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Dispersionless BKP Equation, the Manakov–Santini System and Einstein–Weyl Structures

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Abstract: We construct a map from solutions of the dispersionless BKP (dBKP) equation to solutions of the Manakov–Santini (MS) system. This map defines an Einstein–Weyl structure corresponding to the dBKP equation through the general Lorentzian Einstein–Weyl structure corresponding to the MS system. We give a spectral characterisation of reduction in the MS system, which singles out the image of the dBKP equation solution, and also consider more general reductions of this class. We define the BMS system and extend the map defined above to the map (Miura transformation) of solutions of the BMS system to solutions of the MS system, thus obtaining an Einstein–Weyl structure for the BMS system.

Keywords: dispersionless integrable systems; the Manakov–Santini system; Einstein–Weyl structures; the dispersionless BKP hierarchy

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1. Introduction

The dispersionless BKP hierarchy is a reduction in the dispersionless KP hierarchy by a special symmetry, which is only compatible with odd times of the hierarchy [1,2]. Equations of the hierarchy can be represented as compatibility conditions for certain Hamilton–Jacobi equations. For the first equation of the dispersionless BKP (dBKP) hierarchy, the corresponding Hamilton–Jacobi equations are:

$$\begin{aligned} S_y &= H_1 = p^3 + 3up \\ S_t &= H_2 = p^5 + 5up^3 + vp, \quad p = S_x \end{aligned} \tag{1}$$

The symmetry characterising the reduction from the dKP to dBKP hierarchy is the simple condition on the Hamiltonians $H(-p) = -H(p)$, $x = t_1$, $y = t_3$, $t = t_5$ (in terms of dispersionless KP hierarchy times). Compatibility of the Hamilton–Jacobi Equation (1) requires

$$\partial_t H_1 - \partial_y H_2 + \{H_1, H_2\} = 0, \tag{2}$$

where the Poisson bracket is $\{f, g\} = f_p g_x - f_x g_p$, giving rise to the dispersionless BKP equation (see [1,2])

$$\tfrac{1}{5}u_t + u^2 u_x - \tfrac{1}{3}uu_y - \tfrac{1}{3}u_x\partial_x^{-1}u_y - \tfrac{1}{9}\partial_x^{-1}u_{yy} = 0. \tag{3}$$

In what follows, we rescale the times to simplify the coefficients and use the Hamiltonians $H_1 = \frac{1}{3}p^3 + up$, $H_2 = \frac{1}{5}p^5 + up^3 + vp$, then Equation (3) reads:

$$u_t + u^2 u_x - uu_y - u_x\partial_x^{-1}u_y - \partial_x^{-1}u_{yy} = 0. \tag{4}$$

In potential form, $u = f_x$, we have:

$$\partial_x\left(f_t + \tfrac{1}{3}f_x^3 - f_x f_y\right) = f_{yy}. \tag{5}$$

The Lax pair in terms of Hamilton–Jacobi equations (pseudopotentials) can be represented as commutation relations for Hamiltonian vector fields:

$$\begin{aligned} V_1 &= \partial_y - \{H_1, \dots\}, \\ V_2 &= \partial_t - \{H_2, \dots\}, \end{aligned} \tag{6}$$

where $\{H, \dots\} = (\partial_p H)\partial_x - (\partial_x H)\partial_p$. In this Lax pair, p plays the role of 'spectral parameter', and the commutation relation $[V_1, V_2] = 0$ gives exactly Equation (4). The procedure for integrating equations arising as commutation relations for vector fields is not restricted to Hamiltonian vector fields. Moreover, several interesting examples corresponding to general vector fields were discussed, e.g., the Manakov–Santini (MS) system [3,4], which was recently demonstrated to describe a general ocal form of Einstein–Weyl equations [5].

2. From the dBKP Equation to the MS System

In explicit form, the vector fields (6) read:

$$\begin{aligned} V_1 &= \partial_y - (p^2 + f_x)\partial_x + f_{xx} p \partial_p, \\ V_2 &= \partial_t - (p^4 + 3 f_x p^2 + v)\partial_x, + (f_{xx} p^2 + v_x) p \partial_p, \end{aligned} \tag{7}$$

$v = f_y + f_x^2$. The symmetry of vector fields $V(-p) = V(p)$ characterises the dispersionless BKP hierarchy in the framework of the dKP hierarchy.

Let us transform the spectral parameter $p^2 = \mu$. The commutation relations evidently remain the same:

$$\begin{aligned} V_1 &= \partial_y - (\mu + f_x)\partial_x + 2 f_{xx} \mu \partial_\mu, \\ V_2 &= \partial_t - (\mu^2 + 3 f_x \mu + v)\partial_x + 2(f_{xx}\mu + v_x)\mu \partial_\mu \end{aligned}$$

The vector fields are still Hamiltonian. Now, et us make the change of the spectral parameter depending on times $\lambda = \mu + 2 f_x$ (which preserves the commutation relations):

$$\begin{aligned} V_1 &= \partial_y - (\lambda - f_x)\partial_x + 2(f_{xy} - f_x f_{xx})\partial_\lambda, \\ V_2 &= \partial_t - (\lambda^2 - f_x \lambda + f_y - f_x^2)\partial_x \\ &\quad + 2((f_{xy} - f_x f_{xx})\lambda + (f_{xt} - f_y f_{xx} + f_x^2 f_{xx} - 2 f_{xy} f_x))\partial_\lambda. \end{aligned} \tag{8}$$

The Lax pair V_1, V_2 has the structure of the Manakov–Santini (MS) system Lax pair:

$$\begin{aligned} X_1 &= \partial_y - (\lambda - v_x)\partial_x + u_x \partial_\lambda, \\ X_2 &= \partial_t - (\lambda^2 - v_x \lambda + u - v_y)\partial_x + (u_x \lambda + u_y)\partial_\lambda, \end{aligned} \tag{9}$$

whose compatibility engenders the MS system:

$$\begin{aligned} u_{xt} &= u_{yy} + (u u_x)_x + v_x u_{xy} - u_{xx} v_y, \\ v_{xt} &= v_{yy} + u v_{xx} + v_x v_{xy} - v_{xx} v_y. \end{aligned} \tag{10}$$

A comparison of the Lax pairs (8) and (9) gives the map from solutions of the dBKP Equation (5) to solutions of the MS System (10):

$$v = f, \quad u = 2 f_y - f_x^2, \tag{11}$$

and corresponding solutions for the MS system satisfy the reduction:

$$u = 2 v_y - v_x^2. \tag{12}$$

This map defines the Einstein–Weyl structure corresponding to the dBKP equation.

2.1. Einstein–Weyl Structure for the dBKP Equation

We do not give a detailed description of Einstein–Weyl structures, because it is out of the scope of this work. We just formulate some basic facts we need (see [6,7] for more detail).

Let us start with a Weyl structure. The Weyl structure is defined by a pair (g, ω), where g is a metric tensor and ω is a one-form. It is possible to introduce a symmetric connection D satisfying the relation:

$$Dg = \omega \otimes g; \tag{13}$$

explicit expression for this connection in terms of (g, ω) is given in [6]. Relation (13) is invariant under Weyl transformations:

$$g \to \mathrm{e}^{\rho} g, \quad \omega \to \omega + \mathrm{d}\rho,$$

which represent a combination of a conformal transformation of the metric g and a gauge transformation of the one-form ω (ρ is an arbitrary scalar function). Thus, the connection D only depends on the equivalence class of pairs (g, ω) modulo Weyl transformations, and it preserves a conformal class of the metric. Introducing the curvature tensor and the Ricci tensor of the connection, it is possible to define Einstein equations for this connection, which together with relation (13) constitute the Einstein–Weyl equations system in coordinate form:

$$D_k g_{ij} = \omega_k g_{ij}, \quad R_{(ij)} = \Lambda g_{ij}, \tag{14}$$

where Λ is some function. In the three-dimensional case considered in this work, the Einstein–Weyl equations are integrable by twistor methods [8].

In what follows, we exploit a nice result of the work [5], where it was proved that the Manakov–Santini system (10) defines a general ocal form of the (2 + 1)-dimensional Lorentzian Einstein–Weyl structure (modulo coordinate transformations) with metric g and one-form (covector) ω, defined as:

$$\begin{aligned} g &= -(dy + v_x dt)^2 + 4(dx + (u - v_y)dt)dt, \\ \omega &= v_{xx}dy + (-4u_x + 2v_{xy} + v_x v_{xx})dt, \end{aligned} \tag{15}$$

where u, v sastisfy the MS system. Using the map (11), we obtain the Einstein–Weyl structure corresponding to solutions of the potential dispersionless BKP Equation (5),

$$\begin{aligned} g &= -(dy + f_x dt)^2 + 4(dx + (f_y - f_x^2)dt)dt, \\ \omega &= f_{xx}dy + 3(-2f_{xy} + 3f_x f_{xx})dt. \end{aligned} \tag{16}$$

It could be possible to construct this Einstein–Weyl structure by the methods of the work in [7], starting from the symbol of inearisation of Equation (5). Here, we do it directly, using the map (11).

2.2. dBKP Equation as a Reduction of the MS System

It is possible to obtain condition (12) by means of a reduction in the MS hierarchy, characterised by the existence of wave function of adjoint inear operators of the hierarchy with special analytic properties (with respect to the spectral variable). The technique for constructing this type of reduction was developed in [9]. Here, we carry out an elementary derivation on the evel of the Lax operator for the MS system.

First, we introduce formally adjoint inear operators, defined by the rule $(u\partial)^* = -\partial u$ (for all partial derivatives),

$$-X^* = X + \operatorname{div} X,$$

for the Lax operator of the MS system (9), we obtain

$$-X_1^* = \partial_y - (\lambda - v_x)\partial_x + u_x\partial_\lambda + v_{xx} = X_1 + v_{xx} \tag{17}$$

We should emphasize that adjoint vector fields in general are not vector fields and contain an extra term without derivative, equal to the divergence of the vector field, for zero divergence vector fields are (anti) self-adjoint. However, the commutation of adjoint vector fields gives the same compatibility conditions.

Let us suppose that the adjoint Lax operator (17) possess a wave function of the form

$$\widetilde{\psi} = (\lambda - \eta)^\alpha, \quad X_1^*\widetilde{\psi} = 0, \tag{18}$$

where η is a function of times. This condition is compatible with the dynamics of the hierarchy and defines a reduction, see [9]. This form of the wave function can be found by inspection of the map from the dispersionless BKP equation to the MS system; we skip the details. For the ogarithm of the wave function, we have the equation

$$X_1 \ln \widetilde{\psi} + v_{xx} = (\partial_y - (\lambda - v_x)\partial_x + u_x\partial_\lambda) \ln \widetilde{\psi} + v_{xx} = 0,$$

and, substituting $\widetilde{\psi} = (\lambda - \eta)^\alpha$, we get

$$\begin{aligned} u_x &= \eta_y + \eta_x v_x - \eta\eta_x, \\ v_x &= -\alpha\eta, \end{aligned} \tag{19}$$

implying the condition

$$u = -\alpha^{-1}v_y - \frac{1}{2}(\alpha^{-2} + \alpha^{-1})v_x^2. \tag{20}$$

For $\alpha = -\frac{1}{2}$, this condition coincides with condition (12) and the MS system (10) reduces to the potential dispersionless BKP Equation (5) ($f = v$). For $\alpha = 0$, relations (19) imply that $v = 0$, and we obtain the dKP equation for the function u, $u_x = g_y - gg_x$. For general α, the MS system reduces to the equation

$$v_{xt} = v_{yy} - \alpha^{-1}(v_y + \frac{1}{2}(\alpha^{-1} + 1)v_x^2)v_{xx} + v_x v_{xy} - v_{xx}v_y. \tag{21}$$

An interesting special case corresponds to $\alpha = -1$, then condition (20) takes the form $u = v_y$ and the MS system reduces to the equation

$$v_{xt} = v_{yy} + v_x v_{xy}. \tag{22}$$

This equation is known in the iterature; it belongs to the dKP hierarchy and corresponds to the flow defined by the 'vertex' time.

The Einstein–Weyl structure for Equation (21) is obtained by substitution from (20) for u into the Einstein–Weyl structure (15) for the MS system; thus,

$$\begin{aligned} g &= -(dy + v_x dt)^2 + 4(dx - (\alpha^{-1} + 1)(v_y + \frac{1}{2\alpha}v_x^2)dt)dt, \\ \omega &= v_{xx}dy + (2(1 + 2\alpha^{-1})v_{xy} + (1 + 4(\alpha^{-2} + \alpha^{-1}))v_x v_{xx})dt, \end{aligned}$$

which for Equation (22), reduces to

$$\begin{aligned} g &= -(dy + v_x dt)^2 + 4(dx + dt)dt, \\ \omega &= v_{xx}dy + (-2v_{xy} + v_x v_{xx})dt. \end{aligned}$$

It is natural to expect that, similar to the dBKP case, Equation (21) for arbitrary α could be obtained from some Hamiltonian Lax pairs. Indeed, this is so! Let us consider the dBKP type Lax pair (1) with Hamiltonians

$$H_1 = \frac{1}{1+\beta} p^{\beta+1} + f_x p,$$
$$H_2 = \frac{1}{1+2\beta} p^{2\beta+1} + f_x p^{\beta+1} + wp; \tag{23}$$

$\beta = 2$ corresponds to the dBKP equation case. The corresponding Hamiltonian vector fields are

$$V_1 = \partial_y - (p^\beta + f_x)\partial_x + f_{xx} p \partial_p,$$
$$V_2 = \partial_t - (p^{2\beta} + (\beta+1) f_x p^\beta + w)\partial_x, +(f_{xx} p^\beta + w_x) p \partial_p. \tag{24}$$

Similarly to the dBKP equation Lax pair, we transform this Lax pair to obtain the Lax pair of MS type. The first step is to perform the transformation $\mu = p^\beta$,

$$V_1 = \partial_y - (\mu + f_x)\partial_x + \beta f_{xx} \mu \partial_\mu,$$
$$V_2 = \partial_t - (\mu^2 + (\beta+1) f_x \mu + w)\partial_x, +\beta(f_{xx}\mu + w_x)\mu \partial_\mu.$$

The second step is the transformation $\lambda = \mu + \beta f_x$,

$$V_1 = \partial_y - (\lambda + (1-\beta) f_x)\partial_x + (\beta f_{xy} - \beta f_x f_{xx})\partial_\mu.$$

Comparing this Lax operator with the MS Lax operator (9), we get

$$u_x = \beta f_{xy} - \beta f_x f_{xx},$$
$$v = (\beta - 1) f.$$

After the identification $g = \beta f_x$, $\beta^{-1} = \alpha + 1$, this transformation coincides with the one in (19). Thus, Equation (21) can be obtained from the Hamiltonian Lax pair (23) and (24), upon substituting $\beta^{-1} = \alpha + 1$, $v = \alpha f$.

The Hamiltonians (23) are connected with the Kupershmidt hydrodynamic chains [10].

2.3. BMS System

The symmetry of vector fields $V(-p) = V(p)$ (7) characterising the dispersionless BKP hierarchy in the framework of the dKP hierarchy can be extended to the Manakov–Santini hierarchy. We call the arising hierarchy the BMS hierarchy by analogy with the dispersionless BKP hierarchy. The Lax pair for the first equation of the hierarchy (the BMS system) reads

$$V_1 = \partial_y - (p^2 - v_x)\partial_x + u_x p \partial_p,$$
$$V_2 = \partial_t - (p^4 + (2u - v_x)p^2 + w_1)\partial_x + (u_x p^2 + w_2) p \partial_p, \tag{25}$$

Compatibility conditions imply that $w_2 = u_y + (u^2)_x$, and the BMS system can be written in the form

$$v_{xt} = v_{xx} w_1 - v w_{1x} - w_{1y},$$
$$u_{xt} = u_{yy} + (u^2)_{xy} + u_{xx} w_1 + v u_{xy} + v (u^2)_{xx}, \tag{26}$$
$$w_{1x} = -2u v_{xx} - v_{xy}.$$

For $u = 0$, corresponding to the inearly degenerate case, when vector fields in the Lax pair do not include the derivative with respect to the spectral parameter, the BMS system (26) reduces to the equation

$$v_{xt} = v_x v_{xy} - v_{xx} v_y + v_{yy},$$

coinciding with the inearly degenerate reduction in the MS system (10). This is not unexpected, because in the inearly degenerate case, we have the freedom to make an arbitrary transformation (independent of times) of the spectral variable.

Hamiltonian reduction corresponds to $u = -v_x$, and system (26) reduces to the potential dBKP Equation (5) for the function $f = -v$.

The condition $v = 0$ is evidently also a reduction in system (26); it eads to the equation

$$u_{xt} = u_{yy} + (u^2)_{xy},$$

which coincides with Equation (22) after the identification $v_x = 2u$.

It is rather suprising that, following the steps of transformation of the dBKP Lax pair to the MS-type Lax pair described above, we are able to define a map (Miura transformation) from solutions of the BMS system to solutions of the MS system, thus defining the Einstein–Weyl structure corresponding to the BMS system.

Performing the transformation $\lambda = p^2 + 2u$ in the Lax pair (25), we obtain a Lax pair of MS type with a Lax operator:

$$V_1 = \partial_y - (\lambda - 2u - v_x)\partial_x + 2(u_y + v_x u_x)\partial_\lambda.$$

Comparing this Lax operator to the MS Lax operator (9), we obtain a transformation of solutions of the BMS system to solutions of the MS system:

$$\begin{aligned} &2u + v_x \to v_x, \\ &2(u_y + v_x u_x) \to u_x. \end{aligned}$$

Substituting this transformation into the Einstein–Weyl structure (15), we obtain the Einstein–Weyl structure corresponding to the BMS system (26),

$$\begin{aligned} g &= -(dy + (2u + v_x)dt)^2 + 4(dx + (2\partial_x^{-1}(v_x u_x) - v_y)dt)dt, \\ \omega &= (2u_x + v_{xx})dy + (-8v_x u_x - 4u_y + 2v_{xy} + \frac{1}{2}((2u + v_x)^2)_x)dt. \end{aligned}$$

2.4. Hydrodynamic Type Reductions of the MS System

Let us consider a multicomponent generalisation of reduction (18) $\widetilde{\psi} = \prod(\lambda - \eta^i)^{\alpha_i}$ and complement it with a standard waterbag ansatz [11] for the wave functions of MS inear operators (9) $\psi = \lambda + \sum \gamma_j \ln(\lambda - \phi^j)$. Some special examples of related reductions were considered in [12]. These reductions lead to (1+1)-dimensional systems of hydrodynamic type, defining the dynamics with respect to y,

$$\begin{aligned} &\eta^i_y - (\eta^i - v_x)\eta^i_x - u_x = 0, \\ &\phi^j_y - (\phi^j - v_x)\phi^j_x - u_x = 0, \\ &v_x = -\sum \alpha_i \eta^i, \quad u = -\sum \gamma_j \phi^j \end{aligned}$$

and with respect to t,

$$\eta^i_t - ((\eta^i)^2 - v_x\eta^i + u - v_y)\eta^i_x - (\eta^i u_x + u_y) = 0,$$
$$\phi^j_t - ((\phi^i)^2 - v_x\phi^i + u - v_y)\phi^i_x - (\phi^i u_x + u_y) = 0,$$
$$\partial_x v_y = -\partial_y \sum \alpha_i \eta^i = -\sum \alpha_i((\eta^i - v_x)\eta^i_x + u_x),$$
$$u_y = -\sum \gamma_j((\phi^j - v_x)\phi^j_x + u_x).$$

These (1 + 1)-dimensional systems are compatible and their common solution defines a solution of the MS system

$$v_x = -\sum \alpha_i \eta^i, \quad u = -\sum \gamma_j \phi^j.$$

The Einstein–Weyl structure corresponding to this solution is given by expressions (15).

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