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Quadratic Stabilization of Linear Uncertain Positive Discrete-Time Systems

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Abstract: The paper provides extended methods for control linear positive discrete-time systems that are subject to parameter uncertainties, reflecting structural system parameter constraints and positive system properties when solving the problem of system quadratic stability. By using an extension of the Lyapunov approach, system quadratic stability is presented to become apparent in pre-existing positivity constraints in the design of feedback control. The approach prefers constraints representation in the form of linear matrix inequalities, reflects the diagonal stabilization principle in order to apply to positive systems the idea of matrix parameter positivity, applies observer-based linear state control to assert closed-loop system quadratic stability and projects design conditions, allowing minimization of an undesirable impact on matching parameter uncertainties. The method is utilised in numerical examples to illustrate the technique when applying the above strategy.

Keywords: positive linear systems; diagonal stabilization; linear matrix inequalities; uncertain systems; matching conditions

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1. Introduction

Positive systems cover a special family of systems possessing the property that their states and outputs are inherently non-negative and, as a consequence, are subconsciously connected with such real processes whose internal variables are positive [1,2]. Along this line, since the main task when dealing with control and system state estimation is closely linked to the positivity relations that must be maintained in the system dynamics [3], the existence of positive structures has to accept also limitations in the non-negativity of control law parameters or observer gains. In terms of analysis, stability and performance characterizations, some constrained design approaches were established to solve design problems for dynamical systems with positivity [4,5].

The need for new frameworks in positive system analysis, relying on the practical concept of the linear matrix inequalities (LMIs) feasibility, is reflected in [6–8], whilst the inherent time-delay system properties, exploitable in the control design of positive systems are preferable exposed (see, e.g., [9] and the references therein). The implementability of control structures related to discrete-time systems with time delays is presented in [10,11] and with relation to positive discrete-time systems with time delays in [12]. Stabilization principles for uncertain discrete-time-positive systems are proposed in [13–15], and specific control structure realization is mentioned in [16,17].

Adaptation of the presented new points of view given on the control synthesis of linear positive discrete-time systems in [18,19], as well as their dissemination to positive systems with uncertain parameters, are the main issues of this paper. In considerable order of precedence is LMI formulation in parametric constraint prescription, together with the structural quadratic stability, to handle more general preference of arguments based on the Lyapunov method.

New design conditions are derived related to uncertain discrete-time systems, which ensure both quadratic stability and positiveness performances in controller and observer structures. Such conditions are explicitly represented by feasibility of the proposed LMIs set. The multi-input, multi-output (MIMO) state-space representation is preferred, because the performance specifications used in the design task have to view the controller-related dimensions of matrix parameters when defining the uncertainties by LMIs. Because the objective is intended for diagonal positive matrix variables, it guarantees the diagonal stabilization principle.

For clarity of presentation, following the decisive reason for preference given in Section 1, the paper continues in Section 2 with separate treatments of the design fundamentals related to constraint formulations for uncertain positive discrete-time linear systems. Section 3, in the sense of the above ways in defining design limits, discusses problem of quadratic stability and positiveness in control, preserves design adaptations to state observer synthesis and presents the expression of the design completeness for the observer-based state control of an uncertain system from this class of plants. To illustrate various limits of design, a numerical solution is inserted in Section 4, whilst in Section 5 a summary is presented, and conclusions are drawn.

Throughout this paper, the following notations are used: $\boldsymbol{x}^{\mathrm{T}}$, $\boldsymbol{X}^{\mathrm{T}}$ denotes the transpose of the vector $\boldsymbol{x}$ and the matrix $\boldsymbol{X}$, respectively, diag $[\,\cdot\,]$ characterises the structure of a (block) diagonal matrix, $\rho(\boldsymbol{X})$ indicates the eigenvalue spectrum of $\boldsymbol{X}$ for a square symmetric matrix, by definition $\boldsymbol{X} \prec 0$ means a negative definite matrix, the symbol $\boldsymbol{I}_n$ indicates the n-th order unit matrix, $\mathbb{R}$ $(\mathbb{R}_+)$ is the set of all (non-negative) real numbers, $(\mathbb{R}_+^{n\times r})$, $\mathbb{R}^{n\times r}$ refers to the set of $n \times r$ (non-negative) real matrices, and $\mathbb{R}_{++}^{n\times n}$, $(\mathbb{R}_{++\circ}^{n\times n})$ means the set of strictly (purely) positive square matrices, respectively.

2. Problem Formulation and Starting Preliminaries

Consider the uncertain discrete-time systems of the form

$$\boldsymbol{q}(i+1) = (\boldsymbol{F} + \Delta\boldsymbol{F}(i))\boldsymbol{q}(i) + (\boldsymbol{G} + \Delta\boldsymbol{G}(i))\boldsymbol{u}(i)\,, \tag{1}$$

$$\boldsymbol{y}(i) = (\boldsymbol{C} + \Delta\boldsymbol{C}(i))\boldsymbol{q}(i)\,, \tag{2}$$

$$\begin{bmatrix}\Delta\boldsymbol{F}(i) & \Delta\boldsymbol{G}(i)\end{bmatrix} = \boldsymbol{M}\boldsymbol{H}(i)\begin{bmatrix}\boldsymbol{N}_1 & \boldsymbol{N}_2\end{bmatrix}, \quad \boldsymbol{H}^{\mathrm{T}}(i)\boldsymbol{H}(i) \preceq \boldsymbol{I}_p\,, \tag{3}$$

$$\begin{bmatrix}\Delta\boldsymbol{F}(i)\\ \Delta\boldsymbol{C}(i)\end{bmatrix} = \begin{bmatrix}\boldsymbol{V}_1\\ \boldsymbol{V}_2\end{bmatrix}\boldsymbol{W}(i)\boldsymbol{U}, \quad \boldsymbol{W}^{\mathrm{T}}(i)\boldsymbol{W}(i) \preceq \boldsymbol{I}_p\,, \tag{4}$$

where $\boldsymbol{u}(i) \in \mathbb{R}^r$, $\boldsymbol{q}(t) \in \mathbb{R}^n$ and $\boldsymbol{y}(t) \in \mathbb{R}^m$. The matrix parameters are of the following relations $\boldsymbol{F}, \Delta\boldsymbol{F}(i) \in \mathbb{R}^{n\times n}$, $\boldsymbol{G}, \Delta\boldsymbol{G}(i) \in \mathbb{R}^{n\times r}$, $\boldsymbol{C}, \Delta\boldsymbol{C}(i) \in \mathbb{R}^{m\times n}$, $\boldsymbol{M} \in \mathbb{R}^{n\times p}$, $\boldsymbol{N}_1 \in \mathbb{R}^{p\times n}$, $\boldsymbol{N}_2 \in \mathbb{R}^{p\times r}$ $\boldsymbol{V}_1 \in \mathbb{R}^{n\times p}$, $\boldsymbol{V}_2 \in \mathbb{R}^{m\times p}$ and $\boldsymbol{U} \in \mathbb{R}^{p\times n}$, and the elements of $\boldsymbol{H}(i), \boldsymbol{W}(i) \in \mathbb{R}^{p\times p}$ are Lebesgue measurable [20].

Notice how, in the used context, the above makes use of a square system with $p = m = r$, whilst $rank(\boldsymbol{M}) = p$, $rank(\boldsymbol{V}_1) = p$.

Related to the externally unforced uncertain discrete-time linear system (1) the dual structure of the following theorem is substantial.

Theorem 1. *In the case of the unforced uncertain discrete-time linear system (1), (3) is quadratically stable if and only if there exist a symmetric positive definite matrix $\boldsymbol{P} \in \mathbb{R}^{n\times n}$ and a positive scalar $\delta \in \mathbb{R}$ such that the following inequalities hold*

$$\boldsymbol{P} = \boldsymbol{P}^{\mathrm{T}} \succ 0\,, \quad \delta > 0\,, \tag{5}$$

$$\begin{bmatrix}-\boldsymbol{P} & \boldsymbol{P}\boldsymbol{F}^{\mathrm{T}} & \boldsymbol{P}\boldsymbol{N}_1^{\mathrm{T}}\\ \boldsymbol{F}\boldsymbol{P} & -\boldsymbol{P} + \delta\boldsymbol{M}\boldsymbol{M}^{\mathrm{T}} & \boldsymbol{0}\\ \boldsymbol{N}_1\boldsymbol{P} & \boldsymbol{0} & -\delta\boldsymbol{I}_p\end{bmatrix} \prec 0\,, \tag{6}$$

$$\boldsymbol{P} - \delta\boldsymbol{M}\boldsymbol{M}^{\mathrm{T}} \succ 0\,. \tag{7}$$

In the case of the unforced dual uncertain discrete-time linear system (1), (4) is quadratically stable if and only if there exist a symmetric positive definite matrix $\boldsymbol{Q} \in \mathbb{R}^{n\times n}$ and a positive scalar $\gamma \in \mathbb{R}$ such that the following hold

$$\boldsymbol{Q} = \boldsymbol{Q}^{\mathrm{T}} \succ 0\,, \quad \gamma > 0\,, \tag{8}$$

$$\begin{bmatrix} -\boldsymbol{Q} & \boldsymbol{Q}\boldsymbol{F} & \boldsymbol{Q}\boldsymbol{V}_1 \\ \boldsymbol{F}^{\mathrm{T}}\boldsymbol{Q} & -\boldsymbol{Q}+\gamma\boldsymbol{U}^{\mathrm{T}}\boldsymbol{U} & \boldsymbol{0} \\ \boldsymbol{V}_1^{\mathrm{T}}\boldsymbol{Q} & \boldsymbol{0} & -\gamma\boldsymbol{I}_p \end{bmatrix} \prec 0\,, \tag{9}$$

$$\boldsymbol{Q} - \gamma\boldsymbol{U}^{\mathrm{T}}\boldsymbol{U} \succ 0\,. \tag{10}$$

The proof of the theorem is outlined in Appendix A.

Remark 1. *Theorem 1 is closely related to the method for assigning poles in a specified disk by state feedback for uncertain linear discrete-time systems with norm-bounded uncertainties [21]. The presented formulation extends and specifies the concept mentioned in [12].*

Remark 2. *Both the presented properties are dual in the sense that $\boldsymbol{F}, \boldsymbol{G}$ are replaced by $\boldsymbol{F}^{\mathrm{T}}, \boldsymbol{C}^{\mathrm{T}}$, and $\boldsymbol{M}$ is replaced by $\boldsymbol{U}^{\mathrm{T}}$ with the formally used substitutions $\boldsymbol{P} \leftarrow \boldsymbol{Q}$, $\boldsymbol{N}_1 \leftarrow \boldsymbol{V}_1$, $\boldsymbol{N}_2 \leftarrow \boldsymbol{V}_2$, $\delta \leftarrow \gamma$. The dual view of the representations is useful when the control law parameter design is exploited by the inequality structure (5)–(7) and in the state observer parameter design the inequality structure (8)–(10).*

Working with the uncertain positive discrete-time linear systems, it can be formulated the formally identical representation (1)–(4), but considering a strictly positive $\boldsymbol{F} \in \mathbb{R}_{+}^{n\times n}$ (all its elements are greater than zero) and non-negative $\boldsymbol{G} \in \mathbb{R}_{+}^{n\times r}$, $\boldsymbol{C} \in \mathbb{R}_{+}^{m\times n}$, $\boldsymbol{M} \in \mathbb{R}_{+}^{n\times p}$, $\boldsymbol{N}_1 \in \mathbb{R}_{+}^{p\times n}$, $\boldsymbol{N}_2 \in \mathbb{R}_{+}^{p\times r}$, $\boldsymbol{U} \in \mathbb{R}_{+}^{p\times n}$, $\boldsymbol{V}_1 \in \mathbb{R}_{+}^{n\times p}$ and $\boldsymbol{V}_2 \in \mathbb{R}_{+}^{m\times p}$, where, for generalization, it is considered that $p = r = m$.

Definition 1 ([22])**.** *The nominal autonomous system (1) is said to be a positive system if the corresponding trajectory $\boldsymbol{q}(i) \in \mathbb{R}_{+}^{n}$ is always non-negative for all integers i and non-negative initial conditions $\boldsymbol{q}(0) \in \mathbb{R}_{+}^{n}$.*

Remark 3 (Adapted from [22])**.** *The nominal autonomous system (1) is positive if and only if $\boldsymbol{F}$ is a positive matrix such that element-vise $\boldsymbol{F} \geq 0$. If the nominal autonomous system (1) is positive, then it is asymptotically stable for every initial condition $\boldsymbol{q}(0) \in \mathbb{R}_{+}^{n}$ (implying that $\boldsymbol{F}$ is a Schur matrix).*

Definition 2 ([23])**.** *Matrix $\boldsymbol{L} \in \mathbb{R}^{n\times n}$ is a permutation matrix if exactly one item in each column and row is equal to 1 and all other elements are equal to 0. Permutation matrix $\boldsymbol{L} \in \mathbb{R}^{n\times n}$ is of circulant form if*

$$\boldsymbol{L} = \begin{bmatrix} \boldsymbol{0}^{\mathrm{T}} & 1 \\ \boldsymbol{I}_{n-1} & \boldsymbol{0} \end{bmatrix}, \quad \boldsymbol{L}^{-1} = \boldsymbol{L}^{\mathrm{T}}. \tag{11}$$

Definition 3. *A square matrix $\boldsymbol{F} \in \mathbb{R}_{++}^{n\times n}$ is strictly positive if all its elements are positive. A square matrix $\boldsymbol{F} \in \mathbb{R}_{+}^{n\times n}$ is purely positive if its diagonal elements are positive and its off-diagonal elements are non-negative.*

Remark 4. *Visualizing the square matrix $\boldsymbol{F} \in \mathbb{R}_{++}^{n\times n}$ as*

$$\boldsymbol{F} = \begin{bmatrix} f_{11} & f_{12} & f_{13} & \cdots & f_{1n} \\ f_{21} & f_{22} & f_{23} & \cdots & f_{2n} \\ f_{31} & f_{32} & f_{33} & \cdots & f_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{n1} & f_{n2} & f_{n3} & \cdots & f_{nn} \end{bmatrix}, \tag{12}$$

the strictly positive structure of $\boldsymbol{F}$ implies n^2 structural constraints $f_{ij} > 0 \quad \forall\, i, j = 1, \ldots n$.

To transform this set of structural constraints into a set of LMIs, the two rhombic forms [24] related to (12) are constructed with characterization through circular shifts of columns (rows) of (12) as

$$\boldsymbol{F}_\Theta = \begin{bmatrix} f_{11} & f_{12} & f_{13} & \cdots & f_{1n} & & & & \\ & f_{22} & f_{23} & \cdots & f_{2n} & f_{21} & & & \\ & & f_{33} & \cdots & f_{3n} & f_{31} & f_{32} & & \\ & & & \ddots & \vdots & \vdots & \vdots & \ddots & \\ & & & & f_{nn} & f_{n1} & f_{n2} & \cdots & f_{n,n-1} \end{bmatrix}, \tag{13}$$

$$\boldsymbol{F}_\Sigma = \begin{bmatrix} f_{11} & & & & \\ f_{21} & f_{22} & & & \\ f_{31} & f_{32} & f_{33} & & \\ \vdots & \vdots & \vdots & \ddots & \\ f_{n1} & f_{n2} & f_{n3} & \cdots & f_{nn} \\ & f_{12} & f_{13} & \cdots & f_{1n} \\ & & f_{23} & \cdots & a_{2n} \\ & & & \ddots & \vdots \\ & & & & f_{n-1,n} \end{bmatrix}. \tag{14}$$

It can be underlined that the diagonal matrices, related to these rhombic forms, are defined for $h = 0, \ldots, n-1$ as

$$\boldsymbol{F}_\Sigma(l+h, l) = diag\big[f_{1+h,l} \cdots f_{n,n-h}\, f_{1,n-h+1} \cdots f_{h,n}\big] \succ 0\,, \tag{15}$$

$$\boldsymbol{F}_\Theta(l, l+h) = diag\big[f_{1,1+h} \cdots f_{n-h,n}\, f_{n-h+1,1} \cdots f_{n,h}\big] \succ 0\,, \tag{16}$$

whilst

$$\boldsymbol{F} = \sum_{h=0}^{n-1} \boldsymbol{F}_\Theta(l, l+h)\boldsymbol{L}^{h\mathrm{T}} = \sum_{h=0}^{n-1} \boldsymbol{L}^h \boldsymbol{F}_\Sigma(l+h, l)\,. \tag{17}$$

Once the matrices $\boldsymbol{F}_\Sigma$, $\boldsymbol{F}_\Theta$ are constructed, $\boldsymbol{F}_\Sigma(l+h, l)$, $\boldsymbol{F}_\Theta(l, l+h)$ are defined by the h-th diagonal of $\boldsymbol{F}_\Sigma$, $\boldsymbol{F}_\Theta$, respectively. Moreover, n^2 parametric constraints are given implicitly by positive definiteness of n diagonal matrices (15) or (16).

Note the duality of (14), (13) is also evident. When designing the state observer, (14) has to be used, whilst in the control design, the form (13) has to be applied.

Lemma 1 (Adapted from [18])**.** *If matrix $\boldsymbol{F} \in \mathbb{R}_{++}^{n\times n}$ is strictly positive then it is Schur if and only if there exist positive definite diagonal matrices $\boldsymbol{P}, \boldsymbol{Q} \in \mathbb{R}_{+}^{n\times n}$ such that the following sets of linear matrix inequalities are feasible for $h = 0, 1, \ldots, n-1$,*

i.

$$\boldsymbol{P} \succ 0\,, \quad \boldsymbol{L}^h \boldsymbol{F}_\Theta(l, l+h)\boldsymbol{L}^{h\mathrm{T}}\boldsymbol{P} \succ 0\,, \quad \boldsymbol{F}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{F} - \boldsymbol{P} \prec 0\,, \tag{18}$$

ii.

$$\boldsymbol{Q} \succ 0\,, \quad \boldsymbol{Q}\boldsymbol{L}^h \boldsymbol{F}_\Sigma(l+h, l)\boldsymbol{L}^{h\mathrm{T}} \succ 0\,, \quad \boldsymbol{F}\boldsymbol{Q}\boldsymbol{F}^{\mathrm{T}} - \boldsymbol{Q} \prec 0\,, \tag{19}$$

when computing with the circulant $\boldsymbol{L} \in \mathbb{R}_{+}^{n\times n}$ defined in (11). The LMIs from the above sets guarantee positiveness of the diagonal matrix variables, positive matrix structural constraints and stability of the system matrix.

Remark 5. *Considering a diagonal matrix $\boldsymbol{\Lambda} \in \mathbb{R}^{n\times n}$ of the form*

$$\boldsymbol{\Lambda} = diag\begin{bmatrix}\lambda_1 & \lambda_2 & \cdots & \lambda_n\end{bmatrix}, \tag{20}$$

then, if $\boldsymbol{L} \in \mathbb{R}^{n\times n}$ takes the circulant form (11),

$$\boldsymbol{L}^{\mathrm{T}}\boldsymbol{\Lambda}\boldsymbol{L} = diag\begin{bmatrix}\lambda_2 & \cdots & \lambda_n & \lambda_1\end{bmatrix} = \boldsymbol{\Lambda}_{c1}\,. \tag{21}$$

In the sections to follow, these design approaches will be considered and the supporting constructive methods developed to establish a direct consequence of control or observer parameters and matrix parametric constraints on quadratic stability.

3. Main Results

It is assumed in this section that the state feedback with positive constant gain stabilizes with positiveness in the closed-loop system if it is implemented (i.e., the closed-loop system matrix is strictly positive and Schur), and the positive observer estimates a positive system state trajectory if it is implemented (i.e., the observer system matrix is strictly positive and Schur), meaning that Schur matrix eigenvalues are less than 1 in absolute value. The proposed solutions substantially rely the conditions presented in Theorem 1.

3.1. Parametric Features in Control Design

If the state feedback control

$$\boldsymbol{u}(i) = -\boldsymbol{K}\boldsymbol{q}(i)\,, \tag{22}$$

can be used to control the uncertainty-free positive system (1) the problem is, with respect to diagonal stabilization principle, to formulate the set of LMIs, which guarantees, in a feasible case, a $\boldsymbol{K} \in \mathbb{R}_{+}^{r\times n}$ being positive if the matrix variable $\boldsymbol{P} \in \mathbb{R}_{+}^{n\times n}$ is positive definite diagonal. The main design criterion remains principally the quadratic stability of the positive closed-loop structure.

Lemma 2. *Let the uncertainty-free system (1), where $\boldsymbol{F} \in \mathbb{R}_{++}^{n\times n}$ is strictly positive and $\boldsymbol{G} \in \mathbb{R}_{+}^{n\times r}$ be non-negative, is under the state control (22), then $\boldsymbol{F}_c = \boldsymbol{F} - \boldsymbol{B}\boldsymbol{K} \in \mathbb{R}_{++}^{n\times n}$ is strictly positive if there exists a positive definite diagonal matrix $\boldsymbol{P} \in \mathbb{R}_{+}^{n\times n}$ and a positive $\boldsymbol{K} \in \mathbb{R}_{++}^{r\times n}$ such that for $h = 0, 1, \ldots, n$,*

$$\boldsymbol{P} \succ 0\,, \quad \boldsymbol{L}^h\boldsymbol{F}_{\Theta}(l, l+h)\boldsymbol{L}^{h\mathrm{T}}\boldsymbol{P} - \sum_{j=1}^{r}\boldsymbol{L}^h\boldsymbol{G}_{dj}\boldsymbol{L}^{h\mathrm{T}}\boldsymbol{K}_{dj}\boldsymbol{P} \succ 0\,, \tag{23}$$

where $\boldsymbol{F}_{\Theta}(l, l+h)$ is defined in (16), $\boldsymbol{L}$ in (11) and

$$\boldsymbol{K} = \begin{bmatrix}\boldsymbol{k}_1^{\mathrm{T}} \\ \vdots \\ \boldsymbol{k}_r^{\mathrm{T}}\end{bmatrix}, \; \boldsymbol{K}_{dj} = diag\begin{bmatrix}\boldsymbol{k}_j^{\mathrm{T}}\end{bmatrix} = diag\begin{bmatrix}k_{j1} \cdots k_{jn}\end{bmatrix}, \tag{24}$$

$$\boldsymbol{G} = \begin{bmatrix}\boldsymbol{g}_1 \cdots \boldsymbol{g}_r\end{bmatrix}, \; \boldsymbol{G}_{dj} = diag\begin{bmatrix}\boldsymbol{g}_j\end{bmatrix} = diag\begin{bmatrix}g_{j1} \cdots g_{jn}\end{bmatrix}. \tag{25}$$

Proof. Writing element wise the matrix product $\boldsymbol{GK}$ in the following rhombic form

$$\begin{aligned} \boldsymbol{GK} &= \sum_{j=1}^{r} \boldsymbol{g}_j \boldsymbol{k}_j^{\mathrm{T}} \\ &= \sum_{j=1}^{r} \begin{bmatrix} g_{1j}k_{j1} & g_{1j}k_{j2} & g_{1j}k_{j3} & \cdots & g_{1j}k_{jn} & & & & \\ & g_{2j}k_{j2} & g_{2j}k_{j3} & \cdots & g_{2j}k_{jn} & g_{2j}k_{j1} & & & \\ & & g_{3j}k_{j3} & \cdots & g_{3j}k_{jn} & g_{3j}k_{j1} & g_{3j}k_{j2} & & \\ & & & \ddots & \vdots & \vdots & \vdots & \ddots & \\ & & & & g_{nj}k_{jn} & g_{nj}k_{j1} & g_{nj}k_{j2} & \cdots & g_{nj}k_{j,n-1} \end{bmatrix} \\ &\sim \sum_{j=1}^{r} \begin{bmatrix} \boldsymbol{G}_{dj}\boldsymbol{K}_{dj} & \boldsymbol{G}_{dj}\boldsymbol{K}_{djc1} & \boldsymbol{G}_{dj}\boldsymbol{K}_{djc2} & \cdots & \boldsymbol{G}_{dj}\boldsymbol{K}_{djc,n-1} \end{bmatrix}, \end{aligned} \tag{26}$$

then separating $\boldsymbol{G}$ by the columns $\boldsymbol{g}_j$, $j = 1, \ldots, r$ and representing the column $\boldsymbol{g}_j$ by the diagonal matrix $\boldsymbol{G}_{dj}$ as in (25) and, additionally, separating $\boldsymbol{K}$ by the rows $\boldsymbol{k}_j^{\mathrm{T}}$, $j = 1, \ldots, r$ and representing the row $\boldsymbol{k}_j^{\mathrm{T}}$ by the diagonal matrix $\boldsymbol{K}_{dj}$ as in (24), then, in analogy with (17), it can be written

$$\boldsymbol{F}_c = \boldsymbol{F} - \sum_{j=1}^{r} \boldsymbol{g}_j \boldsymbol{k}_j^{\mathrm{T}} = \sum_{h=0}^{n-1} \left(\boldsymbol{F}_{\Theta}(l, l+h) - \sum_{j=1}^{r} \boldsymbol{G}_{dj} \boldsymbol{K}_{djch} \right) \boldsymbol{L}^{h\mathrm{T}}, \tag{27}$$

where $\boldsymbol{G}_{dj}$, $\boldsymbol{K}_{djch}$ are derived from the rhombic diagonals of (26). Analogously using

$$\boldsymbol{K}_{djch} = \boldsymbol{L}^{h\mathrm{T}} \boldsymbol{K}_{dj} \boldsymbol{L}^{h} \tag{28}$$

and substituting (28) into (27), then

$$\boldsymbol{F}_c = \sum_{h=0}^{n-1} \left(\boldsymbol{F}_{\Theta}(l, l+h) \boldsymbol{L}^{h\mathrm{T}} - \sum_{j=1}^{r} \boldsymbol{G}_{dj} \boldsymbol{L}^{h\mathrm{T}} \boldsymbol{K}_{dj} \right), \tag{29}$$

which implies

$$\boldsymbol{F}_{\Theta}(l, l+h) \boldsymbol{L}^{h\mathrm{T}} - \sum_{j=1}^{r} \boldsymbol{G}_{dj} \boldsymbol{L}^{h\mathrm{T}} \boldsymbol{K}_{dj} \succ 0 \quad \forall\, h\,. \tag{30}$$

Pre-multiplying the left side by $\boldsymbol{L}^{h}$ and post-multiplying the right side by a positive definite diagonal matrix $\boldsymbol{P}$ then (30) implies (23). This concludes the proof. □

Remark 6. *The condition (23) guarantees that the matrix $\boldsymbol{F}_c$ is strictly positive if $\boldsymbol{F}$, $\boldsymbol{K}$ are strictly positive, $\boldsymbol{G}$ is non-negative and a positive definite diagonal matrix $\boldsymbol{P}$ exists.*

If the case of the positive uncertain system (1), (3) under the control (22), where $\boldsymbol{K} \in \mathbb{R}_{++}^{r \times n}$ is strictly positive, the discrete state can be interpreted as follows

$$\begin{aligned} \boldsymbol{q}(i+1) &= (\boldsymbol{F} - \boldsymbol{GK})\boldsymbol{q}(i) + (\Delta \boldsymbol{F}(i)) - \Delta \boldsymbol{G}(i))\boldsymbol{K})\boldsymbol{q}(i) \\ &= (\boldsymbol{F} - \boldsymbol{GK})\boldsymbol{q}(i) + \boldsymbol{MH}(i)(\boldsymbol{N}_1 - \boldsymbol{N}_2\boldsymbol{K})\boldsymbol{q}(i) \\ &= \boldsymbol{F}_c\boldsymbol{q}(i) + \Delta \boldsymbol{F}_c(i)\boldsymbol{q}(i) \end{aligned} \tag{31}$$

where

$$\boldsymbol{F}_c = \boldsymbol{F} - \boldsymbol{GK}\,, \quad \boldsymbol{N}_{c1} = \boldsymbol{N}_1 - \boldsymbol{N}_2\boldsymbol{K}\,, \tag{32}$$

$$\Delta \boldsymbol{F}_c(i) = \boldsymbol{MH}(i)(\boldsymbol{N}_1 - \boldsymbol{N}_2\boldsymbol{K}) = \boldsymbol{MH}(i)\boldsymbol{N}_{c1}\,, \tag{33}$$

whilst the last relation makes sense if $p = r$. Defining a raw vector $\boldsymbol{l}_n = \begin{bmatrix} 1 \, 1 \cdots 1 \end{bmatrix} \in \mathbb{R}_+^n$, it can be written

$$\boldsymbol{F}_c = \boldsymbol{F} - \boldsymbol{G}\boldsymbol{K} = \boldsymbol{F} - \sum_{j=1}^{r} \boldsymbol{g}_j \boldsymbol{k}_j^{\mathrm{T}} = \boldsymbol{F} - \sum_{j=1}^{r} \boldsymbol{G}_{dj} \boldsymbol{l}_n \boldsymbol{l}_n^{\mathrm{T}} \boldsymbol{K}_{dj} \tag{34}$$

and, analogously, defining $\boldsymbol{l}_p = \begin{bmatrix} 1 \, 1 \cdots 1 \end{bmatrix} \in \mathbb{R}_+^p$, it can be obtained

$$\boldsymbol{N}_{c1} = \boldsymbol{N}_1 - \boldsymbol{N}_2 \boldsymbol{K} = \boldsymbol{N}_1 - \sum_{j=1}^{p} \boldsymbol{n}_{2j} \boldsymbol{k}_j^{\mathrm{T}} = \boldsymbol{N}_1 - \sum_{j=1}^{p} \boldsymbol{N}_{2dj} \boldsymbol{l}_p \boldsymbol{l}_n^{\mathrm{T}} \boldsymbol{K}_{dj} \,, \tag{35}$$

$$\boldsymbol{N}_2 = \begin{bmatrix} \boldsymbol{n}_{21} \cdots \boldsymbol{n}_{2p} \end{bmatrix}, \quad \boldsymbol{N}_{2dj} = \mathrm{diag}\begin{bmatrix} n_{2j1} \cdots n_{2jn} \end{bmatrix}. \tag{36}$$

The design conditions can be formulated as an LMI-based task by the following theorem.

Theorem 2. *The uncertain positive discrete-time system (1), (3) under control (22) is quadratically stable if for given strictly positive $\boldsymbol{F} \in \mathbb{R}_{++}^{n \times n}$, non-negative $\boldsymbol{G} \in \mathbb{R}_+^{n \times r}$, $\boldsymbol{M} \in \mathbb{R}_+^{n \times p}$, $\boldsymbol{N}_1 \in \mathbb{R}_+^{p \times N}$, $\boldsymbol{N}_2 \in \mathbb{R}_+^{p \times r}$ and circulant $\boldsymbol{L} \in \mathbb{R}_+^{n \times n}$ there exist positive definite diagonal matrices $\boldsymbol{P}, \boldsymbol{R}_j \in \mathbb{R}_+^{n \times n}$ and a positive scalar $\delta \in \mathbb{R}_+$ such that for $h = 0, \ldots, n-1$, $j = 1, \ldots r$, $p = r$*

$$\boldsymbol{P} \succ 0\,, \quad \boldsymbol{R}_j \succ 0\,, \quad \delta > 0\,, \tag{37}$$

$$\boldsymbol{L}^h \boldsymbol{F}_{\ominus}(l, l+h) \boldsymbol{L}^{\mathrm{T}} \boldsymbol{P} - \sum_{j=1}^{r} \boldsymbol{L}^h \boldsymbol{G}_{dj} \boldsymbol{L}^{h\mathrm{T}} \boldsymbol{R}_j \succ 0\,, \tag{38}$$

$$\begin{bmatrix} -\boldsymbol{P} & * & * \\ \boldsymbol{F}\boldsymbol{P} - \sum\limits_{j=1}^{r} \boldsymbol{G}_{dj} \boldsymbol{l}_n \boldsymbol{l}_n^{\mathrm{T}} \boldsymbol{R}_j & -\boldsymbol{P} + \delta \boldsymbol{M}\boldsymbol{M}^{\mathrm{T}} & * \\ \boldsymbol{N}_1 \boldsymbol{P} - \sum\limits_{j=1}^{p} \boldsymbol{N}_{2dj} \boldsymbol{l}_p \boldsymbol{l}_n^{\mathrm{T}} \boldsymbol{R}_j & \boldsymbol{0} & -\delta \boldsymbol{I}_p \end{bmatrix} \prec 0\,, \tag{39}$$

$$\boldsymbol{P} - \delta \boldsymbol{M}\boldsymbol{M}^{\mathrm{T}} \succ 0\,, \tag{40}$$

with the parameters defined as (16), (26), (32), (36).

If the set of LMIs is feasible, the strictly positive $\boldsymbol{K} \in \mathbb{R}_{++}^{r \times n}$ is computable by the following procedure

$$\boldsymbol{K}_{dj} = \boldsymbol{R}_j \boldsymbol{P}^{-1}, \quad \boldsymbol{k}_j^{\mathrm{T}} = \boldsymbol{l}^{\mathrm{T}} \boldsymbol{K}_{dj}, \quad \boldsymbol{K} = \begin{bmatrix} \boldsymbol{k}_1^{\mathrm{T}} \\ \vdots \\ \boldsymbol{k}_r^{\mathrm{T}} \end{bmatrix}, \tag{41}$$

whilst the realization is $\boldsymbol{F}_c = \boldsymbol{F} - \boldsymbol{G}\boldsymbol{K}$ such that $\boldsymbol{F}_c$ is strictly positive and Schur.

Hereafter, $*$ denotes the symmetric item in a symmetric matrix.

Proof. Considering the relation (23) for the nominal matrix $\boldsymbol{F}_c$ and using the substitution

$$\boldsymbol{R}_j = \boldsymbol{K}_{dj} \boldsymbol{P}\,, \tag{42}$$

then (23) implies (38). Multiplying the right side by a positive definite diagonal matrix $\boldsymbol{P}$ and considering (42), then (34), (35) imply

$$\boldsymbol{F}_c \boldsymbol{P} = \boldsymbol{F}\boldsymbol{P} - \sum_{j=1}^{r} \boldsymbol{G}_{dj} \boldsymbol{l}_n \boldsymbol{l}_n^{\mathrm{T}} \boldsymbol{K}_{dj} \boldsymbol{P} = \boldsymbol{F}\boldsymbol{P} - \sum_{j=1}^{r} \boldsymbol{G}_{dj} \boldsymbol{l}_n \boldsymbol{l}_n^{\mathrm{T}} \boldsymbol{R}_j \,, \tag{43}$$

$$\boldsymbol{N}_{c1} \boldsymbol{P} = \boldsymbol{N}_1 \boldsymbol{P} - \sum_{j=1}^{p} \boldsymbol{N}_{2dj} \boldsymbol{l}_p \boldsymbol{l}_n^{\mathrm{T}} \boldsymbol{K}_{dj} \boldsymbol{P} = \boldsymbol{N}_1 \boldsymbol{P} - \sum_{j=1}^{p} \boldsymbol{N}_{2dj} \boldsymbol{l}_p \boldsymbol{l}_n^{\mathrm{T}} \boldsymbol{R}_j \,. \tag{44}$$

Since the inequality (6) observed above yields also with $\boldsymbol{F}_c$ and $\boldsymbol{N}_{c1}$ and for a positive definite diagonal matrix $\boldsymbol{P}$, then (6) is easily shown to satisfy

$$\begin{bmatrix} -\boldsymbol{P} & * & * \\ \boldsymbol{F}_c\boldsymbol{P} & -\boldsymbol{P}+\delta\boldsymbol{M}\boldsymbol{M}^{\mathrm{T}} & * \\ \boldsymbol{N}_{c1}\boldsymbol{P} & \boldsymbol{0} & -\delta\boldsymbol{I}_p \end{bmatrix} \prec 0 \tag{45}$$

and with (43), (44), then (45) implies (39). This ends the proof of the theorem. □

Note, the set of LMIs (38) reflects parametric constraints for $\boldsymbol{F}_c \in \mathbb{R}^{n\times n}_{++}$, and (39) guaranties quadratic stability of the system under control.

3.2. Parametric Features in Observer Design

This section considers the design problem of the state estimators for uncertain discrete-time linear systems, where an uncertainty-free input matrix $\boldsymbol{G}$ is considered. Within the Lunberger observer scheme, such a structure is applicable for example in the system fault residual generations.

The aim of this observer is to construct the system state estimation $\boldsymbol{q}_e(i) \in \mathbb{R}^n_+$ in such a way that the error of state estimation (residual signal)

$$\boldsymbol{e}(i) = \boldsymbol{q}(i) - \boldsymbol{q}_e(i) \tag{46}$$

is quadratically stable, since the system matrix stays time varying. To such a defined task, the structure considered for the observer is standard

$$\boldsymbol{q}_e(i+1) = \boldsymbol{F}\boldsymbol{q}_e(i) + \boldsymbol{G}\boldsymbol{u}(i) + \boldsymbol{J}(\boldsymbol{y}(i) - \boldsymbol{y}_e(i)), \tag{47}$$

$$\boldsymbol{y}_e(i) = \boldsymbol{C}\boldsymbol{q}_e(i), \tag{48}$$

associated with the system models (1), (2), (4) with uncertainty-free $\boldsymbol{G}$. With respect to the diagonal stabilization principle and given system parametric constraints, a strictly positive observer gain $\boldsymbol{J} \in \mathbb{R}^{n\times m}_{++}$ and the pair $(\boldsymbol{Q}, \boldsymbol{J})$ parametric bound have to be considered, whilst $\boldsymbol{Q} \in \mathbb{R}^{n\times n}_+$ be a positive definite diagonal matrix.

Considering the Luenberger-type state observer (46), (47) associated with the uncertainty-free system (1), (2), (4), then

$$\begin{aligned} \boldsymbol{e}(i+1) &= \boldsymbol{q}(i+1) - \boldsymbol{q}_e(i+1) \\ &= (\boldsymbol{F} + \boldsymbol{G}\boldsymbol{u}(i) - \boldsymbol{F}\boldsymbol{q}_e(i) - \boldsymbol{G}\boldsymbol{u}(i) - \boldsymbol{J}(\boldsymbol{C}\boldsymbol{q}(i) - \boldsymbol{C}\boldsymbol{q}_e(i)) \\ &= (\boldsymbol{F} - \boldsymbol{J}\boldsymbol{C})\boldsymbol{e}(i) \\ &= \boldsymbol{F}_e\boldsymbol{e}(i), \end{aligned} \tag{49}$$

where $\boldsymbol{q}_e(t) \in \mathbb{R}^n_+$, $\boldsymbol{J} \in \mathbb{R}^{n\times m}_+$ and

$$\boldsymbol{F}_e = \boldsymbol{F} - \boldsymbol{J}\boldsymbol{C}. \tag{50}$$

Upon examining (49), it can be seen that the autonomous observer problem being considered is a dual generalization of the time-invariant state control problem, where all of the system matrices are state independent.

Lemma 3. *In the case of the uncertainty-free observer error dynamics given in (49), where $\boldsymbol{F} \in \mathbb{R}^{n\times n}_{++}$ is strictly positive, $\boldsymbol{C} \in \mathbb{R}^{m\times n}_+$ is non-negative and $\boldsymbol{J} \in \mathbb{R}^{n\times m}_+$ is a strictly positive observer gain matrix, then $\boldsymbol{F}_e = \boldsymbol{F} - \boldsymbol{J}\boldsymbol{C} \in \mathbb{R}^{n\times n}_{++}$ is strictly positive if there exists a positive definite diagonal matrix $\boldsymbol{Q} \in \mathbb{R}^{n\times n}_+$ such that for $h = 0, 1, \ldots, n$,*

$$\boldsymbol{Q} \succ 0, \quad \boldsymbol{Q}\boldsymbol{L}^h\boldsymbol{F}_{\Sigma}(l+h, l)\boldsymbol{L}^{h\mathrm{T}} - \sum_{j=1}^{r} \boldsymbol{Q}\boldsymbol{J}_{dj}\boldsymbol{L}^h\boldsymbol{C}_{dj}\boldsymbol{L}^{h\mathrm{T}} \succ 0, \tag{51}$$

where $\boldsymbol{F}_{\Sigma}(l, l+h)$ *is defined in (15),* $\boldsymbol{L}$ *in (11) and*

$$\boldsymbol{J} = [\boldsymbol{j}_1 \cdots \boldsymbol{j}_m], \quad \boldsymbol{J}_{dj} = diag\left[\boldsymbol{j}_j\right] = diag[j_{1j} \cdots j_{nj}], \tag{52}$$

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{c}_1^{\mathrm{T}} \\ \vdots \\ \boldsymbol{c}_r^{\mathrm{T}} \end{bmatrix}, \quad \boldsymbol{C}_{dj} = diag\left[\boldsymbol{c}_j^{\mathrm{T}}\right] = diag[c_{1j} \cdots c_{nj}]. \tag{53}$$

Proof. Writing element wise the matrix product $\boldsymbol{J}\boldsymbol{C}$ in the following rhombic form

$$\begin{aligned} \boldsymbol{J}\boldsymbol{C} &= \sum_{j=1}^{m} \boldsymbol{j}_j \boldsymbol{c}_j^{\mathrm{T}} \\ &= \sum_{j=1}^{r} \begin{bmatrix} j_{1j}c_{j1} & & & \\ j_{2j}c_{j1} & j_{2j}c_{j2} & & \\ \vdots & \vdots & \ddots & \\ j_{nj}c_{j1} & j_{nj}c_{j2} & \cdots & j_{nj}c_{jn} \\ & j_{1j}c_{j2} & \cdots & j_{1j}c_{jn} \\ & & \ddots & \vdots \\ & & & j_{n-1,j}c_{jn} \end{bmatrix} \sim \sum_{j=1}^{r} \begin{bmatrix} \boldsymbol{J}_{dj}\boldsymbol{C}_{dj} \\ \boldsymbol{J}_{djc1}\boldsymbol{C}_{dj} \\ \vdots \\ \boldsymbol{J}_{djc,n-1}\boldsymbol{C}_{dj} \end{bmatrix}, \end{aligned} \tag{54}$$

then separating the matrix $\boldsymbol{J}$ by the columns $\boldsymbol{j}_j$, $j = 1, \ldots, m$ and representing the column $\boldsymbol{j}_j$ by the diagonal matrix $\boldsymbol{J}_{dj}$ as defined in (52) and, additionally, separating the matrix $\boldsymbol{C}$ by the rows $\boldsymbol{c}_j^{\mathrm{T}}$, $j = 1, \ldots, r$ and representing the row $\boldsymbol{c}_j^{\mathrm{T}}$ by the diagonal matrix $\boldsymbol{C}_{dj}$ as defined in (53), in analogy with (17), it can be written

$$\boldsymbol{F}_e = \boldsymbol{F} - \sum_{j=1}^{r} \boldsymbol{j}_j \boldsymbol{c}_j^{\mathrm{T}} = \sum_{h=0}^{n-1} \boldsymbol{L}^h \left(\boldsymbol{F}_{\Sigma}(l+h, l) - \sum_{j=1}^{r} \boldsymbol{J}_{djch} \boldsymbol{C}_{dj} \right), \tag{55}$$

where $\boldsymbol{J}_{djch}$, $\boldsymbol{C}_{dj}$, are derived from the rhombic diagonals of (54).

Analogously using (21) in the following relation

$$\boldsymbol{J}_{djch} = \boldsymbol{L}^{h\mathrm{T}} \boldsymbol{J}_{dj} \boldsymbol{L}^h \tag{56}$$

and substituting (56) into (55), then

$$\boldsymbol{F}_e = \sum_{h=0}^{n-1} \left(\boldsymbol{L}^h \boldsymbol{F}_{\Sigma}(l+h, l) - \sum_{j=1}^{r} \boldsymbol{J}_{dj} \boldsymbol{L}^h \boldsymbol{C}_{dj} \right), \tag{57}$$

which implies for all h

$$\boldsymbol{L}^h \boldsymbol{F}_{\Sigma}(l+h, l) - \sum_{j=1}^{r} \boldsymbol{J}_{dj} \boldsymbol{L}^h \boldsymbol{C}_{dj} \succ 0. \tag{58}$$

Pre-multiplying the inequality left side by a positive definite diagonal matrix $\boldsymbol{Q} \in \mathbb{R}_{+}^{n \times n}$ and post-multiplying the right side by $\boldsymbol{L}^{h\mathrm{T}}$ then (58) implies (51). This ends the proof of the lemma. □

If the state observer (47), (48) is constructed for the positive uncertain system (1), (2), (4) with the uncertainty-free matrix $\boldsymbol{G} \in \mathbb{R}_{+}^{n\times r}$, whilst $\boldsymbol{J} \in \mathbb{R}_{++}^{n\times m}$ is strictly positive, then the error of state variable estimations is expressible as

$$\begin{aligned}\boldsymbol{e}(i+1) &= \boldsymbol{q}(i+1) - \boldsymbol{q}_e(i+1)\\ &= (\boldsymbol{F} + \Delta\boldsymbol{F}(i))\boldsymbol{q}(i) + \boldsymbol{G}\boldsymbol{u}(i) - \boldsymbol{F}\boldsymbol{q}_e(i) - \boldsymbol{G}\boldsymbol{u}(i) -\\ &- \boldsymbol{J}((\boldsymbol{C} + \Delta\boldsymbol{C}(i))\boldsymbol{q}(i) - \boldsymbol{C}\boldsymbol{q}_e(i))\\ &= (\boldsymbol{F} - \boldsymbol{J}\boldsymbol{C})\boldsymbol{e}(i) + (\Delta\boldsymbol{F}(i) - \boldsymbol{J}\Delta\boldsymbol{C}(i))\boldsymbol{q}(i)\\ &= \boldsymbol{F}_e\boldsymbol{e}(i) + (\Delta\boldsymbol{F}(i) - \boldsymbol{J}\Delta\boldsymbol{C}(i))\boldsymbol{q}(i)\,,\end{aligned} \tag{59}$$

where $\boldsymbol{F}_e$ is given in (50). Defining the matrix

$$\boldsymbol{T}^{\diamond} = \begin{bmatrix}\boldsymbol{I}_n & -\boldsymbol{J}\end{bmatrix} \tag{60}$$

and multiplying the left side of (4) by $\boldsymbol{T}^{\diamond}$, then the following relation is obtained

$$\Delta\boldsymbol{F}(i) - \boldsymbol{J}\Delta\boldsymbol{C}(i) = (\boldsymbol{V}_1 - \boldsymbol{J}\boldsymbol{V}_2)\boldsymbol{W}(i)\boldsymbol{U} = \boldsymbol{V}_{e1}\boldsymbol{W}(i)\boldsymbol{U}\,, \tag{61}$$

where the last realization makes use only if $p = m$, whilst

$$\boldsymbol{V}_{e1} = \boldsymbol{V}_1 - \boldsymbol{J}\boldsymbol{V}_2\,. \tag{62}$$

Thus, (59) has the structure

$$\boldsymbol{e}(i+1) = (\boldsymbol{F}_e + \boldsymbol{V}_{1e}\boldsymbol{W}(i)\boldsymbol{U})\boldsymbol{e}(i) + \boldsymbol{V}_{1e}\boldsymbol{W}(i)\boldsymbol{U}\boldsymbol{q}_e(i)\,. \tag{63}$$

Using the defined column vectors $\boldsymbol{l}_n \in \mathbb{R}^n$, $\boldsymbol{l}_p \in \mathbb{R}^p$ it can be written

$$\boldsymbol{F}_e = \boldsymbol{F} - \boldsymbol{J}\boldsymbol{C} = \boldsymbol{F} - \sum_{j=1}^{m} \boldsymbol{j}_j\boldsymbol{c}_j^{\mathrm{T}} = \boldsymbol{F} - \sum_{j=1}^{m} \boldsymbol{J}_{dj}\boldsymbol{l}_n\boldsymbol{l}_n^{\mathrm{T}}\boldsymbol{C}_{dj}\,, \tag{64}$$

$$\boldsymbol{V}_{c1} = \boldsymbol{V}_1 - \boldsymbol{J}\boldsymbol{V}_2 = \boldsymbol{V}_1 - \sum_{j=1}^{p} \boldsymbol{j}_j\boldsymbol{v}_{2j}^{\mathrm{T}} = \boldsymbol{V}_1 - \sum_{j=1}^{p} \boldsymbol{J}_{dj}\boldsymbol{l}_n\boldsymbol{l}_p^{\mathrm{T}}\boldsymbol{V}_{2dj}\,, \tag{65}$$

where the diagonally related expression is

$$\boldsymbol{V}_2 = \begin{bmatrix}\boldsymbol{v}_{21}^{\mathrm{T}}\\ \vdots\\ \boldsymbol{v}_{2m}^{\mathrm{T}}\end{bmatrix}, \quad \boldsymbol{V}_{2dj} = \mathrm{diag}\begin{bmatrix}v_{2j1} \cdots v_{2jn}\end{bmatrix}. \tag{66}$$

The use of these relationships eliminates the introductions of additional structured matrix variables into the solution.

Theorem 3. *The state observer (47), (48) related to uncertain positive discrete-time system (1), (2), (4) with the uncertainty-free input matrix $\boldsymbol{G} \in \mathbb{R}_{+}^{n\times r}$ is quadratically stable if for strictly positive $\boldsymbol{F} \in \mathbb{R}_{++}^{n\times n}$, non-negative $\boldsymbol{C} \in \mathbb{R}_{+}^{m\times n}$, $\boldsymbol{U} \in \mathbb{R}_{+}^{p\times n}$, $\boldsymbol{V}_1 \in \mathbb{R}_{+}^{n\times p}$, $\boldsymbol{V}_2 \in \mathbb{R}_{+}^{m\times p}$ and circulant $\boldsymbol{L} \in \mathbb{R}_{+}^{n\times n}$ there exist positive definite diagonal matrices $\boldsymbol{Q}, \boldsymbol{S}_j \in \mathbb{R}_{+}^{n\times n}$ and a positive scalar $\gamma \in \mathbb{R}_{+}$ such that for $h = 0, \ldots, n-1$, $j = 1, \ldots m$, $p = m$*

$$\boldsymbol{Q} \succ 0\,, \quad \boldsymbol{S}_j \succ 0\,, \quad \gamma > 0\,, \tag{67}$$

$$\boldsymbol{Q}\boldsymbol{L}^h\boldsymbol{F}_{\Sigma}(l+h,l)\boldsymbol{L}^{h\mathrm{T}} - \sum_{j=1}^{m} \boldsymbol{S}_j\boldsymbol{L}^h\boldsymbol{C}_{dj}\boldsymbol{L}^{h\mathrm{T}} \succ 0\,, \tag{68}$$

$$\begin{bmatrix} -\boldsymbol{Q} & \boldsymbol{QF}-\sum_{j=1}^{m}\boldsymbol{S}_j\boldsymbol{l}_n\boldsymbol{l}_n^{\mathrm{T}}\boldsymbol{C}_{dj} & \boldsymbol{QV}_1-\sum_{j=1}^{m}\boldsymbol{S}_j\boldsymbol{l}_n\boldsymbol{l}_m^{\mathrm{T}}\boldsymbol{V}_{2dj} \\ * & -\boldsymbol{Q}+\gamma\boldsymbol{U}^{\mathrm{T}}\boldsymbol{U} & \boldsymbol{0} \\ * & * & -\gamma\boldsymbol{I}_p \end{bmatrix} \prec 0\,, \tag{69}$$

$$\boldsymbol{Q}-\gamma\boldsymbol{U}^{\mathrm{T}}\boldsymbol{U} \succ 0\,, \tag{70}$$

with the parameters from (52), (53), (66).

If the above LMIs are feasible, the strictly positive $\boldsymbol{J} \in \mathbb{R}_{++}^{n\times m}$ can be expressed as

$$\boldsymbol{J}_{dj} = \boldsymbol{Q}^{-1}\boldsymbol{S}_j, \quad \boldsymbol{j}_j = \boldsymbol{J}_{dj}\boldsymbol{l}_n, \quad \boldsymbol{J} = \begin{bmatrix} \boldsymbol{j}_1 & \cdots & \boldsymbol{j}_m \end{bmatrix} \tag{71}$$

and, in dependence on $\boldsymbol{J}$, the matrix $\boldsymbol{F}_e = \boldsymbol{F} - \boldsymbol{JC}$ is strictly positive and Schur.

Proof. Considering the inequalities (51) for the system nominal matrix $\boldsymbol{F}_e$ and using the substitution

$$\boldsymbol{S}_j = \boldsymbol{QJ}_{dj}\,, \tag{72}$$

then (51) implies (38).

Since the inequality (9) yields also for $\boldsymbol{F}_e$ and $\boldsymbol{V}_{e1}$ with relation to a positive definite diagonal matrix of $\boldsymbol{Q}$, LMI (9) takes the following expression

$$\begin{bmatrix} -\boldsymbol{Q} & \boldsymbol{QF}_e & \boldsymbol{QV}_{e1} \\ \boldsymbol{F}_e^{\mathrm{T}}\boldsymbol{Q} & -\boldsymbol{Q}+\gamma\boldsymbol{U}^{\mathrm{T}}\boldsymbol{U} & \boldsymbol{0} \\ \boldsymbol{V}_{e1}^{\mathrm{T}}\boldsymbol{Q} & \boldsymbol{0} & -\gamma\boldsymbol{I}_p \end{bmatrix} \prec 0\,. \tag{73}$$

Considering (72) it can be seen from comparing (64), (65) that

$$\boldsymbol{QF}_e = \boldsymbol{QF} - \sum_{k=1}^{m}\boldsymbol{S}_k\boldsymbol{l}_n\boldsymbol{l}_n^{\mathrm{T}}\boldsymbol{C}_{dk}\,, \tag{74}$$

$$\boldsymbol{QV}_{e1} = \boldsymbol{QV}_1 - \sum_{k=1}^{m}\boldsymbol{S}_k\boldsymbol{l}_n\boldsymbol{l}_m^{\mathrm{T}}\boldsymbol{V}_{2dk}\,, \tag{75}$$

respectively, and (73) under the prescribed elements (74), (75) implies (69). This concludes the proof. □

The observer structure analogy means that (68) reflects parametric constraints of a positive $\boldsymbol{F}_e \in \mathbb{R}_{++}^{n\times n}$, and the LMI (69) guarantees the observer quadratic stability, whilst the diagonal stability principle forces a positive definite diagonal matrix $\boldsymbol{P} \in \mathbb{R}_{+}^{n\times n}$.

3.3. Conjunction with State Observer-Based Control

The most natural way to extend the used basis is to define the control law as the state observer-based

$$\boldsymbol{u}(i) = -\boldsymbol{K}\boldsymbol{q}_e(i)\,, \tag{76}$$

where $\boldsymbol{K} \in \mathbb{R}_{++}^{r\times n}$ is a strictly positive matrix. The associate description of the defined control task is

$$\begin{bmatrix} \boldsymbol{q}(i+1) \\ \boldsymbol{q}_e(i+1) \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}+\Delta\boldsymbol{F}(i) & -(\boldsymbol{G}+\Delta\boldsymbol{G}(i))\boldsymbol{K} \\ \boldsymbol{J}(\boldsymbol{C}+\Delta\boldsymbol{C}(i)) & \boldsymbol{F}-\boldsymbol{JC}-\boldsymbol{GK} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}(i) \\ \boldsymbol{q}_e(i) \end{bmatrix} = \boldsymbol{F}_c^{\bullet} \begin{bmatrix} \boldsymbol{q}(i) \\ \boldsymbol{q}_e(i) \end{bmatrix}, \tag{77}$$

which directly implies from (2), (31), (47), (48) and (76) and where

$$\boldsymbol{F}_{ce}^{\bullet} = \begin{bmatrix} \boldsymbol{F}+\Delta\boldsymbol{F}(i) & -(\boldsymbol{G}+\Delta\boldsymbol{G}(i))\boldsymbol{K} \\ \boldsymbol{J}(\boldsymbol{C}+\Delta\boldsymbol{C}(i)) & \boldsymbol{F}-\boldsymbol{JC}-\boldsymbol{GK} \end{bmatrix}. \tag{78}$$

Since the transform matrix can certainly be chosen

$$T_e = \begin{bmatrix} I_n & 0 \\ I_n & -I_n \end{bmatrix}, \quad T_e^{-1} = T_e\,, \tag{79}$$

then

$$T_e \begin{bmatrix} q(i) \\ q_e(i) \end{bmatrix} = \begin{bmatrix} q(i) \\ q(i) - q_e(i) \end{bmatrix} = \begin{bmatrix} q(i) \\ e(i) \end{bmatrix}, \tag{80}$$

$$\begin{aligned} F_{ce} &= T_e F_{ce}^{\bullet} T_e^{-1} \\ &= \begin{bmatrix} I_n & 0 \\ I_n & -I_n \end{bmatrix} \begin{bmatrix} F + \Delta F(i) & -(G + \Delta G(i))K \\ J(C + \Delta C(i)) & F - JC - GK \end{bmatrix} \begin{bmatrix} I_n & 0 \\ I_n & -I_n \end{bmatrix} \\ &= \begin{bmatrix} F + \Delta F(i) - (G + \Delta G(i))K & (G + \Delta G(i))K \\ \Delta F(i) - J\Delta C(i) - \Delta G(i)K & F - JC + \Delta G(i)K \end{bmatrix} \end{aligned} \tag{81}$$

and the equivalent formulation is obtained in the form

$$\begin{bmatrix} q(i+1) \\ e(i+1) \end{bmatrix} = F_{ce}^{\circ} \begin{bmatrix} q(i) \\ e(i) \end{bmatrix}, \tag{82}$$

where

$$F_{ce}^{\circ} = \begin{bmatrix} F + \Delta F(i) - (G + \Delta G(i))K & (G + \Delta G(i))K \\ \Delta F(i) - J\Delta C(i) - \Delta G(i)K & F - JC + \Delta G(i)K \end{bmatrix}. \tag{83}$$

Since the element in the lower right corner of (83) can be extended as

$$F - JC + \Delta G(i)K = F + \Delta F(i) - J(C + \Delta C(i)) + (-\Delta F(i) + J\Delta C(i) + \Delta G(i)K)\,, \tag{84}$$

it follows directly from (83)

$$\begin{aligned} F_{ce} = &\begin{bmatrix} F + \Delta F(i) - (G + \Delta G(i))K & 0 \\ 0 & F + \Delta F(i) - J(C + \Delta C(i)) \end{bmatrix} + \\ &+ \begin{bmatrix} 0 & (G + \Delta G(i))K \\ \Delta F(i) - J\Delta C(i) - \Delta G(i)K & -(\Delta F(i) - J\Delta C(i) - \Delta G(i)K) \end{bmatrix}. \end{aligned} \tag{85}$$

This can be used to define an equivalent formulation

$$\begin{aligned} \begin{bmatrix} q(i+1) \\ e(i+1) \end{bmatrix} = &\begin{bmatrix} F + \Delta F(i) - (G + \Delta G(i))K & 0 \\ 0 & F + \Delta F(i) - J(C + \Delta C(i)) \end{bmatrix} \begin{bmatrix} q(i) \\ e(i) \end{bmatrix} + \\ &+ \begin{bmatrix} (G + \Delta G(i)) & 0 \\ 0 & \Delta E(i) \end{bmatrix} \begin{bmatrix} Ke(i) \\ q_e(i) \end{bmatrix}, \end{aligned} \tag{86}$$

where

$$\Delta E(i) = \Delta F(i) - J\Delta C(i) - \Delta G(i)K\,. \tag{87}$$

If states of the observer (47), (48) are intended for the state control (22), stabilizing the uncertain system (1), (2), (4), then the state estimation error equation (compare (59)) is

$$\begin{aligned} e(i+1) &= F_e e(i) + (\Delta F(i) - J\Delta C(i) - \Delta G(i)K)q(i) \\ &= F_e e(i) + \Delta E(i) q(i)\,, \end{aligned} \tag{88}$$

where

$$F_e = F - JC\,. \tag{89}$$

If the system (1), (2), (4) is stabilized by the control (76), then (82) implies

$$e(i+1) = F_{eu} e(i) + \Delta E(i) q_e(i)\,, \tag{90}$$

where

$$\boldsymbol{F}_{eu} = \boldsymbol{F} + \Delta\boldsymbol{F}(i) - \boldsymbol{J}(\boldsymbol{C} + \Delta\boldsymbol{C}(i))\,. \tag{91}$$

It is obvious that (88), (89) implies the asymptotically stable estimation error dynamics and, in contrary, the relations (90), (91) mean the quadratically stable estimation error dynamics, related to the same disturbance when $\boldsymbol{q}_e(i) \to \boldsymbol{q}(i)$.

Since (31), (86) implies

$$\boldsymbol{q}(i+1) = \boldsymbol{F}_{cu}\boldsymbol{q}(i)\,, \tag{92}$$

$$\boldsymbol{q}(i+1) = \boldsymbol{F}_{cu}\boldsymbol{q}(i) + (\boldsymbol{G} + \Delta\boldsymbol{G}(i))\boldsymbol{K}\boldsymbol{e}(i)\,, \tag{93}$$

respectively, where

$$\boldsymbol{F}_{cu} = \boldsymbol{F} - \boldsymbol{G}\boldsymbol{K} + \Delta\boldsymbol{F}(i) - \Delta\boldsymbol{G}(i)\boldsymbol{K}\,. \tag{94}$$

As a consequence of the error properties for $\boldsymbol{e}(i) \to \boldsymbol{0}$, the system state trajectories are equivalent, and both system dynamics under control are quadratically stable, taking the same set of eigenvalues, whether the control law (22) or (76) is used.

4. Illustrative Numerical Example

The standard scheme of control structure is considered (see, e.g., [25]), whilst an execution of the task supposed the system model (1)–(4), constructed by these parameters

$$\boldsymbol{F} = \begin{bmatrix} 1.0032 & 0.1047 & 0.1331 \\ 0.0089 & 0.6920 & 0.0224 \\ 0.0354 & 0.0529 & 0.7667 \end{bmatrix}, \quad \boldsymbol{M} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{N}_1 = \begin{bmatrix} 0.01 & 0 & 0 \\ 0 & 0.01 & 0 \end{bmatrix}, \quad \boldsymbol{N}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0.002 \end{bmatrix},$$

$$\boldsymbol{G} = \begin{bmatrix} 0.4805 & 0.8574 \\ 0.6746 & 0.0145 \\ 0.5195 & 0.7832 \end{bmatrix}, \quad \boldsymbol{V}_1 = \begin{bmatrix} 0 & 0 \\ 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}, \quad \boldsymbol{V}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0.02 \end{bmatrix}, \quad \boldsymbol{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{L} = \begin{bmatrix} \boldsymbol{0}^{\mathrm{T}} & 1 \\ \boldsymbol{I}_2 & \boldsymbol{0} \end{bmatrix}, \quad \boldsymbol{l}_n^{\mathrm{T}} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad \boldsymbol{l}_p^{\mathrm{T}} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

where $\boldsymbol{F}$ is strictly positive.

Constructing the rhombic forms of $\boldsymbol{F}$ as

$$\boldsymbol{F}_{\Theta} = \begin{bmatrix} 1.0032 & 0.1047 & 0.1331 & & \\ & 0.6920 & 0.0224 & 0.0089 & \\ & & 0.7667 & 0.0354 & 0.0529 \end{bmatrix}, \quad \boldsymbol{F}_{\Sigma} = \begin{bmatrix} 1.0032 & & \\ 0.0089 & 0.6920 & \\ 0.0354 & 0.0529 & 0.7667 \\ & 0.1047 & 0.1331 \\ & & 0.0224 \end{bmatrix},$$

the matrices $\boldsymbol{F}_{\Theta}$, $\boldsymbol{F}_{\Sigma}$ can be cast into the following diagonal structures

$$\boldsymbol{F}_{\Theta}(l,l) = \operatorname{diag}\begin{bmatrix} 1.0032 & 0.6920 & 0.7667 \end{bmatrix}, \quad \boldsymbol{F}_{\Sigma}(l,l) = \operatorname{diag}\begin{bmatrix} 1.0032 & 0.6920 & 0.7667 \end{bmatrix},$$

$$\boldsymbol{F}_{\Theta}(l,l+1) = \operatorname{diag}\begin{bmatrix} 0.1047 & 0.0244 & 0.0354 \end{bmatrix}, \quad \boldsymbol{F}_{\Sigma}(l+1,1) = \operatorname{diag}\begin{bmatrix} 0.0089 & 0.0529 & 0.1331 \end{bmatrix},$$

$$\boldsymbol{F}_{\Theta}(l,l+2) = \operatorname{diag}\begin{bmatrix} 0.1331 & 0.0089 & 0.0529 \end{bmatrix}, \quad \boldsymbol{F}_{\Sigma}(l+2,l) = \operatorname{diag}\begin{bmatrix} 0.0354 & 0.1047 & 0.0244 \end{bmatrix},$$

while of immediate consequences are the rest system diagonal matrix parameters, when separating $\boldsymbol{G}$, $\boldsymbol{N}_2$ by columns to use (25), (36) and $\boldsymbol{C}$, $\boldsymbol{V}_2$ by rows to use (53), (66),

$$\boldsymbol{G}_{d1} = \operatorname{diag}\begin{bmatrix} 0.4805 & 0.6746 & 0.5195 \end{bmatrix}, \quad \boldsymbol{C}_{d1} = \operatorname{diag}\begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{G}_{d2} = \operatorname{diag}\begin{bmatrix} 0.8574 & 0.0145 & 0.7832 \end{bmatrix}, \quad \boldsymbol{C}_{d2} = \operatorname{diag}\begin{bmatrix} 0 & 1 & 0 \end{bmatrix},$$

$$\boldsymbol{N}_{2d1} = \operatorname{diag}\begin{bmatrix} 0 & 0 \end{bmatrix}, \quad \boldsymbol{V}_{2d1} = \operatorname{diag}\begin{bmatrix} 0 & 0 \end{bmatrix},$$

$$\boldsymbol{N}_{2d2} = \operatorname{diag}\begin{bmatrix} 0 & 0.02 \end{bmatrix}, \quad \boldsymbol{V}_{2d2} = \operatorname{diag}\begin{bmatrix} 0 & 0.02 \end{bmatrix}.$$

Putting (67)–(70) in the program file for the SeDuMi package [26] in the Matlab environment to design the control law parameter, their feasibility admits LMI variables

$$\boldsymbol{P} = \operatorname{diag}\big[1.8136 \;\; 1.9446 \;\; 1.9682\big],$$

$$\boldsymbol{R}_1 = \operatorname{diag}\big[0.0108 \;\; 0.0811 \;\; 0.0309\big], \quad \boldsymbol{R}_2 = \operatorname{diag}\big[0.0566 \;\; 0.0346 \;\; 0.2037\big]$$

and, conditioned by $\delta = 0.5223$, it is satisfied

$$\boldsymbol{P} - \delta \boldsymbol{M}\boldsymbol{M}^{\mathrm{T}} = \begin{bmatrix} 1.8136 & 0 & 0 \\ 0 & 1.4223 & 0 \\ 0 & 0 & 1.4459 \end{bmatrix} \succ 0\,.$$

Since the design conditions are feasible, the prescribed positive gain structure implies

$$\boldsymbol{K} = \begin{bmatrix} 0.0060 & 0.0417 & 0.0157 \\ 0.0312 & 0.0178 & 0.1035 \end{bmatrix}$$

and a direct consequence is the stable positive closed-loop system matrix construction

$$\boldsymbol{F}_c = \begin{bmatrix} 0.9736 & 0.0694 & 0.0368 \\ 0.0044 & 0.6635 & 0.0103 \\ 0.0078 & 0.0173 & 0.6775 \end{bmatrix}, \quad \rho(\boldsymbol{F}_c) = \begin{Bmatrix} 0.9756 \\ 0.6836 \\ 0.6554 \end{Bmatrix}.$$

Programming (37)–(40) into the SeDuMi package in the Matlab environment to design the observer gain, the LMI set admits

$$\boldsymbol{Q} = \operatorname{diag}\big[1.7244 \;\; 1.8336 \;\; 1.7892\big],$$

$$\boldsymbol{S}_1 = \operatorname{diag}\big[1.1011 \;\; 0.0081 \;\; 0.0319\big], \quad \boldsymbol{S}_2 = \operatorname{diag}\big[0.0912 \;\; 0.8867 \;\; 0.0477\big]$$

and for $\gamma = 0.7669 > 0$ it is satisfied

$$\boldsymbol{Q} - \delta \boldsymbol{U}\boldsymbol{U}^{\mathrm{T}} = \begin{bmatrix} 0.9574 & 0 & 0 \\ 0 & 1.0667 & 0 \\ 0 & 0 & 1.7892 \end{bmatrix} \succ 0\,.$$

Taking into account the relation between LMI variables the positive observer gain, guaranteeing observer quadratic stability and positivity, is computed as

$$\boldsymbol{J} = \begin{bmatrix} 0.6385 & 0.0529 \\ 0.0044 & 0.4836 \\ 0.0178 & 0.0267 \end{bmatrix}$$

and the derived observer system matrix is Schur, having the strictly positive form

$$\boldsymbol{F}_e = \begin{bmatrix} 0.3647 & 0.0518 & 0.1331 \\ 0.0045 & 0.2084 & 0.0224 \\ 0.0175 & 0.0263 & 0.7667 \end{bmatrix}, \quad \rho(\boldsymbol{F}_e) = \begin{Bmatrix} 0.7736 \\ 0.3599 \\ 0.2062 \end{Bmatrix}.$$

The presented example documents that the idea of the proposed method consists in using a non-iterative design approach, when applied to the given class of uncertain systems. The method seems to be effective for positive linear discrete-time systems with uncertain parameters if $\Delta \boldsymbol{F}(i)$ is not too complicated.

The potentially comparable method is presented in [13], where the considered system matrix $\boldsymbol{F}$ is non-negative, but the mentioned approach produces a negative gain matrix and signum indefinite closed-loop system matrix. As a result, the authors know no comparison base to the proposed design method, although the conversion of a continuous-time-positive Metzler system usually leads to a strictly positive discrete-time state space description. Using SeDuMi, the computational complexity of this type of algorithm is analyzed in [19]. The interested reader is referred to this reference and references given therein for more details.

5. Summary and Conclusions

The problem of quadratic stability in controller and observer design for uncertain positive discrete-time systems is scrutinized in this paper. The main idea is to maintain the LMI definitions of the incident constraints and quadratic stability. The design condition are derived from the corresponding algorithms for representations of feasible sets of LMIs, a representative of such an equivalence LMI corresponds to a certain choice of positive definite diagonal LMI variables, as a basis for diagonal stabilization. To maintain the correct state estimation by exploiting the observers based on positive system matrices under discrete-time uncertainties, the results are functions of scalar parameters because the system is defined in LPV structures.

The method presented in this paper introduces newly defined LMI structures that improve the feasibility of the method. In particular, it is shown that if there exists an upper bound in the Lyapunov function difference, then there exists a representation of the dynamics such that the feedback action in the control and observer structures is stabilizing. The control effort generated by the proposed method may be enough for a practical application. In the context of the observation and control problems potentially a non-negative structure in the system matrix description can be included adequately [27].

The future scope of study in this field is in the direction of using interval representation of the system parameters in the design of system observers and the functional observers for uncertain positive discrete-time systems with unknown disturbances.

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Abbreviations

The following abbreviations are used in this manuscript:

LMI	Linear Matrix Inequality
LPV	Linear Parameter Varying
MIMO	Multiple-Input Multiple-Output
SeDuMi	Self Dual Minimization

Notations

The following basic notations are used in this manuscript:

$\boldsymbol{q}(i)$, $\boldsymbol{u}(i)$, $\boldsymbol{y}(i)$, $\boldsymbol{e}(i)$	state, input and output vectors of variables, state estimation error
$\boldsymbol{F}$, $\boldsymbol{G}$, $\boldsymbol{C}$	nominal system matrix parameters
$\boldsymbol{F}_c$, $\boldsymbol{F}_e$	nominal matrix parameters of the closed-loop and observer structures
$\boldsymbol{M}$, $\boldsymbol{U}$, $\boldsymbol{N}_1$, $\boldsymbol{N}_2$, $\boldsymbol{V}_1$, $\boldsymbol{V}_2$	matrices which characterize the structure of the uncertainties
$\boldsymbol{H}(i)$, $\boldsymbol{W}(i)$	matrices with Lebesgue measurable elements
$\boldsymbol{F}_{\Theta}$, $\boldsymbol{F}_{\Theta}(l,l+h)$, $\boldsymbol{F}_{\Sigma}$, $\boldsymbol{F}_{\Sigma}(l+h,l)$	rhombic matrices of $\boldsymbol{F}$ and their diagonals
$\boldsymbol{K}$, $\boldsymbol{J}$, $\boldsymbol{L}$	controller gain matrix, observer gain matrix, circulant permutation matrix
$\boldsymbol{B}_{dj}$, $\boldsymbol{C}_{dj}$, $\boldsymbol{K}_{dj}$, $\boldsymbol{J}_{dj}$	associated block diagonal matrix structures
$\boldsymbol{P}$, $\boldsymbol{Q}$, $\boldsymbol{R}_j$, $\boldsymbol{S}_j$	positive definite diagonal matrix variables of LMIs
$\boldsymbol{M}^{\ominus 1}$, $\boldsymbol{V}_1^{\ominus 1}$	left pseudoinverse of $\boldsymbol{M}$, $\boldsymbol{V}_1$, repectively
$\boldsymbol{I}_n$, $\boldsymbol{I}_p$ γ, δ	$(n \times n)$, $(p \times p)$ identity matrices, real positive tuning parameters

All other notations are defined in the given context fluently.

Appendix A

In the Appendix A, Theorem 1 is proven.

Proof. Considering a positive definite matrix $\boldsymbol{X} \in \mathbb{R}_{+}^{n \times r}$, then a positive function can be specified in the Lyapunov sense as

$$v(\boldsymbol{q}(i)) = \boldsymbol{q}^{\mathrm{T}}(i)\boldsymbol{X}\boldsymbol{q}(i) > 0 \tag{A1}$$

and for stable system trajectories, it is required to satisfy

$$\begin{aligned} \Delta v(\boldsymbol{q}(i)) &= \boldsymbol{q}^{\mathrm{T}}(i+1)\boldsymbol{X}\boldsymbol{q}(i+1) - \boldsymbol{q}^{\mathrm{T}}(i)\boldsymbol{X}\boldsymbol{q}(i) \\ &= \boldsymbol{q}^{\mathrm{T}}(i)(\boldsymbol{F}^{\circ\mathrm{T}}(i)\boldsymbol{X}\boldsymbol{F}^{\circ}(i) - \boldsymbol{X})\boldsymbol{q}(i) \\ &< 0\,, \end{aligned} \tag{A2}$$

where $\boldsymbol{F}^{\circ}(i) = \boldsymbol{F} + \Delta\boldsymbol{F}(i)$. Furthermore, it can be assumed that (compare [24])

$$\boldsymbol{F}^{\circ\mathrm{T}}(i)\boldsymbol{X}\boldsymbol{F}^{\circ}(i) \succ 0 \tag{A3}$$

and from the equality (A3) it can be obtained the relation

$$\begin{bmatrix} \boldsymbol{0} & \boldsymbol{F}^{\circ\mathrm{T}}(i) \\ \boldsymbol{F}^{\circ}(i) & -\boldsymbol{X}^{-1} \end{bmatrix} \succ 0\,. \tag{A4}$$

Using a positive semi-definite matrix $\boldsymbol{Z} \in \mathbb{R}^{n \times r}$ such that the inequality

$$\boldsymbol{X}^{\circ} = \boldsymbol{X}^{-1} - \boldsymbol{Z} \succ 0 \tag{A5}$$

is positive definite and regular, then the condition (A4) can be reformulated as

$$\begin{bmatrix} \boldsymbol{0} & \boldsymbol{F}^{\circ\mathrm{T}}(i) \\ \boldsymbol{F}^{\circ}(i) & -\boldsymbol{X}^{-1} \end{bmatrix} \prec \begin{bmatrix} \boldsymbol{0} & \boldsymbol{F}^{\circ\mathrm{T}}(i) \\ \boldsymbol{F}^{\circ}(i) & -(\boldsymbol{X}^{-1} - \boldsymbol{Z}) \end{bmatrix}, \tag{A6}$$

which can be directly employed as an alternative

$$\boldsymbol{F}^{\circ\mathrm{T}}(i)(\boldsymbol{X}^{-1} - \boldsymbol{Z})^{-1}\boldsymbol{F}^{\circ}(i) \succ \boldsymbol{F}^{\circ\mathrm{T}}(i)\boldsymbol{X}\boldsymbol{F}^{\circ}(i) \succ 0\,. \tag{A7}$$

Moreover, using the Sherman–Morrison–Woodbury formula, it can be obtained

$$(\boldsymbol{X}^{-1} - \boldsymbol{Z})^{-1} = \boldsymbol{X} + \boldsymbol{X}(\boldsymbol{Z}^{-1} - \boldsymbol{X})^{-1}\boldsymbol{X} \succ 0 \tag{A8}$$

and considering that

$$\boldsymbol{Z} = \epsilon^{-1}\boldsymbol{M}\boldsymbol{M}^{\mathrm{T}}, \quad \boldsymbol{X}^{\circ} = \boldsymbol{X}^{-1} - \epsilon^{-1}\boldsymbol{M}\boldsymbol{M}^{\mathrm{T}} \succ 0\,, \tag{A9}$$

then the same formula consists directly of assuming that for any $\epsilon > 0$

$$(\boldsymbol{X}^{-1} - \epsilon^{-1}\boldsymbol{M}\boldsymbol{M}^{\mathrm{T}})^{-1} = \boldsymbol{X} + \boldsymbol{X}\boldsymbol{M}^{\mathrm{T}}(\epsilon\boldsymbol{I}_p - \boldsymbol{M}\boldsymbol{X}\boldsymbol{M}^{\mathrm{T}})^{-1}\boldsymbol{M}\boldsymbol{F} \succ 0\,, \tag{A10}$$

which implies

$$\epsilon\boldsymbol{I}_p - \boldsymbol{M}\boldsymbol{X}\boldsymbol{M}^{\mathrm{T}} \succ 0\,. \tag{A11}$$

Substituting (A9) in (A6) then the following conditions holds

$$\begin{aligned}\begin{bmatrix}\boldsymbol{0} & \boldsymbol{F}^{\circ\mathrm{T}}(i)\\ \boldsymbol{F}^{\circ}(i) & -\boldsymbol{X}^{\circ}\end{bmatrix} = \begin{bmatrix}\boldsymbol{0} & \boldsymbol{F}^{\mathrm{T}}\\ \boldsymbol{F} & -\boldsymbol{X}^{-1} + 2\epsilon^{-1}\boldsymbol{M}\boldsymbol{M}^{\mathrm{T}}\end{bmatrix} + \\ + \begin{bmatrix}\boldsymbol{0} & \boldsymbol{N}_1^{\mathrm{T}}\boldsymbol{H}^{\mathrm{T}}(i)\boldsymbol{M}^{\mathrm{T}}\\ \boldsymbol{M}\boldsymbol{H}(i)\boldsymbol{N}_1 & -\epsilon^{-1}\boldsymbol{M}\boldsymbol{M}^{\mathrm{T}}\end{bmatrix} \\ \succ 0\,.\end{aligned} \tag{A12}$$

Denoting for simplicity that

$$\boldsymbol{X}^{\bullet} = \boldsymbol{X}^{\circ} - \epsilon^{-1}\boldsymbol{M}\boldsymbol{M}^{\mathrm{T}} = \boldsymbol{X}^{-1} - 2\epsilon^{-1}\boldsymbol{M}\boldsymbol{M}^{\mathrm{T}} \tag{A13}$$

and prescribing that the given uncertainties are particularly conveyed by the transform matrix

$$\boldsymbol{T} = \mathrm{diag}\begin{bmatrix}\boldsymbol{I}_n & \boldsymbol{M}^{\ominus 1}\end{bmatrix}, \quad \boldsymbol{M}^{\ominus 1} = (\boldsymbol{M}^{\mathrm{T}}\boldsymbol{M})^{-1}\boldsymbol{M}^{\mathrm{T}}, \tag{A14}$$

it is sufficient to verify that the following inequality is satisfied (when by $\boldsymbol{T}$ is pre-multiplied the left side and by $\boldsymbol{T}^{\mathrm{T}}$ post-multiplied the right side of (A12))

$$\begin{aligned}&\begin{bmatrix}\boldsymbol{0} & \boldsymbol{F}^{\mathrm{T}}\boldsymbol{M}^{\ominus\mathrm{T}}\\ \boldsymbol{M}^{\ominus}\boldsymbol{F} & -\boldsymbol{M}^{\ominus 1}\boldsymbol{X}^{\bullet}\boldsymbol{M}^{\ominus\mathrm{T}}\end{bmatrix} + \begin{bmatrix}\boldsymbol{0} & \boldsymbol{N}_1^{\mathrm{T}}\boldsymbol{H}^{\mathrm{T}}(i)\\ \boldsymbol{H}(i)\boldsymbol{N}_1 & -\epsilon^{-1}\boldsymbol{I}_n\end{bmatrix}\\ &= \begin{bmatrix}\epsilon\boldsymbol{N}_1^{\mathrm{T}}\boldsymbol{H}^{\mathrm{T}}(i)\boldsymbol{H}(i)\boldsymbol{N}_1 & \boldsymbol{F}^{\mathrm{T}}\boldsymbol{M}^{\ominus\mathrm{T}}\\ \boldsymbol{M}^{\ominus 1}\boldsymbol{F} & -\boldsymbol{M}^{\ominus 1}\boldsymbol{X}^{\bullet}\boldsymbol{M}^{\ominus\mathrm{T}}\end{bmatrix}\\ &\preceq \begin{bmatrix}\epsilon\boldsymbol{N}_1^{\mathrm{T}}\boldsymbol{N}_1 & \boldsymbol{F}^{\mathrm{T}}\boldsymbol{M}^{\ominus\mathrm{T}}\\ \boldsymbol{M}^{\ominus 1}\boldsymbol{F} & -\boldsymbol{M}^{\ominus 1}\boldsymbol{X}^{\bullet}\boldsymbol{M}^{\ominus\mathrm{T}}\end{bmatrix}\\ &\prec \begin{bmatrix}2\epsilon\boldsymbol{N}_1^{\mathrm{T}}\boldsymbol{N}_1 & \boldsymbol{F}^{\mathrm{T}}\boldsymbol{M}^{\ominus\mathrm{T}}\\ \boldsymbol{M}^{\ominus 1}\boldsymbol{F} & -\boldsymbol{M}^{\ominus 1}\boldsymbol{X}^{\bullet}\boldsymbol{M}^{\ominus\mathrm{T}}\end{bmatrix},\end{aligned} \tag{A15}$$

or, equivalently, using the Schur complement property, the desired result is

$$2\epsilon\boldsymbol{N}_1^{\mathrm{T}}\boldsymbol{N}_1 + \boldsymbol{F}^{\mathrm{T}}\boldsymbol{X}^{\bullet -1}\boldsymbol{F} \succ 0\,. \tag{A16}$$

Thus, setting $\gamma = 2\epsilon$ and combining the inequality (A16) with (A13) it yields

$$\gamma\boldsymbol{N}_1^{\mathrm{T}}\boldsymbol{N}_1 + \boldsymbol{F}^{\mathrm{T}}(\boldsymbol{X}^{-1} - \gamma^{-1}\boldsymbol{M}\boldsymbol{M}^{\mathrm{T}})^{-1}\boldsymbol{F} \succ 0 \tag{A17}$$

and comparing (A3) and (A17), it is fulfilled

$$\begin{aligned}\boldsymbol{F}^{\circ\mathrm{T}}(i)\boldsymbol{X}\boldsymbol{F}^{\circ}(i) &= (\boldsymbol{F} + \boldsymbol{M}\boldsymbol{H}(i)\boldsymbol{N}_1)^{\mathrm{T}}\boldsymbol{X}(\boldsymbol{F} + \boldsymbol{M}\boldsymbol{H}(i)\boldsymbol{N}_1)\\ &\le \boldsymbol{F}^{\mathrm{T}}(\boldsymbol{X}^{-1} - \gamma^{-1}\boldsymbol{M}\boldsymbol{M}^{\mathrm{T}})^{-1}\boldsymbol{F} + \gamma\boldsymbol{N}_1^{\mathrm{T}}\boldsymbol{N}_1\,.\end{aligned} \tag{A18}$$

Therefore it can be substituted (A17) into (A2) to carry out the following property

$$\begin{aligned}\Delta v(\boldsymbol{q}(i)) &= \boldsymbol{q}^{\mathrm{T}}(i)(\boldsymbol{F}^{\circ\mathrm{T}}(i)\boldsymbol{X}\boldsymbol{F}^{\circ}(i) - \boldsymbol{X})\boldsymbol{q}(i)\\ &> \boldsymbol{q}^{\mathrm{T}}(i)\boldsymbol{F}^{\mathrm{T}}(\boldsymbol{X}^{-1} - \gamma^{-1}\boldsymbol{M}\boldsymbol{M}^{\mathrm{T}})^{-1}\boldsymbol{F}\boldsymbol{q}(i) +\\ &+ \boldsymbol{q}^{\mathrm{T}}(i)(\gamma\boldsymbol{N}_1^{\mathrm{T}}\boldsymbol{N}_1 - \boldsymbol{X})\boldsymbol{q}(i)\\ &< 0\,,\end{aligned} \tag{A19}$$

and, with respect to given conditional positivity, the discrete system stability is determined as

$$\boldsymbol{F}^{\mathrm{T}}(\boldsymbol{X}^{-1}-\gamma^{-1}\boldsymbol{M}\boldsymbol{M}^{\mathrm{T}})^{-1}\boldsymbol{F}+\gamma\boldsymbol{N}_1^{\mathrm{T}}\boldsymbol{N}_1-\boldsymbol{X}\prec 0\,, \tag{A20}$$

$$\boldsymbol{X}^{-1}-\gamma^{-1}\boldsymbol{M}\boldsymbol{M}^{\mathrm{T}}\succ 0\,. \tag{A21}$$

Thus, the property of Schur complement then implies that

$$\begin{bmatrix}-\boldsymbol{X}+\gamma\boldsymbol{N}_1^{\mathrm{T}}\boldsymbol{N}_1 & \boldsymbol{F}^{\mathrm{T}}\\ \boldsymbol{F} & -(\boldsymbol{X}^{-1}-\gamma^{-1}\boldsymbol{M}\boldsymbol{M}^{\mathrm{T}})\end{bmatrix}\prec 0\,. \tag{A22}$$

Introducing

$$\boldsymbol{T}^{\circ}=\operatorname{diag}\begin{bmatrix}\boldsymbol{P} & \boldsymbol{I}_n\end{bmatrix},\quad \boldsymbol{P}=\boldsymbol{X}^{-1} \tag{A23}$$

then, by $\boldsymbol{T}^{\circ}$ defined the coordinate transform of (A22) conveys

$$\begin{bmatrix}\gamma\boldsymbol{P}\boldsymbol{N}_1^{\mathrm{T}}\boldsymbol{N}_1\boldsymbol{P}-\boldsymbol{P} & \boldsymbol{P}\boldsymbol{F}^{\mathrm{T}}\\ \boldsymbol{F}\boldsymbol{P} & -(\boldsymbol{P}-\gamma^{-1}\boldsymbol{M}\boldsymbol{M}^{\mathrm{T}})\end{bmatrix}\prec 0\,. \tag{A24}$$

Obviously, with the setting

$$\gamma^{-1}=\delta \tag{A25}$$

the inequality (6) then immediately follows from (A24) and (A21) gives (7).

When reflecting (4) and considering the following expression

$$\boldsymbol{Z}=\epsilon^{-1}\boldsymbol{V}_1\boldsymbol{V}_1^{\mathrm{T}},\quad \boldsymbol{X}^{\circ}=\boldsymbol{X}^{-1}-\epsilon^{-1}\boldsymbol{V}_1\boldsymbol{V}_1^{\mathrm{T}}\succ 0\,, \tag{A26}$$

then, analogously, it is guaranteed that the following holds

$$\begin{aligned}\begin{bmatrix}\boldsymbol{0} & \boldsymbol{F}^{\circ\mathrm{T}}(i)\\ \boldsymbol{F}^{\circ}(i) & -\boldsymbol{X}^{\circ}\end{bmatrix}&=\begin{bmatrix}\boldsymbol{0} & \boldsymbol{F}^{\mathrm{T}}\\ \boldsymbol{F} & -\boldsymbol{X}^{-1}+2\epsilon^{-1}\boldsymbol{V}_1\boldsymbol{V}_1^{\mathrm{T}}\end{bmatrix}+\\ &+\begin{bmatrix}\boldsymbol{0} & \boldsymbol{U}^{\mathrm{T}}\boldsymbol{W}^{\mathrm{T}}(i)\boldsymbol{V}_1^{\mathrm{T}}\\ \boldsymbol{V}_1\boldsymbol{W}(i)\boldsymbol{U} & -\epsilon^{-1}\boldsymbol{V}_1\boldsymbol{V}_1^{\mathrm{T}}\end{bmatrix}\\ &\succ 0\,.\end{aligned} \tag{A27}$$

The solution then becomes, if denoting

$$\boldsymbol{X}^{\bullet}=\boldsymbol{X}^{\circ}-\epsilon^{-1}\boldsymbol{V}_1\boldsymbol{V}_1^{\mathrm{T}}=\boldsymbol{X}^{-1}-2\epsilon^{-1}\boldsymbol{V}_1\boldsymbol{V}_1^{\mathrm{T}} \tag{A28}$$

and defining the composite matrix

$$\boldsymbol{T}_a=\operatorname{diag}\begin{bmatrix}\boldsymbol{I}_n & \boldsymbol{V}_1^{\ominus 1}\end{bmatrix},\quad \boldsymbol{V}_1^{\ominus 1}=(\boldsymbol{V}_1^{\mathrm{T}}\boldsymbol{V}_1)^{-1}\boldsymbol{V}_1^{\mathrm{T}} \tag{A29}$$

that, when using pre-multiplication by $\boldsymbol{T}_a$ and post-multiplication by $\boldsymbol{T}_a^{\mathrm{T}}$, this adjustment determines an improved form in the dependency on (A27)

$$\begin{aligned}&\begin{bmatrix}\boldsymbol{0} & \boldsymbol{F}^{\mathrm{T}}\boldsymbol{V}_1^{\ominus\mathrm{T}}\\ \boldsymbol{V}_1^{\ominus}\boldsymbol{F} & -\boldsymbol{V}_1^{\ominus 1}\boldsymbol{X}^{\bullet}\boldsymbol{V}_1^{\ominus\mathrm{T}}\end{bmatrix}+\begin{bmatrix}\boldsymbol{0} & \boldsymbol{U}^{\mathrm{T}}\boldsymbol{W}^{\mathrm{T}}(i)\\ \boldsymbol{W}(i)\boldsymbol{U} & -\epsilon^{-1}\boldsymbol{I}_n\end{bmatrix}\\ &=\begin{bmatrix}\epsilon\boldsymbol{U}^{\mathrm{T}}\boldsymbol{W}^{\mathrm{T}}(i)\boldsymbol{W}(i)\boldsymbol{U} & \boldsymbol{F}^{\mathrm{T}}\boldsymbol{N}_1^{\ominus\mathrm{T}}\\ \boldsymbol{V}_1^{\ominus 1}\boldsymbol{F} & -\boldsymbol{V}_1^{\ominus 1}\boldsymbol{X}^{\bullet}\boldsymbol{V}_1^{\ominus\mathrm{T}}\end{bmatrix}\\ &\preceq\begin{bmatrix}\epsilon\boldsymbol{U}^{\mathrm{T}}\boldsymbol{U} & \boldsymbol{F}^{\mathrm{T}}\boldsymbol{V}_1^{\ominus\mathrm{T}}\\ \boldsymbol{V}_1^{\ominus 1}\boldsymbol{F} & -\boldsymbol{V}_1^{\ominus 1}\boldsymbol{X}^{\bullet}\boldsymbol{V}_1^{\ominus\mathrm{T}}\end{bmatrix}\\ &\prec\begin{bmatrix}2\epsilon\boldsymbol{U}^{\mathrm{T}}\boldsymbol{U} & \boldsymbol{F}^{\mathrm{T}}\boldsymbol{V}_1^{\ominus\mathrm{T}}\\ \boldsymbol{V}_1^{\ominus 1}\boldsymbol{F} & -\boldsymbol{V}_1^{\ominus 1}\boldsymbol{X}^{\bullet}\boldsymbol{V}_1^{\ominus\mathrm{T}}\end{bmatrix},\end{aligned} \tag{A30}$$

or, equivalently, the problem becomes nontrivial, since

$$2\epsilon \boldsymbol{U}^{\mathrm{T}}\boldsymbol{U} + \boldsymbol{F}^{\mathrm{T}}\boldsymbol{X}^{\bullet -1}\boldsymbol{F} \succ 0\,. \tag{A31}$$

Thus, resetting $\gamma = 2\epsilon$, it yields with respect to (A7)

$$\gamma \boldsymbol{U}^{\mathrm{T}}\boldsymbol{U} + \boldsymbol{F}^{\mathrm{T}}(\boldsymbol{X}^{-1} - \gamma^{-1}\boldsymbol{V}_1\boldsymbol{V}_1^{\mathrm{T}})^{-1}\boldsymbol{F} \succ 0 \tag{A32}$$

and comparing (A3) and (A31), as a consequence of this phenomenon, the following stays positive

$$\begin{aligned} \boldsymbol{F}^{\circ\mathrm{T}}(i)\boldsymbol{X}\boldsymbol{F}^{\circ}(i) &= (\boldsymbol{F} + \boldsymbol{V}_1\boldsymbol{W}(i)\boldsymbol{U})^{\mathrm{T}}\boldsymbol{X}(\boldsymbol{F} + \boldsymbol{V}_1\boldsymbol{W}(i)\boldsymbol{U}) \\ &\leq \boldsymbol{F}^{\mathrm{T}}(\boldsymbol{X}^{-1} - \gamma^{-1}\boldsymbol{V}_1\boldsymbol{V}_1^{\mathrm{T}})^{-1}\boldsymbol{F} + \gamma\boldsymbol{U}^{\mathrm{T}}\boldsymbol{U}\,. \end{aligned} \tag{A33}$$

From this follows that the problem solved is given by

$$\begin{aligned} \Delta v(\boldsymbol{q}(i)) &= \boldsymbol{q}^{\mathrm{T}}(i)(\boldsymbol{F}^{\circ\mathrm{T}}(i)\boldsymbol{X}\boldsymbol{F}^{\circ}(i) - \boldsymbol{X})\boldsymbol{q}(i) \\ &> \boldsymbol{q}^{\mathrm{T}}(i)\boldsymbol{F}^{\mathrm{T}}(\boldsymbol{X}^{-1} - \gamma^{-1}\boldsymbol{V}_1\boldsymbol{V}_1^{\mathrm{T}})^{-1}\boldsymbol{F}\boldsymbol{q}(i) + \\ &+ \boldsymbol{q}^{\mathrm{T}}(i)(\gamma\boldsymbol{U}^{\mathrm{T}}\boldsymbol{U} - \boldsymbol{X})\boldsymbol{q}(i) \\ &< 0\,, \end{aligned} \tag{A34}$$

$$\boldsymbol{F}^{\mathrm{T}}(\boldsymbol{X}^{-1} - \gamma^{-1}\boldsymbol{V}_1\boldsymbol{V}_1^{\mathrm{T}})^{-1}\boldsymbol{F} + \gamma\boldsymbol{U}^{\mathrm{T}}\boldsymbol{U} - \boldsymbol{X} \prec 0\,, \tag{A35}$$

respectively. Thus, the complement of a symmetric block-matrix provides the reasonable specification

$$\begin{bmatrix} -\boldsymbol{X} + \gamma\boldsymbol{U}^{\mathrm{T}}\boldsymbol{U} & \boldsymbol{F}^{\mathrm{T}} \\ \boldsymbol{F} & -\boldsymbol{X}^{-1} + \gamma^{-1}\boldsymbol{V}_1\boldsymbol{V}_1^{\mathrm{T}} \end{bmatrix} \prec 0\,. \tag{A36}$$

Introducing the matrix

$$\boldsymbol{T}_a^{\circ} = \mathrm{diag}\begin{bmatrix}\boldsymbol{I}_n & \boldsymbol{Q}\end{bmatrix}, \quad \boldsymbol{Q} = \boldsymbol{X}^{-1} \tag{A37}$$

pre- and post-multiplication of (A36) by $\boldsymbol{T}_a^{\circ}$ entails that (A36) can be expressed in the following matrix inequality form

$$\begin{bmatrix} -\boldsymbol{Q} + \gamma\boldsymbol{U}^{\mathrm{T}}\boldsymbol{U} & \boldsymbol{F}^{\mathrm{T}}\boldsymbol{Q} \\ \boldsymbol{Q}\boldsymbol{F} & -\boldsymbol{Q} + \gamma^{-1}\boldsymbol{Q}\boldsymbol{V}_1\boldsymbol{V}_1^{\mathrm{T}}\boldsymbol{Q} \end{bmatrix} \prec 0\,. \tag{A38}$$

The new formulation of the problem can then be stated as

$$\begin{bmatrix} -\boldsymbol{Q} + \gamma\boldsymbol{U}^{\mathrm{T}}\boldsymbol{U} & \boldsymbol{F}^{\mathrm{T}}\boldsymbol{Q} \\ \boldsymbol{Q}\boldsymbol{F} & -\boldsymbol{Q} \end{bmatrix} + \gamma^{-1}\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{Q}\boldsymbol{V}_1 \end{bmatrix}\begin{bmatrix} \boldsymbol{0} & \boldsymbol{V}_1^{\mathrm{T}}\boldsymbol{Q} \end{bmatrix} \prec 0 \tag{A39}$$

and this inequality merely expresses the fact that

$$\begin{bmatrix} -\boldsymbol{Q} + \gamma\boldsymbol{U}^{\mathrm{T}}\boldsymbol{U} & \boldsymbol{F}^{\mathrm{T}}\boldsymbol{Q} & \boldsymbol{0} \\ \boldsymbol{Q}\boldsymbol{F} & -\boldsymbol{Q} & \boldsymbol{Q}\boldsymbol{V}_1 \\ \boldsymbol{0} & \boldsymbol{V}_1^{\mathrm{T}}\boldsymbol{Q} & -\gamma\boldsymbol{I}_p \end{bmatrix} \prec 0\,. \tag{A40}$$

On the other hand, defining the following transform matrix

$$\boldsymbol{T}^{\bullet} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I}_n & \boldsymbol{0} \\ \boldsymbol{I}_n & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_p \end{bmatrix} \tag{A41}$$

to have a similar structure to (6), pre- and post-multiplication of (A40) by $\boldsymbol{T}^{\bullet}$ implies (9) and, as a consequence, (10). This concludes the proof. □

It can now be ready to state that the algebraic conditions in Theorem 1 mean feasibility of the set of LMIs.

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