i+1}, y_j)|i \in [1, n-1]$ and $j \in [1,2]\} \cup \{(x_i, y_1)(x_i, y_2)|i \in [1,n]\}$. Illustrations of a wheel, a fan, and a ladder can be seen in Figure 4.

Theorem 5 ([16,17])**.** *For $n \geq 4$, the wheel W_n is C_3-supermagic.*

Theorem 6 ([16])**.** *For any two integers $k \geq 2$ and $r \geq 3$, the windmill $W(r,k)$ is C_r-supermagic.*

Theorem 7 ([18])**.** *Let $n \geq 4$ be a positive integer.*

1. *The fan F_n is C_m-supermagic for any integer $4 \leq m \leq \lfloor \frac{n+4}{2} \rfloor$;*
2. *The ladder L_n is C_m-supermagic for any positive integer $3 \leq m \leq \lfloor \frac{n}{2} \rfloor + 1$.*

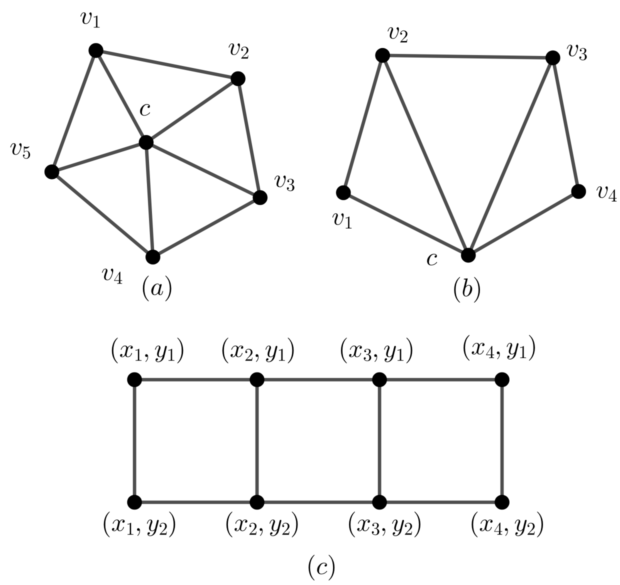

Figure 4. (**a**) The wheel W_5, (**b**) the fan F_4, and (**c**) the ladder L_4.

One important observation on H-magicness is the following.

Observation 1. *If G is H-magic then $G \bigcup nK_1$ is also H-magic.*

The converse of Observation 1 is not true. For example, $2P_3 \cup K_1$ is P_3-supermagic (as shown in Figure 5) but $2P_3$ is not P_3-magic (as a result of Theorem 1 in [14]).

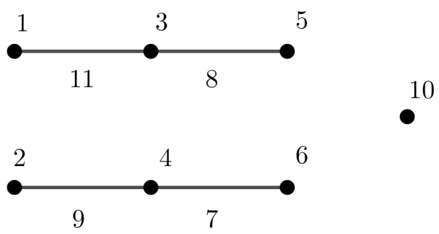

Figure 5. P_3-magic of $2P_3 \cup K_1$.

Due to this fact, in the rest of the paper, we restrict our observation to graphs with components not isomorphic to K_1.

3. (K_2, P_n)-Sim-Supermagic Labelings

In this section, we provide the collection of forbidden subgraphs and characterize a (K_2, P_n)-sim-supermagic graph.

Let $m \geq 3$ and $n \geq 2$ be two integers. We denote the edge sets of a path P_n and a cycle C_m as $E(P_n) = \{w_i w_{i+1} | i \in [1, n-1]\}$ and $E(C_m) = \{v_i v_{i+1} \,| i \in [1, m-1] \cup \{v_1 v_m\}$, respectively. An (m, n)-*tadpole* is a graph obtained by joining the end vertex w_1 of P_{n-1} to the vertex v_1 of C_m. Figure 6 shows the $(4, 5)$-tadpole graph.

We denote the star with n pendant edges as S_n. Consider the star S_3 with three pendant edges denoted by e_1, e_2, e_3. We define $S(S_3; e_1, e_2, e_3; n, 3, 3)$ as a subdivision of the star S_3 by replacing the edge e_1 with a path on n vertices and the remaining edges by paths on three vertices. Figure 7 illustrates the subdivided star $S(S_3; e_1, e_2, e_3; 5, 3, 3)$.

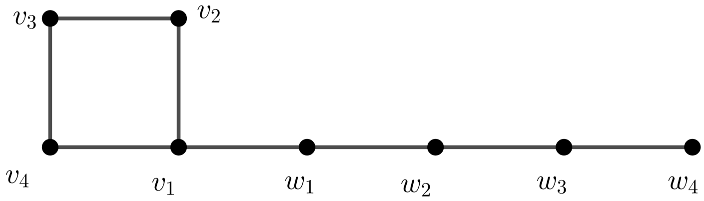

Figure 6. The $(4, 5)$-tadpole graph.

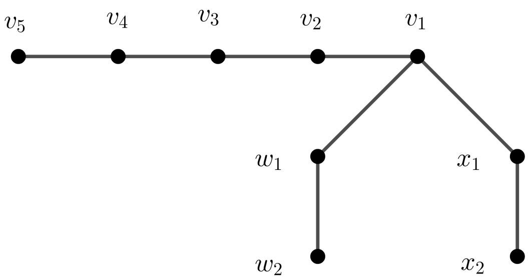

Figure 7. The $S(S_3; e_1, e_2, e_3; 5, 3, 3)$ graph.

In [19], Maryati et al. introduced a subgraph-amalgamation. For $n \geq 2$, let $\{G_i\}_{i=1}^n$ be a collection of graphs G_is where each G_i contains $H_i^* \cong H$ as a fixed subgraph and let $\mathcal{H} = \{H_i^*\}_{i=1}^n$ be the collection of H_i^*s. The H-amalgamation of $\{G_i\}_{i=1}^n$, denoted by $Amal(G_1, G_2, \ldots, G_n; \mathcal{H}; n)$, is a graph constructed from identifying the H_i^* of each G_i. If G_i is isomorphic to a given graph G, we write the H-amalgamation as $Amal(G; \mathcal{H}; n)$.

Let G_1 be an (m, n)-tadpole containing a subgraph $H_1^* = v_2v_3$ isomorphic to P_2; let G_2 be a path P_k, whose edge set is $E(P_k) = \{x_ix_{i+1} | i \in [1, k-1]\}$, containing a subgraph $H_2^* = x_3x_4$ isomorphic to P_2; and $\mathcal{H} = \{H_i^*\}_{i=1}^2$. Figure 8 illustrates the $Amal((5,3)$-tadpole,$P_8; \mathcal{H}; 2)$.

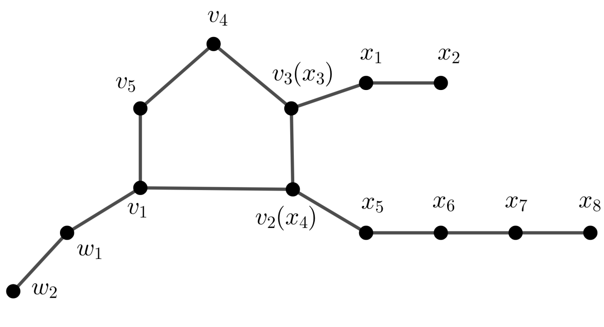

Figure 8. The $Amal((5,3)$-tadpole,$P_8; \mathcal{H}; 2)$ graph.

The next theorem stated forbidden subgraphs of (K_2, P_n)-sim-(super)magic graphs.

Theorem 8. *If G is (K_2, P_n)-sim-(super)magic, then G is H-free where*

1. $H \cong C_m$, *for any $n \geq 4$ and $m \in [n-1, n+1]$;*
2. $H \cong H_{n+2}$, *for any $n \geq 3$;*
3. $H \cong P_{n+1}$, *for any $n \geq 3$;*
4. $H \cong S(S_3; e_1, e_2, e_3; n-2, 3, 3)$, *for any $n \geq 5$;*
5. $H \cong (k, n-k)$*-tadpole, for any $n > 4$ and $k \in [3, n-2]$;*
6. $H \cong Amal((m, 3)$*-tadpole, $P_{n-m}; \mathcal{H}; 2)$, for any $n \geq 7$, and $m \in [3, n-4]$.*

Proof. The case where $H \cong C_m$, for any $n \geq 4$ and $m \in [n-1, n+1]$, is an immediate consequence of Theorems 1 and 2; and the case where $H \cong H_{n+2}$, for any $n \geq 3$, is an immediate consequence of Theorem 3. The rest of the cases are proven as follows.

Case 3. $H \cong P_{n+1}$, for any $n \geq 3$.

Suppose that G is a (K_2, P_n)-sim-(super)magic graph and G is not P_{n+1}-free. Let f be a (K_2, P_n)-sim-(super)magic labeling of G. Consider two subgraphs isomorphic to P_n with edges $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n$ and $v_2v_3, v_3v_4, \ldots, v_nv_{n+1}$. Since G is (K_2, P_n)-magic,

$$\sum_{i=1}^{n} f(v_i) + \sum_{i=1}^{n-1} f(v_iv_{i+1}) = \sum_{i=2}^{n+1} f(v_i) + \sum_{i=2}^{n} f(v_iv_{i+1}). \tag{1}$$

By eliminating $\sum_{i=2}^{n} f(v_i) + \sum_{i=2}^{n-1} f(v_iv_{i+1})$ in both sides of Equation (1), we have

$$f(v_1) + f(v_1v_2) = f(v_nv_{n+1}) + f(v_{n+1}).$$

However, $\sum_{i=1}^{2} f(v_i) + f(v_1v_2) = \sum_{i=n}^{n+1} f(v_i) + f(v_nv_{n+1})$. This clearly forces $f(v_2) = f(v_n)$, a contradiction.

Case 4. $H \cong S(S_3; e_1, e_2, e_3; n-2, 3, 3)$, for any $n \geq 5$.

Assume to the contrary that G is (K_2, P_n)-sim-(super)magic and G contains $S(S_3; e_1, e_2, e_3;\ n-2, 3, 3)$ as a subgraph. Let f be a (K_2, P_n)-sim-(super)magic labeling of G. Consider a subgraph H isomorphic to $S(S_3; e_1, e_2, e_3; n-2, 3, 3)$. Label the vertex set $V(H) = \{v_i | i \in [1, n-2]\} \cup \{w_i, x_i | i \in [1, 2]\}$ and the edge set $E(H) = \{v_iv_{i+1} | i \in [1, n-3]\} \cup \{v_1w_1, v_1x_1, w_1w_2, x_1x_2\}$. There exist two paths isomorphic to P_n with edges $v_{n-2}v_{n-3}, v_{n-3}v_{n-4}, \ldots, v_2v_1, v_1w_1, w_1w_2$ and $v_{n-2}v_{n-3}, v_{n-3}v_{n-4}, \ldots, v_2v_1, v_1x_1, x_1x_2$. As f is a (K_2, P_n)-sim-(super)magic labeling, we have $\sum_{i=1}^{n-2} f(v_i) + \sum_{i=1}^{2} f(w_i) + f(v_1w_1) + \sum_{i=1}^{n-3} f(v_iv_{i+1}) + f(w_1w_2) = \sum_{i=1}^{n-2} f(v_i) + \sum_{i=1}^{2} f(x_i) + \sum_{i=1}^{n-3} f(v_iv_{i+1}) + f(v_1x_1) + f(x_1x_2)$. Thus, we obtain $f(v_1w_1) = f(v_1x_1)$, a contradiction.

Case 5. $H \cong (k, n-k)$-tadpole, for any $n > 4$ and $k \in [3, n-2]$.

Suppose that G is (K_2, P_n)-sim-(super)magic and contains $(k, n-k)$-tadpole as a subgraph. Let f be a (K_2, P_n)-sim-(super)magic labeling of G. Next, let k be an arbitrary positive integer with $k \in [3, n-2]$. Consider a subgraph H isomorphic to $(k, n-k)$-tadpole. Denote the vertex set $V(H) = \{v_i, w_j | i \in [1, k], j \in [1, n-k]\}$ and the edge set $E(H) = \{v_iv_{i+1} | i \in [1, k-1]\} \cup \{v_1v_k, v_1w_1\} \cup \{w_jw_{j+1} | j \in [1, n-k-1]\}$. Consider two paths isomorphic to P_n with edges $w_{n-k}w_{n-k-1}, w_{n-k-1}w_{n-k-2}, \ldots, w_2w_1, w_1v_1, v_1v_2, v_2v_3, \ldots, v_{k-1}v_k$ and $w_{n-k}w_{n-k-1}, w_{n-k-1}w_{n-k-2}, \ldots, w_2w_1,\ w_1v_1,\ v_1v_k, v_kv_{k-1}, \ldots,\ v_3v_2$. Since G is (K_2, P_n)-sim-(super)magic, $\sum_{i=1}^{n-k} f(w_i) + \sum_{i=1}^{k} f(v_i) + \sum_{i=1}^{n-k-1} f(w_iw_{i+1}) + f(v_1w_1) + \sum_{i=1}^{k-1} f(v_iv_{i+1}) = \sum_{i=1}^{n-k} f(w_i) + \sum_{i=1}^{k} f(v_i) + \sum_{i=1}^{n-k-1} f(w_iw_{i+1}) + f(v_1w_1) + f(v_1v_k) + \sum_{i=2}^{k-1} f(v_iv_{i+1})$. As a result, we have $f(v_1v_k) = f(v_1v_2)$, a contradiction.

Case 6. $H \cong Amal((m, 3)\text{-(tadpole)}, P_{n-m}; \mathcal{H}; 2)$, for any $n \geq 7$ and $m \in [3, n-4]$.

Assume to the contrary that G is (K_2, P_n)-sim-(super)magic and contains a subgraph isomorphic to $Amal((m, 3)\text{-(tadpole)}, P_{n-m}; \mathcal{H}; 2)$. Let f be a (K_2, P_n)-sim-(super)magic labeling of G. Then, let m be an arbitrary positive integer with $m \in [3, n-4]$. Consider a subgraph H of G isomorphic to $Amal((m, 3)\text{-(tadpole)}, P_{n-m}; \mathcal{H}; 2)$. Denote the vertex set $V(H) = \{v_i | i \in [1, m]\} \cup \{w_i | i \in [1, n-m-2]\} \cup \{x_i, y_i | i \in [1, 2]\}$ and the edge set $E(H) = \{v_iv_{i+1} | i \in [1, m-1]\} \cup \{w_iw_{i+1} | i \in [1, n-m-3]\} \cup \{v_1v_m, v_1w_1, v_2x_1, v_my_1, x_1x_2, y_1y_2\}$. Consider two paths isomorphic to P_n with edges $w_{n-m-2}w_{n-m-3}$, $w_{n-m-3}w_{n-m-4}$, $\ldots$, w_1v_1, v_1v_2, v_2v_3, $\ldots$, $v_{m-1}v_m$, v_my_1, y_1y_2 and $w_{n-m-2}w_{n-m-3}, w_{n-m-3}w_{n-m-4}, \ldots, w_1v_1, v_1v_m, v_mv_{m-1}, \ldots, v_3v_2, v_2x_1, x_1x_2$. As G is (K_2, P_n)-sim-(super)magic, we have $\sum_{i=1}^{n-m-2} f(w_i) + f(w_1v_1) + \sum_{i=1}^{m} f(v_i) + \sum_{i=1}^{2} f(y_i) + \sum_{i=1}^{n-m-3} f(w_iw_{i+1}) + \sum_{i=1}^{m-1} f(v_iv_{i+1}) + f(v_my_1) + f(y_1y_2) = \sum_{i=1}^{2} f(x_i) + \sum_{i=1}^{m} f(v_i) + \sum_{i=1}^{n-m-2} f(w_i) + \sum_{i=1}^{n-m-3} f(w_iw_{i+1}) + \sum_{i=2}^{m-1} f(v_iv_{i+1}) + f(v_1v_m) + f(v_2x_1) + f(x_1x_2) + f(w_1v_1)$. Thus, we have $f(x_1) = f(y_1)$, a contradiction. □

We remark that if G is (K_2, P_n)-sim-(super)magic, then P_n is the longest path of G. Notice that, for $n \in [3, 4]$, $S(S_3; e_1, e_2, e_3; n-2, 3, 3)$ contains P_5 as a subgraph. By Theorem 8, such graphs are not (K_2, P_n)-sim-supermagic. The converse of Theorem 8 is not true as shown in the following example.

Example 1. *The graph mP_3 is not (K_2, P_3)-sim-(super)magic for any integer $m \geq 3$.*

Proof. Suppose that there exists a (K_2, P_3)-sim-(super)magic labeling on mP_3. Let X_i be the set of the internal vertex label in a P_3^i for $i \in [1, m]$. Clearly $|X_i| = 1$. For each edge $xy \in P_3^i$, the xy-weight, $f(xy) + f(x) + f(y) = k$. Thus, the P_3-weight of P_3^i is $2k - \sum X_i$ for every $i \in [1, m]$. Consequently, $\sum X_i$ should be a constant for every $i \in [1, m]$, a contradiction. □

Problem 1. *What are the other forbidden subgraphs of (K_2, P_n)-sim-(super)magic graph?*

As a consequence of Theorem 8 where $H \cong P_{n+1}$, for any integer $n \geq 3$, we have the following two results.

Corollary 2. *Let $n \geq 3$ be a positive integer and G be a graph that admits P_n-covering. If G is (K_2, P_n)-sim-(super)magic, then $n \geq diam(G) + 1$.*

Corollary 3. *Let $n \geq 3$ be a positive integer and G be a graph that admits P_n-covering. If G is (K_2, P_n)-sim-(super)magic, then G is C_h-free for any $h > n$.*

By the previous two corollaries, Theorem 8, and Example 1, we have the following corollaries.

Corollary 4. *Let $n \in \{3, 4\}$ and G be a graph that admits P_n-covering. If G is (K_2, P_n)-sim-(super)magic, then G is a forest. In particular, if G is (K_2, P_n)-sim-(super)magic, then G is a tree.*

Let $n \geq 2$ be a positive integer. In [12], it is proved that the star S_n is S_m-supermagic for each $m < n$. Moreover, the S_m-supermagic labeling of S_n in [12] is also an SEMT labeling of S_n. Combining with Example 1 and Corollary 4, we obtain the following.

Corollary 5. *A graph G is (K_2, P_3)-sim-(super)magic if and only if G is isomorphic to the star S_n for any positive integer $n \geq 3$.*

A *caterpillar* $S_{n_1,n_2,\ldots,n_k}$ is a graph derived from a path P_k, $k \geq 2$, where the vertex $w_i \in V(P_k)$ is adjacent to $m_i \geq 0$ leaves, $i \in [1, k]$. A special case of caterpillars when $k = 2$, $m_1 \geq 1$, and $m_2 \geq 1$ is called a *double star* S_{m_1,m_2}. An illustration of the double star $S_{5,3}$ and a (K_2, P_4)-sim-supermagic labeling on $S_{5,3}$ can be seen in Figure 9. Since Kotzig and Rosa [1] have proved that all caterpillars are SEMT, utilizing Corollary 4, we have the following.

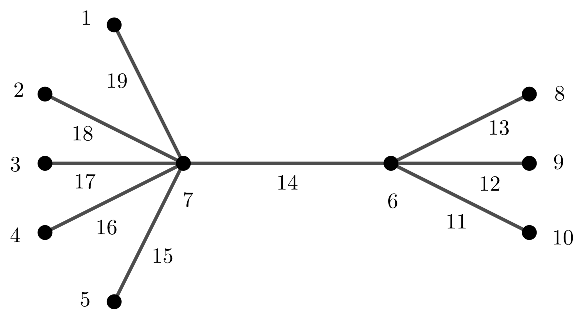

Figure 9. A (K_2, P_4)-sim-supermagic labeling of the double star $S_{5,3}$.

Corollary 6. *A connected graph G is (K_2, P_4)-sim-(super)magic if and only if G is isomorphic to a double star $S_{m,n}$ for any two positive integers m and n.*

Problem 2. *Characterize (K_2, P_n)-sim-(super)magic graphs for any $n \geq 5$.*

We conclude this section by presenting sufficient conditions for an (S)EMT graph to be (K_2, P_n)-sim-(super)magic.

Lemma 3. *Let k and n be two positive integers. Let G be a graph of order at least $n+1$ that admits P_n-covering. Let $\{P_n^i\}_{i=1}^k$ be the family of all subgraphs of G isomorphic to P_n and let $\sum X_i$ be the sum of all internal vertices labels in P_n^i for every $i \in [1, k]$. If f is an (S)EMT labeling in G such that $\sum X_i$ is constant, for each $i \in [1, k]$, then G is (K_2, P_n)-sim-(super)magic.*

Proof. Let m_f be the magic sum of the labeling. Let $i \neq j$ be two positive integers in $[1, k]$. Consider two arbitrary paths P_n^i and P_n^j in $\{P_n^i\}_{i=1}^k$. Thus, $\sum X_i = \sum X_j$. Hence, we have the following:

$$\begin{aligned} w(P_n^i) &= \sum_{v \in V(P_n^i)} f(v) + \sum_{e \in E(P_n^i)} f(e) \\ &= (n-1)\, m_f - \sum X_i \\ &= (n-1)\, m_f - \sum X_j \\ &= \sum_{v \in V(P_n^j)} f(v) + \sum_{e \in E(P_n^j)} f(e) \\ &= w(P_n^j) \end{aligned}$$

As a result, the sum of all edges and vertices labels associated to a subgraph of G isomorphic to P_n is a constant. Therefore, G is a P_n-(super)magic. Since f is simultaneously SEMT and P_n-(super)magic, G is (K_2, P_n)-sim-(super)magic. □

As an immediate consequence of Lemma 3, we have the following special cases of caterpillars that are (K_2, P_n)-sim-magic. The *broom* $B_{m,n}$ is defined as a graph isomorphic to the caterpillar $S_{n_1, n_2, \ldots, n_{m-n}}$ where $n_1 = n_2 = \cdots = n_{m-n-1} = 0$ and $n_{m-n} = n$. The *double broom* DB_{m,k_1,k_2} is a graph isomorphic to the caterpillar $S_{n_1, n_2, \ldots, n_{m-k_1-k_2}}$ where $n_1 = k_1$, $n_2 = n_3 = \ldots = n_{m-k_1-k_2-1} = 0$, and $n_{m-k_1-k_2} = k_2$. Figure 10 illustrates the broom $B_{11,6}$ and the double broom $DB_{14,3,6}$.

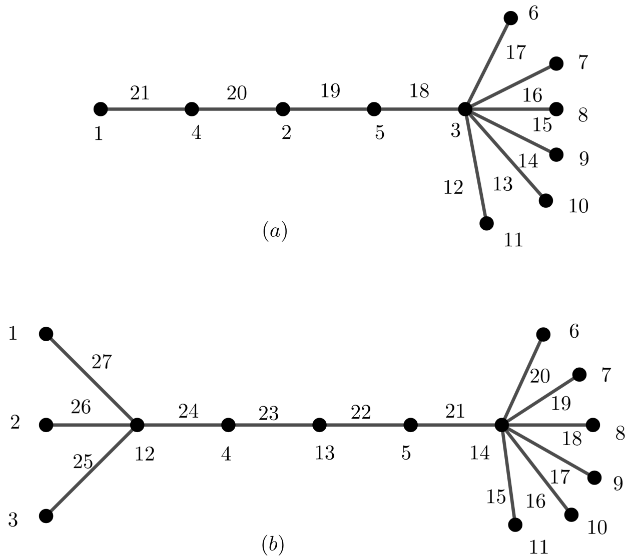

Figure 10. (**a**) A (K_2, P_6)-sim-supermagic labeling of the broom $B_{11,6}$ and (**b**) A (K_2, P_7)-sim-supermagic labeling of the double broom $DB_{14,3,6}$.

Corollary 7. *Let n_1, n_2, and m be three positive integers at least two and $n \geq 3$. The broom $B_{m+n-1,m}$ and the double broom $DB_{n+n_1+n_2-2,n_1,n_2}$ are (K_2, P_n)-sim-magic.*

Proof. It is known that all caterpillars are edge magic [1]. Moreover, all subgraphs isomorphic to P_k have the same internal vertices. This completes the proof. □

Figure 10 illustrates (K_2, P_n)-sim-supermagic labelings of the broom $B_{11,6}$ and the double broom $DB_{14,3,6}$ for $n = 6$ and $n = 7$, respectively.

4. A (K_2, S_n)-Sim-Supermagic Labelings

In this section, we characterize (K_2, S_n)-sim-(super)magic graphs. Clearly, necessary conditions of S_n-magic graphs in Theorem 4 and Corollary 1 are also necessary conditions of (K_2, S_n)-sim-(super)magic graphs. In the following Lemma, we strengthen the degree condition of Corollary 1 for (K_2, S_n)-sim-(super)magic graphs.

Lemma 4. *Let $n \geq 2$ be a positive integer and G be a (K_2, S_n)-sim-(super)magic. Then, there is only one vertex x of G with $deg(x) \geq n$.*

Proof. Suppose that there are two vertices v and w in $V(G)$ such that $deg(v) \geq n$ and $deg(w) \geq n$. Let f be a (K_2, S_n)-sim-(super)magic labeling of G. Hence, there exist two positive integers k_1 and k_2 such that each edge $xy \in E(G)$ satisfies $f(x) + f(y) + f(xy) = k_1$ and each subgraph H of G isomorphic to S_n satisfies $\sum_{u \in V(H)} f(u) + \sum_{e \in E(H)} f(e) = k_2$. Consider two arbitrary stars with center v and w that are isomorphic to S_n as S^1 and S^2. Thus,

$$\begin{aligned} \sum_{v \in V(S^1)} f(v) + \sum_{e \in S^1} f(e) &= \sum_{w \in V(S^2)} f(w) + \sum_{e \in E(S^2)} f(e) \\ nk_1 - (n-1)f(v) &= nk_1 - (n-1)f(w). \end{aligned}$$

As a result, we have $f(v) = f(w)$, a contradiction. □

Recall that Gutiérrez and Lladó [12] proved the following theorem. The labeling in the proof of the theorem will be utilized to characterize (K_2, S_n)-sim-supermagic graphs.

Theorem 9 ([12]). *The star S_m is S_n-supermagic for any $n \in [1, m]$.*

Proof. Denote the vertex set of S_m by $V(S_m) = \{v_1, v_2, \ldots, v_m, v_{m+1}\}$, where v_{m+1} is the maximum degree vertex, and the edge set of S_m by $E(S_m) = \{v_{m+1}v_i | i \in [1, m]\}$. Define a bijection $f : V(S_m) \cup E(S_m) \to [1, 2m+1]$ with $f(v_i) = i$ and $f(v_{m+1}v_i) = 2(m+1) - i$, for any $i \in [1, m]$, and $f(v_{m+1}) = m+1$. Thus, $f(V(S_m)) = [1, m+1]$. We can verify that $w(H) = \sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e) = (m+1) + n(i + (2(m+1) - i)) = (m+1)(2n+1)$ (constant) for every subgraph H of S_m isomorphic to S_n. Therefore, S_m is S_n-supermagic for each $n \in [1, m]$. □

Now we are ready to characterize (K_2, S_n)-sim-supermagic graphs.

Theorem 10. *Let $n \geq 1$ be a positive integer. A graph G is (K_2, S_n)-sim-supermagic if and only if G is isomorphic to the star S_m for $m > n$.*

Proof. ($\Leftarrow$) First, we prove that, for $m > n$, the star S_m is (K_2, S_n)-sim-supermagic. Recall the S_n-supermagic labeling of S_m in the proof of Theorem 9, where $w(v_{n+1}v_i) = f(v_{n+1}) + f(v_i) + f(v_{n+1}v_i) = n + 1 + i + 2(m+1) - i = n + 1 + 2(m+1)$ (constant), for each edge $v_{n+1}v_i$ in $E(S_m)$. Hence, S_m is (K_2, S_n)-sim-supermagic for $m > n$.

($\Rightarrow$) Conversely, we prove that if G is (K_2, S_n)-sim-supermagic, then G is isomorphic to the star S_m for $m > n$. Clearly, a connected graph G with order two and three is isomorphic to S_1 and S_2, respectively. Then, consider G with order at least four. Suppose to the contrary that G is not isomorphic to any star $S_m, m > n$. Let e be an arbitrary edge in G. Suppose that e belongs to S^*, a subgraph isomorphic to S_n, where e is incident with c, the center of S^*. Since G is not isomorphic to a star, there exists another edge e' which is not incident with c. Since G admits S_n-covering, then e' belongs to a subgraph that is isomorphic to S_n where the center is not c, a contradiction by Lemma 4. □

We remark that by considering $n = 2$, we can derive another proof of Corollary 5 from Theorem 10.

5. A (K_2, C_n)-Sim-Supermagic Labelings

In this section, we list some forbidden subgraphs and some (K_2, C_n)-sim-(super)-magic graphs. We start by presenting results for (K_2, C_n)-sim-(super)magic graphs of order n by considering the relation between two well-known magic labelings: (S)EMT and C_n-(super)magic.

Lemma 5. *Let G be a graph of order n admitting a C_n covering. If G is (S)EMT then G is C_n-(super)magic.*

Proof. Let f be an (S)EMT labeling of G. Thus, there exists a positive integer k_1 such that $f(x) + f(y) + f(xy) = k_1$ for each edge xy in $E(G)$. Denote $\{x_i | i \in [1, n]\}$ as the set of vertices in G. Define a bijection $g : V(G) \cup E(G) \rightarrow [1, |V(G)| + |E(G)|]$ with $g(x) = f(x)$ for all $x \in$V(G)∪$E(G)$. Consider an arbitrary subgraph C isomorphic to C_n. Since the label of each vertex x is counted twice in $w(C) = \sum_{v \in V(C)} f(v) + \sum_{e \in E(C)} f(e)$, then $w(C) = nk_1 - \sum_{i=1}^{n} f(x_i)$, a constant. Therefore, G is C_n-(super)magic. □

The converse of Lemma 5 is not true since K_4 is C_4-(super)magic, although it is known that K_4 is neither EMT [1,3] nor SEMT [3] (See Figure 11). However, it is clear that we have the following necessary and sufficient condition for a graph of order n to admit a (K_2, C_n)-sim-(super)magic labeling.

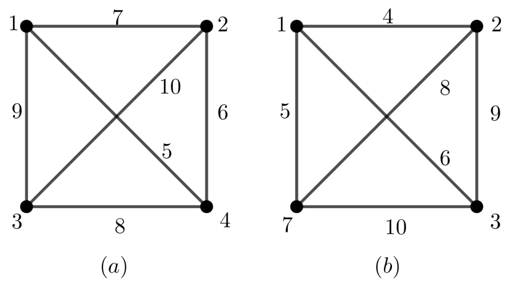

Figure 11. (**a**) A C_4-supermagic labeling of K_4 and (**b**) A C_4-magic labeling of K_4.

Corollary 8. *Let G be a graph order n admitting a C_n covering. G is (S)EMT if and only if G is (K_2, C_n)-sim-(super)magic.*

It is known that the complete graph K_n is EMT if and only if $n = 3, 5, 6$ [1]. Since each pair of vertices in K_n are adjacent, the number of subgraphs of K_n isomorphic to C_n is the number of n-cycles in the symmetric group S_n, which is $\frac{n!}{n} = (n-1)!$ Thus, the number of subgraphs of K_5 and K_6 isomorphic to C_5 and C_6 is 24 and 120, respectively.

Corollary 9. *Let $n > 3$ be a positive integer. A complete graph K_n is (K_2, C_n)-sim-magic if and only if $n = 5$ or $n = 6$.*

Proof. $(\Leftarrow)$ Recall the known EMT labeling f in K_n for $n = 5$ or 6 [1]. By Lemma 5, f is a C_n-magic labeling. This gives K_n as (K_2, C_n)-sim-magic for $n = 5$ or 6.

$(\Rightarrow)$ Conversely, it is immediately known from the fact that K_n is not EMT according to Kotzig and Rosa [1]. □

Figure 12 shows (K_2, C_5)-sim-supermagic and (K_2, C_6)-sim-supermagic graphs.

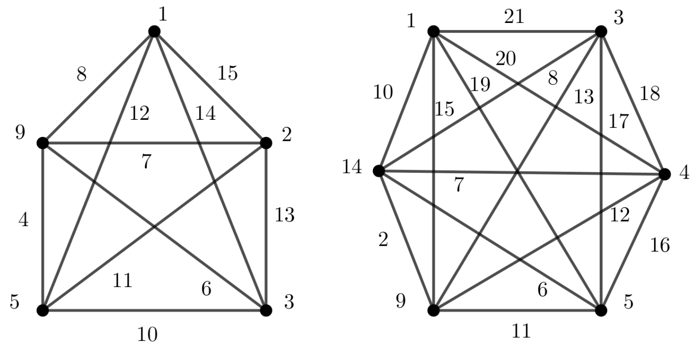

Figure 12. A (K_2, C_n)-sim-supermagic labeling for $n \in [5, 6]$.

Kotzig and Rosa [1] proved that the complete bipartite graph $K_{m,n}$ is EMT for all m and n. Philips et al. [20] constructed an EMT labeling of the wheel W_n for $n \equiv 0, 1$, or $2 (\mathrm{mod}\ 4)$. By Lemma 5, we have the following Corollary.

Corollary 10. *Let $n \geq 3$ be a positive integer.*

1. *$K_{n,n}$ is (K_2, C_{2n})-sim-magic;*
2. *W_n is (K_2, C_{n+1})-sim-magic for $n \equiv 0, 1$, or $2 (\mathrm{mod}\ 4)$.*

In the next two theorems, we consider a (K_2, C_m)-sim-supermagic labeling of a cycle with chords. A *chord* is an edge joining two non-adjacent vertices in a cycle. An n-power of graph G^n is a graph with the vertex set $V(G^n) = V(G)$ and any two vertices are adjacent when their distance in G is at most n. Recall from Lemma 2 that C_n^2 is not SEMT, so it is if we remove at most two edges from C_n^2. Thus, it is interesting to construct a maximal SEMT graph, where the number of edges is equal to the upper bound of inequality in Lemma 2, from C_n^2.

Let $n \geq 3$ be a positive integer and $\{x_i | i \in [1, n]\}$ be the vertex set of the cycle C_n. Let $E = \{x_{\lfloor \frac{n}{2} \rfloor} x_{\lfloor \frac{n}{2} \rfloor + 2}, x_{n-1}\, x_1, x_n\, x_2\}$ be the set of three edges in C_n^2. We define the cycle with chords CC_n^1 where the vertex set is $V(C_n)$ and the edge set is $E(C_n^2) \setminus E$. It is clear that CC_n^1 admits a C_n-covering for every odd integer $n \geq 7$ and we have the following theorem.

Theorem 11. *Let $n \geq 7$ be an odd integer. A cycle with chords CC_n^1 is (K_2, C_n)-sim-supermagic.*

Proof. Let $\{x_i | i \in [1, n]\}$ be the vertex set of CC_n^1. Define a bijection $f : V(CC_n^1) \to [1, |V(CC_n^1)|]$ as $f(x_i) = i$, for $i \in [1, n]$. Thus, for each edge $x_i x_j \in E(CC_n^1)$, we have

1. $f(x_i) + f(x_{i+1}) = 2i + 1$, for each $i \in [1, n-1]$;
2. $f(x_n) + f(x_1) = n + 1$;
3. $f(x_i) + f(x_j) = i + i + 2 = 2i + 2$, for $j = (i + 2) \mod n$ and $i \in [1, n-2]$.

Consequently, $3 \leq f(x_i) + f(x_j) \leq 2n - 1$ and the set $S = \{f(x_i) + f(x_j) | x_i x_j \in E(G)\}$ consists of $|E(G)|$ consecutive integers. By Lemma 1, CC_n^1 is SEMT and f is the SEMT labeling with magic sum $k = |V(G)| + |E(G)| + min(S) = n + 2n - 3 + 3 = 3n$. By Lemma 5, f is also a C_n-supermagic labeling of CC_n^1. This concludes that CC_n^1 is (K_2, C_n)-sim-supermagic. □

Figure 13a illustrates a (K_2, C_7)-sim-supermagic labeling of CC_7^1.

Let $n \geq 8$ be an even integer. Let $E^* = \{\, x_{\frac{n}{2}-1} x_{\frac{n}{2}+1},\ x_{\frac{n}{2}} x_{\frac{n}{2}+2},\ x_{\frac{n}{2}+2}\, x_{\frac{n}{2}+4},\ x_{n-1}\, x_1,$ $x_n\, x_2\}$. We define the cycle with chords CC_n^2 as a graph where the vertex set is $V(C_n)$ and the edge set is $E(C_n^2) \setminus E^* \bigcup \{x_{\frac{n}{2}-1} x_{\frac{n}{2}+2},\ x_{\frac{n}{2}+1} x_{\frac{n}{2}+4}\}$. Such a cycle with chords admits C_n-covering for each $n \geq 8$ an even integer.

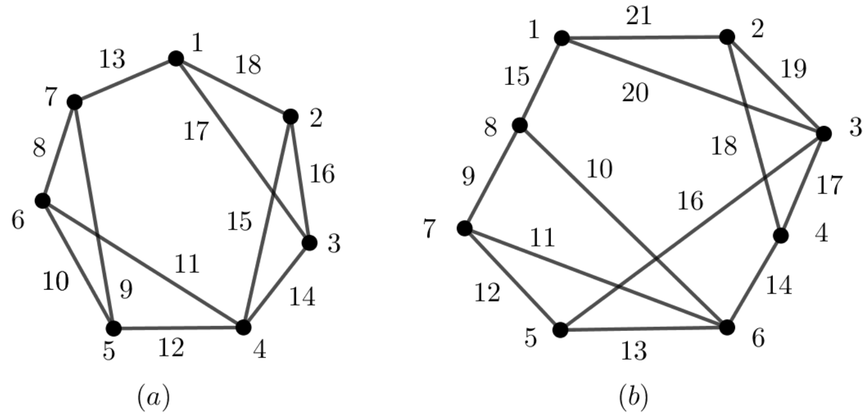

Figure 13. (**a**) A (K_2, C_7)-sim-supermagic labeling CC_7^1 and (**b**) A (K_2, C_8)-sim-supermagic labeling of CC_8^2.

Theorem 12. *Let $n \geq 8$ be an even integer. A cycle with chords CC_n^2 is (K_2, C_n)-sim-supermagic.*

Proof. Let $\{x_i | i \in [1, n]\}$ be the vertex set of CC_n^2. Define a bijection $f : V(CC_n^2) \to [1, |V(CC_n^2)|]$ as follows.

1. $f(x_i) = i$, for every $i \in [1, \frac{n}{2}]$ and $i \in [\frac{n}{2} + 3, n]$;
2. $f(x_{\frac{n}{2}+1}) = \frac{n}{2} + 2$;
3. $f(x_{\frac{n}{2}+2}) = \frac{n}{2} + 1$.

For each $x_i x_j \in E(CC_n^2)$, we have

1. $f(x_i) + f(x_{i+1}) = 2i + 1$, for each $i \in [1, \frac{n}{2} - 1]$ and $i \in [\frac{n}{2} + 2, n - 1]$;
2. $f(x_{\frac{n}{2}}) + f(x_{\frac{n}{2}+1}) = n + 2$;
3. $f(x_{\frac{n}{2}+1}) + f(x_{\frac{n}{2}+2}) = n + 3$;
4. $f(x_n) + f(x_1) = n + 1$;
5. $f(x_i) + f(x_{i+2}) = 2i + 2$, for each $x_i x_{i+2} \in E(CC_n^2)$, $i \in [1, \frac{n}{2}]$ and $i \in [\frac{n}{2} + 3, n - 2]$;
6. $f(x_{\frac{n}{2}+1}) + f(x_{\frac{n}{2}+3}) = n + 5$;
7. $f(x_{\frac{n}{2}-1}) + f(x_{\frac{n}{2}+2}) = n$;
8. $f(x_{\frac{n}{2}+1}) + f(x_{(\frac{n}{2}+4)}) = n + 6$.

It can be counted that $3 \leq f(x_i) + f(x_j) \leq 2n - 1$ and the set $S = \{f(x_i) + f(x_j) | x_i x_j \in E(G)\}$ consists of $|E(G)|$ consecutive integers. By Lemma 1, CC_n^2 is SEMT and f is the SEMT labeling with magic sum $k = |V(CC_n^2)| + |E(CC_n^2)| + min(S) = n + 2n - 3 + 3 = 3n$. By Lemma 5, f is also a C_n-supermagic labeling of CC_n^2. This concludes that CC_n^2 is (K_2, C_n)-sim-supermagic. □

Figures 13b shows a (K_2, C_8)-sim-supermagic labeling of cycle with chords CC_8^2.

In addition to maximal SEMT graphs construction, we remark that Theorems 11 and 12 also enlarge the classes of graphs known to be C_n-supermagic and SEMT.

Notice that up to Theorem 12 we only consider (K_2, C_n)-sim-supermagic graphs of order n. Therefore, it is interesting to ask whether an (S)EMT graph G of order n can admit a C_m-(super)magic labeling, for $m < n$. We start by presenting some forbidden subgraphs of (K_2, C_n)-sim-(super)magic graphs, for $n \geq 3$.

Theorem 13. *If G is (K_2, C_n)-sim-(super)magic, then G is H-free, where*

1. *$H \cong Amal(C_n; P_{n-1}; 2)$, for any $n \geq 3$;*
2. *$H \cong Amal(C_n; P_{n-2}; 2)$, for any $n \geq 3$.*

Proof. Suppose that G is (K_2, C_n)-sim-(super)magic and G is not H-free. Then, G contains a subgraph that is isomorphic to H. Let f be a (K_2, C_n)-sim-(super)magic labeling of G, such that there exist two positive integers k_1 and k_2, satisfying $f(x) + f(y) + f(xy) = k_1$ and $\sum_{v\in V(C)} f(v) + \sum_{e\in E(C)} f(e) = k_2$, for each edge $xy \in E(G)$ and for each subgraph C isomorphic to C_n, respectively. We consider the following two cases.

Case 1. $H \cong Amal(C_n; P_{n-1}; 2)$.

Consider a subgraph $H \cong Amal(C_n; P_{n-1}; 2)$ of a graph G. Denote the vertices in $Amal(C_n; P_{n-1}; 2)$ by $\{v_n\} \cup \{u_i | i \in [1, n-1]\} \cup \{w_n\}$ such that the edge set is $\{u_iu_{i+1} | i \in [1, n-2]\} \cup \{v_nu_i, w_nu_i | i \in \{1, n-1\}\}$. There are two cycles C^1 and C^2 isomorphic to C_n with $V(C^1) = v_n, u_1, u_2, \ldots, u_{n-2}, u_{n-1}$ and $V(C^2) = w_n, u_1, u_2, \ldots, u_{n-2}, u_{n-1}$. Then,

$$\sum_{i=1}^{n-1} f(u_i) + f(v_n) + \sum_{i=1}^{n-2} f(u_iu_{i+1}) + f(v_nu_1) + f(v_nu_{n-1}) = \sum_{i=1}^{n-1} f(u_i) + f(w_n) + \sum_{i=1}^{n-2} f(u_iu_{i+1}) + f(w_nu_1) + f(w_nu_{n-1})$$

or

$$f(v_n) + f(v_nu_1) + f(v_nu_{n-1}) = f(w_n) + f(w_nu_1) + f(w_nu_{n-1}).$$

Since $f(x) + f(y) + f(xy) = k_1$ for each edge $xy \in E(G)$, $f(v_n) + f(v_nu_1) = f(w_n) + f(w_nu_1)$. Hence, $f(v_nu_{n-1}) = f(w_nu_{n-1})$, a contradiction.

Case 2. $H \cong Amal(C_n; P_{n-2}; 2)$.

Consider a subgraph $H \cong Amal(C_n; P_{n-2}; 2)$ of a graph G. Denote the vertices in $Amal(C_n; P_{n-2}; 2)$ by $\{v_{n-1}, v_n\} \cup \{u_i | i \in [1, n-2]\} \cup \{w_{n-1}, w_n\}$ such that the edge set is $\{u_iu_{i+1} | i \in [1, n-3]\} \cup \{u_1v_{n-1}, v_{n-1}v_n, v_nu_{n-2}, u_1w_{n-1}, w_{n-1}w_n, w_nu_{n-2}\}$. There are two cycles C^1 and C^2 isomorphic to C_n with $V(C^1) = v_n, v_{n-1}, u_1, u_2, \ldots, u_{n-2}$ and $V(C^2) = w_n, w_{n-1}, u_1, u_2, \ldots, u_{n-2}$. Then

$$\sum_{i=1}^{n-2} f(u_i) + \sum_{i=n-1}^{n} f(v_i) + \sum_{i=1}^{n-3} f(u_iu_{i+1}) + f(v_{n-1}u_1) + f(v_nu_{n-2}) + f(v_nv_{n-1}) = \sum_{i=1}^{n-2} f(u_i) + \sum_{i=n-1}^{n} f(w_i) + \sum_{i=1}^{n-3} f(u_iu_{i+1}) + f(w_{n-1}u_1) + f(w_nu_{n-2}) + f(w_nw_{n-1})$$

or

$$\sum_{i=n-1}^{n} f(v_i) + f(v_nv_{n-1}) + f(v_{n-1}u_1) + f(v_nu_{n-2}) = \sum_{i=n-1}^{n} f(w_i) + f(w_nw_{n-1}) + f(w_{n-1}u_1) + f(w_nu_{n-2}).$$

Thus $f(v_{n-1}) + f(v_{n-1}u_1) = f(w_{n-1}) + f(w_{n-1}u_1)$ and $f(v_n) + f(v_nu_{n-2}) = f(w_n) + f(w_nu_{n-2})$. Hence, $f(v_{n-1}v_n) = f(w_{n-1}w_n)$, a contradiction. □

The converse of Theorem 13 is not true. Consider m copies of isomorphic cycles of order n, mC_n. It is clear that mC_n admits C_n-covering and is H-free, for H isomorphic to the forbidden subgraphs in Theorem 13. However mC_n is SEMT if and only if m and n are odd [21], and so mC_n, for even $m \geq 2$, is not (K_2, C_n)-sim-supermagic. Therefore, the two subgraphs in Theorem 13 are not the only forbidden subgraphs of (K_2, C_n)-sim-supermagic graphs.

Problem 3. *What are the other forbidden subgraphs of (K_2, C_n)-sim-(super)magic graphs?*

In the following lemma, we state sufficient conditions for an (S)EMT graph to be a (K_2, C_n)-sim-(super)magic graph.

Lemma 6. *Let $k \geq 2$ and $n \geq 3$ be two positive integers. Let G be a graph order at least $n+1$ that admits C_n-covering. Let $\{C_n^i\}_{i=1}^k$ be the family of all subgraph of G isomorphic to C_n and $\sum Y_i$ be* *the sum of all vertices labels in C_n^i, for each $i \in [1,k]$. If f is an (S)EMT labeling in G such that $\sum Y_i$ is constant, for every $i \in [1,k]$, then G is (K_2, C_n)-sim-(super)magic.*

Proof. Let m_f as the magic sum of the labeling. Let $i \neq j$ be two positive integers in $[1,k]$. Consider two arbitrary cycles C_n^i and C_n^j in $\{C_n^i\}_{i=1}^k$. Thus, $\sum Y_i = \sum Y_j$. Hence, we have that

$$\begin{aligned} w(C_n^i) &= \sum_{v \in V(C_n^i)} f(v) + \sum_{e \in E(C_n^i)} f(e) \\ &= nm_f - \sum Y_i = nm_f - \sum Y_j \\ &= \sum_{v \in V(C_n^j)} f(v) + \sum_{e \in E(C_n^j)} f(e) \\ &= w(C_n^j). \end{aligned}$$

Hence, the sum of all edges and vertices labels associated to a subgraph of G isomorphic to C_n is a constant. Therefore, G is a C_n-(super)magic for each $n \geq 3$. Since f is simultaneously an (S)EMT and C_n-(super)magic, G is (K_2, C_n)-sim-(super)magic. □

Consequently, by Lemma 6, we have the following corollary.

Corollary 11. *Let $m \geq 3$ be an odd integer. The disjoint copies of cycle on 3 vertices, mC_3, is (K_2, C_3)-sim-supermagic.*

Proof. Recall an SEMT labeling of mC_3, for odd m, from [21]. We denote $V(mC_3) = \{u_{i,j} | i \in [1,m] \text{ and } j \in [1,3]\}$ and $E(mC_3) = \{u_{i,j}u_{i,j+1} | i \in [1,m] \text{ and } j \in [1,2]\} \cup \{u_{i,1}u_{i,3} | i \in [1,m]\}$ and define

$$f(u_{i,j}) = \begin{cases} i, & \text{if } i \in [1,m] \text{ and } j = 1; \\ 2m + \frac{2i+1+m}{2}, & \text{if } i \in [1, \frac{m-1}{2}] \text{ and } j = 2; \\ 2m + \frac{2i+1-m}{2}, & \text{if } i \in [\frac{m+1}{2}, m] \text{ and } j = 2; \\ 2m + 1 - 2i, & \text{if } i \in [1, \frac{m-1}{2}] \text{ and } j = 3; \\ 3m + 1 - 2i, & \text{if } i \in [\frac{m+1}{2}, m] \text{ and } j = 3. \end{cases}$$

Let C_3^i be a subgraph of mC_3 isomorphic to C_3 and $\sum Y_i$ be the sum of all vertices labels in C_3^i. Hence, for $1 \leq i \leq \frac{m-1}{2}$, we have

$$\begin{aligned} \sum Y_i &= i + 2m + \frac{2i+1+m}{2} + 2m + 1 - 2i \\ &= i + 2m + i + \frac{1+m}{2} + 2m + 1 - 2i \\ &= \frac{3}{2}(3m+1) \end{aligned}$$

and, for $\frac{m+1}{2} \leq i \leq m$, we have

$$\begin{aligned} \sum Y_i &= i + 2m + \frac{2i+1-m}{2} + 3m + 1 - 2i \\ &= i + 2m + i + \frac{1-m}{2} + 3m + 1 - 2i \\ &= \frac{3}{2}(3m+1). \end{aligned}$$

Therefore, $\sum Y_i$ is constant for $1 \leq i \leq m$. By Lemma 6, mC_3 is (K_2, C_3)-sim-supermagic. □

6. Open Problems

We conclude by listing open problems from the previous sections that could be interesting for further investigation.

1. What are the other forbidden subgraphs of (K_2, P_n)-sim-(super)magic graph?
2. What are the other forbidden subgraphs of (K_2, C_n)-sim-(super)magic graphs?
3. Characterize (K_2, P_n)-sim-(super)magic graphs for any $n \geq 5$.

Author Contributions: Conceptualization, Y.F.A., A.N.M.S. and R.S.; methodology, Y.F.A. and R.S.; formal analysis, Y.F.A., R.S. and A.N.M.S.; writing—original draft preparation, Y.F.A. and R.S.; writing—review and editing, Y.F.A., A.N.M.S. and R.S.; supervision, A.N.M.S. and R.S.; funding acquisition, Y.F.A., A.N.M.S. and R.S. All authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

Funding: Y.F.A. and A.N.M.S. were funded by PMDSU Scholarship, Directorate General of Higher Education, Ministry of Education and Culture of Indonesia; R.S. was partially funded by Penelitian Dasar 2018–2021 (Indonesian National Research and Innovation Agency), Grant Number 2/E1/KP.PTNBH/2021.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

SEMT	Super edge magic total
EMT	Edge magic total

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