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Linearizability of 2:−3 Resonant Systems with Quadratic Nonlinearities

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Abstract: In this paper, the linearizability of a 2:−3 resonant system with quadratic nonlinearities is studied. We provide a list of the conditions for this family of systems having a linearizable center. The conditions for linearizablity are obtained by computing the ideal generated by the linearizability quantities and its decomposition into associate primes. To successfully perform the calculations, we use an approach based on modular computations. The sufficiency of the obtained conditions is proven by several methods, mainly by the method of Darboux linearization.

Keywords: polynomial systems of ODEs; linearizability problem; linearizability quantities; Darboux linearization; p:$-q$ resonant system

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1. Introduction

The linearizability problem of a 1:−1 resonant system is intimately connected to the isochronicity problem, where we assume that the singularity in question is known to be a center and determine whether it is isochronous, that is, whether every periodic solution in a neighborhood of the origin has the same period [1]. This phenomena is closely related to the mirror symmetry if considered in the physical plane. The history of isochronicity goes back to the clocks based on some periodic motion (such as the swinging of a pendulum). In the 17th century, Huygens designed a pendulum clocks with cycloidal "cheeks", which is probably the earliest example of a nonlinear isochronous system. Then, in second half of 20th century, the interest of isochronicity in planar systems of ODEs was renewed. Both the isochronicity problem and linearizability problem represent very important problems in dynamical systems, which describes real-life phenomena in different branches of science, such as biochemistry, biology, physics, engineering, etc.

There are several methods on how to study the linearizability problem. An interesting approach is based on Lie symmetries [2]. Lie proved that the necessary and sufficient conditions for a scalar nonlinear ODEs to be linearizable is that it must have eight Lie point symmetries [3,4]. In [5], the authors considered linearizability of systems of ODEs obtained by complex symmetry analysis. In this paper, for a certain type of systems, i.e., p:$-q$ resonant systems, we use approach based on computation of linearizability quantities.

The linearzability problem is one of the main problems in the qualitative theory of ODEs, and it is closely connected to the center problem. The p:$-q$ resonant center problem is a generalization of the center problem to complex systems: in the context that classical center problem for complex systems may be referred as the 1:−1 resonant center problem, which can be obtained as a complexification [1] of the real systems. Note that the complexification is based on the involution of complex numbers for which the geometric meaning is the mirror reflection of the complex plane over the real axis.

A generalization of the classical center problem to a $p{:}-q$ resonant center problem was proposed in [6] to the following differential systems in $\mathbb{C}^2$

$$\dot{x} = px + P(x,y), \quad \dot{y} = -qy + Q(x,y), \tag{1}$$

where $p, q \in \mathbb{N}$, $\gcd(p,q) = 1$, and $P(x,y)$ and $Q(x,y)$ are polynomials of the form

$$P(x,y) = P(\mathbf{a}, x, y) = \sum_{\substack{j+k\geq 1 \\ j\geq -1,\ k\geq 0}} a_{jk} x^{j+1} y^k$$

and

$$Q(x,y) = Q(\mathbf{b}, x, y) = \sum_{\substack{j+k\geq 1 \\ j\geq -1,\ k\geq 0}} b_{kj} x^k y^{j+1}.$$

The elementary singular point $O(0,0)$ of system (1) is called a $p{:}-q$ resonant center if there exists a formal first integral of the form

$$\Psi(x,y) = x^q y^p + \sum_{\substack{j+k\geq p+q+1 \\ j,k\in \mathbb{Z},\ j,k\geq 0}} \phi_{j-q,k-p} x^j y^k. \tag{2}$$

In [6], system (1) is considered for $p = 1$ and $q = 2$ and with $P(x,y)$ and $Q(x,y)$ being quadratic polynomials. The case when $P(x,y)$ and $Q(x,y)$ are quadratic polynomials has been studied also by several other authors; in particular, the solution of the $1{:}-3$ resonant center problem can be found in [7] and that of the $1{:}-4$ resonant center problem can be found in [8]. Some results are also obtained for $P(x,y)$ and $Q(x,y)$ for cubic, (homogeneous) quartic, and quintic polynomials.

System (1) is linearizable if and only if there exist a formal change of coordinates that transforms system (1) to the linear system

$$\dot{x} = px, \quad \dot{y} = -qy.$$

One of the methods to obtain conditions for linearizizability of the classical center (i.e., $p = q = 1$) is to compute the so-called linearizability quantities (see [1]). A similar idea (see preliminaries for details) can be applied for systems (1). The computations for linearizability quantities for systems (1) become demanding with increasing values of p and q, and degrees of polynomials P and Q. Therefore, the linearizability problem for the system (1) only for some special families of polynomial systems have been investigated.

In [9], the linearizability problem for a $1{:}-2$ resonant quadratic system (1) was considered and the authors also generalized the results to the linearizability of $1{:}-\lambda$ resonant systems for continuous values of λ, but they did not present the exhaustive list of possibilities. The linearizability problem of $1{:}-3$ resonant quadratic systems of the form (1) was solved in [10], and the linearizablity problem of the $1{:}-3$ resonant system with homogeneous cubic nonlinearities was solved in [11]. In [12], the authors listed the necessary and sufficient conditions for linearizable $2{:}-q$ and $p{:}-2$ resonant centers for $q, p \in \mathbb{N}^+$, but they considered only Lotka–Volterra systems with quadratic nonlinearities. In [13], the authors gave some new sufficient conditions for linearizable Lotka–Volterra systems. Lotka–Volterra systems were investigated also in [14], where all conditions for linearizable systems with $3{:}-4$ and $3{:}-5$ resonance were listed. Later in [15], the authors considered a $3{:}-q$ linearizable resonant system, where q is an arbitrary positive integer.

In this paper, the linearizability problem for system (1) with $p = 2$, $q = 3$, and $\deg(P(x,y), Q(x,y)) = 2$, i.e.,

$$\begin{aligned} \dot{x} &= 2x - a_{10}x^2 - a_{01}xy - a_{-12}y^2, \\ \dot{y} &= -3y + b_{2,-1}x^2 + b_{10}xy + b_{01}y^2 \end{aligned} \tag{3}$$

is considered. In the rest of the paper, we first consider some theoretic results needed for proving the main results. We present the theory of the linearizability problem, which also includes the computation of linearizability quantities. Next, we state the Darboux theory of linearizability for $p{:}-q$ resonant systems, which is an straightforward generalization from $1{:}-1$ resonant systems, and give the proofs to justify the proposed generalization. Finally, we present the conditions for linearizability of systems (3).

2. Preliminaries

By means of a change of coordinates

$$\begin{aligned} x &= x_1 + \sum_{k_1+k_2>1} h_1^{(k_1,k_2)} x_1^{k_1} y_1^{k_2}, \\ y &= y_1 + \sum_{k_1+k_2>1} h_2^{(k_1,k_2)} x_1^{k_1} y_1^{k_2} \end{aligned} \tag{4}$$

system (1) can be transformed into its normal form

$$\begin{aligned} \dot{x}_1 &= px_1 + x_1 \sum_{j=1}^{\infty} X^{(jq+1,jp)} (x_1^q y_1^p)^j, \\ \dot{y}_1 &= -qy_1 + y_1 \sum_{j=1}^{\infty} Y^{(jq,jp+1)} (x_1^q y_1^p)^j. \end{aligned} \tag{5}$$

Definition 1. *System (1) is linearizable (i.e., there is a linearizable center at the origin), if and only if there exists an analytic change of coordinates of the form (4) that reduces (1) to the system*

$$\dot{x}_1 = px_1, \quad \dot{y}_1 = -qy_1. \tag{6}$$

We see that system (1) is linearizable at the origin if the coefficients $X^{(jq+1,jp)}$, $Y^{(jq,jp+1)}$ of the normal form (5) are zero for all $j \in \mathbb{N}$.

Instead of directly constructing a transformation that changes system (1) to its normal form (5) and then imposing that $X^{(jq+1,jp)} = Y^{(jq,jp+1)} = 0$, we look for an inverse of such transformation (see [1])

$$\begin{aligned} x_1 &= x + \sum_{j+k=2}^{\infty} c_{j-1,k} x^j y^k, \\ y_1 &= y + \sum_{j+k=2}^{\infty} d_{j,k-1} x^j y^k, \end{aligned} \tag{7}$$

which changes the linear system (6) into system (1).

After computing derivatives with respect to t in each part of (7), applying (1) and (6), we equate coefficients of the terms $x^{k_1+1}y^{k_2}$ and $x^{k_1}y^{k_2+1}$ on each side of the equalities. This yields the following recurrence formulas:

$$(pk_1 - qk_2)c_{k_1,k_2} = \sum_{n+\ell=0}^{k_1+k_2-1} \left[(n+1)a_{k_1-n,k_2-\ell} - \ell \cdot b_{k_1-n,k_2-\ell} \right] c_{n,\ell}, \tag{8}$$

$$(pk_1 - qk_2)d_{k_1,k_2} = \sum_{n+\ell=0}^{k_1+k_2-1} \left[n \cdot a_{k_1-n,k_2-\ell} - (\ell+1)b_{k_1-n,k_2-\ell}\right] d_{n,\ell}. \quad (9)$$

To initialize the calculations, we set $c_{0,0} = d_{0,0} = 1$, $c_{-1,1} = d_{-1,1} = 0$, and $a_{jk} = b_{kj} = 0$ if $j + k < 1$. The recurrence formulas (8) and (9) are used to compute coefficients c_{k_1,k_2} and d_{k_1,k_2} for $(k_1, k_2) \in \mathbb{N}_{-q} \times \mathbb{N}_{-p}$ of the transformation (7). At the first step, we find all c_{k_1,k_2} and d_{k_1,k_2} for which $k_1 + k_2 = 1$; at the second step, we find all coefficients for which $k_1 + k_2 = 2$, etc. As long as $pk_1 - qk_2 \neq 0$, the process is successful and c_{k_1,k_2} and d_{k_1,k_2} are uniquely determined by (8) and (9). We note that c_{k_1,k_2} and d_{k_1,k_2} are polynomial functions of the coefficients $(\mathbf{a}, \mathbf{b})$ of system (1), i.e., $c_{k_1,k_2} = c_{k_1,k_2}(\mathbf{a}, \mathbf{b})$ and $d_{k_1,k_2} = d_{k_1,k_2}(\mathbf{a}, \mathbf{b})$.

At every $(p + q)$th stage, when $pk_1 - qk_2 = 0$ ($k_1 = kq$ and $k_2 = kp$), coefficients c_{k_1,k_2} and d_{k_1,k_2} cannot be computed using (8) and (9). In this case, we can chose them arbitrarily, and usually, the choice is $c_{kq,kp} = d_{kq,kp} = 0$. Then, we denote the polynomials on the right-hand side of (8) by $I_{kq,kp}$ and on the right-hand side of (9) by $J_{kq,kp}$, i.e.,

$$I_{kq,kp} = \sum_{n+\ell=0}^{kq+kp-1} \Big[(n+1)a_{kq-n,kp-\ell} - \ell \cdot b_{kq-n,kp-\ell}\Big] c_{n,\ell}, \quad (10)$$

$$J_{kq,kp} = \sum_{n+\ell=0}^{kq+kp-1} \Big[n \cdot a_{kq-n,kp-\ell} - (\ell+1)b_{kq-n,kp-\ell}\Big] d_{n,\ell}. \quad (11)$$

We call $I_{kq,kp}$ and $J_{kq,kp}$ *the kth linearizability quantities* of polynomial system (1). Clearly, $I_{kq,kp}$ and $J_{kq,kp}$ are polynomials in the coefficients $(\mathbf{a}, \mathbf{b})$, $I_{kq,kp} = I_{kq,kp}(\mathbf{a}, \mathbf{b})$, and $J_{kq,kp} = J_{kq,kp}(\mathbf{a}, \mathbf{b})$, implying that system (1) is linearizable on condition that

$$I_{kq,kp}(\mathbf{a}, \mathbf{b}) = J_{kq,kp}(\mathbf{a}, \mathbf{b}) = 0, \quad \forall k \in \mathbb{N}.$$

Thus, we need to find the affine variety (variety of the ideal generated by polynomials $f_1, \dots f_s$ is the set $\mathbf{V}(\langle f_1, \dots f_s \rangle) = \{(a_1, \dots a_n) \in k^n : f_j(a_1, \dots a_n) = 0, \text{ for every } j = 1, \dots, s\}$) $\mathbf{V}(\mathcal{L})$ of the ideal

$$\mathcal{L} = \langle I_{q,p}, J_{q,p}, I_{2q,2p}, J_{2q,2p}, I_{3q,3p}, J_{3q,3p}, \dots \rangle.$$

Hilbert Basis Theorem (see, e.g., [16]) ensures that every ideal $\mathcal{L}$ is finitely generated, which means that every ascending chain of ideals eventually stabilizes. Consequently, to attain p:$-q$ resonant systems (1) with linearizable center at the origin we first compute some linearizability quantities and thereafter find irreducible decomposition of the variety of the ideal generated by obtained linearizability quantities.

Once the decomposition of variety associated with the ideal of $\mathcal{L}$ is determined in order to prove the sufficiency of the obtained conditions, for each system satisfying the conditions of the corresponding component of the decomposition of the variety, one needs to find the linearizing transformation that transforms the system into the linear system (6). One of the most efficient tools to find the linearizing transformation is the Darboux linearization, which is well known for 1:$-$1 systems [9]. In the next section, we give a generalization of this theory to p:$-q$ resonant systems based on analogy, where we summarize some results from [9,11] and generalize some results of Darboux linearization theory for 1:$-$1 resonant systems from [1].

3. Darboux Linearization for p:$-q$ Resonant Systems

First, recall some notation related to the Darboux theory. We consider systems

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \quad (12)$$

where $x, y \in \mathbb{C}$, P and Q are polynomials without constant terms having no nonconstant common factor. By D, we denote the corresponding vector field of system (12)

$$D := \frac{\partial}{\partial x}P + \frac{\partial}{\partial y}Q.$$

Definition 2. *A nonconstant polynomial $f(x, y) \in \mathbb{C}[x, y]$ is called an* algebraic partial integral *(also* Darboux factor*) of system* (12) *if there exists a polynomial $K(x, y) \in \mathbb{C}[x, y]$ such that*

$$Df = \frac{\partial f}{\partial x}P + \frac{\partial f}{\partial y}Q = Kf.$$

The polynomial K is called a cofactor of f, *and it is of the degree at most $m - 1$, where $m = \max(\deg(P), \deg(Q))$.*

A simple computation shows that, if there are algebraic partial integrals $f_1, f_2, \ldots, f_k$ with the cofactors $K_1, K_2, \ldots, K_k$ satisfying

$$\sum_{i=1}^{k} \alpha_i K_i = 0,$$

then $H = f_1^{\alpha_1} \cdots f_k^{\alpha_k}$, is a Darboux first integral of (12) and, if

$$\sum_{i=1}^{k} \alpha_i K_i + P'_x + Q'_y = 0,$$

then

$$M = f_1^{\alpha_1} \cdots f_k^{\alpha_k} \tag{13}$$

is the Darboux integrating factor of (12).

When proving the integrability of a 1:$-q$ resonant system, one can also use the following subresult proven in ([9], Theorem 4.13).

Remark 1. *Let M be of the form (13) and denotes a local (reciprocal) integrating factor with f_i being analytic in x and y and $\alpha_i \neq 0$. System* (12) *with $p = 1$ and $q \in \mathbb{R}^+$ admitting M is integrable if $q \neq 0$ is rational and if at most one $f_i(0, 0)$ vanishes and the corresponding algebraic partial integral has one of the forms: $f_i(x, y) = x + o(x, y)$ or $f_i(x, y) = y + o(x, y)$.*

Next, we consider Darboux linearization for p:$-q$ resonant systems (1).

Definition 3. *A Darboux linearization of a polynomial system of the form* (1) *is an analytic change of coordinates*

$$x_1 = Z(x, y), \quad y_1 = W(x, y)$$

in which inverse linearizes (1), i.e., it changes system (1) to the linear system

$$\dot{x}_1 = px_1, \quad \dot{y}_1 = -qy_1,$$

and $Z(x, y)$ and $W(x, y)$ are of the form

$$Z(x, y) = \prod_{j=0}^{m} f_j^{\alpha_j}(x, y) = x + Z_1(x, y),$$

$$W(x, y) = \prod_{j=0}^{n} g_j^{\beta_j}(x, y) = y + W_1(x, y),$$

where $f_j, g_j \in \mathbb{C}[x,y]$, $\alpha_j, \beta_j \in \mathbb{C}$ and polynomials Z_1 and W_1 begin with terms of order at least two. A system is Darboux linearizable if it admits a Darboux linearization.

Theorem 1. *A fixed system of the form (1) is called Darboux linearizable if and only if there exist $k+1$, for $k \geq 0$, algebraic partial integrals $f_0, f_1, \ldots, f_k$ with corresponding cofactors $K_0, K_1, \ldots, K_k$ and $l+1$, for $l \geq 0$, algebraic partial integrals $g_0, g_1, \ldots, g_l$ with corresponding cofactors $L_0, L_1, \ldots, L_l$ with the properties listed below:*

(i) $f_0(x,y) = x + \ldots$ *and* $f_j(0,0) = 1$ *for* $j \geq 1$;

(ii) $g_0(x,y) = y + \ldots$ *and* $g_j(0,0) = 1$ *for* $j \geq 1$;

(iii) *There are $k+l$ constants $\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, \beta_2, \ldots, \beta_l \in \mathbb{C}$, s.t.*

$$K_0 + \alpha_1 K_1 + \ldots + \alpha_k K_k = p \text{ and } L_0 + \beta_1 L_1 + \ldots + \beta_l L_l = -q. \tag{14}$$

Then, the Darboux linearization is given by

$$x_1 = Z(x,y) = f_0 f_1^{\alpha_1} \ldots f_k^{\alpha_k}, \qquad y_1 = W(x,y) = g_0 g_1^{\beta_1} \ldots g_l^{\beta_l}.$$

Proof. For two smooth functions f and g vector field D is a derivation

$$\begin{aligned} D(fg) &= gD(f) + fD(g), \\ D\left(\frac{f}{g}\right) &= \frac{gD(f) - fD(g)}{g^2}, \end{aligned} \tag{15}$$

which may be verified using straightforward computations.

Now, we assume that, for a system (1), there are $k+1 \geq 1$ algebraic partial integrals $f_0, f_1, \ldots, f_k$ and $l+1 \geq 1$ algebraic partial integrals $g_0, g_1, \ldots, g_l$ fulfilling conditions (i)–(iii). The mapping

$$x_1 = Z(x,y) = f_0 f_1^{\alpha_1} \ldots f_k^{\alpha_k}, \quad y_1 = W(x,y) = g_0 g_1^{\beta_1} \ldots g_l^{\beta_l}, \tag{16}$$

is constructed to be analytic. By the Inverse function Theorem, (16) admits an analytic inverse $x = \overline{Z}(x_1, y_1), y = \overline{W}(x_1, y_1)$ in some neighborhood of the origin in $\mathbb{C}^2$.

The differentiation of the first equation in (16) with the respect to t gives

$$\begin{aligned} \dot{x}_1 &= D(f_0 f_1^{\alpha_1} \ldots f_k^{\alpha_k}) = \\ &= D(f_0) f_1^{\alpha_1} \ldots f_k^{\alpha_k} + \sum_{i=1}^{k} f_0 f_1^{\alpha_1} \ldots f_{i-1}^{\alpha_{i-1}} \alpha_i f_i^{\alpha_i - 1} D(f_i) f_{i+1}^{\alpha_{i+1}} \ldots f_k^{\alpha_k} = \\ &= K_0 f_0 f_1^{\alpha_1} \ldots f_k^{\alpha_k} + \sum_{i=1}^{k} \alpha_i K_i f_0 f_1^{\alpha_1} \ldots f_k^{\alpha_k} = \\ &= f_0 f_1^{\alpha_1} \ldots f_k^{\alpha_k} (K_0 + K_1 \alpha_1 + \ldots + \alpha_k K_k) = \\ &= x_1 (K_0 + \alpha_1 K_1 + \ldots + \alpha_k K_k). \end{aligned} \tag{17}$$

Condition $\dot{x}_1 = p x_1$ yields the left expression of (14).

Similarly, the differentiation of the second equation in (16) with the respect to t gives $\dot{y}_1 = -q y_1$. Therefore, the system (1) can be linearized by the transformation (16). □

Note that some algebraic partial integrals may be used as f_j or as g_j.

Next two theorems help us to construct the linearizing transformation for polynomial system (1) even, if one is not able to find enough (see Theorem 1) algebraic partial integrals. This is possible, if the existence of the first integral of the system is ensured. This makes sense, since a resonant center at the origin is assumed. However, we have to find enough algebraic partial integrals satisfying one or the other conditions of (14).

Theorem 2. *Suppose system* (1) *has a resonant center at the origin; hence, it admits a formal first integral* $\psi(x, y)$ *of the form* (2) *and there exist algebraic partial integrals* $f_0, f_1, \ldots, f_k$ *that meet condition (i) in Theorem 1 and that satisfy the first equation of* (14)*. The system* (1) *is then linearized by the transformation*

$$\begin{aligned} x_1 &= Z(x, y) = f_0 \prod_{i=1}^{k} f_i^{\alpha_i} = x + \ldots \\ y_1 &= W(x, y) = \left(\frac{\psi}{(Z(x, y))^q} \right)^{\frac{1}{p}} = y + \ldots \end{aligned} \tag{18}$$

Proof. Recall that, if there exists a formal first integral ψ of the form (2), then there exists an analytic first integral of the same form ([1], Corollary 3.2.6), which we still denote by ψ. By the condition (i) of Theorem 1, transformation (18) is analytic and it admits an analytic inverse in some neighborhood of the origin. The computation (17) is valid and yields $\dot{x}_1 = D(Z) = pZ = px_1$. Then, (15) and the fact that ψ is a first integral yield

$$\begin{aligned} \dot{y}_1 &= \frac{1}{p} \left(\frac{\psi}{Z^q} \right)^{\frac{1}{p}-1} \frac{Z^q D(\psi) - \psi q Z^{q-1} D(Z)}{Z^{2q}} = \\ &= \frac{1}{p} \left(\frac{\psi}{Z^q} \right)^{\frac{1}{p}} \frac{Z^q}{\psi} \cdot \frac{-\psi q Z^{q-1} pZ}{Z^{2q}} = -q \left(\frac{\psi}{Z^q} \right)^{\frac{1}{p}} = -qy_1. \end{aligned}$$

□

For the analogous theorem, we omit the proof.

Theorem 3. *Suppose that system* (1) *has a resonant center at the origin; hence, it admits a formal first integral* $\psi(x, y)$ *of the form* (2) *and that there exist algebraic partial integrals* $g_0, g_1, \ldots, g_l$ *that meet condition (ii) in Theorem 1 and that satisfy the second equation in* (14)*. The system* (1) *is then linearized by the transformation*

$$\begin{aligned} x_1 &= Z(x, y) = \left(\frac{\psi}{(W(x, y))^p} \right)^{\frac{1}{q}} = x + \ldots \\ y_1 &= W(x, y) = g_0 \prod_{i=1}^{l} g_i^{\beta_i} = y + \ldots \end{aligned} \tag{19}$$

4. The Linearizability Conditions

In this section, we present conditions for linearizability of the 2:−3 resonant family of quadratic systems (3).

Theorem 4. *System (3) is linearizable if one of the following conditions is fulfilled:*

(1.) $b_{10} = b_{2,-1} = 0$;
(2.) $b_{01} + 3a_{01} = a_{-12} = b_{10} = 0$;
(3.) $b_{10} = a_{-12} = a_{10} = 2a_{01} + b_{01} = 0$;
(4.) $b_{10} = a_{-12} = a_{10} = a_{01} + \frac{5b_{01}}{6} = 0$;
(5.) $b_{2,-1} = a_{-12} = a_{10} + b_{10} = a_{01} + b_{01} = 0$;
(6.) $256b_{01}b_{2,-1} - 189b_{10}^2 = 243b_{10}a_{-12} + 64b_{01}^2 = 16a_{10} + 17b_{10} = 9a_{01} + 7b_{01} = 0$;
(7.) $a_{-12} = a_{10} + 2b_{10} = a_{01} + \frac{4b_{01}}{3} = 0$;
(8.) $b_{2,-1} = a_{10} + 2b_{10} = 0$;
(9.) $a_{-12} = a_{01} = 0$;
(10.) $b_{2,-1} = a_{10} - b_{10} = a_{-12}b_{10} + 6a_{01}^2 - 2a_{01}b_{01} = 0$;
(11.) $b_{2,-1} = a_{10} - 2b_{10} = a_{01} = 0$;
(12.) $b_{2,-1} = a_{-12}b_{10} - \frac{b_{01}^2}{3} = a_{10} + \frac{2b_{10}}{3} = a_{01} + \frac{b_{01}}{3} = 0$;

(13.) $\frac{3b_{10}^2}{4} + b_{01}b_{2,-1} = a_{-12} = a_{10} + \frac{b_{10}}{2} = a_{01} - b_{01} = 0;$

(14.) $b_{2,-1}b_{01} + \frac{112b_{10}^2}{27} = a_{-12}b_{10} + \frac{3b_{01}^2}{16} = a_{10} + 2b_{10} = a_{01} - \frac{b_{01}}{3} = 0;$

(15.) $b_{2,-1}b_{01} - \frac{7b_{10}^2}{3} = a_{-12}b_{10} + \frac{12b_{01}^2}{49} = a_{10} - 3b_{10} = a_{01} - \frac{2b_{01}}{21} = 0;$

(16.) $-\frac{108b_{10}^2}{7} + b_{01}b_{21} = -\frac{7b_{01}^2}{81} + a_{-12}b_{10} = a_{10} - \frac{26b_{10}}{7} = a_{01} + \frac{2b_{01}}{9} = 0;$

(17.) $b_{2,-1}b_{01} - \frac{63b_{10}^2}{2} = a_{-12}b_{10} + \frac{b_{01}^2}{27} = a_{10} + \frac{19b_{10}}{2} = a_{01} + \frac{b_{01}}{2} = 0;$

(18.) $36b_{01}b_{2,-1} + 91b_{10}^2 = 169b_{10}a_{-12} - 33b_{01}^2 = 12a_{10} + 29b_{10} = 13a_{01} + 9b_{01} = 0;$

(19.) $36b_{01}b_{2,-1} + 91b_{10}^2 = 169b_{10}a_{-12} - 48b_{01}^2 = a_{10} + 2b_{10} = 13a_{01} + 4b_{01} = 0;$

(20.) $676b_{01}b_{2,-1} + 231b_{10}^2 = 121b_{10}a_{-12} - 52b_{01}^2 = 13a_{10} + 11b_{10} = 33a_{01} + 4b_{01} = 0;$

(21.) $256b_{01}b_{2,-1} + 161b_{10}^2 = 529b_{10}a_{-12} - 48b_{01}^2 = 16a_{10} + 27b_{10} = 23a_{01} + 24b_{01} = 0;$

(22.) $7b_{01}b_{2,-1} + 192b_{10}^2 = 64b_{10}a_{-12} + 7b_{01}^2 = 7a_{10} - 46b_{10} = 2a_{01} + b_{01} = 0.$

(23.) $a_{01}b_{2,-1} - \frac{7a_{10}b_{10}}{9} + \frac{8b_{01}b_{2,-1}}{15} + \frac{14b_{10}^2}{45} = a_{10}^2 - 8a_{10}b_{10} + \frac{27b_{01}b_{2,-1}}{35} + \frac{239b_{10}^2}{20} = a_{01}a_{10} - a_{01}b_{10} + \frac{a_{10}b_{01}}{6} + \frac{11a_{-12}b_{2,-1}}{14} - \frac{2b_{01}b_{10}}{3} = -\frac{63a_{01}^2b_{10}}{5} - \frac{21a_{01}b_{01}b_{10}}{10} + \frac{14a_{10}a_{-12}b_{10}}{15} - \frac{7a_{10}b_{01}^2}{15} + a_{-12}b_{01}b_{2,-1} - \frac{427a_{-12}b_{10}^2}{30} + \frac{28b_{01}^2b_{10}}{15} = -\frac{7a_{10}b_{01}b_{2,-1}}{15} + \frac{98a_{10}b_{10}^2}{15} + a_{-12}b_{2,-1}^2 - \frac{14b_{01}b_{10}b_{2,-1}}{15} - \frac{343b_{10}^3}{30} = -\frac{63a_{01}b_{10}^2}{10} + a_{10}a_{-12}b_{2,-1} - \frac{7a_{10}b_{01}b_{10}}{5} - 7a_{-12}b_{10}b_{2,-1} + \frac{9b_{01}^2b_{2,-1}}{25} + \frac{371b_{01}b_{10}^2}{100} = a_{01}^3 + \frac{a_{01}^2b_{01}}{3} + \frac{19a_{01}a_{-12}b_{10}}{18} - \frac{a_{01}b_{01}^2}{12} - \frac{5a_{10}a_{-12}b_{01}}{81} + \frac{11a_{-12}^2b_{2,-1}}{189} + \frac{53a_{-12}b_{01}b_{10}}{324} = 0;$

(24.) $-15a_{10}b_{10} + b_{01}a_{2,-1} + 51b_{10}^2 = \frac{37a_{01}b_{10}}{7} + a_{10}b_{01} - \frac{10b_{01}b_{10}}{3} = a_{01}b_{01} - \frac{37a_{-12}b_{10}}{2} + \frac{5b_{01}^2}{3} = 6a_{01}b_{10} + a_{-12}a_{2,-1} = a_{01}a_{2,-1} + 4a_{10}b_{10} - 15b_{10}^2 = a_{10}a_{-12} + 4a_{-12}b_{10} - \frac{2b_{01}^2}{3} = a_{10}^2 - \frac{57a_{10}b_{10}}{7} + \frac{349b_{10}^2}{21} = a_{01}a_{10} - \frac{101a_{01}b_{10}}{21} - \frac{b_{01}b_{10}}{9} = a_{01}^2 + \frac{31a_{-12}b_{10}}{6} - \frac{4b_{01}^2}{9} = \frac{111a_{01}a_{-12}b_{10}}{14} - 11a_{-12}b_{01}b_{10} + b_{01}^3 = 0;$

(25.) $-\frac{170a_{01}b_{01}}{231} + a_{10}a_{-12} + \frac{29a_{-12}b_{10}}{21} - \frac{18b_{01}^2}{77} = a_{01}a_{10} + \frac{3a_{01}b_{10}}{2} + \frac{a_{10}b_{01}}{6} + \frac{11a_{-12}a_{2,-1}}{24} - \frac{2b_{01}b_{10}}{3} = a_{01}^2 + \frac{25a_{01}b_{01}}{42} - \frac{121a_{-12}b_{10}}{84} + \frac{b_{01}^2}{14} = a_{01}b_{01}a_{2,-1} + \frac{15a_{01}b_{10}^2}{7} + -\frac{11}{7}a_{10}b_{01}b_{10} + \frac{33a_{-12}b_{10}a_{2,-1}}{4} + \frac{3b_{01}^2a_{2,-1}}{7} + \frac{8b_{01}b_{10}^2}{7} = -\frac{1020a_{01}b_{10}a_{2,-1}}{77} + \frac{24a_{10}^2b_{10}}{7} - \frac{4a_{10}b_{01}a_{2,-1}}{7} + \frac{24a_{10}b_{10}^2}{7} + a_{-12}a_{2,-1}^2 - \frac{544b_{01}b_{10}a_{2,-1}}{77} - \frac{48b_{10}^3}{7} = \frac{20a_{01}a_{-12}b_{10}}{77} - \frac{96a_{01}b_{01}^2}{539} + a_{-12}^2a_{2,-1} + \frac{192a_{-12}b_{01}b_{10}}{539} - \frac{16b_{01}^3}{539} = a_{01}a_{-12}a_{2,-1} + \frac{46a_{01}b_{01}b_{10}}{147} + \frac{3a_{-12}b_{01}a_{2,-1}}{7} + \frac{55a_{-12}b_{10}^2}{147} - \frac{6b_{01}^2b_{10}}{49} = -6a_{10}^3b_{10} + a_{10}^2b_{01}a_{2,-1} - \frac{100a_{10}^2b_{10}^2}{7} + \frac{55}{7}a_{10}b_{01}b_{10}a_{2,-1} + \frac{26a_{10}b_{10}^3}{7} + \frac{b_{01}^2a_{2,-1}^2}{7} + 34b_{01}b_{10}^2a_{2,-1} + \frac{116b_{10}^4}{7} = 0.$

Proof. Following the approach described in Section 2 using computer algebra system Mathematica, we compute the first four pairs of linearizability quantities $\{I_{3k,2k}, J_{3k,2k} : k = 1,2,3,4\}$. We present only the first pair of linearizability quantities:

$$
\begin{aligned}
I_{3,2} = \frac{1}{155232}(&-21252a_{01}^3a_{10}b_{2,-1} - 31416a_{01}^3b_{10}b_{2,-1} - 6468a_{01}^2a_{10}^2b_{10} \\
&- 18172a_{01}^2a_{10}b_{01}b_{2,-1} - 25872a_{01}^2a_{10}b_{10}^2 - 25857a_{01}^2a_{-12}b_{2,-1}^2 \\
&- 16016a_{01}^2b_{01}b_{10}b_{2,-1} - 25872a_{01}^2b_{10}^3 + 1113a_{01}a_{10}^2a_{-12}b_{2,-1} \\
&+ 6468a_{01}a_{10}^2b_{01}b_{10} - 39382a_{01}a_{10}a_{-12}b_{10}b_{2,-1} \\
&- 3696a_{01}a_{10}b_{01}^2b_{2,-1} + 12936a_{01}a_{10}b_{01}b_{10}^2 \\
&- 19812a_{01}a_{-12}b_{01}b_{2,-1}^2 - 75236a_{01}a_{-12}b_{10}^2b_{2,-1} \\
&+ 1617a_{10}^3a_{-12}b_{10} - 84a_{10}^2a_{-12}b_{01}b_{2,-1} + 1617a_{10}^2a_{-12}b_{10}^2 \\
&+ 396a_{10}a_{-12}^2b_{2,-1}^2 + 4704a_{10}a_{-12}b_{01}b_{10}b_{2,-1} - 6468a_{10}a_{-12}b_{10}^3 \\
&- 22308a_{-12}^2b_{10}b_{2,-1}^2 - 3648a_{-12}b_{01}^2b_{2,-1}^2 - 2016a_{-12}b_{01}b_{10}^2b_{2,-1} \\
&- 6468a_{-12}b_{10}^4),
\end{aligned}
$$

$$
\begin{aligned}
J_{3,2} =& \frac{1}{232848}(27720a_{01}^3 a_{10} b_{2,-1} + 51282a_{01}^3 b_{10} b_{2,-1} \\
&+ 33264a_{01}^2 a_{10} b_{01} b_{2,-1} + 29106a_{01}^2 a_{10} b_{10}^2 + 42012a_{01}^2 a_{-12} b_{2,-1}^2 \\
&+ 62370a_{01}^2 b_{01} b_{10} b_{2,-1} + 58212a_{01}^2 b_{10}^3 - 10080a_{01} a_{10}^2 a_{-12} b_{2,-1} \\
&- 19404a_{01} a_{10}^2 b_{01} b_{10} + 27405a_{01} a_{10} a_{-12} b_{10} b_{2,-1} \\
&+ 8008a_{01} a_{10} b_{01}^2 b_{2,-1} - 29106a_{01} a_{10} b_{01} b_{10}^2 \\
&+ 55332a_{01} a_{-12} b_{01} b_{2,-1}^2 + 166950a_{01} a_{-12} b_{10}^2 b_{2,-1} \\
&+ 17864a_{01} b_{01}^2 b_{10} b_{2,-1} + 19404a_{01} b_{01} b_{10}^3 - 1680a_{10}^2 a_{-12} b_{01} b_{2,-1} \\
&- 4851a_{10}^2 a_{-12} b_{10}^2 - 10032a_{10} a_{-12}^2 b_{2,-1}^2 \\
&- 25424a_{10} a_{-12} b_{01} b_{10} b_{2,-1} + 49236a_{-12}^2 b_{10} b_{2,-1}^2 \\
&+ 14688a_{-12} b_{01}^2 b_{2,-1}^2 + 34832a_{-12} b_{01} b_{10}^2 b_{2,-1} + 19404a_{-12} b_{10}^4),
\end{aligned}
$$

and the others are too long to be presented in this paper.

Next, we compute the irreducible decomposition of the variety of the ideal

$$I = \langle I_{3k,2k}, J_{3k,2k} : k = 1,2,3,4 \rangle$$

in order to obtain a set of necessary conditions for linearizability. The computational tool that we use is the routine `minAssGTZ` [17] (which finds the minimal associate primes of a polynomial ideal using the algorithm by Gianni, Trager, and Zacharias [18]) of the computer algebra system SINGULAR [19]. Since computations are too laborious they can not be completed in the field of rational numbers. Therefore, we follow the algorithm based on modular computations [10,20], and we replace the ring $\mathbb{Q}[\mathbf{a},\mathbf{b}]$ by the ring $\mathbb{Z}_p[\mathbf{a},\mathbf{b}]$, where $\mathbf{a} = (a_{10}, a_{01}, a_{-12})$, $\mathbf{b} = (b_{2,-1}, b_{10}, b_{01})$, and p is some prime number. The most difficult step of this algorithm is to realize that all points of the variety $\mathbf{V}(I)$ were found. All of the encountered points belong to the decomposition of $\mathbf{V}(I)$, but we do not know whether the given decomposition is complete. Thus, the proof of the completeness of the decomposition is not given exactly (over the field of characteristic zero), but the probability of the opposite event is extremely low (see the estimation in [6], where the Faugére method [21] is used).

We choose prime $p = 32,003$ and compute the decomposition of I over $\mathbb{Z}_{32,003}$. We obtain 25 ideals. Then, we perform rational reconstruction algorithm to obtain ideals $P_1, \dots, P_{25}$ in $\mathbb{Q}[\mathbf{a},\mathbf{b}]$. For instance, we find that one of the component is $b_{10} = a_{-12} = a_{01} + 10,668b_{01} = 0$. Performing the rational reconstruction using Mathematica, we obtain $10,668 \equiv \frac{1}{3} \mod 32,003$. Therefore, the corresponding component over characteristic zero is $b_{10} = a_{-12} = a_{01} + \frac{1}{3}b_{01} = 0$. Then, we check by a direct computation using Mathematica if the 8 linearizability quantities from I are zero under each of the obtained conditions. It turns out that they are all zero. To check if some conditions in computations over the field of characteristic "32,003" were lost, we first compute intersection $P = \bigcap_{i=1}^{25} P_i$ over the field of characteristic zero. We obtain 50 polynomials $p_1, \dots, p_{50}$. We want to check if $\sqrt{I} = \sqrt{P}$. Computing over the field of characteristic zero Gröbner basis of each ideal $\langle 1 - wI_{3k,2k}, P : k = 1,2,3,4 \rangle$ and $\langle 1 - wJ_{3k,2k}, P : k = 1,2,3,4 \rangle$ with w, a new variable, we find that they are all equal to $\{1\}$, implying that $\sqrt{I} \subset \sqrt{P}$. To check the opposite inclusion, $\sqrt{P} \subset \sqrt{I}$, we need to use computations with modular arithmetic. We choose prime "32,003", and after computing the Gröbner basis of each ideal $\langle 1 - wp_k, I : k = 1, \dots, 50 \rangle$, where $p_k \in P$ over the field $\mathbb{Z}_{32,003}$, we find that they are not all $\{1\}$. Thus, we assume that not all points of $\mathbf{V}(I)$ were found. In such a case, we have to repeat the calculations with another prime. We repeat the calculations with three more (larger) primes, "1,548,586", "179,595,127", and "479,001,599". It turns out that, in the case of "1,548,586" and "179,595,127", we arrive at the same troubles as with "32,003", i.e., some points of $\mathbf{V}(I)$ are lost in the computation. Anyway, computing the decomposition over

$\mathbb{Z}_{479,001,599}$ yields 25 components. After performing rational reconstruction algorithm, we obtain 25 ideals $P_1, \ldots, P_{25}$ that give components of Theorem 4 and we check that, under each component, all linearizability quantities from I are zero. Then, computing intersection $P = \bigcap_{i=1}^{25} P_i$ over the field $\mathbb{Q}$ yields 85 polynomials $p_1, \ldots, p_{85}$. Similar to that above for "32,003", we easily check $\sqrt{I} \subset \sqrt{P}$. To check the opposite inclusion $\sqrt{P} \subset \sqrt{I}$, we need to use computations with modular arithmetics. We performed computations using two primes, $p = 32,003$ and $p = 479,001,599$, and in both cases, we find that the Gröbner basis of ideal $\langle 1 - wp_k, I : k = 1, \ldots, 85 \rangle$ is $\{1\}$ for each $k = 1, \ldots, 85$. We can conclude that equality $\sqrt{P} = \sqrt{I}$ holds with high probability. However, since still there is a small probability that some points are lost, we say in the statement of Theorem 4 "if" and not "if and only if".

Next, we prove the sufficiency of each component.

Case (1) In the first case, the corresponding system is written as

$$\begin{aligned} \dot{x} &= 2x - a_{10}x^2 - a_{01}xy - a_{-12}y^2, \\ \dot{y} &= -3y + b_{01}y^2, \end{aligned} \tag{20}$$

for which we obtain two algebraic partial integrals

$$g_1 = y \text{ and } g_2 = 1 - \frac{b_{01}y}{3}$$

with corresponding cofactors $K_1 = -3 + b_{01}y$ and $K_2 = b_{01}y$. Apparently, the equation $K_1 + aK_2 = -3$ holds for $a = -1$; hence, transformation

$$y_1 = g_1 g_2^{-1} = \frac{3y}{3 - b_{01}y}$$

linearizes the second equation of the system. The integrability of the above quadratic system has been proven in [22]. Using Theorem 3, we can also linearize the first equation; thus, system (20) is linearizable.

Case (2) We prove the linearizability of the following system:

$$\begin{aligned} \dot{x} &= 2x - a_{10}x^2 - a_{01}xy, \\ \dot{y} &= -3y + b_{-2,1}x^2 - 3a_{01}y^2. \end{aligned} \tag{21}$$

To find linearizing transformations, we use four algebraic partial integrals:

$$\begin{aligned} &f_1 = x, \quad f_2 = 1 - \frac{a_{10}x}{2} - \frac{1}{2}a_{01}b_{-2,1}x^2 + 2a_{01}y - \frac{1}{2}a_{01}a_{10}xy + a_{01}^2y^2, \\ &f_3 = 1 + \frac{1}{4}\left(-a_{10} - \sqrt{a_{10}^2 + 8a_{01}b_{2,-1}}\right)x + a_{01}y \end{aligned}$$

and f_4, which is too long to be written here.

Their corresponding cofactors are

$$\begin{aligned} K_1 &= 2 - a_{10}x - a_{01}y, \quad K_2 = -a_{10}x - 6a_{01}y, \\ K_3 &= \frac{1}{2}\left(\left(-a_{10} - \sqrt{a_{10}^2 + 8a_{01}b_{-2,1}}\right)x - 6a_{01}y\right), \\ K_4 &= 2 - 2a_{10}x + \sqrt{a_{10}^2 + 8a_{01}b_{-2,1}}x - 7a_{01}y. \end{aligned}$$

The first equation of system (21) is linearizable by the change of coordinates:

$$x_1 = f_1 f_2^a f_3^b,$$

where
$$a=-\frac{1}{6}+\frac{5a_{10}}{6\sqrt{a_{10}^2+8a_{01}b_{-2,1}}}\quad \text{and}\quad b=-\frac{5a_{10}}{3\sqrt{a_{10}^2+8a_{01}b_{-2,1}}}.$$

We are able to construct the Darboux integrating factor $M=f_2^c f_3^d f_4^e$ for $c=-\frac{-5a_{10}+9\sqrt{a_{10}^2+8a_{01}b_{21}}}{4\sqrt{a_{10}^2+8a_{01}b_{21}}}$, $d=\frac{-5a_{10}+2\sqrt{a_{10}^2+8a_{01}b_{21}}}{2\sqrt{a_{10}^2+8a_{01}b_{21}}}$, and $e=\frac{1}{2}$. By Remark 1, system (21) is integrable, and according to Theorem 2, it is linearizable at the origin.

For all cases where we use the method of Darboux linearization, the procedure and reasoning is similar to that of Case (2). Therefore, in the following cases, where we construct a Darboux linearization, we list only the algebraic partial integrals, one of the linearizations and a Darboux integrating factor, or in some cases, linearizations of both equations.

Case (3) In this case, we find three algebraic partial integrals:
$$f_1=x,\quad f_2=1+\frac{1}{12}b_{01}b_{2,-1}x^2-\frac{b_{01}y}{3}\quad \text{and}\quad f_3=-\frac{b_{2,-1}x^2}{7}+y.$$

The first equation is linearizable by the change of coordinates:
$$x_1=f_1f_2^{-\frac{1}{2}}=\frac{x}{\sqrt{\frac{1}{12}b_{01}b_{2,-1}x^2-\frac{b_{01}y}{3}+1}}.$$

The second equation is linearizable by the transformation:
$$y_1=f_3f_2^{-1}=-\frac{12\left(b_{2,-1}x^2-7y\right)}{7(b_{01}b_{2,-1}x^2-4b_{01}y+12)}.$$

Case (4) The system is written as
$$\begin{aligned}\dot{x}&=2x+\frac{5}{6}b_{01}xy=P(x,y),\\ \dot{y}&=-3y+b_{2,-1}x^2+b_{01}y^2=Q(x,y).\end{aligned}\tag{22}$$

To construct the linearizability transformation, we use an approach from [23]. We look for a transformation $y_1(x,y)=y+\dots$ that linearizes the second equation of system (22). In system (22), we perform the blow-up transformation $(x,y)\to(z,y)=(\frac{x}{y},y)$ and attain the following system
$$\begin{aligned}\dot{z}&=5z-\frac{b_{01}}{6}yz-b_{2,-1}yz^3=\tilde{P}(x,y),\\ \dot{y}&=-3y+b_{2,-1}y^2z^2+b_{01}y^2=\tilde{Q}(x,y),\end{aligned}\tag{23}$$

for which we search for a transformation $Y(z,y)$ of the form
$$Y(z,y)=\sum_{i=1}^{\infty}f_i(z)y^i,\tag{24}$$

where f_i is a polynomial of degree at most i and $Y(z,y)$ linearizes the second equation of system (23), i.e.,
$$\frac{\partial Y}{\partial x}\tilde{P}(x,y)+\frac{\partial Y}{\partial y}\tilde{Q}(x,y)=-3y_1.\tag{25}$$

We insert some initial terms of (24) into Equation (25) and equate the coefficients of the same powers on both sides of the equations and obtain differential equations with unknown functions $f_i(z)$. The first differential equation for $f_1(z)$ is $5zf_1'(z)=0$, which gives solution $f_1(z) = C$. Choosing $C = 1$, we have $f_1(z) = 1$. Next, we see that, for $k > 1$, we obtain recursive differential equations of the form

$$(k-1)(b_{01} + b_{2,-1}z^2)f_{k-1}(z) - \left(\frac{b_{01}}{6}z + b_{2,-1}z^3\right)f'_{k-1}(z) \\ - 3(k-1)f_k(z) + 5zf'_k(z) = 0. \tag{26}$$

If we solve the differential equation in (26) in turn and insert the solution of each one into the next equation, we obtain

$$\begin{aligned} f_2(z) &= \frac{b_{01}}{3} + \frac{b_{2,-1}}{7}z^2 + C_2 z^{\frac{3}{5}}, \\ f_3(z) &= \frac{b_{01}^2}{9} + \frac{3}{28}b_{2,-1}b_{01}z^2 + C_3 z^{\frac{6}{5}}, \\ f_4(z) &= \frac{b_{01}^3}{27} - \frac{1}{21}b_{2,-1}b_{01}^2 z^2 + \frac{3}{308}b_{01}b_{2,-1}^2 z^4 + C_4 z^{\frac{9}{5}}, \\ f_5(z) &= \frac{b_{01}^4}{81} - \frac{5}{378}b_{2,-1}b_{01}^3 z^2 + \frac{29}{3696}b_{01}^2 b_{2,-1}^2 z^4 + C_5 z^{\frac{12}{5}}. \end{aligned}$$

We set the integration constant to zero for all $k \geq 2$. Next, we obtain $f_6(z) = P_6(z)$, $f_7(z) = P_6(z)$, $f_8(z) = P_8(z)$, $f_9(z) = P_8(z)$, and $f_{10}(z) = P_{10}(z)$, where for $k = 6, 7, 8, \ldots$, function $P_k(z)$ denotes a polynomial of degree k and contains only terms with even powers. Then, we note that $f_{11}(z) = P_{10}(z)$ without the term with z^6.

Choosing $C_k = 0$ in each f_k, for $k = 2, 3, \ldots$, we assume that

$$f_k(z) = \begin{cases} P_k(z) & ; \text{if } k \text{ is even} \\ P_{k-1}(z) & ; \text{if } k \text{ is odd} \end{cases}$$

where $P_k(z)$ denotes a polynomial of degree at most k.

Moreover, we assume also that polynomials $P_k(z)$ contain only terms with even powers, i.e.,

$$P_k(z) = \begin{cases} a_0 + a_2z^2 + a_4z^4 + \ldots + a_kz^k & ; \text{if } k \text{ is even} \\ a_0 + a_2z^2 + a_4z^4 + \ldots + a_{k-1}z^{k-1} & ; \text{if } k \text{ is odd.} \end{cases} \tag{27}$$

We prove this using mathematical induction. Suppose that the assumption is true for $k = 1, 2, \ldots, n-1$ and we compute f_k for $k = n$, solving the differential equation

$$f'_n(z) - \frac{3(n-1)}{5z}f_n(z) = h(z), \tag{28}$$

where

$$h(z) = \frac{1}{5z}\Big[-(n-1)(b_{01} + b_{2,-1}z^2)f_{n-1}(z) \\ + \left(\frac{b_{01}}{6}z + b_{2,-1}z^3\right)f'_{n-1}(z)\Big]. \tag{29}$$

First, let n be even, and we want to prove that $f_n(z) = P_n(z)$, where P_n is a polynomial of degree at most n containing only terms of even powers. If n is even, then $n-1$ is odd and f_{n-1} is of the form

$$f_{n-1} = P_{n-2}(z) = A_0 + A_2z^2 + \ldots + A_{n-2}z^{n-2}$$

and

$$f'_{n-1} = 2A_2z + 4A_4z^3 + \ldots + (n-2)A_{n-2}z^{n-3}.$$

Then,

$$h(z) = \frac{1}{5z}(B_0 + B_2 z^2 + B_4 z^4 + \ldots + B_n z^n),$$

for some constants $B_0, B_1, \ldots, B_n$.

Since the linear equation

$$f'(x) + g(x)f(x) = h(x) \tag{30}$$

has the solution

$$f(x) = Ce^{-\int g(x)dx} + e^{-\int g(x)dx} \int e^{\int g(x)dx} h(x)dx, \tag{31}$$

the solution of differential Equation (28) is

$$\begin{aligned} f_n(z) &= Ce^{\int \frac{3(n-1)}{5z}dz} + e^{\int \frac{3(n-1)}{5z}dz} \int e^{-\int \frac{3(n-1)}{5z}dz} \frac{1}{5z}(B_0 + B_2 z^2 + \ldots + B_n z^n)dz \\ &= Cz^{\frac{3(n-1)}{5}} + z^{\frac{3(n-1)}{5}} \int z^{\frac{3(1-n)}{5}} z^{-1} \frac{1}{5}(B_0 + B_2 z^2 + \ldots + B_n z^n)dz \\ &= Cz^{\frac{3(n-1)}{5}} + z^{\frac{3(n-1)}{5}} \frac{1}{5} \int \left(B_0 z^{\frac{-2-3n}{5}} + B_2 z^{\frac{8-3n}{5}} + \ldots + B_n z^{\frac{-2+2n}{5}}\right)dz \\ &= Cz^{\frac{3(n-1)}{5}} + \frac{1}{5} z^{\frac{3(n-1)}{5}} \left(B_0 \frac{z^{\frac{3-3n}{5}}}{\frac{3-3n}{5}} + B_2 \frac{z^{\frac{13-3n}{5}}}{\frac{13-3n}{5}} + \ldots + B_n \frac{z^{\frac{3-2n}{5}}}{\frac{3-2n}{5}}\right) \\ &= Cz^{\frac{3(n-1)}{5}} + \widetilde{B}_0 + \widetilde{B}_2 z^2 + \widetilde{B}_4 z^4 + \ldots + \widetilde{B}_n z^n. \end{aligned}$$

Choosing $C = 0$, we obtain that $f_n(z) = P_n(z)$, where $P_n(z)$ is a polynomial of degree n and it contains only terms with even powers.

Now, we assume that n is odd and we want to prove that $f_n(z) = P_{n-1}(z)$, where P_{n-1} is a polynomial of degree at most $n-1$ and contains only terms of even powers.

If n is odd, then $n-1$ is even and f_{n-1} is of the form

$$f_{n-1} = P_{n-1}(z) = A_0 + A_2 z^2 + \ldots + A_{n-1} z^{n-1}$$

with

$$f'_{n-1} = 2A_2 z + 4A_4 z^3 + \ldots + (n-1)A_{n-1} z^{n-2}.$$

Then,

$$\begin{aligned} h(z) =& \frac{1}{5z}\Big[(B_0 + B_2 z^2 + \ldots + B_{n-1} z^{n-1} - (n-1)b_{2,-1} A_{n-1} z^{n+1} \\ &+ (n-1)b_{2,-1} A_{n-1} z^{n+1})\Big] \\ =& \frac{1}{5z}\Big[(B_0 + B_2 z^2 + \ldots + B_{n-1} z^{n-1})\Big] \end{aligned}$$

for some $B_0, \ldots, B_{n-1}$.

Differential Equation (28) has a solution:

$$\begin{aligned} f_n(z) &= Ce^{\int \frac{3(n-1)}{5z}dz} + e^{\int \frac{3(n-1)}{5z}dz} \int e^{-\int \frac{3(n-1)}{5z}dz} \frac{1}{5z}(B_0 + B_2 z^2 + B_4 z^4 + \\ &\ldots + B_{n-1} z^{n-1})dz \\ &= Cz^{\frac{3(n-1)}{5}} + z^{\frac{3(n-1)}{5}} \left(\widetilde{B}_0 z^{\frac{3-3n}{5}} + \widetilde{B}_2 z^{\frac{13-3n}{5}} + \ldots + \widetilde{B}_{n-1} z^{\frac{2n-2}{5}}\right) \\ &= Cz^{\frac{3(n-1)}{5}} + \widetilde{B}_0 + \widetilde{B}_2 z^2 + \widetilde{B}_4 z^4 + \ldots + \widetilde{B}_{n-1} z^{n-1}. \end{aligned}$$

Choosing $C = 0$, we obtain that $f_n(z) = P_{n-1}(z)$, as assumed. Therefore, transformation (24) takes the form

$$Y(z,y) = \sum_{i=1}^{\infty} f_i(z)y^i = y + \sum_{i=2}^{\infty} P_i(z)y^i, \tag{32}$$

where polynomials $P_i(z)$ are defined by (27).

Thus, the transformation $y_1(x,y)$, which linearizes the second equation of system (22), is

$$y_1(x,y) = Y\left(\frac{x}{y}, y\right) = y + \sum_{i=2}^{\infty} P_i\left(\frac{x}{y}\right)y^i.$$

The obstacle in the proof could appear if we obtain logarithmic terms after integration in the computation of $f_n(z)$. We see that this is the case if one of the powers is -1. We observe that all powers of z in the integral are of the form $\frac{-2+10k-3n}{5}$ for $k = 0,1,2,\ldots,n-1$ (n odd) and $k = 0,1,2,\ldots,n$ (n even). We check when

$$\frac{-2+10k-3n}{5} = -1$$

and obtain $k = \frac{3n-3}{10}$. Since $k \in \{0,1,2,\ldots,n\}$, this is the case if $3n - 3 \equiv 0 \pmod{10}$.

However, we can prove that this can never happen. Logarithmic terms could appear after integration only in functions $f_{11}, f_{21}, \ldots, f_{10\ell+1}, \ldots$ for $\ell = 1,2,3,\ldots$, but we prove by induction that functions $f_{10\ell+1}$ do not have a term $z^{6\ell}$, which could yield a logarithmic term after integration. We already proved by induction that functions $f_{10\ell+1}$ are of the form $P_{10\ell}(z)$, where $P_i(z)$ denotes a polynomial of degree at most i.

Suppose that functions $f_{10\ell+1}$ are polynomials of degree 10ℓ without the monomial term $z^{6\ell}$ for $\ell = 1,2,\ldots,m$, and we compute $f_{10(m+1)+1}$ by solving the differential Equation (28).

Suppose

$$f_{10\ell+1}(z) = \tilde{A}_0 + \ldots + \tilde{A}_{6\ell-2}z^{6\ell-2} + \tilde{A}_{6\ell+2}z^{6\ell+2} + \ldots + \tilde{A}_{10\ell}z^{10\ell}$$

for all $\ell = 1,2,3,\ldots,m$. Functions $f_{10m}(z)$ and $f'_{10m}(z)$ are of the form

$$\begin{aligned} f_{10m}(z) &= A_0 + A_2z^2 + \ldots + A_{10m}z^{10m}, \\ f'_{10m}(z) &= 2A_2z + \ldots + (10m)A_{10m}z^{10m-1}. \end{aligned}$$

Moreover, we assume that the coefficients of polynomial f_{10m} satisfy the condition

$$A_{6m-2}b_{2,-1}(5m-1) + A_{6m}b_{01} = 0. \tag{33}$$

If we insert expressions $f_{10m}(z)$ and $f'_{10m}(z)$ into differential Equation (28), condition (33) assures that f_{10m+1} does not have a term with z^{6m}. In functions $f_{10m+i}(z)$, for $i = 3,4,\ldots,10$, no logarithmic term can appear and we already proved that these functions are of the required form. Detailed analysis is needed for $f_{10m+11}(z) = f_{10(m+1)+1}$, since in this function logarithmic terms after integration could appear. We compute the function $f_{10m+11}(z)$ and obtain $f_{10m+11}(z) = P_{10m+10}(z)$, where $P_{10m+10}(z)$ is a polynomial of degree $10m + 10$ and we see that the coefficient of $z^{6(m+1)}$ is given by

$$A_{6m+4}b_{2,-1}(5m+4) + A_{6m+6}b_{01}. \tag{34}$$

No logarithmic term appears in $f_{10(m+1)}(z)$ after integration only if expression (34) is zero. We observe that condition (34) corresponds to condition (33) for $m+1$; hence, it is zero, and thus ,the proof of induction is completed.

Now, we also have to prove that the first equation is linearizable. To do so, we prove that system (22) is integrable and then use Theorem 3.

We are not able to form the first integral of the required form using the Darboux theory of integrability and we can not construct it using power series, so we use the method based on blow-up (see [24]). According to transformation $(x,y) \longmapsto (z,y) = \left(\frac{x}{y}, y\right)$, we attain system (23). Now, we search for the formal series

$$\Psi(z,y) = \sum_{k=5}^{\infty} f_k(z) y^k,$$

which will be the first integral of system (23) only if $D\Psi \equiv 0$. Setting the coefficients of term y^k to zero, for each $k \geq 5$ and set $f_4(z) = 0$, generate for $k \geq 5$ the subsequent differential equation

$$(k-1)\left(b_{01} + b_{2,-1}z^2\right) f_{k-1}(z) - 3k f_k(z) + 5z f_k'(z) - \left(\frac{1}{6} b_{01} z + b_{2,-1} z^3\right) f_{k-1}'(z) = 0.$$

We compute

$$\begin{aligned}
f_5(z) &= z^3, \\
f_6(z) &= \frac{3}{2} b_{01} z^3 - \frac{2}{7} b_{2,-1} z^5, \\
f_7(z) &= \frac{11 b_{01}^2 z^3}{8} - \frac{127}{168} b_{01} b_{2,-1} z^5 + \frac{b_{2,-1}^2 z^7}{49}, \\
f_8(z) &= \frac{143 b_{01}^3 z^3}{144} - \frac{845 b_{01}^2 b_{2,-1} z^5}{1008} + \frac{39}{308} b_{01} b_{2,-1}^2 z^7,
\end{aligned}$$

where we chose an integration constant equal to one for f_5, and in all other functions (f_k, $k = 6,7,\ldots$), we chose the integration constant to be zero. Next, $f_9(z) = P_9(z)$, $f_{10}(z) = P_9(z)$, $f_{11}(z) = P_{11}(z)$, $f_{12}(z) = P_{11}(z)$, $f_{13}(z) = P_{13}(z)$, $f_{14}(z) = P_{13}(z)$, and $f_{15}(z) = P_{15}(z)$ without the term with z^9, where for $k = 9,\ldots,15$ function $P_k(z)$ denotes a polynomial of degree k.

We notice that functions f_k for $k = 6,7\ldots,15$ are polynomials that contain only terms with odd powers:

$$f_k(z) = \begin{cases} a_3 z^3 + a_5 z^5 + \ldots + a_k z^k & ; \text{if } k \text{ is odd} \\ b_3 z^3 + b_5 z^5 + \ldots + b_{k-1} z^{k-1} & ; \text{if } k \text{ is even} \end{cases} \tag{35}$$

which can be proven inductively very similar to the above (simply replace n with $n-1$), and to compute f_n, we solve the differential equation

$$f_n'(z) - \frac{3n}{5z} f_n(z) = h(z), \tag{36}$$

where $h(z)$ is the same as in (28).

Additionally, in this case, we have to prove that no logarithmic terms appear in any solution f_k. We can easily check that only in functions $f_{15}, f_{25}, \ldots, f_{10\ell+5}, \ldots$, for $\ell = 1,2,3,\ldots$, logarithmic terms could appear. We prove by induction that functions $f_{10\ell+5}$ do not have a term $z^{6\ell+3}$, which could yield logarithmic terms after integration. We already proved by induction that functions $f_{10\ell+5}$ are of the form $f_{10\ell+5}(z) = P_{10\ell+5}(z)$, where $P_i(z)$ denotes polynomial of degree i.

Suppose that functions $f_{10\ell+5}$ are polynomials of degree $10\ell+5$ without the monomial term $z^{6\ell+3}$, for $\ell = 1,2,\ldots,m$, and we compute $f_{10(m+1)+5}$ by solving the differential Equation (36).

Suppose

$$f_{10\ell+5}(z) = \\ = \tilde{A}_3 z^3 + \ldots + \tilde{A}_{6\ell+1} z^{6\ell+1} + \tilde{A}_{6\ell+5} z^{6\ell+5} + \ldots + \tilde{A}_{10\ell+5} z^{10\ell+5}$$

for all $\ell = 1, 2, 3, \ldots, m$. Functions $f_{10m+4}(z)$ and $f'_{10m+4}(z)$ are of the form

$$f_{10m+4}(z) = A_3 z^3 + \ldots + A_{10m+3} z^{10m+3},$$
$$f'_{10m+4}(z) = 3A_3 z^2 + \ldots + (10m+3) A_{10m+3} z^{10m+2}.$$

We assume that coefficients of polynomial f_{10m+4} satisfy the condition

$$A_{6m+1} b_{2,-1}(5m+2) + \frac{3}{2} A_{6m+3} b_{01} = 0. \tag{37}$$

Condition (37) assures that f_{10m+5} does not have a term with z^{6m+3}. Similar as before, we have to prove only for $f_{10m+15}(z) = f_{10(m+1)+5}$, that it does not contain any logarithmic term. We compute the function $f_{10m+15}(z)$ and obtain a polynomial of degree $10m+15$, denoted by $P_{10m+15}(z)$, and we see that the coefficient of $z^{6(m+1)+3}$ is given by

$$A_{6m+7} b_{2,-1}(5m+7) + \frac{3}{2} A_{6m+9} b_{01}. \tag{38}$$

We can see that condition (38) coincides with condition (37) if m is replaced by $m+1$; hence, it is zero. Therefore, in function $f_{10m+15}(z)$, no logarithmic terms appear after integration, and thus, the induction is proven.

We have proven that the formal first integral of system (23) is of the form

$$\Psi(z, y) = \sum_{k=5}^{\infty} f_k(z) y^k = z^3 y^5 + \sum_{k=6}^{\infty} P_k(z) y^k,$$

where polynomials $P_k(z)$ are defined by (35). Substituting $z \mapsto \frac{x}{y}$, $y \mapsto y$, yields

$$\widetilde{\Psi}(x, y) = \Psi\left(\frac{x}{y}, y\right) = \frac{x^3}{y^3} \cdot y^5 + \sum_{k=6}^{\infty} P_k\left(\frac{x}{y}\right) y^k = x^3 y^2 + \ldots,$$

which is a formal first integral of (22) of the required form, and due to ([1], Corollary 3.2.6), it is also the analytic first integral of (22). Finally, we can form the linearizing transformation of the first equation of system (22), as stated in Theorem 3.

Case (5) We find three algebraic partial integrals:

$$f_1 = x, \quad f_2 = y \quad \text{and} \quad f_3 = 1 + \frac{b_{10} x}{2} - \frac{b_{01} y}{3}.$$

The first equation of the system is linearizable by the change of coordinates

$$x_1 = f_1 f_3^{-1} = \frac{6x}{-2b_{01} y + 3b_{10} x + 6}$$

and the second equation by the transformation

$$y_1 = f_2 f_3^{-1} = \frac{6y}{-2b_{01} y + 3b_{10} x + 6}.$$

Case (6) In this case, system (3) has the form:

$$\dot{x} = 2x + \frac{17b_{10}x^2}{16} + \frac{7b_{01}xy}{9} + \frac{64b_{01}^2y^2}{243b_{10}},$$
$$\dot{y} = -3y + \frac{189b_{10}^2x^2}{256b_{01}} + b_{10}xy + b_{01}y^2. \quad (39)$$

Using algebraic partial integrals

$$f_1 = x + \frac{3b_{10}x^2}{32} - \frac{b_{01}xy}{9} + \frac{8b_{01}^2y^2}{243b_{10}},$$
$$f_2 = y - \frac{27b_{10}^2x^2}{256b_{01}} + \frac{b_{10}xy}{8} - \frac{b_{01}y^2}{27},$$
$$f_3 = 1 + \frac{5b_{10}x}{8} + \frac{25b_{10}^2x^2}{256} - \frac{10b_{01}y}{27} - \frac{25}{216}b_{01}b_{10}xy + \frac{25b_{01}^2y^2}{729},$$

we find the transformation

$$x_1 = f_1f_3^{-1} = \frac{24\left(243b_{10}x(32+3b_{10}x) - 864b_{01}b_{10}xy + 256b_{01}^2y^2\right)}{b_{10}(432+135b_{10}x-80b_{01}y)^2}$$

that linearizes the first equation of the system, and the transformation

$$y_1 = f_2f_3^{-1} = \frac{27\left(6912b_{01}y - (27b_{10}x - 16b_{01}y)^2\right)}{b_{01}(432+135b_{10}x-80b_{01}y)^2}$$

that linearizes the second equation of the system.

We see that linearizing transformations x_1, y_1, and the expressions on the right-hand side of system (39) are not defined for $b_{10} = 0$ or $b_{01} = 0$. By V we denote the variety $\mathbf{V}(I)$. Here

$$I = \langle 256b_{01}b_{2,-1} - 189b_{10}^2, 243b_{10}a_{-12} + 64b_{01}^2, 16a_{10} + 17b_{10}, 9a_{01} + 7b_{01}\rangle$$

is the ideal which is generated by polynomials arising from conditions of Case (6) of Theorem 4, and $J_1 = \langle b_{10}\rangle$ and $J_2 = \langle b_{01}\rangle$. We see that x_1 and y_1 are defined properly for all points from $\mathbf{V}(I)$ except perhaps for points with $b_{10} = 0$ or $b_{01} = 0$, i.e., they are defined for points from $\mathbf{V}\backslash\mathbf{V}(J)$, where $J = J_1 \cap J_2$. The set $\mathbf{V}\backslash\mathbf{V}(J)$ is not necessarily a variety. Note that the smallest variety which contains this set is its Zariski closure $\overline{\mathbf{V}\backslash\mathbf{V}(J)}$. We know (see [16]) that

$$\mathbf{V}(I:J) = \overline{\mathbf{V}(I)\backslash\mathbf{V}(J)}.$$

It is easy to see that $J = \langle b_{10}b_{01}\rangle$. Applying SINGULAR (using the routine "quotient") one easily obtains the quotient

$$I:J = \langle 256b_{01}b_{2,-1} - 189b_{10}^2, 243b_{10}a_{-12} + 64b_{01}^2, 16a_{10} + 17b_{10}, 9a_{01} + 7b_{01}\rangle.$$

Obviously $I:J = I$. Thus $\overline{\mathbf{V}\backslash\mathbf{V}(J)} = \mathbf{V}$ and consequently for each point of the variety V, the corresponding system (39) is linearizable. (With similar computations, one can prove the linearizability for systems, which we study in cases below when some functions (integrating factors, analytic first integrals, and linearizing transformations) are not defined for specific values of parameters.)

Case (7) Using algebraic partial integrals

$$\begin{aligned} f_1 &= x,\\ f_2 &= 1 + 3b_{10}x + 3b_{10}^2x^2 + \frac{1}{3}b_{01}b_{2,-1}x^2 + b_{10}^3x^3 + \frac{1}{3}b_{01}b_{10}b_{2,-1}x^3\\ &\quad + \frac{1}{9}b_{01}^2b_{2,-1}^2x^4 - \frac{4b_{01}y}{3} - \frac{4}{3}b_{01}b_{10}xy - \frac{8}{9}b_{01}^2b_{2,-1}x^2y\\ &\quad - \frac{2}{9}b_{01}^2b_{10}b_{2,-1}x^3y + \frac{2b_{01}^2y^2}{3} - \frac{1}{9}b_{01}^2b_{10}xy^2 + \frac{1}{9}b_{01}^2b_{10}^2x^2y^2\\ &\quad - \frac{2}{27}b_{01}^3b_{2,-1}x^2y^2 - \frac{4b_{01}^3y^3}{27} + \frac{2}{27}b_{01}^3b_{10}xy^3 + \frac{b_{01}^4y^4}{81}, \end{aligned}$$

we construct the linearizing transformation $x_1 = f_1f_2^{-1/3}$ of the first equation of system and the Darboux integrating factor $M = f_1^{1/2}f_2^{-1}$.

Case (8) In this case, the corresponding system is

$$\begin{aligned} \dot{x} &= 2x + 2b_{10}x^2 - a_{01}xy - a_{-12}y^2,\\ \dot{y} &= -3y + b_{10}xy + b_{01}y^2. \end{aligned} \tag{40}$$

We attempt to linearize the second equation of system (40); therefore, we look for a transformation

$$y_1 = \sum_{k=1}^{\infty} f_k(x)y^k \tag{41}$$

such that equation

$$\frac{\partial y_1}{\partial x}\dot{x} + \frac{\partial y_1}{\partial y}\dot{y} = -3y_1 \tag{42}$$

is satisfied for all $x, y \in \mathbb{C}$.

After inserting some initial terms of (41) into (42) and equating coefficients of the same powers on both sides of equation, we see that functions f_k are determined recursively by the differential equation

$$\begin{aligned} &2x(1 + b_{10}x)f_k'(x) - a_{01}xf_{k-1}'(x) - a_{-12}f_{k-2}'(x)\\ &\qquad + (kb_{10}x - 3(k-1))f_k(x) + (k-1)b_{01}f_{k-1}(x) = 0. \end{aligned}$$

For $k = 1, 2, 3, 4, 5, 6$, we find

$$\begin{aligned} f_1(x) &= \frac{1}{(1 + b_{10}x)^{\frac{1}{2}}}, & f_2(x) &= \frac{p_1(x)}{(1 + b_{10}x)^{\frac{5}{2}}},\\ f_3(x) &= \frac{p_2(x)}{(1 + b_{10}x)^{\frac{9}{2}}}, & f_4(x) &= \frac{p_3(x)}{(1 + b_{10}x)^{\frac{13}{2}}},\\ f_5(x) &= \frac{p_4(x)}{(1 + b_{10}x)^{\frac{17}{2}}}, & f_6(x) &= \frac{p_5(x)}{(1 + b_{10}x)^{\frac{21}{2}}}, \end{aligned}$$

where $p_i(x)$ denotes a polynomial of degree i. Suppose that $f_k(x) = \frac{p_{k-1}(x)}{(1+b_{10}x)^{\frac{4k-3}{2}}}$, where $k = 1, 2, \ldots, n-1$. We compute $f_k(x)$ for $k = n$. To this end, we solve the differential equation

$$f_n'(x) + \frac{nb_{10}x - 3(n-1)}{2x(1 + b_{10}x)}f_n(x) = H(x), \tag{43}$$

where $H(x) = \frac{a_{01}xf_{n-1}'(x) + a_{-12}f_{n-2}'(x) + (1-k)b_{01}f_{n-1}(x)}{2x(1+b_{10}x)}$ using the induction assumption about f_{n-1}, f_{n-1}' and f_{n-2}'.

As the general solution of linear differential equation of the form (30) is (31) and, in our case, we have $g(x) = \frac{nb_{10}x - 3(n-1)}{2x(1+b_{10}x)}$ and $h(x) = \frac{q_{n-1}(x)}{x(1+b_{10}x)^{\frac{4n-3}{2}}}$, where q_{n-1} is polynomial of degree at most $n-1$, it follows that $e^{\int g(x)dx} = \frac{(1+b_{10}x)^{\frac{4n-3}{2}}}{x^{\frac{3(n-1)}{2}}}$ and the solution of differential Equation (43) is

$$f_n(x) = \frac{Cx^{\frac{3(n-1)}{2}}}{(1+b_{10}x)^{\frac{4n-3}{2}}} + \frac{\tilde{A}_0 + \tilde{A}_1 x + \ldots + \tilde{A}_{n-1}x^{n-1}}{(1+b_{10}x)^{\frac{4n-3}{2}}}.$$

Choosing $C = 0$, we obtain $f_n(x) = \frac{p_{n-1}(x)}{(1+b_{10}x)^{\frac{4n-3}{2}}}$, where p_{n-1} denotes a polynomial of degree at most $n-1$. We can check that the power series expansion of transformation $y_1(x,y) = \sum_{k=1}^{\infty} f_k(x)y^k$, which linearizes the second equation of system (40) is $y + \sum_{i+j>1}^{\infty} \alpha_{ij}x^i y^j$.

Note that the same method cannot be used to linearize the first equation of system (40). Therefore, we want to find a formal first integral of the form $\Psi(x,y) = \sum_{k=2}^{\infty} f_k(x)y^k$. We compute $D\Psi \equiv 0$ and set the coefficients of the terms y^k to zero, for $k \geq 2$ and thus generate the differential equation:

$$\begin{aligned} 2x(1+b_{10}x)f_k'(x) - a_{01}xf_{k-1}'(x) - a_{-12}f_{k-2}'(x) + k(b_{10}x-3)f_k(x) \\ + (k-1)b_{01}f_{k-1}(x) = 0. \end{aligned}$$

Setting the integration constant equal to 1 and considering $f_0(x) = f_1(x) = 0$, we obtain $f_2(x) = \frac{x^3}{(1+b_{10}x)^4}$ for $k = 2$, and for $k = 3,4,5,6$ (setting the integration constant for each k equal to 0), we obtain the following:

$$\begin{aligned} f_3(x) &= \frac{p_4(x)}{(1+b_{10}x)^6}, & f_4(x) &= \frac{p_5(x)}{(1+b_{10}x)^8}, \\ f_5(x) &= \frac{p_6(x)}{(1+b_{10}x)^{10}}, & f_6(x) &= \frac{p_7(x)}{(1+b_{10}x)^{12}}, \end{aligned}$$

where $p_k(x)$ denotes a polynomial of degree at most k. Suppose by induction that, for $k = 2,\ldots,n-1$, we have $f_k(x) = \frac{p_{k+1}(x)}{(1+b_{10}x)^{2k}}$ and that integration constant is zero. We solve the differential equation:

$$f_n'(x) - \frac{n(3-b_{10}x)}{2x(1+b_{10}x)} f_n(x) = H(x) \tag{44}$$

where $H(x)$ is the same as in (43) and for f_{n-1} and f_{n-2} we use the induction assumption.

Considering Formula (31), we obtain the general solution of linear differential Equation (44):

$$\begin{aligned} f_n(x) &= C \cdot \frac{x^{\frac{3n}{2}}}{(1+b_{10}x)^{2n}} + \frac{B_0x^1 + B_1x^2 + \ldots + B_nx^{n+1}}{(1+b_{10}x)^{2n}} \\ &= C \cdot \frac{x^{\frac{3n}{2}}}{(1+b_{10}x)^{2n}} + \frac{p_{n+1}(x)}{(1+b_{10}x)^{2n}}. \end{aligned}$$

Choosing $C = 0$, we obtain $f_n(x) = \frac{p_{n+1}(x)}{(1+b_{10}x)^{2n}}$, where $p_{n+1}(x)$ is a polynomial of degree at most $n+1$. Therefore, the first integral of the form $\Psi(x,y) = \sum_{k=2}^{\infty} f_k(x)y^k$ in which the power series expansion is of the form $x^3y^2 + \sum_{i+j>6}^{\infty} \alpha_{ij}x^i y^j$ exists.

Case (9) If $a_{-12} = a_{01} = 0$, system (3) has the form

$$\begin{aligned} \dot{x} &= 2x - a_{10}x^2, \\ \dot{y} &= -3y + b_{2,-1}x^2 + b_{10}xy + b_{01}y^2. \end{aligned} \tag{45}$$

We find two algebraic partial integrals:

$$f_1 = x \quad \text{and} \quad f_2 = 1 - \frac{a_{10}x}{2}.$$

Transformation

$$x_1 = f_1 f_2^{-1} = \frac{2x}{2 - a_{10}x}$$

linearizes the first equation of system (45). System (45) has an analytic first integral of the form $\Psi(x, y) = x^3y^2 + \ldots$ [22]. According to Theorem 2, the second equation of system is linearized by transformation $y_1 = \sqrt{\frac{\Psi}{x_1^3}}$.

Case (10) The linearizability of the corresponding system is proven in a similar way as in Case (4), using the method based on blow-up.

Case (11) We are able to linearize the second equation using three algebraic partial integrals:

$$f_1 = y,$$

$$f_2 = 1 - 2b_{10}x + b_{10}^2x^2 - \frac{b_{01}y}{3} + \frac{1}{3}b_{01}b_{10}xy + \frac{1}{3}a_{-12}b_{10}y^2y^3,$$

$$\begin{aligned} f_3 = {} & 1 - 3b_{10}x + 3b_{10}^2x^2 - b_{10}^3x^3 + \frac{1}{6}\left(-3b_{01} - \sqrt{b_{01}^2 - 12a_{-12}b_{10}}\right)y \\ & + \frac{1}{3}\left(3b_{01}b_{10} + b_{10}\sqrt{b_{01}^2 - 12a_{-12}b_{10}}\right)xy + \frac{1}{6}(-3b_{01}b_{10}^2 \\ & - b_{10}^2\sqrt{b_{01}^2 - 12a_{-12}b_{10}})x^2y + \frac{1}{18}\left(b_{01}^2 + 6a_{-12}b_{10}\right. \\ & \left.+ b_{01}\sqrt{b_{01}^2 - 12a_{-12}b_{10}}\right)y^2 + \frac{1}{18}\left(-b_{01}^2b_{10} - 6a_{-12}b_{10}^2\right. \\ & \left.- b_{01}b_{10}\sqrt{b_{01}^2 - 12a_{-12}b_{10}}\right)xy^2 + \frac{1}{1296}(-72a_{-12}b_{01}b_{10} + \\ & + b_{01}^2\sqrt{b_{01}^2 - 12a_{-12}b_{10}} - 84a_{-12}b_{10}\sqrt{b_{01}^2 - 12a_{-12}b_{10}} \\ & \left.-\left(b_{01}^2 - 12a_{-12}b_{10}\right)^{3/2}\right)y^3. \end{aligned}$$

The equation $K_1 + aK_2 + bK_3 = -3$ holds for

$$a = \frac{1}{4} + \frac{15b_{01}}{4\sqrt{b_{01}^2 - 12a_{-12}b_{10}}} \quad \text{and} \quad b = -\frac{5b_{01}}{2\sqrt{b_{01}^2 - 12a_{-12}b_{10}}};$$

thus, the second equation is linearizable and we construct a Darboux integrating factor of the form $M = f_1^c f_2^d f_3^e$ for $c = -\frac{1}{3}$, $d = -\frac{5\left(-3b_{01}+\sqrt{b_{01}^2-12a_{12}b_{10}}\right)}{6\sqrt{b_{01}^2-12a_{-12}b_{10}}}$ and $e = -\frac{5b_{01}}{3\sqrt{b_{01}^2-12a_{12}b_{10}}}$.

Case (12) There are two algebraic partial integrals:

$$f_1 = y,$$
$$f_2 = 1 + b_{10}x + \frac{b_{10}^2x^2}{3} + \frac{b_{10}^3x^3}{27} - \frac{2b_{01}y}{3} - b_{01}b_{10}xy + \frac{1}{9}b_{01}b_{10}^2x^2y$$
$$+ \frac{b_{01}^2y^2}{18} + \frac{1}{9}b_{01}^2b_{10}xy^2 + \frac{b_{01}^3y^3}{27}.$$

The second equation is linearizable by the change of coordinates $y_1 = f_1f_2^{-1/2}$ and the Darboux integrating factor is of the form $M = f_1^{-1/3}f_2^{-1}$.

Case (13) The algebraic partial integrals are

$$f_1 = x, \quad f_2 = 1 - \frac{b_{01}y}{3} - \frac{b_{10}x}{4},$$
$$f_3 = 1 - \frac{5}{4}b_{01}b_{10}xy + \frac{5b_{10}^2x^2}{8} + \frac{5b_{10}x}{4},$$

the linearization of the first equation is

$$x_1 = f_1f_2 = x - \frac{b_{01}xy}{3} - \frac{b_{10}x^2}{4},$$

and teh Darboux integrating factor is $M = f_1^{1/2}f_2^{-1/2}f_3^{-1}$.

Case (14) We use three algebraic partial integrals

$$f_1 = \frac{16b_{10}^2x^2}{27b_{01}} + y + \frac{b_{10}xy}{3} - \frac{b_{01}y^2}{12}, \quad f_2 = 1 - \frac{5b_{10}x}{9} - \frac{5b_{01}y}{12},$$
$$f_3 = 1 + \frac{25b_{10}x}{9} + \frac{200b_{10}^2x^2}{81} - \frac{25}{27}b_{01}b_{10}xy + \frac{25b_{01}^2y^2}{288},$$

the second equation is linearizable by the change of coordinates

$$y_1 = f_1f_2^{-1}f_3^{-1/2}$$

and the Darboux integrating factor is of the form $M = f_1^{-1/3}f_2^{-1}f_3^{-1}$.

Case (15) We find three algebraic partial integrals

$$f_1 = x - \frac{b_{10}x^2}{6} - \frac{4b_{01}xy}{21} + \frac{3b_{01}^2y^2}{98b_{10}}, \quad f_2 = 1 - \frac{5b_{10}x}{3} + \frac{5b_{01}y}{21},$$
$$f_3 = 1 - \frac{5b_{10}x}{6} - \frac{10b_{01}y}{21},$$

the first equation is linearizable by the change of coordinates

$$x_1 = f_1(f_2f_3)^{-2/3}$$

and we construct Darboux integrating factor $M = f_1^{1/2}f_2^{-1}f_3^{-2}$.

Case (16) We compute the following algebraic partial integrals:

$$
\begin{aligned}
f_1 &= y - \frac{108b_{10}^2x^2}{49b_{01}} + \frac{480b_{10}^3x^3}{343b_{01}} - \frac{11b_{10}xy}{7} + \frac{40}{49}b_{10}^2x^2y - \frac{2b_{01}y^2}{9} \\
&+ \frac{10}{63}b_{01}b_{10}xy^2 + \frac{5b_{01}^2y^3}{486}, \\
f_2 &= 1 - \frac{20b_{10}x}{7} + \frac{25b_{10}^2x^2}{7} - \frac{5b_{01}y}{9} + \frac{25}{63}b_{01}b_{10}xy + \frac{25b_{01}^2y^2}{243}, \\
f_3 &= 1 - \frac{15}{7}i\left(-2ib_{10} + \sqrt{3}b_{10}\right)x + \frac{75}{49}\left(b_{10}^2 + 4i\sqrt{3}b_{10}^2\right)x^2 \\
&+ \frac{125}{343}\left(10b_{10}^3 - 9i\sqrt{3}b_{10}^3\right)x^3 + \frac{5}{18}i\left(3ib_{01} + \sqrt{3}b_{01}\right)y \\
&+ \frac{25}{63}\left(9b_{01}b_{10} + i\sqrt{3}b_{01}b_{10}\right)xy + \frac{25}{162}\left(b_{01}^2 - i\sqrt{3}b_{01}^2\right)y^2 \\
&- \frac{125}{882}i\left(-15ib_{01}b_{10}^2 + 11\sqrt{3}b_{01}b_{10}^2\right)x^2y \\
&+ \frac{125i\left(5ib_{01}^2b_{10} + \sqrt{3}b_{01}^2b_{10}\right)xy^2}{1134} + \frac{125ib_{01}^3y^3}{2187\sqrt{3}}.
\end{aligned}
$$

Transformation

$$y_1 = f_1f_2^af_3^b$$

with $a = \frac{1}{4}i\left(3i + \sqrt{3}\right)$ and $b = \frac{-i+\sqrt{3}}{2\left(3i+\sqrt{3}\right)}$ linearizes the second equation of considered system and we construct the Darboux integrating factor $M = f_1^{-1/3}f_2^{-1}$.

Case (17) We obtain three algebraic partial integrals

$$
\begin{aligned}
f_1 &= y - \frac{9b_{10}^2x^2}{2b_{01}} + 2b_{10}xy - \frac{2b_{01}y^2}{9}, \quad f_2 = 1 + \frac{5b_{10}x}{4} - \frac{5b_{01}y}{18}, \\
f_3 &= 1 + \frac{35b_{10}x}{4} + \frac{175b_{10}^2x^2}{8} - \frac{5b_{01}y}{9} - \frac{25}{24}b_{01}b_{10}xy + \frac{25b_{01}^2y^2}{216},
\end{aligned}
$$

the second equation is linearizable by the change of coordinates

$$y_1 = f_1f_2^{-2} = -\frac{72\left(-18b_{01}y + (9b_{10}x - 2b_{01}y)^2\right)}{b_{01}(36 + 45b_{10}x - 10b_{01}y)^2}$$

and the Darboux integrating factor is of the form $M = f_1^{-1/3}f_2^{-1/3}f_3^{-1}$.

Case (18) The linearizability of the corresponding system is proven using the method based on blow-up in a similar way as in Case (4).

Case (19) Using algebraic partial integrals

$$
\begin{aligned}
f_1 &= x - \frac{b_{10}x^2}{6} - \frac{2b_{01}xy}{13} - \frac{6b_{01}^2y^2}{169b_{10}}, \\
f_2 &= y + \frac{13b_{10}^2x^2}{36b_{01}} + \frac{b_{10}xy}{3} + \frac{b_{01}y^2}{13}, \\
f_3 &= 1 + \frac{5b_{10}x}{6} - \frac{25b_{10}^2x^2}{108} - \frac{10b_{01}y}{39} - \frac{25}{117}b_{01}b_{10}xy - \frac{25b_{01}^2y^2}{507}
\end{aligned}
$$

we are able to linearize both equations of the system by the transformations

$$x_1 = f_1f_3^{-1} \quad \text{and} \quad y_1 = f_2f_3^{-1}.$$

Case (20) The linearizability of the obtained system is proven in a similar way as in Case (4), using method based on blow-up.

Case (21) Using algebraic partial integrals

$$\begin{aligned} f_1 &= x + \frac{33b_{10}x^2}{32} + \frac{15b_{10}^2x^3}{1024} - \frac{12b_{01}xy}{23} \\ &\quad + \frac{45b_{01}b_{10}x^2y}{1472} - \frac{6b_{01}^2y^2}{529b_{10}} + \frac{45b_{01}^2xy^2}{2116} + \frac{60b_{01}^3y^3}{12167b_{10}}, \\ f_2 &= \frac{13b_{10}^2x^2}{36b_{01}} + y + \frac{b_{10}xy}{3} + \frac{b_{01}y^2}{13}, \\ f_3 &= 1 + \frac{5b_{10}x}{6} - \frac{25b_{10}^2x^2}{108} - \frac{10b_{01}y}{39} - \frac{25}{117}b_{01}b_{10}xy - \frac{25b_{01}^2y^2}{507}, \end{aligned}$$

we find linearizing transformation

$$x_1 = f_1f_3^{-1} \quad \text{and} \quad y_1 = f_2f_3^{-1}.$$

Case (22) We use the following three algebraic partial integrals:

$$\begin{aligned} f_1 &= \frac{192b_{10}^2x^2}{49b_{01}} + y - \frac{11b_{10}xy}{7} + \frac{b_{01}y^2}{8}, \quad f_2 = 1 + \frac{5b_{10}x}{7} - \frac{5b_{01}y}{24} \\ f_3 &= 1 - 15b_{10}x + 75b_{10}^2x^2 - 125b_{10}^3x^3 - \frac{15b_{01}y}{8} + \frac{75}{4}b_{01}b_{10}xy \\ &\quad - \frac{375}{8}b_{01}b_{10}^2x^2y + \frac{75b_{01}^2y^2}{64} - \frac{375}{64}b_{01}^2b_{10}xy^2 - \frac{125b_{01}^3y^3}{512}. \end{aligned}$$

The second equation of the system is linearizable by

$$y_1 = f_1f_2^{-1/4}f_3^{-1/12}$$

and system has a Darboux integrating factor $M = f_1^{-1/3}f_2^{-1/6}f_3^{-7/18}$.

Cases (23), (24) and (25) As for the conditions that arise in cases (23), (24), and (25) we observe that they are quite long. We can find some solutions that satisfy to the particular cases (23), (24), or (25). For instance, using Mathematica, we can solve the system of equations in case (23) and we obtain 15 solutions. If we pick up one solution, for example

$$a_{01} = -\frac{8b_{01}}{15}, \; a_{10} = -\frac{9b_{01}^2}{25a_{-12}}, \; b_{10} = 0, \; b_{2,-1} = -\frac{21b_{01}^3}{125a_{-12}^2},$$

we can easily check that, under this solution, all four pairs of linearizability quantities are zero and the system corresponding to this solution is

$$\begin{aligned} \dot{x} &= 2x + \frac{9b_{01}^2}{25a_{-12}}x^2 + \frac{8b_{01}}{15}xy - a_{-12}y^2, \\ \dot{y} &= -3y - \frac{21b_{01}^3}{125a_{-12}^2}x^2 + b_{01}y^2. \end{aligned} \tag{46}$$

To prove the linearizability of this system, we can use a similar approach based on blow-up transformation as in Case (4). Similarly, we can use the approach also in cases (24) and (25). □

Remark 2. *Some functions appearing in the proof of Theorem 4 are not defined for specific values of parameters. The existence of analytic first integrals or analytic linearizing transformations for these specific values is following from the fact that the set of all systems (3) with a complex resonant (linearizable) center is a closed set with Zariski topology. The computations are similar as in the proof of Case (6) of Theorem 4.*

The proofs of cases (10), (18), (20), (23), (24) and (25) are very long and similar to Case (4); thus they are omitted in the paper, but the interested reader can requent them from the authors.

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References

1. Romanovski, V.G.; Shafer, D.S. *The Center and Cyclicity Problems: A Computational Algebra Approach*; Birkhäuser: Boston, MA, USA, 2009.
2. Oliveri, F. Lie symmetries of differential equations: Classical results and recent contributions. *Symmetry* **2010**, *2*, 658–706. [CrossRef]
3. Mahomed, F.M. Symmetry group classification of ordinary differential equations: Survey of some results. *Math. Methods Appl. Sci.* **2007**, *30*, 1995–2012. [CrossRef]
4. Mahomed, F.M.; Leach, P.G.L. Symmetry Lie algebras of nth order ordinary differential equations. *J. Math. Anal. Appl.* **1990**, *151*, 80–107. [CrossRef]
5. Safdar, M.; Qadir, A.; Ali, S. Linearizability of Systems of Ordinary Differential Equations Obtained by Complex Symmetry Analysis. *Math. Probl. Eng.* **2011**, *2011*, 171834. [CrossRef]
6. Fronville, A.; Sadovskiĭ, A.P.; Żoładek, H. Solution of the 1:−2 resonant center problem in the quadratic case. *Fund. Math.* **1998**, *157*, 191–207. [CrossRef]
7. Dukarić, M.; Ferčec, B.; Giné, J. The solution of the 1:−3 resonant center problem in the quadratic case. *Appl. Math. Comput.* **2014**, *237*, 501–511. [CrossRef]
8. Ferčec, B.; Giné, J.; Mencinger, M.; Oliveira, R. The center problem for a 1:−4 resonant quadratic system. *Math. Anal. Appl.* **2014**, *420*, 1568–1591. [CrossRef]
9. Christopher, C.; Mardešić, P.; Rousseau, C. Normalizable, integrable and linerizable saddle points in complex quadratic systems in $\mathbb{C}^2$. *J. Dyn. Control Syst.* **2003**, *9*, 311–363 [CrossRef]
10. Romanovski, V.G.; Prešern, M. An approach to solving systems of polynomials via modular arithmetics with application. *J. Comput. Appl. Math.* **2011**, *236*, 196–208. [CrossRef]
11. Kadyrsizova, Z.; Romanovski, V.G. Linearizablity of 1:−3 resonant system with homogeneous cubic nonlinearities. In Proceedings of the Twenty-First International Symposium on Symbolic and Algebraic Computation, Linz/Hagenberg, Austria, 20–23 July 2008; pp. 255–260.
12. Gravel, S.; Pierre T. Integrability and linearizability of the Lotka–Volterra system with a saddle point with rational hyperbolicity ratio. *J. Differ. Equ.* **2002**, *184*, 20–47. [CrossRef]
13. Liu, C.; Chen, G.; Li, C. Integrability and linearizability of the Lotka-Volterra systems. *J. Differ. Equ.* **2004**, *198*, 301–320. [CrossRef]
14. Wang, Q.; Liu, Y. Linearizability of the polynomial differential systems with a resonant singular point. *Bull. Sci. Math.* **2008**, *132*, 97–111. [CrossRef]
15. Wang, Q.; Huang, W. Integrability and linearizability for Lotka-Volterra systems with the 3:$-q$ resonant saddle point. *Adv. Differ. Equ.* **2014**, *2014*, 23. [CrossRef]
16. Cox, D.; Little, J.; O'Shea, D. *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*, 3rd ed.; Springer: New York, NY, USA, 2007.
17. Decker, W.; Laplagne, S.; Pfister, G.; Schonemann, H.A. *SINGULAR* 3-1 Library for Computing the Prime Decomposition and Radical of Ideals, Primdec.lib. 2010. Available online: https://www.singular.uni-kl.de (accessed on 16 August 2021).
18. Gianni, P.; Trager, B.; Zacharias, G. Gröbner bases and primary decomposition of polynomial ideals. *J. Symb. Comput.* **1988**, *6*, 149–167. [CrossRef]
19. Greuel, G.M.; Pfister, G.; Schönemann, H.A. `SINGULAR 3.0`. In *A Computer Algebra System for Polynomial Computations*; Centre for Computer Algebra, University of Kaiserlautern: Kaiserslautern, Germany, 2005. Available online: http://www.singular.uni-kl.de (accessed on 16 August 2021).
20. Wang, P.S.; Guy, M.J.T.; Davenport, J.H. P-adic reconstruction of rational numbers. *ACM Sigsam Bull.* **1982**, *16*, 2–3. [CrossRef]
21. Faugére, J.C. FGb: A library for computing Gröbner bases. In *Mathematical Software—ICMS 2010, Lecture Notes in Computer Science*; Fukuda, K., Hoeven, J., Joswig, M., Takayama, N., Eds.; Springer: Berlin/Heidelberg, Germany, 2010; Volume 6327, pp. 84–87.

22. Žulj, M.; Ferčec, B.; Mencinger, M. The solution of some persistent p:$-q$ resonant center problems. *Electron. J. Qual. Theory Differ. Equ.* **2018**, *99*, 1–21. [CrossRef]
23. Giné, J.; Romanovski, V.G. Linearizability conditions for Lotka–Volterra planar complex cubic systems. *J. Phys. Math. Theor.* **2009**, *42*, 225206. [CrossRef]
24. Ferčec, B.; Giné, J. Blow-up method to prove formal integrability for planar differential systems. *J. Appl. Anal. Comput.* **2018**, *8*, 1833–1850.