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On the Parallel Subgradient Extragradient Rule for Solving Systems of Variational Inequalities in Hadamard Manifolds

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Abstract: In a Hadamard manifold, let the VIP and SVI represent a variational inequality problem and a system of variational inequalities, respectively, where the SVI consists of two variational inequalities which are of symmetric structure mutually. This article designs two parallel algorithms to solve the SVI via the subgradient extragradient approach, where each algorithm consists of two parts which are of symmetric structure mutually. It is proven that, if the underlying vector fields are of monotonicity, then the sequences constructed by these algorithms converge to a solution of the SVI. We also discuss applications of these algorithms for approximating solutions to the VIP. Our theorems complement some recent and important ones in the literature.

Keywords: parallel subgradient extragradient rule; Hadamard manifold; system of variational inequalities; monotone vector fields; convex set

MSC: 47J20; 51H25; 65C10; 65C15; 90C33

Citation: Wang, C.-Y.; Ceng, L.-C.; He, L.; Hu, H.-Y.; Zhao, T.-Y.; Wang, D.-Q.; Fan, H.-L. On the Parallel Subgradient Extragradient Rule for Solving Systems of Variational Inequalities in Hadamard Manifolds. *Symmetry* **2021**, *13*, 1496. https://doi.org/10.3390/sym13081496

Academic Editors: Yagub Sharifov and Nazim Mahmudov

Received: 26 July 2021
Accepted: 10 August 2021
Published: 15 August 2021

1. Introduction

Suppose that the operator F is a self-mapping on a real Hilbert space $(H, \langle\cdot,\cdot\rangle)$. Let the set $C \subset H$ be nonempty, convex, and closed. Consider the classical variational inequality problem (VIP) of finding a point $\bar{z} \in C$ s.t.:

$$\langle F\bar{z}, x - \bar{z}\rangle \geq 0 \quad \forall x \in C. \tag{1}$$

It is well known that variational inequalities like VIP (1) have played an important role in the study of economics, transportation, mathematical programming, engineering mechanics, etc. Let F be L-Lipschitzian with constant $L > 0$. Given $\ell \in (0, \frac{1}{L})$. In 1976, Korpelevich's extragradient rule was first introduced in [1] for solving VIP (1). For any initial $v_0 \in C$, let the sequence $\{v_l\}$ be generated by

$$\begin{cases} z_l = P_C(v_l - \ell F v_l), \\ v_{l+1} = P_C(v_l - \ell F z_l) \quad \forall l \geq 0, \end{cases} \tag{2}$$

where P_C is the metric projection of H onto C. To the most of our knowledge, Korpelevich's extragradient rule has become one of the best effective numerical methods for the VIP and related optimization problems. Moreover, many authors improved it in various kinds of ways; see, e.g., [2–11] and references therein, to name but a few.

In 2008, Ceng et al. [8] considered the following system of variational inequalities (SVI): find $(p^*, q^*) \in C \times C$ s.t.

$$\begin{cases} \langle p^* - q^* + \ell_1 F_1 q^*, p - p^*\rangle \geq 0 \quad \forall p \in C, \\ \langle q^* - p^* + \ell_2 F_2 p^*, q - q^*\rangle \geq 0 \quad \forall q \in C, \end{cases} \tag{3}$$

where F_k is a self-mapping on H and ℓ_k is a positive constant for $k = 1, 2$. It is clear that the SVI (3) consists of two variational inequalities which are of symmetric structure mutually. It is worth mentioning that the SVI (3) has been transformed into the following fixed-point problem (FPP).

Lemma 1 (see [8], [Lemma 2.1])**.** *A pair $(p^*, q^*) \in C \times C$, is a solution of SVI (3) if and only if p^* is a fixed point of the mapping $G := P_C(I - \ell_1 F_1) P_C(I - \ell_2 F_2)$, i.e., $p^* \in \mathrm{Fix}(G)$, where $q^* = P_C(I - \ell_2 F_2) p^*$.*

In terms of Lemma 1, Ceng et al. [8] suggested and analyzed a relaxed extragradient algorithm for solving SVI (3). In 2011, the subgradient extragradient rule was first proposed in [6] for solving VIP (1), where the second projection onto C is replaced by the projection onto a half-space:

$$\begin{cases} q_l = P_C(p_l - \xi F p_l), \\ C_l = \{y \in H : \langle p_l - \xi F p_l - q_l, y - q_l \rangle \le 0\}, \\ p_{l+1} = P_{C_l}(p_l - \xi F q_l) \quad \forall l \ge 0, \end{cases}$$

with constant $\xi \in (0, \frac{1}{L})$. The above rule is more advantageous and more subtle than the rule (2) in the case when C is a feasible set with a complex structure and the calculation of projection onto C is oppressively time-squandering.

In 2018, Yang et al. [12] designed the modified subgradient extragradient rule for solving VIP (1). For any given $\lambda_0 > 0$, $u_0 \in H$ and $\mu \in (0, 1)$, let the sequences $\{u_l\}$ and $\{v_l\}$ be generated by

$$\begin{cases} v_l = P_C(u_l - \varsigma_l F u_l), \\ C_l = \{y \in H : \langle u_l - \varsigma_l F u_l - v_l, y - v_l \rangle \le 0\}, \\ u_{l+1} = P_{C_l}(u_l - \varsigma_l F v_l) \quad \forall l \ge 0, \end{cases}$$

where ς_{l+1} is chosen as

$$\varsigma_{l+1} = \begin{cases} \min\{\frac{\mu(\|u_l - v_l\|^2 + \|u_{l+1} - v_l\|^2)}{2\langle F u_l - F v_l, u_{l+1} - v_l \rangle}, \varsigma_l\}, & \text{if } \langle F u_l - F v_l, u_{l+1} - v_l \rangle > 0, \\ \varsigma_l, & \text{otherwise.} \end{cases}$$

It was proven in [12] that $\{u_l\}$ and $\{v_l\}$ converge weakly to a solution of VIP (1).

On the other hand, suppose that C is a nonempty, convex and closed subset of a Hadamard manifold $\mathcal{M}$, and $A : \mathcal{M} \to T\mathcal{M}$ is a vector field, that is, $Au \in T_u \mathcal{M}\ \forall u \in \mathcal{M}$. In 2003, Németh [13] introduced the new VIP of finding $u^* \in C$ s.t.:

$$\langle A u^*, \exp_{u^*}^{-1} u \rangle \ge 0 \quad \forall u \in C, \tag{4}$$

where $\exp^{-1}$ is the inverse of an exponential map. The solution set of VIP (4) is denoted by S. Subsequently, some rules and methods are extended from Euclidean spaces to Riemannian manifolds because of some important advantages of the extension; see, e.g., [14–17]. Furthermore, inspired by the SVI (3) and the multiobjective optimization problem in [17], Ceng et al. [18] introduced a system of multiobjective optimization problems (SMOP) in a Hadamard manifold and invented a parallel proximal point rule for solving the SMOP.

It is remarkable that the research works on the algorithms for VIP (4) are mainly focused on a proximal point algorithm [19] and Korpelevich's extragradient rule [20]. Very recently, Chen et al. [9] suggested the modified Tseng's extragradient method to solve VIP (4). Moreover, their results gave an affirmative answer to the open question put forth in [21].

Let C be a nonempty closed convex subset of a Hadamard manifold $\mathcal{M}$, and $A_k : \mathcal{M} \to T\mathcal{M}$ be a vector field for $k = 1, 2$, i.e., $A_k u \in T_u\mathcal{M}\ \forall u \in \mathcal{M}$. According to problems (3) and (4), Ceng et al. [22] introduced the new SVI of finding $(u^*, v^*) \in C \times C$ s.t.

$$\begin{cases} \langle \exp_{v^*}^{-1} u^* + \mu_1 A_1 v^*, \exp_{u^*}^{-1} u \rangle \geq 0 & \forall u \in C, \\ \langle \exp_{u^*}^{-1} v^* + \mu_2 A_2 u^*, \exp_{v^*}^{-1} v \rangle \geq 0 & \forall v \in C, \end{cases} \tag{5}$$

where constants $\mu_1, \mu_2 \in (0, \infty)$, and $\exp^{-1}$ is the inverse of exponential map. In particular, if $A_1 = A_2 = A$ and $u^* = v^*$, then SVI (5) reduces to VIP (4).

In this paper, we design two parallel algorithms to solve the SVI (5) via the subgradient extragradient approach, where each algorithm consists of two parts which have a mutually symmetric structure. It is proven that, if the underlying vector fields are of monotonicity, then the sequences constructed by these algorithms converge to a solution of the SVI (5). We also discuss applications of these algorithms to the approximation of solutions to the VIP (4). Our results improve and extend the corresponding results announced in [8,9,12,22].

The remainder of the paper is arranged below. Some preliminary concepts, notations, important lemmas, and propositions in Riemannian geometry are recalled in Section 2. It is remarkable that one can find most of them in every textbook about Riemannian geometry (e.g., [23]). Two new parallel algorithms based on the modified subgradient extragradient approach [12] are proposed for SVI (5), and some convergence theorems are proved in Section 3.

2. Preliminaries

Let $\mathcal{M}$ indicate a simply connected and finite-dimensional differentiable manifold. A differentiable manifold $\mathcal{M}$ endowed with a Riemannian metric is called a Riemannian manifold. We denote by $T_v\mathcal{M}$ the tangent space of $\mathcal{M}$ at $v \in \mathcal{M}$, by $\langle \cdot, \cdot \rangle_v$ the scalar product on $T_v\mathcal{M}$ with the associated norm $\| \cdot \|_v$, where the subscript v is sometimes omitted, and by $T\mathcal{M} := \cup_{v \in \mathcal{M}} T_v\mathcal{M}$ the tangent bundle of $\mathcal{M}$, which is actually a manifold. Let $\gamma : [a, b] \to \mathcal{M}$ be a piecewise smooth curve joining v to ω (i.e., $\gamma(a) = v$ and $\gamma(b) = \omega$), we define the length $l(\gamma) = \int_a^b \|\gamma'(t)\| dt$. Then, the Riemannian distance $d(v, \omega)$, which induces the original topology on $\mathcal{M}$, is defined by minimizing this length over the set of all such curves joining v to ω.

Suppose that the Levi–Civita connection ∇ is associated with the Riemannian metric and the smooth curve γ lies in $\mathcal{M}$. A vector field X is referred to as being parallel along γ iff $\nabla_{\gamma'} X = 0$. In case γ' itself is parallel along γ, γ is known as a geodesic, and, in this case, $\|\gamma'\|$ is constant. It is remarkable that this notion is different from the corresponding one in the calculus of variations—in particular, if $\|\gamma'\| = 1$, γ is referred to as being normalized. A geodesic joining v to ω in $\mathcal{M}$ is called minimal if its length equals $d(v, \omega)$.

Let $\mathcal{M}$ be a Riemannian manifold. $\mathcal{M}$ is referred to as being complete iff for each $v \in \mathcal{M}$ all geodesics emanating from v are defined for all $t \in \mathbf{R} := (-\infty, \infty)$. Using the Hopf–Rinow Theorem, we infer that, if $\mathcal{M}$ is complete, each pair of points in $\mathcal{M}$ can be joined by a minimal geodesic. In the meantime, $(\mathcal{M}, d)$ becomes a complete metric space and bounded closed subsets are compact ones in $\mathcal{M}$.

We denote by $P_{\gamma,\cdot,\cdot}$ the parallel transport on the tangent bundle $T\mathcal{M}$ along γ w.r.t. ∇, defined by

$$P_{\gamma,\gamma(b),\gamma(a)}(v) = V(\gamma(b)) \quad \forall a, b \in \mathbf{R},\ v \in T_{\gamma(a)}\mathcal{M},$$

where V is the unique vector field such that $\nabla_{\gamma'(t)} V = 0$ for each t and $V(\gamma(a)) = v$. Then, for any $a, b \in \mathbf{R}$, $P_{\gamma,\gamma(b),\gamma(a)}$ is an isometry from $T_{\gamma(a)}\mathcal{M}$ to $T_{\gamma(b)}\mathcal{M}$. For the convenience, we will write $P_{\omega,v}$ instead of $P_{\gamma,\omega,v}$ in the case where γ is a minimal geodesic joining v to ω.

Let $\mathcal{M}$ be complete. An exponential map $\exp_v : T_v\mathcal{M} \to \mathcal{M}$ at v is defined by $\exp_v \omega = \gamma_\omega(1, v)$ for each $\omega \in T_v\mathcal{M}$, where $\gamma(\cdot) = \gamma_\omega(\cdot, v)$ is the geodesic starting at v with velocity v. Then, $\exp_v t\omega = \gamma_\omega(t, v)$ for each real number t. It is worth emphasizing that the mapping $\exp_v$ is differentiable on $T_v\mathcal{M}$ for each $v \in \mathcal{M}$. The exponential map has inverse $\exp_v^{-1} : \mathcal{M} \to T_v\mathcal{M}$, i.e., $\phi = \exp_v^{-1} \omega$, and the geodesic is the unique shortest path with $\| \exp_v^{-1} \omega \| = \| \exp_\omega^{-1} v \| = d(v, \omega)$, where $d(v, \omega)$ is the geodesic distance between v and ω in $\mathcal{M}$.

A set $D \subset \mathcal{M}$ is referred to as being convex if, for every $y, z \in K$, the geodesic joining y to z lies in D, i.e., if $\gamma : [a, b] \to \mathcal{M}$ is a geodesic satisfying $y = \gamma(a)$ and $z = \gamma(b)$, then $\gamma((1-t)a + tb) \in D \; \forall t \in [0, 1]$. From now on, we denote by D a nonempty closed convex set in $\mathcal{M}$, and by P_D the projection of $\mathcal{M}$ onto D, i.e.,

$$P_D(y) = \{y_0 \in D : d(y, y_0) \leq d(y, z) \text{ for all } z \in D\} \quad \forall y \in \mathcal{M}.$$

A real-valued function f defined on $\mathcal{M}$ is referred to as being convex if, for each geodesic γ of $\mathcal{M}$, the composite function $f \circ \gamma : \mathbf{R} \to \mathbf{R}$ is convex, i.e.,

$$(f \circ \gamma)(sa + (1-s)b) \leq s(f \circ \gamma)(a) + (1-s)(f \circ \gamma)(b) \quad \forall a, b \in \mathbf{R},\ s \in [0, 1].$$

A Hadamard manifold $\mathcal{M}$ is a complete simply connected Riemannian manifold of non-positive sectional curvature. If $\mathcal{M}$ is a Hadamard manifold, then $\exp_v^{-1} : \mathcal{M} \to T_v\mathcal{M}$ is a diffeomorphism for each $v \in \mathcal{M}$ and, if $v, \omega \in \mathcal{M}$, then there exists a unique minimal geodesic joining v to ω. Next, we always assume that $\mathcal{M}$ is a Hadamard manifold.

Proposition 1 (see [23]). *Let $v \in \mathcal{M}$. Then, $\exp_v : T_v\mathcal{M} \to \mathcal{M}$ is a diffeomorphism, and, for any points $v, \omega \in \mathcal{M}$, there exists a unique normalized geodesic joining v to ω, which is actually a minimal geodesic.*

The above proposition shows that $\mathcal{M}$ is diffeomorphic to the Euclidean space $\mathbf{R}^m$. Then, $\mathcal{M}$ has the same topology and differential structure as $\mathbf{R}^m$. Moreover, Hadamard manifolds and Euclidean spaces have some similar geometrical properties.

Definition 1 (see [20]). *Let $\mathcal{X}(\mathcal{M})$ be the set of all single-valued vector fields $V : \mathcal{M} \to T\mathcal{M}$ s.t. $V(v) \in T_v\mathcal{M} \; \forall v \in \mathcal{M}$ and the domain $\mathcal{D}(V)$ of V is defined by $\mathcal{D}(V) = \{v \in \mathcal{M} : V(v) \neq \emptyset\}$. Let $V \in \mathcal{X}(\mathcal{M})$. Then, V is referred to as being pseudomonotone if, for each $v, \omega \in \mathcal{D}(V)$,*

$$\langle V(v), \exp_v^{-1} \omega \rangle \geq 0 \;\Rightarrow\; \langle V(\omega), \exp_\omega^{-1} v \rangle \leq 0.$$

A geodesic triangle $\Delta(p_1, p_2, p_3)$ of a Riemannian manifold is a set consisting of three points p_1, p_2 and p_3, and three minimal geodesics γ_i joining p_i to p_{i+1}, with $i = 1, 2, 3(\text{mod}3)$.

Proposition 2 (see [23] (Comparison theorem for triangles)). *Suppose that $\Delta(p_1, p_2, p_3)$ is a geodesic triangle. Ones denote, for each $i = 1, 2, 3(\text{mod}3)$, by $\gamma_i : [0, l_i] \to \mathcal{M}$, the geodesic joining p_i to p_{i+1}, and put $l_i = L(\gamma_i)$, and $\alpha_i := \angle(\gamma_i'(0), -\gamma_{i-1}'(l_{i-1}))$. Then,*

(i) $\alpha_1 + \alpha_2 + \alpha_3 \leq \pi$;

(ii) $l_i^2 + l_{i+1}^2 - 2 l_i l_{i+1} \cos \alpha_{i+1} \leq l_{i-1}^2$;

(iii) $l_{i+1} \cos \alpha_{i+2} + l_i \cos \alpha_i \geq l_{i+2}$.

According to the distance and the exponential map, inequality (ii) in Proposition 2 can be rewritten as

$$d^2(p_i, p_{i+1}) + d^2(p_{i+1}, p_{i+2}) - 2\langle \exp_{p_{i+1}}^{-1} p_i, \exp_{p_{i+1}}^{-1} p_{i+2} \rangle \leq d^2(p_{i-1}, p_i),$$

owing to the fact that

$$\langle \exp_{p_{i+1}}^{-1} p_i, \exp_{p_{i+1}}^{-1} p_{i+2} \rangle = d(p_i, p_{i+1}) d(p_{i+1}, p_{i+2}) \cos \alpha_{i+1}. \tag{6}$$

Lemma 2 (see [24]). *Let $u_0 \in \mathcal{M}$ and $\{u_n\} \subset \mathcal{M}$ s.t. $u_n \to u_0$. Then, the following holds:*

(i) $\exp_{u_n}^{-1} v \to \exp_{u_0}^{-1} v$ and $\exp_v^{-1} u_n \to \exp_v^{-1} u_0$ for all $v \in \mathcal{M}$.

(ii) If $y_n \in T_{u_n}\mathcal{M}$ and $y_n \to y_0$, then $y_0 \in T_{u_0}\mathcal{M}$.

(iii) Given $p_n, q_n \in T_{u_n}\mathcal{M}$ and $p_0, q_0 \in T_{u_0}\mathcal{M}$, if $p_n \to p_0$ and $q_n \to q_0$, then $\langle p_n, q_n \rangle \to \langle p_0, q_0 \rangle$.

(iv) For each $v \in T_{u_0}M$, the map $F : \mathcal{M} \to T\mathcal{M}$, defined by $F(u) = P_{u,u_0}v \ \forall u \in \mathcal{M}$, is continuous on $\mathcal{M}$.

For each $u \in \mathcal{M}$ and $C \subset \mathcal{M}$, there is only a point $u_0 \in C$ satisfying $d(u, u_0) \leq d(u, v) \ \forall v \in C$. Then, the unique point is known as the projection of u onto the convex set C, denoted by $P_C(u)$.

Proposition 3 (see [25]). *For each $u \in \mathcal{M}$, the following inequality holds:*

$$\langle \exp^{-1}_{P_C(u)} u, \exp^{-1}_{P_C(u)} v \rangle \leq 0 \quad \forall v \in C.$$

Proposition 4 (see [20]). *Let $C \subset \mathcal{M}$ be closed and convex. Then, the metric projection P_C is nonexpansive, i.e., $d(P_C(p), P_C(q)) \leq d(p, q) \ \forall p, q \in \mathcal{M}$.*

Lemma 3 (see [21]). *Assume that $\mathcal{M}$ is of constant curvature, $u \in \mathcal{M}$ and $\varrho \in T_u\mathcal{M}$. Then, $L_{u,\varrho} := \{v \in \mathcal{M} : \langle \exp^{-1}_u v, \varrho \rangle \leq 0\}$ is convex.*

Lemma 4 (see [20]). *Suppose that C is a nonempty closed convex subset of a Hadamard manifold $\mathcal{M}$. Then, $d^2(P_C(p), q) \leq d^2(p, q) - d^2(p, P_C(p)) \quad \forall p \in \mathcal{M}, q \in C$.*

Lemma 5 (see [13]). *Let A be a continuous and monotone vector field on C, given $z \in C$. Then, $\langle Az, \exp^{-1}_z v \rangle \geq 0 \ \forall v \in C \ \Leftrightarrow \ \langle Av, \exp^{-1}_v z \rangle \leq 0 \ \forall v \in C$.*

It is easy from Proposition 3 to see that the following hold:

Proposition 5 (see [20]). *The following assertions are equivalent:*
(i) u^* solves the VIP (4);
(ii) $u^* = P_C(\exp_{u^*}(-\beta_0 Au^*))$ for some $\beta_0 > 0$;
(iii) $u^* = P_C(\exp_{u^*}(-\beta Au^*))$ for all $\beta > 0$;
(iv) $r(u^*, \beta) = 0$, with $r(u^*, \beta) = \exp^{-1}_{u^*}[P_C(\exp_{u^*}(-\beta Au^*))]$.

The following two lemmas play a crucial role in the convergence derivation of the algorithms.

Lemma 6 (see [26]). *Suppose that $\Delta(u, v, w)$ is a geodesic triangle in $\mathcal{M}$, a Hadamard manifold. Then, $\exists u', v', w' \in \mathbf{R}^2$ s.t.*

$$d(u, v) = \|u' - v'\|, \quad d(v, w) = \|v' - w'\| \quad \text{and} \quad d(w, u) = \|w' - u'\|.$$

The triangle $\Delta(u', v', w')$ is called the comparison triangle of $\Delta(u, v, w)$, which is unique up to isometry of $\mathcal{M}$. The following result can be proved by using element geometry. This is also a direct application of the Alexandrov's Lemma in $\mathbf{R}^2$ (see [27]). It explains the relationship between two triangles $\Delta(u, v, w)$ and $\Delta(u', v', w')$ involving angles and distances between points.

Lemma 7 (see [28]). *Let $\Delta(u, v, w)$ be a geodesic triangle in a Hadamard manifold $\mathcal{M}$ and $\Delta(u', v', w')$ its comparison triangle.*
(i) Assume that α, β, γ (resp., α', β', γ') are three angles of $\Delta(u, v, w)$ (resp., $\Delta(u', v', w')$) at three vertices u, v, w (resp., u', v', w'). Then, the inequalities hold: $\alpha \leq \alpha'$, $\beta \leq \beta'$ and $\gamma \leq \gamma'$.
(ii) Assume that the point z lies in the geodesic joining u to v and z' is its comparison point in the interval $[u', v']$ satisfying $d(z, u) = \|z' - u'\|$ and $d(z, v) = \|z' - v'\|$. Then, the inequality holds: $d(z, w) \leq \|z' - w'\|$.

Definition 2 (see [9]). *A vector field f defined on a complete Riemannian manifold $\mathcal{M}$ is referred to as being Lipschitzian if $\exists L(\mathcal{M}) = L > 0$ s.t.*

$$d(f(v), f(\omega)) \leq Ld(v, \omega) \quad \forall v, \omega \in \mathcal{M}. \tag{7}$$

Besides the above concept, if for each $v_0 \in \mathcal{M}$, $\exists L(v_0) > 0$ and $\exists \sigma = \sigma(v_0) > 0$ s.t. (7) holds, with $L = L(v_0)$, for all $v, \omega \in B_\sigma(v_0) := \{z \in \mathcal{M} : d(v_0, z) < \delta\}$, then f is said to be locally Lipschitzian.

Finally, by the similar inference to that of transforming SVI (3) into the FPP in [8], we obtain the following.

Lemma 8 (see [22], [Lemma 5]). *A pair $(p^*, q^*) \in C \times C$ is a solution of SVI (5) if and only if p^* is a fixed point of the mapping $G := P_C(\exp_I(-\mu_1 A_1))P_C(\exp_I(-\mu_2 A_2))$, i.e., $p^* \in \mathrm{Fix}(G)$, where $q^* = P_C(\exp_I(-\mu_2 A_2))p^*$.*

3. Algorithms and Convergence Criteria

In this section, inspired by the algorithms in [9], we suggest two new parallel algorithms for solving VIP (5) on Hadamard manifolds via the modified subgradient extragradient approach in [12].

From now on, the following assumptions are always adopted:

Hypothesis 1 (H1). *The solution set of SVI (5), denoted by $\mathcal{S}$, is nonempty.*

Hypothesis 2 (H2). *$A_1, A_2 : \mathcal{M} \to T\mathcal{M}$ are vector fields, i.e., $A_k u \in T_u\mathcal{M}$ $\forall u \in \mathcal{M}$ for $k = 1, 2$.)*

Hypothesis 3 (H3). *A_1 and A_2 both are monotone, i.e., for $k = 1, 2$, $\langle A_k x - A_k y, \exp_y^{-1} x\rangle \geq 0 \quad \forall x, y \in \mathcal{M}$.*

Hypothesis 4 (H4). *A_1 and A_2 both are Lipschitzian with constants $L_1, L_2 > 0$, i.e., for $k = 1, 2$, $\exists L_k > 0$ s.t.*

$$d(A_k x, A_k y) \leq L_k d(x, y) \quad \forall x, y \in \mathcal{M}.$$

Next, we recall the notion of Fejér convergence and related result.

Definition 3 (see [29]). *Suppose that X is a complete metric space and $C \subset X$ is a nonempty set. Then, a sequence $\{x_l\} \subset X$ is referred to as being Fejér convergent to C, if $d(x_{l+1}, y) \leq d(x_l, y)$ $\forall y \in C$, $l \geq 0$.*

Proposition 6 (see [24]). *Suppose that X is a complete metric space and $C \subset X$ is a nonempty set. Let $\{x_l\} \subset X$ be Fejér convergent to C and assume that any cluster point of $\{x_l\}$ belongs in C. Then, $\{x_l\}$ converges to a point of C.*

3.1. The First Parallel Algorithm

Algorithm 1 is the first parallel algorithm for the SVI.

Algorithm 1: The first parallel algorithm for the SVI.

Initialization: Given $x_0 \in \mathcal{M}$ arbitrarily. Let $\mu_{k,0} > 0$ and $\lambda_k \in (0,1)$ for $k = 1,2$.
Iteration Steps: Compute x_{n+1} below:
Step 1. Compute
$$\begin{cases} \tilde{z}_n = P_C(\exp_{x_n}(-\mu_{2,n} A_2 x_n)), \\ y_n = P_C(\exp_{z_n}(-\mu_{1,n} A_1 z_n)). \end{cases}$$
Step 2. Construct
$$\begin{cases} C_{2,n} = \{x \in \mathcal{M} : \langle \exp_{\tilde{z}_n}^{-1} x_n - \mu_{2,n} A_2 x_n, \exp_{\tilde{z}_n}^{-1} x \rangle \leq 0\}, \\ C_{1,n} = \{x \in \mathcal{M} : \langle \exp_{y_n}^{-1} z_n - \mu_{1,n} A_1 z_n, \exp_{y_n}^{-1} x \rangle \leq 0\}, \end{cases}$$
and calculate
$$\begin{cases} z_n = P_{C_{2,n}}(\exp_{x_n}(-\mu_{2,n} A_2 \tilde{z}_n)), \\ x_{n+1} = P_{C_{1,n}}(\exp_{z_n}(-\mu_{1,n} A_1 y_n)). \end{cases}$$
Step 3. Calculate
$$\begin{cases} \mu_{2,n+1} = \begin{cases} \min\{\frac{\lambda_2(d^2(x_n,\tilde{z}_n)+d^2(z_n,\tilde{z}_n))}{2\langle A_2x_n - A_2\tilde{z}_n, \exp_{\tilde{z}_n}^{-1} z_n\rangle}, \mu_{2,n}\} & \text{if } \langle A_2 x_n - A_2\tilde{z}_n, \exp_{\tilde{z}_n}^{-1} z_n\rangle > 0, \\ \mu_{2,n} & \text{otherwise.} \end{cases} \\ \mu_{1,n+1} = \begin{cases} \min\{\frac{\lambda_1(d^2(z_n,y_n))+d^2(x_{n+1},y_n)}{2\langle A_1 z_n - A_1 y_n, \exp_{y_n}^{-1} x_{n+1}\rangle}, \mu_{1,n}\} & \text{if } \langle A_1 z_n - A_1 y_n, \exp_{y_n}^{-1} x_{n+1}\rangle > 0, \\ \mu_{1,n} & \text{otherwise.} \end{cases} \end{cases} \tag{8}$$
Again, put $n := n+1$ and go to Step 1.

In particular, putting $A_1 = A_2 = A$ in Algorithm 1, we obtain the following algoritm (Algorithm 2) for solving VIP (4).

Algorithm 2: The first parallel algorithm for the VIP.

Initialization: Given $x_0 \in \mathcal{M}$ arbitrarily, let $\mu_{k,0} > 0$ and $\lambda_k \in (0,1)$ for $k = 1,2$.
Iteration Steps: Compute x_{n+1} below:
Step 1. Compute
$$\begin{cases} \tilde{z}_n = P_C(\exp_{x_n}(-\mu_{2,n} A x_n)), \\ y_n = P_C(\exp_{z_n}(-\mu_{1,n} A z_n)). \end{cases}$$
Step 2. Construct
$$\begin{cases} C_{2,n} = \{x \in \mathcal{M} : \langle \exp_{\tilde{z}_n}^{-1} x_n - \mu_{2,n} A x_n, \exp_{\tilde{z}_n}^{-1} x \rangle \leq 0\}, \\ C_{1,n} = \{x \in \mathcal{M} : \langle \exp_{y_n}^{-1} z_n - \mu_{1,n} A z_n, \exp_{y_n}^{-1} x \rangle \leq 0\}, \end{cases}$$
and calculate
$$\begin{cases} z_n = P_{C_{2,n}}(\exp_{x_n}(-\mu_{2,n} A \tilde{z}_n)), \\ x_{n+1} = P_{C_{1,n}}(\exp_{z_n}(-\mu_{1,n} A y_n)). \end{cases}$$
Step 3. Calculate
$$\begin{cases} \mu_{2,n+1} = \begin{cases} \min\{\frac{\lambda_2(d^2(x_n,\tilde{z}_n)+d^2(z_n,\tilde{z}_n))}{2\langle Ax_n - A\tilde{z}_n, \exp_{\tilde{z}_n}^{-1} z_n\rangle}, \mu_{2,n}\} & \text{if } \langle A x_n - A\tilde{z}_n, \exp_{\tilde{z}_n}^{-1} z_n\rangle > 0, \\ \mu_{2,n} & \text{otherwise.} \end{cases} \\ \mu_{1,n+1} = \begin{cases} \min\{\frac{\lambda_1(d^2(z_n,y_n))+d^2(x_{n+1},y_n)}{2\langle A z_n - A y_n, \exp_{y_n}^{-1} x_{n+1}\rangle}, \mu_{1,n}\} & \text{if } \langle A z_n - A y_n, \exp_{y_n}^{-1} x_{n+1}\rangle > 0, \\ \mu_{1,n} & \text{otherwise.} \end{cases} \end{cases}$$
Again, put $n := n+1$ and go to Step 1.

Lemma 9. *For $k = 1,2$, the sequence $\{\mu_{k,n}\}$ generated by Algorithm 1 is a monotonically decreasing one with lower bound* $\min\{\frac{\lambda_k}{L_k}, \mu_{k,0}\}$.

Proof. It is clear that $\{\mu_{k,n}\}$ is a monotonically decreasing sequence for $k = 1,2$. Since A_k is a Lipschitzian mapping with constant $L_k > 0$ for $k = 1,2$, in the case of $\langle A_2 x_n - A_2\tilde{z}_n, \exp_{\tilde{z}_n}^{-1} z_n\rangle > 0$, we have
$$\frac{\lambda_2(d^2(x_n,\tilde{z}_n) + d^2(z_n,\tilde{z}_n))}{2\langle A_2 x_n - A_2\tilde{z}_n, \exp_{\tilde{z}_n}^{-1} z_n\rangle} \geq \frac{2\lambda_2 d(x_n,\tilde{z}_n) d(z_n,\tilde{z}_n))}{2d(A_2x_n, A_2\tilde{z}_n) d(z_n,\tilde{z}_n)} \geq \frac{\lambda_2 d(x_n,\tilde{z}_n)}{L_2 d(x_n,\tilde{z}_n)} = \frac{\lambda_2}{L_2}. \tag{9}$$

Thus, the sequence $\{\mu_{2,n}\}$ has the lower bound $\min\{\frac{\lambda_2}{L_2}, \mu_{2,0}\}$. In a similar way, we can show that $\{\mu_{1,n}\}$ has the lower bound $\min\{\frac{\lambda_1}{L_1}, \mu_{1,0}\}$. $\square$

Corollary 1. *For $k = 1,2$, the sequence $\{\mu_{k,n}\}$ generated by Algorithm 2 is a monotonically decreasing one with lower bound* $\min\{\frac{\lambda_k}{L}, \mu_{k,0}\}$.

Lemma 10. *Let $\{x_n\}$ and $\{z_n\}$ be the sequences generated by Algorithm 1. Then, the sequences $\{x_n\}$ and $\{z_n\}$ are bounded, provided for all $(p,q) \in \mathcal{S}$ and $n \geq n_0$,*

$$\begin{cases} (1 - \frac{\mu_{2,n}}{\mu_{2,n+1}}\lambda_2)d^2(x_n, \tilde{z}_n) + (1 - \frac{\mu_{1,n}}{\mu_{1,n+1}}\lambda_1)d^2(z_n, y_n) \\ \quad + 2\mu_{2,n}\langle A_2 p, \exp_p^{-1} \tilde{z}_n\rangle + 2\mu_{1,n}\langle A_1 p, \exp_p^{-1} y_n\rangle \geq 0, \\ (1 - \frac{\mu_{2,n+1}}{\mu_{2,n+2}}\lambda_2)d^2(x_{n+1}, \tilde{z}_{n+1}) + (1 - \frac{\mu_{1,n}}{\mu_{1,n+1}}\lambda_1)d^2(z_n, y_n) \\ \quad + 2\mu_{2,n+1}\langle A_2 q, \exp_q^{-1} \tilde{z}_{n+1}\rangle + 2\mu_{1,n}\langle A_1 q, \exp_q^{-1} y_n\rangle \geq 0. \end{cases}$$

Proof. Take a fixed $(p,q) \in C \times C$ arbitrarily. Then, from the monotonicity of A_2, we get $\langle A_2\tilde{z}_n - A_2 p, \exp_p^{-1} \tilde{z}_n\rangle \geq 0$, which hence yields $\langle A_2\tilde{z}_n, \exp_p^{-1} \tilde{z}_n\rangle \geq \langle A_2 p, \exp_p^{-1} \tilde{z}_n\rangle$. That is, $\langle A_2\tilde{z}_n, \exp_{\tilde{z}_n}^{-1} \tilde{z}_n + \exp_p^{-1} z_n\rangle \geq \langle A_2 p, \exp_p^{-1} \tilde{z}_n\rangle \ \forall n \geq 0$. Thus, it immediately follows that

$$\langle A_2\tilde{z}_n, \exp_{z_n}^{-1} p\rangle \leq \langle A_2\tilde{z}_n, \exp_{z_n}^{-1} \tilde{z}_n\rangle - \langle A_2 p, \exp_p^{-1} \tilde{z}_n\rangle \quad \forall n \geq 0. \tag{10}$$

By the definition of $C_{2,n}$, we have $\langle \exp_{\tilde{z}_n}^{-1} x_n - \mu_{2,n}A_2 x_n, \exp_{\tilde{z}_n}^{-1} z_n\rangle \leq 0$. Then,

$$\begin{aligned} &\langle \exp_{\tilde{z}_n}^{-1} x_n - \mu_{2,n}A_2\tilde{z}_n, \exp_{\tilde{z}_n}^{-1} z_n\rangle \\ &= \langle \exp_{\tilde{z}_n}^{-1} x_n - \mu_{2,n}A_2 x_n, \exp_{\tilde{z}_n}^{-1} z_n\rangle + \mu_{2,n}\langle A_2 x_n - A_2\tilde{z}_n, \exp_{\tilde{z}_n}^{-1} z_n\rangle \\ &\leq \mu_{2,n}\langle A_2 x_n - A_2\tilde{z}_n, \exp_{\tilde{z}_n}^{-1} z_n\rangle. \end{aligned} \tag{11}$$

Now, by fixing $n \geq 0$, we consider the geodesic triangle $\Delta(x_n, \tilde{z}_n, p)$ and its comparison triangle $\Delta(x'_n, \tilde{z}'_n, p')$. Then, $d(x_n, p) = d(x'_n, p')$, $d(\tilde{z}_n, p) = d(\tilde{z}'_n, p')$, and $d(x_n, \tilde{z}_n) = d(x'_n, \tilde{z}'_n)$. Recall from Algorithm 1 that $z_n = P_{C_{2,n}}(\exp_{x_n}(-\mu_{2,n}A_2\tilde{z}_n))$. The comparison point of z'_n is $P_{C_{2,n}}(x'_n - \mu_{2,n}A_2\tilde{z}'_n)$. Thus, in $\Delta(x'_n, \tilde{z}'_n, p')$, (10) and (11) can be rewritten as

$$\langle A_2\tilde{z}'_n, p' - z'_n\rangle + \langle A_2 p', \tilde{z}'_n - p'\rangle \leq \langle A_2\tilde{z}'_n, \tilde{z}'_n - z'_n\rangle, \tag{12}$$

$$\langle x'_n - \mu_{2,n}A_2\tilde{z}'_n - \tilde{z}'_n, z'_n - \tilde{z}'_n\rangle \leq \mu_{2,n}\langle A_2 x'_n - A_2\tilde{z}'_n, z'_n - \tilde{z}'_n\rangle. \tag{13}$$

Then, by Lemma 7 (ii), (10) and Lemma 4, we have

$$\begin{aligned} d^2(z_n, p) &\leq d^2(z'_n, p') = \|P_{C_{2,n}}(x'_n - \mu_{2,n}A_2\tilde{z}'_n) - p'\|^2 \\ &\leq \|x'_n - \mu_{2,n}A_2\tilde{z}'_n - p'\|^2 - \|x'_n - \mu_{2,n}A_2\tilde{z}'_n - z'_n\|^2 \\ &= \|x'_n - p'\|^2 - \|x'_n - z'_n\|^2 + 2\mu_{2,n}\langle A_2\tilde{z}'_n, p' - z'_n\rangle \\ &\leq \|x'_n - p'\|^2 - \|x'_n - z'_n\|^2 + 2\mu_{2,n}(\langle A_2\tilde{z}'_n, \tilde{z}'_n - z'_n\rangle - \langle A_2 p', \tilde{z}'_n - p'\rangle) \\ &= \|x'_n - p'\|^2 - \|x'_n - \tilde{z}'_n + \tilde{z}'_n - z'_n\|^2 + 2\mu_{2,n}(\langle A_2\tilde{z}'_n, \tilde{z}'_n - z'_n\rangle - \langle A_2 p', \tilde{z}'_n - p'\rangle) \\ &= \|x'_n - p'\|^2 - \|x'_n - \tilde{z}'_n\|^2 - \|\tilde{z}'_n - z'_n\|^2 \\ &\quad + 2\langle x'_n - \mu_{2,n}A_2\tilde{z}'_n - \tilde{z}'_n, z'_n - \tilde{z}'_n\rangle - 2\mu_{2,n}\langle A_2 p', \tilde{z}'_n - p'\rangle \\ &= d^2(x'_n, p') - d^2(x'_n, \tilde{z}'_n) - d^2(\tilde{z}'_n, z'_n) \\ &\quad + 2\langle x'_n - \mu_{2,n}A_2\tilde{z}'_n - \tilde{z}'_n, z'_n - \tilde{z}'_n\rangle - 2\mu_{2,n}\langle A_2 p', \tilde{z}'_n - p'\rangle \\ &= d^2(x_n, p) - d^2(x_n, \tilde{z}_n) - d^2(\tilde{z}_n, z_n) \\ &\quad + 2\langle x'_n - \mu_{2,n}A_2\tilde{z}'_n - \tilde{z}'_n, z'_n - \tilde{z}'_n\rangle - 2\mu_{2,n}\langle A_2 p', \tilde{z}'_n - p'\rangle. \end{aligned} \tag{14}$$

Consider the geodesic triangle $\Delta(a, b, c)$ and its comparison triangle $\Delta(a', b', c')$. Then, set $a = \exp_{\tilde{z}_n}^{-1} x_n - \mu_{2,n}A_2\tilde{z}_n$ and $b = \exp_{\tilde{z}_n}^{-1} z_n$, (resp., $a' = x'_n - \mu_{2,n}A_2\tilde{z}'_n - \tilde{z}'_n$ and

$b' = z'_n - \tilde{z}'_n$). Let β and β' denote the angles at c and c', respectively. Then, $\beta \leq \beta'$ by Lemma 7 and so $\cos\beta' \leq \cos\beta$. Then, by Proposition 2 and Lemma 6, we have

$$\langle a', b' \rangle = \|a'\|\|b'\|\cos\beta' \leq \|a'\|\|b'\|\cos\beta = \|a\|\|b\|\cos\beta = \langle a, b \rangle.$$

Hence,

$$\langle\langle x'_n - \mu_{2,n}A_2\tilde{z}'_n - \tilde{z}'_n, z'_n - \tilde{z}'_n \rangle \leq \langle \exp^{-1}_{\tilde{z}_n} x_n - \mu_{2,n}A_2\tilde{z}_n, \exp^{-1}_{\tilde{z}_n} z_n \rangle. \tag{15}$$

Similarly, we get $\langle -2\mu_{2,n}A_2p', \tilde{z}'_n - p' \rangle \leq \langle -2\mu_{2,n}A_2p, \exp^{-1}_p \tilde{z}_n \rangle$. This together with (14) and (15) imply that

$$\begin{aligned} d^2(z_n, p) &\leq d^2(x_n, p) - d^2(x_n, \tilde{z}_n) - d^2(\tilde{z}_n, z_n) \\ &\quad + 2\langle x'_n - \mu_{2,n}A_2\tilde{z}'_n - \tilde{z}'_n, z'_n - \tilde{z}'_n \rangle - 2\mu_{2,n}\langle A_2p', \tilde{z}'_n - p' \rangle \\ &\leq d^2(x_n, p) - d^2(x_n, \tilde{z}_n) - d^2(\tilde{z}_n, z_n) \\ &\quad + \langle \exp^{-1}_{\tilde{z}_n} x_n - \mu_{2,n}A_2\tilde{z}_n, \exp^{-1}_{\tilde{z}_n} z_n \rangle - 2\mu_{2,n}\langle A_2p, \exp^{-1}_p \tilde{z}_n \rangle. \end{aligned} \tag{16}$$

Combining (11) and (16), we get

$$\begin{aligned} d^2(z_n, p) &\leq d^2(x_n, p) - d^2(x_n, \tilde{z}_n) - d^2(\tilde{z}_n, z_n) \\ &\quad + \langle \exp^{-1}_{\tilde{z}_n} x_n - \mu_{2,n}A_2\tilde{z}_n, \exp^{-1}_{\tilde{z}_n} z_n \rangle - 2\mu_{2,n}\langle A_2p, \exp^{-1}_p \tilde{z}_n \rangle \\ &\leq d^2(x_n, p) - d^2(x_n, \tilde{z}_n) - d^2(\tilde{z}_n, z_n) \\ &\quad + 2\mu_{2,n}\langle A_2x_n - A_2\tilde{z}_n, \exp^{-1}_{\tilde{z}_n} z_n \rangle - 2\mu_{2,n}\langle A_2p, \exp^{-1}_p \tilde{z}_n \rangle \\ &= d^2(x_n, p) - d^2(x_n, \tilde{z}_n) - d^2(\tilde{z}_n, z_n) \\ &\quad + 2\tfrac{\mu_{2,n}}{\mu_{2,n+1}}\mu_{2,n+1}\langle A_2x_n - A_2\tilde{z}_n, \exp^{-1}_{\tilde{z}_n} z_n \rangle - 2\mu_{2,n}\langle A_2p, \exp^{-1}_p \tilde{z}_n \rangle. \end{aligned} \tag{17}$$

By the definition of $\mu_{2,n}$, if $\langle A_2x_n - A_2\tilde{z}_n, \exp^{-1}_{\tilde{z}_n} z_n \rangle > 0$, then

$$2\frac{\mu_{2,n}}{\mu_{2,n+1}}\mu_{2,n+1}\langle A_2x_n - A_2\tilde{z}_n, \exp^{-1}_{\tilde{z}_n} z_n \rangle \leq \frac{\mu_{2,n}}{\mu_{2,n+1}}\lambda_2(d^2(x_n, \tilde{z}_n) + d^2(z_n, \tilde{z}_n));$$

in the case of $\langle A_2x_n - A_2\tilde{z}_n, \exp^{-1}_{\tilde{z}_n} z_n \rangle \leq 0$, it is clear that

$$2\frac{\mu_{2,n}}{\mu_{2,n+1}}\mu_{2,n+1}\langle A_2x_n - A_2\tilde{z}_n, \exp^{-1}_{\tilde{z}_n} z_n \rangle \leq 0 \leq \frac{\mu_{2,n}}{\mu_{2,n+1}}\lambda_2(d^2(x_n, \tilde{z}_n) + d^2(z_n, \tilde{z}_n)).$$

Thus,

$$\begin{aligned} &d^2(z_n, p) \leq d^2(x_n, p) - d^2(x_n, \tilde{z}_n) - d^2(\tilde{z}_n, z_n) \\ &\quad + \tfrac{\mu_{2,n}}{\mu_{2,n+1}}\lambda_2(d^2(x_n, \tilde{z}_n) + d^2(z_n, \tilde{z}_n)) - 2\mu_{2,n}\langle A_2p, \exp^{-1}_p \tilde{z}_n \rangle \\ &= d^2(x_n, p) - (1 - \tfrac{\mu_{2,n}}{\mu_{2,n+1}}\lambda_2)d^2(x_n, \tilde{z}_n) - (1 - \tfrac{\mu_{2,n}}{\mu_{2,n+1}}\lambda_2)d^2(\tilde{z}_n, z_n) - 2\mu_{2,n}\langle A_2p, \exp^{-1}_p \tilde{z}_n \rangle. \end{aligned} \tag{18}$$

In a similar way, we get

$$\begin{aligned} &d^2(x_{n+1}, q) \\ &\leq d^2(z_n, q) - (1 - \tfrac{\mu_{1,n}}{\mu_{1,n+1}}\lambda_1)d^2(z_n, y_n) - (1 - \tfrac{\mu_{1,n}}{\mu_{1,n+1}}\lambda_1)d^2(y_n, x_{n+1}) - 2\mu_{1,n}\langle A_1q, \exp^{-1}_q y_n \rangle. \end{aligned} \tag{19}$$

Note that the limit $\lim_{n\to\infty} \frac{\mu_{k,n}}{\mu_{k,n+1}}\lambda_k = \lambda_k \in (0,1)$ for $k = 1,2$. Hence, there exists $n_0 \geq 0$ such that $\frac{\mu_{k,n}}{\mu_{k,n+1}}\lambda_k \in (0,1)$ $\forall n \geq n_0$ for $k = 1,2$.

Next, we restrict $(p,q) \in \mathcal{S}$. Then, substituting (18) for (19) with $q := p$, we obtain that, for all $n \geq n_0$,

$$\begin{aligned} d^2(x_{n+1}, p) &\leq d^2(x_n, p) - (1 - \tfrac{\mu_{2,n}}{\mu_{2,n+1}}\lambda_2)d^2(\tilde{z}_n, x_n) - 2\mu_{2,n}\langle A_2p, \exp^{-1}_p \tilde{z}_n \rangle \\ &\quad - (1 - \tfrac{\mu_{1,n}}{\mu_{1,n+1}}\lambda_1)d^2(y_n, z_n) - 2\mu_{1,n}\langle A_1p, \exp^{-1}_p y_n \rangle \\ &= d^2(x_n, p) - (1 - \tfrac{\mu_{2,n}}{\mu_{2,n+1}}\lambda_2)d^2(\tilde{z}_n, x_n) - (1 - \tfrac{\mu_{1,n}}{\mu_{1,n+1}}\lambda_1)d^2(y_n, z_n) \\ &\quad - 2\mu_{2,n}\langle A_2p, \exp^{-1}_p \tilde{z}_n \rangle - 2\mu_{1,n}\langle A_1p, \exp^{-1}_p y_n \rangle. \end{aligned}$$

This, together with the assumptions, guarantees that $d(x_{n+1}, p) \le d(x_n, p)\ \forall n \ge n_0$. Thus, the sequence $\{x_n\}$ is bounded.

In the same way, substituting (19) for (18) with $n := n+1$ and $p := q$, we obtain that, for all $n \ge n_0$,

$$\begin{aligned} d^2(z_{n+1}, q) &\le d^2(z_n, q) - (1 - \tfrac{\mu_{1,n}}{\mu_{1,n+1}}\lambda_1)d^2(y_n, z_n) - 2\mu_{1,n}\langle A_1 q, \exp_q^{-1} y_n\rangle \\ &\quad - (1 - \tfrac{\mu_{2,n+1}}{\mu_{2,n+2}}\lambda_2)d^2(\tilde{z}_{n+1}, x_{n+1}) - 2\mu_{2,n+1}\langle A_2 q, \exp_q^{-1} \tilde{z}_{n+1}\rangle \\ &= d^2(z_n, q) - (1 - \tfrac{\mu_{2,n+1}}{\mu_{2,n+2}}\lambda_2)d^2(\tilde{z}_{n+1}, x_{n+1}) - (1 - \tfrac{\mu_{1,n}}{\mu_{1,n+1}}\lambda_1)d^2(y_n, z_n) \\ &\quad - 2\mu_{2,n+1}\langle A_2 q, \exp_q^{-1} \tilde{z}_{n+1}\rangle - 2\mu_{1,n}\langle A_1 q, \exp_q^{-1} y_n\rangle. \end{aligned}$$

This, together with the assumptions, guarantees that $d(z_{n+1}, q) \le d(z_n, q)$. Thus, the sequence $\{z_n\}$ is bounded. □

Corollary 2. *Let the sequences $\{x_n\}$ and $\{z_n\}$ be generated by Algorithm 2. Then, $\{x_n\}$ and $\{z_n\}$ both are bounded sequences.*

Proof. We denote by S the solution set of VIP (4). Take a fixed $p \in S$ arbitrarily. Noticing $A_1 = A_2 = A$, we deduce from (18) and (19) that, for each $n \ge n_0$, $d(x_{n+1}, p) \le d(x_n, p)$, $d(z_{n+1}, p) \le d(z_n, p)$, and

$$\begin{aligned} d^2(x_{n+1}, p) &\le d^2(x_n, p) - (1 - \tfrac{\mu_{2,n}}{\mu_{2,n+1}}\lambda_2)d^2(x_n, \tilde{z}_n) - (1 - \tfrac{\mu_{2,n}}{\mu_{2,n+1}}\lambda_2)d^2(\tilde{z}_n, z_n) \\ &\quad - (1 - \tfrac{\mu_{1,n}}{\mu_{1,n+1}}\lambda_1)d^2(z_n, y_n) - (1 - \tfrac{\mu_{1,n}}{\mu_{1,n+1}}\lambda_1)d^2(y_n, x_{n+1}). \end{aligned}$$

Hence, $\{x_n\}$ and $\{z_n\}$ both are bounded sequences. Moreover, it is clear that, for all $n \ge n_0$,

$$\begin{aligned} &(1 - \tfrac{\mu_{2,n}}{\mu_{2,n+1}}\lambda_2)d^2(x_n, \tilde{z}_n) + (1 - \tfrac{\mu_{2,n}}{\mu_{2,n+1}}\lambda_2)d^2(\tilde{z}_n, z_n) \\ &\quad + (1 - \tfrac{\mu_{1,n}}{\mu_{1,n+1}}\lambda_1)d^2(z_n, y_n) + (1 - \tfrac{\mu_{1,n}}{\mu_{1,n+1}}\lambda_1)d^2(y_n, x_{n+1}) \\ &\le d^2(x_n, p) - d^2(x_{n+1}, p). \end{aligned}$$

Since $\lim_{n\to\infty} \frac{\mu_{i,n}}{\mu_{i,n+1}}\lambda_i = \lambda_i \in (0,1)$ for $i = 1,2$, we conclude that $d(x_n, \tilde{z}_n) \to 0$, $d(\tilde{z}_n, z_n) \to 0$, $d(z_n, y_n) \to 0$ and $d(y_n, x_{n+1}) \to 0$ as $n \to \infty$. □

Theorem 1. *Let the sequences $\{x_n\}$ and $\{z_n\}$ be generated by Algorithm 1. Suppose that the conditions in Lemma 10 hold. Then, $\{(x_n, z_n)\}$ converges to a solution of SVI (5) provided $\lim_{n\to\infty}\{d(x_n, y_n) + d(z_n, \tilde{z}_n)\} = 0$.*

Proof. First of all, by Lemma 9, we have $\mu_j := \lim_{n\to\infty} \mu_{j,n} \ge \min\{\frac{\lambda_j}{L_j}, \mu_{j,0}\}$ for $j = 1,2$. Moreover, by Lemma 10, we know that $\{z_n\}$ and $\{x_n\}$ both are bounded, and that, for all $n \ge n_0$,

$$d(z_{n+1}, q) \le d(z_n, q) \quad \text{and} \quad d(x_{n+1}, p) \le d(x_n, p) \quad \forall (p, q) \in \mathcal{S}.$$

Noticing $\lim_{n\to\infty}\{d(x_n, y_n) + d(z_n, \tilde{z}_n)\} = 0$ (due to the assumption), we deduce that $\{\tilde{z}_n\}$ and $\{y_n\}$ both are bounded. We define the sets S_1, S_2 as follows:

$$S_1 = \{p \in C : \exists q \in C \text{ s.t. } (p, q) \in \mathcal{S}\} \ \text{ and } \ S_2 = \{q \in C : \exists p \in C \text{ s.t. } (p, q) \in \mathcal{S}\}.$$

From Definition 3, we know that $\{x_n\}$ and $\{z_n\}$ are Fejér convergent to S_1 and S_2, respectively. Let $\bar{p}$ be a cluster point of $\{x_n\}$. Then, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = \bar{p}$. From the boundedness of $\{z_n\}$, we might assume that $z_{n_k} \to \bar{q}$ as $k \to \infty$. Since $\lim_{n\to\infty}\{d(x_n, y_n) + d(z_n, \tilde{z}_n)\} = 0$, we obtain that $\tilde{z}_{n_k} \to \bar{q}$ and $y_{n_k} \to \bar{p}$. In addition, noticing

$$\begin{cases} \tilde{z}_{n_k} = P_C(\exp_{x_{n_k}}(-\mu_{2,n_k} A_2 x_{n_k})), \\ y_{n_k} = P_C(\exp_{z_{n_k}}(-\mu_{1,n_k} A_1 z_{n_k})), \end{cases}$$

by Proposition 3, we infer that, for all $x \in C$,

$$\begin{aligned} 0 &\leq \langle \exp^{-1}_{x_{n_k}} \tilde{z}_{n_k} + \mu_{2,n_k} A_2 x_{n_k}, \exp^{-1}_{\tilde{z}_{n_k}} x \rangle \\ &= \langle \exp^{-1}_{x_{n_k}} \tilde{z}_{n_k}, \exp^{-1}_{\tilde{z}_{n_k}} x \rangle + \mu_{2,n_k} \langle A_2 x_{n_k}, \exp^{-1}_{\tilde{z}_{n_k}} x \rangle \\ &= \langle \exp^{-1}_{x_{n_k}} \tilde{z}_{n_k}, \exp^{-1}_{\tilde{z}_{n_k}} x \rangle + \mu_{2,n_k} \langle A_2 x_{n_k}, \exp^{-1}_{\tilde{z}_{n_k}} x_{n_k} \rangle + \mu_{2,n_k} \langle A_2 x_{n_k}, \exp^{-1}_{x_{n_k}} x \rangle, \end{aligned} \tag{20}$$

and

$$\begin{aligned} 0 &\leq \langle \exp^{-1}_{z_{n_k}} y_{n_k} + \mu_{1,n_k} A_1 z_{n_k}, \exp^{-1}_{y_{n_k}} x \rangle \\ &= \langle \exp^{-1}_{z_{n_k}} y_{n_k}, \exp^{-1}_{y_{n_k}} x \rangle + \mu_{1,n_k} \langle A_1 z_{n_k}, \exp^{-1}_{y_{n_k}} x \rangle \\ &= \langle \exp^{-1}_{z_{n_k}} y_{n_k}, \exp^{-1}_{y_{n_k}} x \rangle + \mu_{1,n_k} \langle A_1 z_{n_k}, \exp^{-1}_{y_{n_k}} z_{n_k} \rangle + \mu_{1,n_k} \langle A_1 z_{n_k}, \exp^{-1}_{z_{n_k}} x \rangle. \end{aligned} \tag{21}$$

Note that $\lim_{k\to\infty}\{d(x_{n_k}, y_{n_k}) + d(z_{n_k}, \tilde{z}_{n_k})\} = 0$, the subsequences $\{y_{n_k}\}, \{\tilde{z}_{n_k}\}$ are bounded, and $\lim_{n\to\infty} \mu_{j,n} = \mu_j > 0$ for $j = 1, 2$. Letting $k \to \infty$, we take the limits in (20) and (21) and hence get

$$\begin{cases} 0 \leq \langle \exp^{-1}_{\bar{p}} \bar{q}, \exp^{-1}_{\bar{q}} x \rangle + \mu_2 \langle A_2 \bar{p}, \exp^{-1}_{\bar{q}} \bar{p} \rangle + \mu_2 \langle A_2 \bar{p}, \exp^{-1}_{\bar{p}} x \rangle, \\ 0 \leq \langle \exp^{-1}_{\bar{q}} \bar{p}, \exp^{-1}_{\bar{p}} x \rangle + \mu_1 \langle A_1 \bar{q}, \exp^{-1}_{\bar{p}} \bar{q} \rangle + \mu_1 \langle A_1 \bar{q}, \exp^{-1}_{\bar{q}} x \rangle. \end{cases}$$

Therefore,

$$\begin{cases} \langle \exp^{-1}_{\bar{q}} \bar{p} + \mu_1 \langle A_1 \bar{q}, \exp^{-1}_{\bar{p}} x \rangle \geq 0 \quad \forall x \in C, \\ \langle \exp^{-1}_{\bar{p}} \bar{q} + \mu_2 \langle A_2 \bar{p}, \exp^{-1}_{\bar{q}} x \rangle \geq 0 \quad \forall x \in C. \end{cases} \tag{22}$$

This leads to $(\bar{p}, \bar{q}) \in \mathcal{S}$, and hence $\bar{p} \in S_1$. Thus, by Proposition 6, we obtain that $x_n \to \bar{p}$ as $n \to \infty$.

Next, let $\hat{q}$ be a cluster point of $\{z_n\}$. It is known that there exists a subsequence $\{z_{m_k}\} \subset \{z_n\}$ such that $\lim_{k\to\infty} z_{m_k} = \hat{q}$. Using the boundedness of $\{x_n\}$, we might assume that $x_{m_k} \to \hat{p}$ as $k \to \infty$. Thanks to $\lim_{n\to\infty}\{d(x_n, y_n) + d(z_n, \tilde{z}_n)\} = 0$, we obtain that $\tilde{z}_{m_k} \to \hat{q}$ and $y_{m_k} \to \hat{p}$. Noticing

$$\begin{cases} \tilde{z}_{m_k} = P_C(\exp_{x_{m_k}}(-\mu_{2,m_k} A_2 x_{m_k})), \\ y_{m_k} = P_C(\exp_{z_{m_k}}(-\mu_{1,m_k} A_1 z_{m_k})), \end{cases}$$

by similar arguments to those of (22), we deduce that

$$\begin{cases} \langle \exp^{-1}_{\hat{q}} \hat{p} + \mu_1 \langle A_1 \hat{q}, \exp^{-1}_{\hat{p}} x \rangle \geq 0 \quad \forall x \in C, \\ \langle \exp^{-1}_{\hat{p}} \hat{q} + \mu_2 \langle A_2 \hat{p}, \exp^{-1}_{\hat{q}} x \rangle \geq 0 \quad \forall x \in C. \end{cases} \tag{23}$$

This yields $(\hat{p}, \hat{q}) \in \mathcal{S}$, and hence $\hat{q} \in S_2$. Thus, by Proposition 6, we obtain that $z_n \to \hat{q}$ as $n \to \infty$. Consequently, using the uniqueness of the limit, we infer that $\{(x_n, z_n)\}$ is convergent to a solution $(\hat{p}, \hat{q}) \in \mathcal{S}$ of SVI (5). □

Theorem 2. *Suppose that the sequences $\{x_n\}$ and $\{z_n\}$ both are generated by Algorithm 2. Then, $\{x_n\}$ and $\{z_n\}$ both converge to a solution of VIP (4).*

Proof. Using Definition 3 and Corollary 2, we deduce that $\{z_n\}$ and $\{x_n\}$ both are Fejér convergent to the same S. Let $\bar{p}$ be a cluster point of $\{x_n\}$. It is known that $\exists\{x_{n_k}\} \subset \{x_n\}$ s.t. $\lim_{k\to\infty} x_{n_k} = \bar{p}$. Then, using $\lim_{k\to\infty} d(x_{n_k}, \tilde{z}_{n_k}) = 0$, we have $\lim_{k\to\infty} \tilde{z}_{n_k} = \bar{p}$. Since $\lim_{k\to\infty} \mu_{2,n_k} = \mu_2$ and $\tilde{z}_{n_k} = P_C(\exp_{x_{n_k}}(-\mu_{2,n_k} A x_{n_k}))$, we obtain $\bar{p} = P_C(\exp_{\bar{p}}(-\mu_2 A \bar{p}))$. Hence, by Proposition 3, we get $\bar{p} \in S$. Thus, from Proposition 6, it follows that $x_n \to \bar{p}$ as $n \to \infty$. Similarly, we can infer that $z_n \to \bar{q}$ as $n \to \infty$ for some $\bar{q} \in S$. Using $\lim_{n\to\infty}\{d(x_n, \tilde{z}_n) + d(\tilde{z}_n, z_n)\} = 0$, we obtain the desired result. □

3.2. The Second Parallel Algorithm

Algorithm 3 is the second parallel algorithm for the SVI.

Algorithm 3: The second parallel algorithm for the SVI.

Initialization: Given $x_0, y_0, z_0, \tilde{z}_0 \in \mathcal{M}$ arbitrarily. Let $\mu_{k,0} = \mu_{k,1} > 0$ and $\lambda_k \in (0, \sqrt{2}-1)$ for $k = 1, 2$, and compute

$$\begin{cases} z_1 = P_C(\exp_{x_0}(-\mu_{2,0} A_2 \tilde{z}_0)), \\ x_1 = P_C(\exp_{z_0}(-\mu_{1,0} A_1 y_0)) \end{cases} \text{and} \quad \begin{cases} \tilde{z}_1 = P_C(\exp_{x_1}(-\mu_{2,1} A_2 y_0)), \\ y_1 = P_C(\exp_{z_1}(-\mu_{1,1} A_1 \tilde{z}_0)). \end{cases}$$

Iteration Steps: Compute x_{n+1} and z_{n+1} $(n \geq 1)$ below:
Step 1. Construct

$$\begin{cases} C_{2,n} = \{x \in \mathcal{M} : \langle \exp^{-1}_{\tilde{z}_n} x_n - \mu_{2,n} A_2 \tilde{z}_{n-1}, \exp^{-1}_{\tilde{z}_n} x \rangle \leq 0\}, \\ C_{1,n} = \{x \in \mathcal{M} : \langle \exp^{-1}_{y_n} z_n - \mu_{1,n} A_1 y_{n-1}, \exp^{-1}_{y_n} x \rangle \leq 0\}, \end{cases}$$

and calculate

$$\begin{cases} z_{n+1} = P_{C_{2,n}}(\exp_{x_n}(-\mu_{2,n} A_2 \tilde{z}_n)), \\ x_{n+1} = P_{C_{1,n}}(\exp_{z_n}(-\mu_{1,n} A_1 y_n)). \end{cases}$$

Step 2. Calculate

$$\begin{cases} \tilde{z}_{n+1} = P_C(\exp_{x_{n+1}}(-\mu_{2,n+1} A_2 y_n)), \\ y_{n+1} = P_C(\exp_{z_{n+1}}(-\mu_{1,n+1} A_1 \tilde{z}_n)), \end{cases} \text{where}$$

$$\begin{cases} \mu_{2,n+1} = \begin{cases} \min\{\frac{\lambda_2 d(\tilde{z}_n, \tilde{z}_{n-1})}{d(A_2 \tilde{z}_n, A_2 \tilde{z}_{n-1})}, \mu_{2,n}\} & \text{if } d(A_2 \tilde{z}_n, A_2 \tilde{z}_{n-1}) \neq 0, \\ \mu_{2,n} & \text{otherwise.} \end{cases} \\ \mu_{1,n+1} = \begin{cases} \min\{\frac{\lambda_1 d(y_n, y_{n-1})}{d(A_1 y_n, A_1 y_{n-1})}, \mu_{1,n}\} & \text{if } d(A_1 y_n, A_1 y_{n-1}) \neq 0, \\ \mu_{1,n} & \text{otherwise.} \end{cases} \end{cases} \tag{24}$$

Again, put $n := n + 1$ and go to Step 1.

In particular, putting $A_1 = A_2 = A$ in Algorithm 3, we obtain the following algorithm (Algorithm 4) for solving VIP (4).

Algorithm 4: The second parallel algorithm for the VIP.

Initialization: Given $x_0, y_0, z_0, \tilde{z}_0 \in \mathcal{M}$ arbitrarily. Let $\mu_{k,0} = \mu_{k,1} > 0$ and $\lambda_k \in (0, \sqrt{2}-1)$ for $k = 1, 2$, and compute

$$\begin{cases} z_1 = P_C(\exp_{x_0}(-\mu_{2,0} A \tilde{z}_0)), \\ \tilde{z}_1 = P_C(\exp_{z_1}(-\mu_{2,1} A \tilde{z}_0)) \end{cases} \text{and} \quad \begin{cases} x_1 = P_C(\exp_{z_0}(-\mu_{1,0} A y_0)), \\ y_1 = P_C(\exp_{x_1}(-\mu_{1,1} A y_0)). \end{cases}$$

Iteration Steps: Compute x_{n+1} and z_{n+1} $(n \geq 1)$ below:
Step 1. Construct

$$\begin{cases} C_{2,n} = \{x \in \mathcal{M} : \langle \exp^{-1}_{\tilde{z}_n} x_n - \mu_{2,n} A \tilde{z}_{n-1}, \exp^{-1}_{\tilde{z}_n} x \rangle \leq 0\}, \\ C_{1,n} = \{x \in \mathcal{M} : \langle \exp^{-1}_{y_n} z_n - \mu_{1,n} A y_{n-1}, \exp^{-1}_{y_n} x \rangle \leq 0\} \end{cases}$$

and calculate

$$\begin{cases} z_{n+1} = P_{C_{2,n}}(\exp_{x_n}(-\mu_{2,n} A \tilde{z}_n)), \\ x_{n+1} = P_{C_{1,n}}(\exp_{z_n}(-\mu_{1,n} A y_n)). \end{cases}$$

Step 2. Calculate

$$\begin{cases} \tilde{z}_{n+1} = P_C(\exp_{x_{n+1}}(-\mu_{2,n+1} A y_n)), \\ y_{n+1} = P_C(\exp_{z_{n+1}}(-\mu_{1,n+1} A \tilde{z}_n)), \end{cases}$$

where

$$\begin{cases} \mu_{2,n+1} = \begin{cases} \min\{\frac{\lambda_2 d(\tilde{z}_n, \tilde{z}_{n-1})}{d(A \tilde{z}_n, A \tilde{z}_{n-1})}, \mu_{2,n}\} & \text{if } d(A \tilde{z}_n, A \tilde{z}_{n-1}) \neq 0, \\ \mu_{2,n} & \text{otherwise.} \end{cases} \\ \mu_{1,n+1} = \begin{cases} \min\{\frac{\lambda_1 d(y_n, y_{n-1})}{d(A y_n, A y_{n-1})}, \mu_{1,n}\} & \text{if } d(A y_n, A y_{n-1}) \neq 0, \\ \mu_{1,n} & \text{otherwise.} \end{cases} \end{cases}$$

Again, put $n := n + 1$ and go to Step 1.

Lemma 11. *For $k = 1, 2$, the sequence $\{\mu_{k,n}\}$ generated by Algorithm 3 is monotonically decreasing with lower bound* $\min\{\frac{\lambda_k}{L_k}, \mu_{k,0}\}$.

Proof. It is clear that $\{\mu_{k,n}\}$ is monotonically decreasing for $k = 1, 2$. Note that A_k is a Lipschitzian mapping with constant $L_k > 0$ for $k = 1, 2$. Then, in the case of $d(A_2\tilde{z}_n, A_2\tilde{z}_{n-1}) \neq 0$, we have

$$\frac{\lambda_2 d(\tilde{z}_n, \tilde{z}_{n-1})}{d(A_2\tilde{z}_n, A_2\tilde{z}_{n-1})} \geq \frac{\lambda_2 d(\tilde{z}_n, \tilde{z}_{n-1})}{L_2 d(\tilde{z}_n, \tilde{z}_{n-1})} = \frac{\lambda_2}{L_2}.$$

Consequently, $\{\mu_{2,n}\}$ is the sequence with lower bound $\min\{\frac{\lambda_2}{L_2}, \mu_{2,0}\}$. Similarly, we can show that $\{\mu_{1,n}\}$ is the sequence with lower bound $\min\{\frac{\lambda_1}{L_1}, \mu_{1,0}\}$. $\square$

Corollary 3. *For $k = 1, 2$, the sequence $\{\mu_{k,n}\}$ generated by Algorithm 4 is monotonically decreasing with lower bound* $\min\{\frac{\lambda_k}{L}, \mu_{k,0}\}$.

Lemma 12. *Let $\{x_n\}$ and $\{z_n\}$ be the sequences generated by Algorithm 3. Then, the sequences $\{x_n\}$ and $\{z_n\}$ are bounded, provided for all $(p, q) \in \mathcal{S}$ and $n \geq n_0$,*

$$\begin{cases} (1 - \frac{(1+\sqrt{2})\lambda_2\mu_{2,n}}{\mu_{2,n+1}})d^2(z_n, \tilde{z}_n) + (1 - \frac{(1+\sqrt{2})\lambda_1\mu_{1,n}}{\mu_{1,n+1}})d^2(z_n, y_n) \\ \qquad + 2\mu_{2,n}\langle A_2 p, \exp_p^{-1} \tilde{z}_n\rangle + 2\mu_{1,n}\langle A_1 p, \exp_p^{-1} y_n\rangle \geq 0, \\ (1 - \frac{(1+\sqrt{2})\lambda_2\mu_{2,n}}{\mu_{2,n+1}})d^2(z_n, \tilde{z}_n) + (1 - \frac{(1+\sqrt{2})\lambda_1\mu_{1,n}}{\mu_{1,n+1}})d^2(z_n, y_n) \\ \qquad + 2\mu_{2,n}\langle A_2 q, \exp_q^{-1} \tilde{z}_n\rangle + 2\mu_{1,n}\langle A_1 q, \exp_q^{-1} y_n\rangle \geq 0. \end{cases}$$

Proof. Take $(p, q) \in C \times C$ arbitrarily. Utilizing the similar arguments to those in the proof of Lemma 10, we can deduce the following inequality:

$$\begin{aligned} d^2(z_{n+1}, p) \quad &\leq d^2(x_n, p) - d^2(z_{n+1}, \tilde{z}_n) - d^2(z_n, \tilde{z}_n) \\ &\quad + 2\mu_{2,n}\langle A_2\tilde{z}_{n-1} - A_2\tilde{z}_n, \exp_{\tilde{z}_n}^{-1} z_{n+1}\rangle - 2\mu_{2,n}\langle A_2 p, \exp_p^{-1} \tilde{z}_n\rangle. \end{aligned} \tag{25}$$

We now estimate the term $\langle A_2\tilde{z}_{n-1} - A_2\tilde{z}_n, \exp_{\tilde{z}_n}^{-1} z_{n+1}\rangle$ in (25). From (6), the definition of $\mu_{2,n+1}$ in Algorithm 3, we have

$$\begin{aligned} &2\mu_{2,n}\langle A_2\tilde{z}_{n-1} - A_2\tilde{z}_n, \exp_{\tilde{z}_n}^{-1} z_{n+1}\rangle \leq 2\mu_{2,n} d(A_2\tilde{z}_{n-1}, A_2\tilde{z}_n) d(z_{n+1}, \tilde{z}_n) \\ &\leq 2\mu_{2,n}\frac{\lambda_2 d(\tilde{z}_{n-1}, \tilde{z}_n)}{\mu_{2,n+1}} d(z_{n+1}, \tilde{z}_n) = \frac{2\lambda_2\mu_{2,n}}{\mu_{2,n+1}} d(\tilde{z}_{n-1}, \tilde{z}_n) d(z_{n+1}, \tilde{z}_n) \\ &\leq \frac{\lambda_2\mu_{2,n}}{\mu_{2,n+1}}(\frac{1}{\sqrt{2}} d^2(\tilde{z}_{n-1}, \tilde{z}_n) + \sqrt{2} d^2(z_{n+1}, \tilde{z}_n)). \end{aligned} \tag{26}$$

In the meantime, by the fact $(a + b)^2 \leq (2 + \sqrt{2})a^2 + \sqrt{2}b^2$, we get

$$d^2(\tilde{z}_{n-1}, \tilde{z}_n) \leq (d(\tilde{z}_n, z_n) + d(z_n, \tilde{z}_{n-1}))^2 \leq (\sqrt{2} + 2) d^2(\tilde{z}_n, z_n) + \sqrt{2} d^2(z_n, \tilde{z}_{n-1}). \tag{27}$$

From (26) and (27), it follows that

$$\begin{aligned} &2\mu_{2,n}\langle A_2\tilde{z}_{n-1} - A_2\tilde{z}_n, \exp_{\tilde{z}_n}^{-1} z_{n+1}\rangle \\ &\leq \frac{(1+\sqrt{2})\lambda_2\mu_{2,n}}{\mu_{2,n+1}} d^2(\tilde{z}_n, z_n) + \frac{\lambda_2\mu_{2,n}}{\mu_{2,n+1}} d^2(z_n, \tilde{z}_{n-1}) + \frac{\sqrt{2}\lambda_2\mu_{2,n}}{\mu_{2,n+1}} d^2(z_{n+1}, \tilde{z}_n). \end{aligned} \tag{28}$$

Substituting (28) for (25), we obtain

$$\begin{aligned} d^2(z_{n+1}, p) \quad &\leq d^2(x_n, p) + \frac{\lambda_2\mu_{2,n}}{\mu_{2,n+1}} d^2(z_n, \tilde{z}_{n-1}) - (1 - \frac{(1+\sqrt{2})\lambda_2\mu_{2,n}}{\mu_{2,n+1}}) d^2(z_n, \tilde{z}_n) \\ &\quad - (1 - \frac{\sqrt{2}\lambda_2\mu_{2,n}}{\mu_{2,n+1}}) d^2(z_{n+1}, \tilde{z}_n) - 2\mu_{2,n}\langle A_2 p, \exp_p^{-1} \tilde{z}_n\rangle. \end{aligned} \tag{29}$$

Adding $\frac{\lambda_2\mu_{2,n+1}}{\mu_{2,n+2}}d^2(z_{n+1},\tilde{z}_n)$ to both sides of (29), we get

$$\begin{aligned}&[d^2(z_{n+1},p)+\tfrac{\lambda_2\mu_{2,n+1}}{\mu_{2,n+2}}d^2(z_{n+1},\tilde{z}_n)]\\ &\leq [d^2(x_n,p)+\tfrac{\lambda_2\mu_{2,n}}{\mu_{2,n+1}}d^2(z_n,\tilde{z}_{n-1})]-(1-\tfrac{(1+\sqrt{2})\lambda_2\mu_{2,n}}{\mu_{2,n+1}})d^2(z_n,\tilde{z}_n)\\ &\quad-(1-\tfrac{\sqrt{2}\lambda_2\mu_{2,n}}{\mu_{2,n+1}}-\tfrac{\lambda_2\mu_{2,n+1}}{\mu_{2,n+2}})d^2(z_{n+1},\tilde{z}_n)-2\mu_{2,n}\langle A_2p,\exp_p^{-1}\tilde{z}_n\rangle.\end{aligned} \tag{30}$$

In a similar way, we get

$$\begin{aligned}&[d^2(x_{n+1},q)+\tfrac{\lambda_1\mu_{1,n+1}}{\mu_{1,n+2}}d^2(x_{n+1},y_n)]\\ &\leq [d^2(z_n,q)+\tfrac{\lambda_1\mu_{1,n}}{\mu_{1,n+1}}d^2(x_n,y_{n-1})]-(1-\tfrac{(1+\sqrt{2})\lambda_1\mu_{1,n}}{\mu_{1,n+1}})d^2(z_n,y_n)\\ &\quad-(1-\tfrac{\sqrt{2}\lambda_1\mu_{1,n}}{\mu_{1,n+1}}-\tfrac{\lambda_1\mu_{1,n+1}}{\mu_{1,n+2}})d^2(x_{n+1},y_n)-2\mu_{1,n}\langle A_1q,\exp_q^{-1}y_n\rangle.\end{aligned} \tag{31}$$

From $\lim_{n\to\infty}\mu_{2,n}=\mu_2>0$ (due to Lemma 11) and $\lambda_2\in(0,\sqrt{2}-1)$ (due to Algorithm 3), we get

$$\lim_{n\to\infty}(1-\frac{(1+\sqrt{2})\lambda_2\mu_{2,n}}{\mu_{2,n+1}})=\lim_{n\to\infty}(1-\frac{\sqrt{2}\lambda_2\mu_{2,n}}{\mu_{2,n+1}}-\frac{\lambda_2\mu_{2,n+1}}{\mu_{2,n+2}})=1-\lambda_2(1+\sqrt{2})>0. \tag{32}$$

Hence, there exists an integer $n_0\geq 0$ such that

$$1-\frac{(1+\sqrt{2})\lambda_2\mu_{2,n}}{\mu_{2,n+1}}>0 \quad\text{and}\quad 1-\frac{\sqrt{2}\lambda_2\mu_{2,n}}{\mu_{2,n+1}}-\frac{\lambda_2\mu_{2,n+1}}{\mu_{2,n+2}}>0 \quad \forall n\geq n_0. \tag{33}$$

Next, we restrict $(p,q)\in\mathcal{S}$. Assume that, for all $n\geq n_0$,

$$\begin{aligned}&(1-\tfrac{(1+\sqrt{2})\lambda_2\mu_{2,n}}{\mu_{2,n+1}})d^2(z_n,\tilde{z}_n)+(1-\tfrac{(1+\sqrt{2})\lambda_1\mu_{1,n}}{\mu_{1,n+1}})d^2(z_n,y_n)\\ &\quad+2\mu_{2,n}\langle A_2p,\exp_p^{-1}\tilde{z}_n\rangle+2\mu_{1,n}\langle A_1p,\exp_p^{-1}y_n\rangle\geq 0.\end{aligned} \tag{34}$$

Adding (30) to (31) with $q:=p$, we obtain that, for all $n\geq n_0$,

$$\begin{aligned}&[d^2(z_{n+1},p)+\tfrac{\lambda_2\mu_{2,n+1}}{\mu_{2,n+2}}d^2(z_{n+1},\tilde{z}_n)]+[d^2(x_{n+1},p)+\tfrac{\lambda_1\mu_{1,n+1}}{\mu_{1,n+2}}d^2(x_{n+1},y_n)]\\ &\leq [d^2(x_n,p)+\tfrac{\lambda_2\mu_{2,n}}{\mu_{2,n+1}}d^2(z_n,\tilde{z}_{n-1})]+[d^2(z_n,p)+\tfrac{\lambda_1\mu_{1,n}}{\mu_{1,n+1}}d^2(x_n,y_{n-1})].\end{aligned}$$

This implies that there exists the limit

$$\lim_{n\to\infty}\{[d^2(z_n,p)+\frac{\lambda_2\mu_{2,n}}{\mu_{2,n+1}}d^2(z_n,\tilde{z}_{n-1})]+[d^2(x_n,p)+\frac{\lambda_1\mu_{1,n}}{\mu_{1,n+1}}d^2(x_n,y_{n-1})]\}.$$

Hence, $\{d^2(z_n,p)\}$ and $\{d^2(x_n,p)\}$ both are bounded. Therefore, $\{z_n\}$ and $\{x_n\}$ both are bounded. In addition, again from (30), (31), and (34), we deduce that, for all $n\geq n_0$,

$$\begin{aligned}&(1-\tfrac{\sqrt{2}\lambda_2\mu_{2,n}}{\mu_{2,n+1}}-\tfrac{\lambda_2\mu_{2,n+1}}{\mu_{2,n+2}})d^2(z_{n+1},\tilde{z}_n)+(1-\tfrac{\sqrt{2}\lambda_1\mu_{1,n}}{\mu_{1,n+1}}-\tfrac{\lambda_1\mu_{1,n+1}}{\mu_{1,n+2}})d^2(x_{n+1},y_n)\\ &\leq [d^2(z_n,p)+\tfrac{\lambda_2\mu_{2,n}}{\mu_{2,n+1}}d^2(z_n,\tilde{z}_{n-1})]+[d^2(x_n,p)+\tfrac{\lambda_1\mu_{1,n}}{\mu_{1,n+1}}d^2(x_n,y_{n-1})]\\ &\quad-\{[d^2(z_{n+1},p)+\tfrac{\lambda_2\mu_{2,n+1}}{\mu_{2,n+2}}d^2(z_{n+1},\tilde{z}_n)]+[d^2(x_{n+1},p)+\tfrac{\lambda_1\mu_{1,n+1}}{\mu_{1,n+2}}d^2(x_{n+1},y_n)]\},\end{aligned}$$

which, together with (32), leads to

$$\lim_{n\to\infty}d(z_{n+1},\tilde{z}_n)=\lim_{n\to\infty}d(x_{n+1},y_n)=0. \tag{35}$$

Consequently, from the boundedness of $\{z_n\}$ and $\{x_n\}$, we infer that $\{\tilde{z}_n\}$ and $\{y_n\}$ both are bounded. Moreover, it follows that there exists the limit $\lim_{n\to\infty}(d^2(x_n,p)+$

$d^2(z_n, p))$ for each $(p, q) \in \mathcal{S}$. In a similar way, we also infer that there exists the limit $\lim_{n\to\infty}(d^2(x_n, q) + d^2(z_n, q))$ for each $(p, q) \in \mathcal{S}$. $\square$

Corollary 4. *Let $\{x_n\}$ and $\{z_n\}$ be the sequences generated by Algorithm 4. Then, the sequences $\{x_n\}$ and $\{z_n\}$ are bounded.*

Proof. Let S indicate the solution set of VIP (4) and fix $p \in S$ arbitrarily. Noticing $A_1 = A_2 = A$, we deduce from (30) and (31) that

$$\begin{cases} [d^2(z_{n+1}, p) + \frac{\lambda_2\mu_{2,n+1}}{\mu_{2,n+2}} d^2(z_{n+1}, \tilde{z}_n)] \le [d^2(x_n, p) + \frac{\lambda_2\mu_{2,n}}{\mu_{2,n+1}} d^2(z_n, \tilde{z}_{n-1})] \\ \quad - (1 - \frac{(1+\sqrt{2})\lambda_2\mu_{2,n}}{\mu_{2,n+1}}) d^2(z_n, \tilde{z}_n) - (1 - \frac{\sqrt{2}\lambda_2\mu_{2,n}}{\mu_{2,n+1}} - \frac{\lambda_2\mu_{2,n+1}}{\mu_{2,n+2}}) d^2(z_{n+1}, \tilde{z}_n), \\ [d^2(x_{n+1}, p) + \frac{\lambda_1\mu_{1,n+1}}{\mu_{1,n+2}} d^2(x_{n+1}, y_n)] \le [d^2(z_n, p) + \frac{\lambda_1\mu_{1,n}}{\mu_{1,n+1}} d^2(x_n, y_{n-1})] \\ \quad - (1 - \frac{(1+\sqrt{2})\lambda_1\mu_{1,n}}{\mu_{1,n+1}}) d^2(z_n, y_n) - (1 - \frac{\sqrt{2}\lambda_1\mu_{1,n}}{\mu_{1,n+1}} - \frac{\lambda_1\mu_{1,n+1}}{\mu_{1,n+2}}) d^2(x_{n+1}, y_n). \end{cases}$$

Since $\lim_{n\to\infty}(1 - \frac{(1+\sqrt{2})\lambda_k\mu_{k,n}}{\mu_{k,n+1}}) = \lim_{n\to\infty}(1 - \frac{\sqrt{2}\lambda_k\mu_{k,n}}{\mu_{k,n+1}} - \frac{\lambda_k\mu_{k,n+1}}{\mu_{k,n+2}}) = 1 - \lambda_k(1+\sqrt{2}) > 0$ for $k = 1, 2$, we know that there exists an integer $n_0 \ge 0$ such that $1 - \frac{(1+\sqrt{2})\lambda_k\mu_{k,n}}{\mu_{k,n+1}} > 0$ and $1 - \frac{\sqrt{2}\lambda_k\mu_{k,n}}{\mu_{k,n+1}} - \frac{\lambda_k\mu_{k,n+1}}{\mu_{k,n+2}} > 0$ for all $n \ge n_0$. Thus, it follows that, for all $n \ge n_0$,

$$\begin{aligned} &[d^2(z_{n+1}, p) + \frac{\lambda_2\mu_{2,n+1}}{\mu_{2,n+2}} d^2(z_{n+1}, \tilde{z}_n)] + [d^2(x_{n+1}, p) + \frac{\lambda_1\mu_{1,n+1}}{\mu_{1,n+2}} d^2(x_{n+1}, y_n)] \\ &\le [d^2(z_n, p) + \frac{\lambda_2\mu_{2,n}}{\mu_{2,n+1}} d^2(z_n, \tilde{z}_{n-1})] + [d^2(x_n, p) + \frac{\lambda_1\mu_{1,n}}{\mu_{1,n+1}} d^2(x_n, y_{n-1})]. \end{aligned}$$

This implies that there exists the limit

$$\lim_{n\to\infty}\{[d^2(z_n, p) + \frac{\lambda_2\mu_{2,n}}{\mu_{2,n+1}} d^2(z_n, \tilde{z}_{n-1})] + [d^2(x_n, p) + \frac{\lambda_1\mu_{1,n}}{\mu_{1,n+1}} d^2(x_n, y_{n-1})]\}.$$

Therefore, $\{z_n\}$ and $\{x_n\}$ both are bounded. Moreover, it is easy to see that $\lim_{n\to\infty} d(z_n, \tilde{z}_n)$ $=\lim_{n\to\infty} d(z_{n+1}, \tilde{z}_n) = 0$ and $\lim_{n\to\infty} d(z_n, y_n) = \lim_{n\to\infty} d(x_{n+1}, y_n) = 0$. $\square$

Theorem 3. *Let the sequences $\{x_n\}$, $\{z_n\}$ be generated by Algorithm 3. Assume that the conditions in Lemma 12 hold. Then, $\{(x_n, z_n)\}$ converges to a solution of SVI (5) provided $\lim_{n\to\infty}\{d(x_n, y_n) + d(z_n, \tilde{z}_n)\} = 0$ and $\lim_{n\to\infty}\{d^2(z_n, p) + d^2(x_n, q)\} < +\infty$ for all $(p, q) \in \mathcal{S}$.*

Proof. First of all, by Lemma 11, we have $\lim_{n\to\infty} \mu_{k,n} = \mu_k > 0$ for $k = 1, 2$. Using Lemma 12, we obtain the boundedness of the sequences $\{x_n\}$, $\{z_n\}$, and the existence of the limits $\lim_{n\to\infty}(d^2(x_n, p) + d^2(z_n, p))$ and $\lim_{n\to\infty}(d^2(x_n, q) + d^2(z_n, q))$ for each $(p, q) \in \mathcal{S}$. We observe that, for each $(p, q) \in \mathcal{S}$,

$$\begin{aligned} &\lim_{n\to\infty}(d^2(x_n, p) + d^2(z_n, q)) \\ &= \lim_{n\to\infty}[d^2(x_n, p) + d^2(z_n, p) + d^2(x_n, q) + d^2(z_n, q) - (d^2(z_n, p) + d^2(x_n, q))] \\ &= \lim_{n\to\infty}(d^2(x_n, p) + d^2(z_n, p)) + \lim_{n\to\infty}(d^2(x_n, q) + d^2(z_n, q)) - \lim_{n\to\infty}(d^2(z_n, p) + d^2(x_n, q)) \\ &< +\infty. \end{aligned}$$

We claim that each cluster point of $\{(x_n, z_n)\}$ belongs to $\mathcal{S}$. Indeed, since $\{(x_n, z_n)\}$ is bounded, there exists a subsequence $\{(x_{m_k}, z_{m_k})\}$ of $\{(x_n, z_n)\}$ converging to $(x^*, y^*) \in \mathcal{M} \times \mathcal{M}$. This means that $x_{m_k} \to x^*$ and $z_{m_k} \to y^*$. It is clear that $y_{m_k} \to x^*$ and $\tilde{z}_{m_k} \to y^*$ because $d(x_{m_k}, y_{m_k}) \to 0$ and $d(z_{m_k}, \tilde{z}_{m_k}) \to 0$ as $k \to \infty$. Since C is closed and convex in $\mathcal{M}$, from $\{(y_{m_k}, \tilde{z}_{m_k})\} \subset C \times C$, we get $(x^*, y^*) \in C \times C$. Taking into account that $d(z_n, \tilde{z}_n) \to 0$ and $d(x_n, y_n) \to 0$ as $n \to \infty$, we infer from (35) that $d(\tilde{z}_n, \tilde{z}_{n+1}) \to 0$ and $d(y_n, y_{n+1}) \to 0$ as $n \to \infty$.

Noticing that $\tilde{z}_n = P_C(\exp_{x_n}(-\mu_{2,n}A_2y_{n-1}))$ and $y_n = P_C(\exp_{z_n}(-\mu_{1,n}A_1\tilde{z}_{n-1}))$, from Proposition 3, we get

$$\begin{cases} \langle \exp_{x_n}^{-1}\tilde{z}_n + \mu_{2,n}A_2y_{n-1}, \exp_{\tilde{z}_n}^{-1}x\rangle \geq 0 \quad \forall x \in C, \\ \langle \exp_{z_n}^{-1}y_n + \mu_{1,n}A_1\tilde{z}_{n-1}, \exp_{y_n}^{-1}x\rangle \geq 0 \quad \forall x \in C. \end{cases}$$

Hence, we have

$$\begin{cases} 0 \leq \langle \exp_{x_n}^{-1}\tilde{z}_n, \exp_{\tilde{z}_n}^{-1}x\rangle + \mu_{2,n}\langle A_2y_{n-1}, \exp_{\tilde{z}_n}^{-1}x\rangle \\ \quad = \langle \exp_{x_n}^{-1}\tilde{z}_n, \exp_{\tilde{z}_n}^{-1}x\rangle + \mu_{2,n}\langle A_2y_{n-1}, \exp_{\tilde{z}_{n-1}}^{-1}x\rangle + \mu_{2,n}\langle A_2y_{n-1}, \exp_{\tilde{z}_n}^{-1}\tilde{z}_{n-1}\rangle, \\ 0 \leq \langle \exp_{z_n}^{-1}y_n, \exp_{y_n}^{-1}x\rangle + \mu_{1,n}\langle A_1\tilde{z}_{n-1}, \exp_{y_n}^{-1}x\rangle \\ \quad = \langle \exp_{z_n}^{-1}y_n, \exp_{y_n}^{-1}x\rangle + \mu_{1,n}\langle A_1\tilde{z}_{n-1}, \exp_{y_{n-1}}^{-1}x\rangle + \mu_{1,n}\langle A_1\tilde{z}_{n-1}, \exp_{y_n}^{-1}y_{n-1}\rangle. \end{cases} \tag{36}$$

Passing to the limits in two inequalities of (36) as $n := m_k \to \infty$, we get

$$\begin{cases} 0 \leq \langle \exp_{x^*}^{-1}y^*, \exp_{y^*}^{-1}x\rangle + \mu_2\langle A_2x^*, \exp_{y^*}^{-1}x\rangle \quad \forall x \in C, \\ 0 \leq \langle \exp_{y^*}^{-1}x^*, \exp_{x^*}^{-1}x\rangle + \mu_1\langle A_1y^*, \exp_{x^*}^{-1}x\rangle \quad \forall x \in C. \end{cases}$$

This means that (x^*, y^*) is a solution to the SVI (5), i.e., $(x^*, y^*) \in \mathcal{S}$.

For the rest of the proof, it is sufficient to show that the sequence $\{(x_n, z_n)\}$ only has a cluster point. Indeed, suppose that $\{(x_n, z_n)\}$ has at least two cluster points $(\bar{x}, \bar{y}), (\hat{x}, \hat{y}) \in \mathcal{S}$. Then, there exist two subsequences $\{(x_{n_i}, z_{n_i})\}$ and $\{(x_{m_i}, z_{m_i})\}$ of $\{(x_n, z_n)\}$ such that $\{(x_{n_i}, z_{n_i}) \to (\bar{x}, \bar{y})$ and $(x_{m_i}, z_{m_i}) \to (\hat{x}, \hat{y})$ as $i \to \infty$. By Proposition 2, we get

$$\begin{aligned} &\lim_{n\to\infty}(d^2(x_n, \hat{x}) + d^2(z_n, \hat{y})) = \lim_{i\to\infty}(d^2(x_{n_i}, \hat{x}) + d^2(z_{n_i}, \hat{y})) \\ &\geq \lim_{i\to\infty}[d^2(x_{n_i}, \bar{x}) + d^2(\bar{x}, \hat{x}) - 2\langle \exp_{\bar{x}}^{-1}x_{n_i}, \exp_{\bar{x}}^{-1}\hat{x}\rangle \\ &\quad + d^2(z_{n_i}, \bar{y}) + d^2(\bar{y}, \hat{y}) - 2\langle \exp_{\bar{y}}^{-1}z_{n_i}, \exp_{\bar{y}}^{-1}\hat{y}\rangle] \\ &= \lim_{n\to\infty}(d^2(x_n, \bar{x}) + d^2(z_n, \bar{y})) + d^2(\bar{x}, \hat{x}) + d^2(\bar{y}, \hat{y}), \end{aligned} \tag{37}$$

and

$$\begin{aligned} &\lim_{n\to\infty}(d^2(x_n, \bar{x}) + d^2(z_n, \bar{y})) = \lim_{i\to\infty}(d^2(x_{m_i}, \bar{x}) + d^2(z_{m_i}, \bar{y})) \\ &\geq \lim_{i\to\infty}[d^2(x_{m_i}, \hat{x}) + d^2(\hat{x}, \bar{x}) - 2\langle \exp_{\hat{x}}^{-1}x_{m_i}, \exp_{\hat{x}}^{-1}\bar{x}\rangle \\ &\quad + d^2(z_{m_i}, \bar{y}) + d^2(\hat{y}, \bar{y}) - 2\langle \exp_{\hat{y}}^{-1}z_{m_i}, \exp_{\hat{y}}^{-1}\bar{y}\rangle] \\ &= \lim_{n\to\infty}(d^2(x_n, \hat{x}) + d^2(z_n, \hat{y})) + d^2(\hat{x}, \bar{x}) + d^2(\hat{y}, \bar{y}). \end{aligned} \tag{38}$$

Combining (37) and (38), we have $\bar{x} = \hat{x}$ and $\bar{y} = \hat{y}$. □

Theorem 4. *Suppose that the sequences $\{x_n\}$ and $\{z_n\}$ both are generated by Algorithm 4. Then, $\{x_n\}$ and $\{z_n\}$ both converge to a solution of VIP (4).*

Proof. By Corollary 4, we know that $\{x_n\}$ and $\{z_n\}$ are bounded. Putting $A_1 = A_2 = A$ and $p = q \in S$ in (30) and (31), we deduce that

$$\begin{cases} \lim_{n\to\infty} d(z_n, \tilde{z}_n) = \lim_{n\to\infty} d(z_{n+1}, \tilde{z}_n) = 0, \\ \lim_{n\to\infty} d(z_n, y_n) = \lim_{n\to\infty} d(x_{n+1}, y_n) = 0. \end{cases}$$

Thus, it follows that $\lim_{n\to\infty} d(z_n, z_{n+1}) = \lim_{n\to\infty} d(z_n, x_{n+1}) = 0$. Note that $d(y_{n+1}, y_n) \leq d(y_{n+1}, z_{n+1}) + d(z_{n+1}, z_n) + d(z_n, y_n) \to 0$ $(n \to \infty)$. Thus, we have $d(x_{n+1}, y_{n+1}) \leq d(x_{n+1}, y_n) + d(y_n, y_{n+1})$ $(n \to \infty)$, and hence $\lim_{n\to\infty} d(x_n, y_n) = 0$. In addition, since $d(x_{n+1}, z_{n+1}) \leq d(x_{n+1}, z_n) + d(z_n, z_{n+1})$ $(n \to \infty)$, we get $\lim_{n\to\infty} d(x_{n+1}, z_{n+1}) = 0$, and hence $\lim_{n\to\infty} d(x_n, z_n) = 0$. Note that the SVI (5) with $A_1 = A_2 = A$ has a solution $(p, p) \in C \times C$ if and only if the VIP (4) has solution $p \in C$. Therefore, by Theorem 3, we know that $\{(x_n, z_n)\}$ converges to a solution $(x^*, y^*) \in C \times C$

to the SVI (5) with $A_1 = A_2 = A$. Thus, from $\lim_{n\to\infty} d(x_n, z_n) = 0$, it follows that $\{x_n\}$ and $\{z_n\}$ both are convergent to a solution $x^* = y^* \in C$ to the VIP (4) $\square$

Author Contributions: Conceptualization, L.H. and H.-L.F.; methodology, L.H.; software, H.-L.F.; validation, H.-Y.H., T.-Y.Z. and D.-Q.W.; formal analysis, L.H.; investigation, C.-Y.W. and L.-C.C.; resources, L.-C.C.; data curation, H.-Y.H.; writing original draft preparation, C.-Y.W.; writing review and editing, L.-C.C.; supervision, L.-C.C.; project administration, L.-C.C.; funding acquisition, L.-C.C. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the 2020 Shanghai Leading Talents Program of the Shanghai Municipal Human Resources and Social Security Bureau, 20LJ2006100; the Innovation Program of Shanghai Municipal Education Commission, 15ZZ068; and the Program for Outstanding Academic Leaders in Shanghai City, 15XD1503100.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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