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Some Trapezoid Intuitionistic Fuzzy Linguistic Maclaurin Symmetric Mean Operators and Their Application to Multiple-Attribute Decision Making

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Abstract: In order to solve multiple-attribute group decision-making (MAGDM) problems under a trapezoid intuitionistic fuzzy linguistic (TIFL) environment and the relationships between multiple input parameters needed, in this paper, we extend the Maclaurin symmetric mean (MSM) operators to TIFL numbers (TIFLNs). Some new aggregation operators are proposed, including the trapezoid intuitionistic fuzzy linguistic Maclaurin symmetric mean (TIFLMSM) operator, trapezoid intuitionistic fuzzy linguistic generalized Maclaurin symmetric mean (TIFLGMSM) operator, trapezoid intuitionistic fuzzy linguistic weighted Maclaurin symmetric mean (TIFLWMSM) operator and trapezoid intuitionistic fuzzy linguistic weighted generalized Maclaurin symmetric mean (TIFLWGMSM) operator. Next, based on the TIFLWMSM and TIFLWGMSM operators, two methods are presented to deal with MAGDM problems. Finally, there is a numerical example to verify the effectiveness and feasibility of the proposed approaches.

Keywords: Maclaurin symmetric mean operation operators; trapezoid intuitionistic fuzzy linguistic; multiple-attribute group decision making

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1. Introduction

As an important branch of modern decision science, the MAGDM problems have widely existed in various areas, such as economics, management, military, society and so on. With the development of society, economy and information technology, decision makers (DMs) are confronted with the complexity of decision-making problems and the limitations of their knowledge; thus, sometimes, they cannot make the correct judgment in the decision-making process. In order to obtain more reasonable and comprehensive decision-making results, Zadeh [1] first developed the concept of a fuzzy set (FS). However, the FS contains only the membership degree, which makes it difficult to accurately express the uncertainty and fuzziness in the decision-making process. To solve the above problems, Atanassov [2,3] introduced the non-membership degree into FS and presented the intuitionistic fuzzy sets (IFSs), which are the generalizations of FS. Fan and Xiao [4] presented two-dimensional IFSs (TDIFSs), which integrate the uncertainty and reliability expressions of IFSs. Through the continuous research by researchers, a new concept of linguistic variables (LVs) was presented by Zadeh [5], which plays an important role in qualitative information analysis. After that, LVs have attracted the attention of many experts and been thoroughly studied. Xu [6] defined uncertain LVs (ULVs) based on LVs and proposed two uncertain linguistic aggregation operators to deal with MAGDM problems under the uncertain linguistic environment. Chai et al. [7] proposed Z-uncertain probabilistic linguistic variables (Z-UPLVs) and defined the operational rules, normalization, distance and similarity measures, as well as a comparison method of Z-UPLVs. Herrera and Martínez [8] developed a 2-tuple linguistic representation model that can overcome the loss of information in the decision-making process. Ju et al. [9] presented the trapezoid fuzzy 2-tuple linguistic approach and applied it to multi-attribute decision-making (MADM) problems with trapezoid fuzzy linguistic information. Subsequently, on the basis of LVs and ULVs, Xu [10] defined the trapezoid fuzzy LVs (TFLVs), which has great advantages in dealing with vague data.

In the real decision-making process, DMs have difficultly expressing their ideas clearly by utilizing LVs or IFSs. Therefore, Wang and Li [11] further extended LVs and IFSs to the intuitionistic fuzzy linguistic set (ILS). Furthermore, Liu and Jin [12] presented intuitionistic ULVs (IULVs) based on ILS. In addition, Liu [13] defined the interval-valued intuitionistic uncertain linguistic set (IVIULS) by combing ULVs and interval-valued intuitionistic fuzzy sets (IVIFSs) [14]. In order to make the decision-making process more precise and objective, Ju et al. [15] further proposed TIFLVs, which are composed of a trapezoid fuzzy linguistic part and an intuitionistic fuzzy part.

The aggregation operators are an efficient and practical tool in coping with the MAGDM issues. These operators can not only combine the massive input individual data into aggregate data but also rank the alternatives on the basis of the comprehensive value. Therefore, the study of aggregation operators has aroused great interest among researchers; for example, the Bonferroni mean (BM) operator [16], Heronian mean (HM) operator [17,18], Maclaurin symmetric mean (MSM) operator [19] and so forth. The BM operator was initially developed by Bonferroni [16], which can effectively capture the interrelationships between two input arguments. Chu et al. [20] proposed some TIFLBM aggregation operators and utilized them for solving MAGDM problems. Moreover, Beliakov [21] introduced the HM aggregation operator, which has the same characteristic of being able to consider the correlation between any two input parameters as the BM operator. Some new generalizations of HM operators have been presented [22–24]. However, both the BM operator and the HM operator have the weakness, that is, they can only reflect the relevance between two arguments and ignore the interrelationships among multiple parameters. Therefore, in order to avoid the above defect, Maclaurin [19] firstly presented the MSM operator, which has the characteristic of considering the correlations among multiple input parameters. Then, the extended MSM operator was developed further by Detemple and Robertson [25]. In addition, based on the above merit, researchers have conducted in-depth research on the MSM operator and achieved many accomplishments. Qin and Liu [26] extended the MSM operator to the intuitionistic fuzzy environment and then applied it to deal with intuitionistic fuzzy MAGDM problems. Liu and Qin [27] further investigated the uncertain linguistic dual MSM (ULDMSM) operator. Liu and zhang [28] developed some single-valued trapezoidal neutrosophic MSM (SVTNMSM) operators. Geng et al. [29] presented the MSM operator for interval neutrosophic linguistic numbers (INLNs). Ali and Mahmood [30] initiated some complex q-rung orthopair fuzzy MSM (Cq-ROFMSM) operators.

With the increasing complexity of the current MAGDM issues, DMs can hardly process problems existing in decision making and obtain reasonable results. In order to cope with the ambiguity and uncertainty in complex problems more precisely and flexibly, we can take the following steps. Firstly, we aim at uncertain and vague information; the form of TIFL numbers (TIFLNs) can be used to represent attribute values. Moreover, a significant feature of the MSM operator is that it can capture the relationships among multiple parameters. So far, there is a lack of literature on the MSM operator dealing with MAGDM problems under TIFL environments. Therefore, in this paper, we extend the MSM operator to TIFLNs and further propose the TIFLMSM operator, TIFLGMSM operator, TIFLWMSM operator and TIFLWGMSM operator. Then, some properties and some peculiar cases of these operators mentioned above are analyzed. In addition, an MAGDM approach for TIFLNs is proposed, and we demonstrate its validity and superiority by comparing it with other methods.

The structure of this paper is shown below. Section 2 briefly introduces the basic notions of TFLVs, IFLS, TIFLS and MSM operators. Section 3 proposes TIFLMSM, TIFLGMSM, TIFLWMSM and TIFLGWMSM operators; furthermore, some of their properties and special cases are analyzed. In Section 4 of this paper, two MAGDM methods based on the TIFLWMSM operator and TIFLWGMSM operator are introduced. In Section 5, we utilize illustrative examples of the proposed approaches and show its validity and usability.

2. Preliminaries

2.1. Trapezoid Fuzzy Linguistic Variables (TFLVs)

Suppose that a linguistic term (LT) set (LTS) $S = \{s_0, s_1, s_2, \ldots, s_{t-1}\}$ is totally ordered and contains limited discrete items, in which l is an odd number. For example, if t = 5, the LTS S could be shown as follows:

$$S = \{s_0 = \text{very bad}, s_1 = bad, s_2 = fair, s_3 = good, s_4 = \text{very good}\}.$$

For any linguistic variables (LVs) in LTS S, the following features should be satisfied [6,31–33]:

(1) if and only if $i > j$, then $s_i > s_j$;
(2) There is the negation operator: $\text{neg}(s_i) = s_{l-i-1}$;
(3) Max operator: If $i \geq j$, then $\max(s_i, s_j) = s_i$;
(4) Min operator: If $i \leq j$, then $\min(s_i, s_j) = s_i$.

Moreover, in order to keep all the given information, the LTS $S = \{s_0, s_1, s_2, \ldots, s_{t-1}\}$ needs to be converted into a continuous LTS $\tilde{S} = \{S_\alpha \mid \alpha \in R\}$. The following related operations are defined [34,35]:

$$\beta s_i = s_{\beta \times i}, \tag{1}$$

$$s_i \oplus s_j = s_{i+j}, \tag{2}$$

$$s_i / s_j = s_{i/j}, \tag{3}$$

$$(s_i)^n = s_{i^n}, \tag{4}$$

Definition 1 ([10])**.** *Let LTS $\tilde{S} = \{s_\theta \mid s_0 \leq s_\theta \leq s_{t-1}, \theta \in [0, t-1]\}$, which takes a continuous form. $s_\alpha, s_\beta, s_\theta$ and s_τ are the LTs in $\tilde{S}$ and meet the following conditions: $s_0 \leq s_\alpha \leq s_\beta \leq s_\theta \leq s_\tau \leq s_{t-1}$, then TFLVs are definded as $\bar{s} = [s_\alpha, s_\beta, s_\theta, s_\tau]$, and $\bar{S}$ represents a set of all TFLVs.*

Let $\bar{s}_1 = [s_{\alpha 1}, s_{\beta 1}, s_{\theta 1}, s_{\tau 1}]$ and $\bar{s}_2 = [s_{\alpha 2}, s_{\beta 2}, s_{\theta 2}, s_{\tau 2}]$ be any two TFLVs in $\bar{S}$. The following operational laws are defined [36,37]:

$$\bar{s}_1 \oplus \bar{s}_2 = [s_{\alpha 1+\alpha 2}, s_{\beta 1+\beta 2}, s_{\theta 1+\theta 2}, s_{\tau 1+\tau 2}], \tag{5}$$

$$\bar{s}_1 \otimes \bar{s}_2 = [s_{\alpha 1\times\alpha 2}, s_{\beta 1\times\beta 2}, s_{\theta 1\times\theta 2}, s_{\tau 1\times\tau 2}], \tag{6}$$

$$\bar{s}_1 / \bar{s}_2 = [s_{\alpha 1/\alpha 2}, s_{\beta 1/\beta 2}, s_{\theta 1/\theta 2}, s_{\tau 1/\tau 2}], \tag{7}$$

$$\lambda \bar{s}_1 = [s_{\lambda*\alpha 1}, s_{\lambda*\beta 1}, s_{\lambda*\theta 1}, s_{\lambda*\tau 1}] \tag{8}$$

2.2. Intuitionistic Fuzzy Linguistic Set (IFLS)

Based on the intuitionistic fuzzy set and LTS, IFLS was proposed by Wang and Li [9], as shown below.

Definition 2 ([11])**.** *An IFLS A in X is expressed as*

$$A = \{\langle x, [h_{\theta(x)}, (\mu(x), \nu(x))] \mid x \in X\rangle\}. \tag{9}$$

where $h_{\theta(x)} \in S, \mu(x), \nu(x) \in [0,1]$, and $\mu(x) + \nu(x) \leq 1, \forall_x \in X$. $h_{\theta(x)}$ is an LT. $\mu(x)$ and $\nu(x)$ denote the membership degree and the non-membership degree of the element x to the $LTh_{\theta(x)}$, respectively.

2.3. Trapezoid Intuitionistic Fuzzy Linguistic Set (TIFLS)

Definition 3 ([15]). *An TIFL set $\bar{A}$ in X is expressed as*

$$\bar{A} = \{\langle x, [[s_{\sigma(x)}, s_{\zeta(x)}, s_{\eta(x)}, s_{\varphi(x)}], (\mu(x), \nu(x))]\rangle \mid x \in X\}. \tag{10}$$

where $s_{\sigma(x)}, s_{\zeta(x)}, s_{\eta(x)}, s_{\varphi(x)} \in \bar{S}, \mu(x), \nu(x) \in [0,1]$, and $\mu(x) + \nu(x) \leq 1, \forall_x \in X$. $[s_{\sigma(x)}, s_{\zeta(x)}, s_{\eta(x)}, s_{\varphi(x)}]$ is a TFLV. $\mu(x)$ and $\nu(x)$ denote the membership degree and the non-membership degree of the element x to the TFLV$[s_{\sigma(x)}, s_{\zeta(x)}, s_{\eta(x)}, s_{\varphi(x)}]$, respectively. Let $\rho(x) = 1 - \mu(x) - \nu(x), \rho(x) \in [0,1], \forall_x \in X$, then $\rho(x)$ is called the hesitancy degree of the element x to the TFLV$[s_{\sigma(x)}, s_{\zeta(x)}, s_{\eta(x)}, s_{\varphi(x)}]$.

In Equation (10), $\langle [s_{\sigma(x)}, s_{\zeta(x)}, s_{\eta(x)}, s_{\varphi(x)}], (\mu(x), \nu(x))\rangle$ is a TIFLN. Clearly, $\bar{A}$ can be considered as a collection of TIFLNs. Therefore, $\bar{a} = \langle [s_{\sigma(x)}, s_{\zeta(x)}, s_{\eta(x)}, s_{\varphi(x)}], (\mu(x), \nu(x))\rangle$ can denote a TIFLN.

Definition 4 ([15]). *Let $\bar{a}_1 = \langle [s_{\sigma(\bar{a}_1)}, s_{\zeta(\bar{a}_1)}, s_{\eta(\bar{a}_1)}, s_{\varphi(\bar{a}_1)}], (\mu(\bar{a}_1), \nu(\bar{a}_1))\rangle$ and $\bar{a}_2 = \langle [s_{\sigma(\bar{a}_2)}, s_{\zeta(\bar{a}_2)}, s_{\eta(\bar{a}_2)}, s_{\varphi(\bar{a}_2)}], (\mu(\bar{a}_2), \nu(\bar{a}_2))\rangle$ be any two TIFLNs, and let $\lambda \geq 0$. Then, the operational rules of TIFLNs can be defined as follows:*

$$\begin{aligned} \bar{a}_1 \oplus \bar{a}_2 = &\langle [s_{\sigma(\bar{a}_1)+\sigma(\bar{a}_2)}, s_{\zeta(\bar{a}_1)+\zeta(\bar{a}_2)}, s_{\eta(\bar{a}_1)+\eta(\bar{a}_2)}, s_{\varphi(\bar{a}_1)+\varphi(\bar{a}_2)}], \\ &(\mu(\bar{a}_1) + \mu(\bar{a}_2) - \mu(\bar{a}_1) \times \mu(\bar{a}_2), \nu(\bar{a}_1) \times \nu(\bar{a}_2))\rangle \end{aligned} \tag{11}$$

$$\begin{aligned} \bar{a}_1 \otimes \bar{a}_2 = &\langle [s_{\sigma(\bar{a}_1)\times\sigma(\bar{a}_2)}, s_{\zeta(\bar{a}_1)\times\zeta(\bar{a}_2)}, s_{\eta(\bar{a}_1)\times\eta(\bar{a}_2)}, s_{\varphi(\bar{a}_1)\times\varphi(\bar{a}_2)}], \\ &(\mu(\bar{a}_1) \times \mu(\bar{a}_2), \nu(\bar{a}_1) + \nu(\bar{a}_2) - \nu(\bar{a}_1) \times \nu(\bar{a}_2))\rangle \end{aligned} \tag{12}$$

$$\lambda\bar{a}_1 = \langle [s_{\lambda\times\sigma(\bar{a}_1)}, s_{\lambda\times\zeta(\bar{a}_1)}, s_{\lambda\times\eta(\bar{a}_1)}, s_{\lambda\times\varphi(\bar{a}_1)}], (1 - (1 - \mu(\bar{a}_1))^{\lambda}, (\nu(\bar{a}_1))^{\lambda})\rangle \tag{13}$$

$$\bar{a}_1^{\lambda} = \langle [s_{(\sigma(\bar{a}_1))^{\lambda}}, s_{(\zeta(\bar{a}_1))^{\lambda}}, s_{(\eta(\bar{a}_1))^{\lambda}}, s_{(\varphi(\bar{a}_1))^{\lambda}}], ((\mu(\bar{a}_1))^{\lambda}, 1 - (1 - \nu(\bar{a}_1))^{\lambda})\rangle \tag{14}$$

Theorem 1. *Let $\bar{a}_1 = \langle [s_{\sigma(\bar{a}_1)}, s_{\zeta(\bar{a}_1)}, s_{\eta(\bar{a}_1)}, s_{\varphi(\bar{a}_1)}], (\mu(\bar{a}_1), \nu(\bar{a}_1))\rangle$ and $\bar{a}_2 = \langle [s_{\sigma(\bar{a}_2)}, s_{\zeta(\bar{a}_2)}, s_{\eta(\bar{a}_2)}, s_{\varphi(\bar{a}_2)}], (\mu(\bar{a}_2), \nu(\bar{a}_2))\rangle$ be any two TIFLNs, then the following rules can be obtained.*

$$\bar{a}_1 \oplus \bar{a}_2 = \bar{a}_2 \oplus \bar{a}_1, \tag{15}$$

$$\bar{a}_1 \otimes \bar{a}_2 = \bar{a}_2 \otimes \bar{a}_1, \tag{16}$$

$$\gamma(\bar{a}_1 \oplus \bar{a}_2) = \gamma\bar{a}_1 \oplus \gamma\bar{a}_2, \gamma \geq 0, \tag{17}$$

$$\gamma_1\bar{a}_1 \oplus \gamma_2\bar{a}_1 = (\gamma_1 + \gamma_2)\bar{a}_1, \gamma_1, \gamma_2 \geq 0, \tag{18}$$

$$\bar{a}_1^{\gamma_1} \otimes \bar{a}_1^{\gamma_2} = (\bar{a}_1)^{\gamma_1+\gamma_2}, \gamma_1, \gamma_2 \geq 0, \tag{19}$$

$$\bar{a}_1^{\gamma} \otimes \bar{a}_2^{\gamma} = (\bar{a}_1 \otimes \bar{a}_2)^{\gamma}, \gamma \geq 0. \tag{20}$$

Definition 5 ([15]). *Let $\bar{a} = \langle [s_{\sigma(x)}, s_{\zeta(x)}, s_{\eta(x)}, s_{\varphi(x)}], (\mu(x), \nu(x))\rangle$ be a TIFLN, then the score function $S(\tilde{\alpha})$ and the accuracy function $H(\tilde{\alpha})$ of TILFN $\tilde{\alpha}$ are defined as follows, respectively:*

$$S(\tilde{\alpha}) = \frac{1 + \mu(\tilde{\alpha}) - \nu(\tilde{\alpha})}{2} \times s_{\frac{\sigma(\tilde{\alpha})+\zeta(\tilde{\alpha})+\eta(\tilde{\alpha})+\varphi(\tilde{\alpha})}{4}} = s_{\frac{(\sigma(\tilde{\alpha})+\zeta(\tilde{\alpha})+\eta(\tilde{\alpha})+\varphi(\tilde{\alpha}))\times(1+\mu(\tilde{\alpha})-\nu(\tilde{\alpha}))}{8}} \tag{21}$$

$$H(\tilde{\alpha}) = (\mu(\tilde{\alpha}) + \nu(\tilde{\alpha})) \times s_{\frac{\sigma(\tilde{\alpha})+\zeta(\tilde{\alpha})+\eta(\tilde{\alpha})+\varphi(\tilde{\alpha})}{4}} = s_{\frac{(\sigma(\tilde{\alpha})+\zeta(\tilde{\alpha})+\eta(\tilde{\alpha})+\varphi(\tilde{\alpha}))\times(\mu(\tilde{\alpha})+\nu(\tilde{\alpha}))}{4}} \tag{22}$$

Definition 6 ([15]). *Let $\bar{a}_1 = \langle [s_{\sigma(\bar{a}_1)}, s_{\zeta(\bar{a}_1)}, s_{\eta(\bar{a}_1)}, s_{\varphi(\bar{a}_1)}], (\mu(\bar{a}_1), \nu(\bar{a}_1))\rangle$ and $\bar{a}_2 = \langle [s_{\sigma(\bar{a}_2)}, s_{\zeta(\bar{a}_2)}, s_{\eta(\bar{a}_2)}, s_{\varphi(\bar{a}_2)}], (\mu(\bar{a}_2), \nu(\bar{a}_2))\rangle$ be any two TIFLNs, then the comparison means of $\bar{a}_1$ and $\bar{a}_2$ are shown as follows:*

(*a*) *if $S(\bar{a}_1) > S(\bar{a}_2)$, then $\bar{a}_1 \succ \bar{a}_2$;*
(*b*) *if $S(\bar{a}_1) = S(\bar{a}_2)$ and $H(\bar{a}_1) > H(\bar{a}_2)$, then $\bar{a}_1 \succ \bar{a}_2$;*
(*c*) *if $S(\bar{a}_1) = S(\bar{a}_2)$ and $H(\bar{a}_1) = H(\bar{a}_2)$, then $\bar{a}_1 = \bar{a}_2$;*
(*d*) *if $S(\bar{a}_1) = S(\bar{a}_2)$ and $H(\bar{a}_1) < H(\bar{a}_2)$, then $\bar{a}_1 \prec \bar{a}_2$.*

2.4. Maclaurin Symmetric Mean Operator (MSM Operator)

Definition 7 ([19,38]). *Let $a_i(i = 1,2,\ldots,n)$ be a set of nonnegative real numbers, and $k = 1,2,\ldots,n$, then the $MSM^{(k)}$ operator is given as*

$$MSM^{(k)}(a_1,a_2,\ldots,a_n) = \left(\frac{\sum_{\substack{1\le i_1<\ldots \\ <i_k\le n}} \prod_{j=1}^{k} a_{i_j}}{C_n^k}\right)^{\frac{1}{k}} \tag{23}$$

where $(i_1,i_2,\ldots,i_k)$ traversal all the k-tuple combination of $(1,2,\ldots,n)$ and C_n^k is the binomial coefficient.

The important attributes of $MSM^{(k)}$ are shown as follows:

(1) Idempotency. If $a_i = a$ for each i, $MSM^{(k)}(a,a,\ldots,a) = a$;
(2) Monotonicity. If $a_i \le b_i$ for each i, $MSM^{(k)}(a_1,a_2,\ldots,a_n) \le MSM^{(k)}(b_1,b_2,\ldots,b_n)$;
(3) Boundedness. $min(a_1,a_2,\ldots,a_n) \le MSM^{(k)}(a_1,a_2,\ldots,a_n) \le max(a_1,a_2,\ldots,a_n)$.

Definition 8 ([38]). *Let $a_i(i = 1,2,\ldots,n)$ be a set of non-negative real numbers, and $p_1,p_2,\ldots,p_k \ge 0$, then the $GMSM^{(k,p_1,p_2,\ldots,p_k)}$ operator is given as*

$$GMSM^{(k,p_1,p_2,\ldots,p_k)}(a_1,a_2,\ldots,a_n) = \left(\frac{\sum_{\substack{1\le i_1<\ldots \\ <i_k\le n}} \prod_{j=1}^{k} a_{i_j}^{p_j}}{C_n^k}\right)^{\frac{1}{(p_1+p_2+\cdots+p_k)}} \tag{24}$$

where $(i_1,i_2,\ldots,i_k)$ the traversal pf all the k-tuple combinations of $(1,2,\ldots,n)$, and C_n^k is the binomial coefficient.

The properties of $\mathrm{GMSM}^{(k,p_1,p_2,\ldots,p_k)}$ are shown as follows:

(1) Idempotency. If

$$a_i = a \text{ for each i, } GMSM^{(k,p_1,p_2,\ldots,p_k)}(a,a,\ldots,a) = a;$$

(2) Monotonicity. If

$$a_i \le b_i \text{ for each i, } GMSM^{(k,p_1,p_2,\ldots,p_k)}(a_1,a_2,\ldots,a_n) \le GMSM^{(k,p_1,p_2,\ldots,p_k)}(b_1,b_2,\ldots,b_n);$$

(3) Boundedness.

$$min(a_1,a_2,\ldots,a_n) \le GMSM^{(k,p_1,p_2,\ldots,p_k)}(a_1,a_2,\ldots,a_n) \le max(a_1,a_2,\ldots,a_n).$$

3. Some MSM Operators Based on TIFLNs

In this section, based on TIFLNs and MSM operators, we developed the TILFMSM operator, TIFLGMSM operator, TIFLWMSM operator and TIFLWGMSM operator.

3.1. The TILFMSM and TIFLGMSM Operators

Definition 9. *Let $\bar{a}_i = \langle[s_{\sigma(x_i)}, s_{\zeta(x_i)}, s_{\eta(x_i)}, s_{\varphi(x_i)}],(\mu(x_i),\nu(x_i))\rangle(i = 1,2,\ldots,n)$ be a set of the TIFLNs, then the TILFMSM operator: $\Omega^n \to \Omega$ is given as follows.*

$$TILFMSM^{(k)}(\bar{a}_1,\bar{a}_2,\ldots,\bar{a}_n) = \left(\frac{\sum_{\substack{1\le i_1<\ldots \\ <i_k\le n}} \prod_{j=1}^{k} \bar{a}_{i_j}}{C_n^k}\right)^{\frac{1}{k}}, \tag{25}$$

where Ω is a collection of TIFLNs and $k = 1,2,\ldots,n$.

According to the calculation laws of TIFLNs, we can obtain the following result of the TIFLMSM operator as follows.

Theorem 2. *Let $\bar{a}_i = \langle [s_{\sigma(a_i)}, s_{\zeta(a_i)}, s_{\eta(a_i)}, s_{\varphi(a_i)}], (\mu(a_i), \nu(a_i))\rangle (i = 1, 2, \ldots, n)$ be a set of the TIFLNs, then the aggregated result from Definition 9 is still a TIFLN.*

$$
\begin{aligned}
TIFLMSM^{(k)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) &= \left(\frac{\sum_{\substack{1 \le i_1 < \ldots \\ < i_k \le n}} \prod_{j=1}^{k} \bar{a}_{i_j}}{C_n^k} \right)^{\frac{1}{k}} \\
&= \Bigg\langle \Bigg[s_{\left(\frac{\sum_{\substack{1 \le i_1 < \ldots \\ < i_k \le n}} \prod_{j=1}^{k} \sigma(\bar{a}_{i_j})}{C_n^k} \right)^{\frac{1}{k}}}, s_{\left(\frac{\sum_{\substack{1 \le i_1 < \ldots \\ < i_k \le n}} \prod_{j=1}^{k} \zeta(\bar{a}_{i_j})}{C_n^k} \right)^{\frac{1}{k}}}, \\
&\quad s_{\left(\frac{\sum_{\substack{1 \le i_1 < \ldots \\ < i_k \le n}} \prod_{j=1}^{k} \eta(\bar{a}_{i_j})}{C_n^k} \right)^{\frac{1}{k}}}, s_{\left(\frac{\sum_{\substack{1 \le i_1 < \ldots \\ < i_k \le n}} \prod_{j=1}^{k} \varphi(\bar{a}_{i_j})}{C_n^k} \right)^{\frac{1}{k}}} \Bigg], \\
&\quad \Bigg(\Bigg(1 - \Big(\prod_{1 \le i_1 < \cdots < i_k \le n} (1 - \prod_{j=1}^{k} \mu(\bar{a}_{i_j})) \Big)^{\frac{1}{C_n^k}} \Bigg)^{\frac{1}{k}}, \\
&\quad 1 - \Bigg(1 - \Big(\prod_{1 \le i_1 < \cdots < i_k \le n} \big(1 - \prod_{j=1}^{k} (1 - \nu(\bar{a}_{i_j})) \big) \Big)^{\frac{1}{C_n^k}} \Bigg)^{\frac{1}{k}} \Bigg) \Bigg\rangle
\end{aligned}
\tag{26}
$$

Proof. By the operational laws of TIFLNs, we can obtain

$$
\begin{aligned}
\prod_{j=1}^{k} \bar{a}_{i_j} = & \Big\langle \Big[s_{\left(\prod_{j=1}^{k} \sigma(\bar{a}_{i_j}) \right)}, s_{\left(\prod_{j=1}^{k} \zeta(\bar{a}_{i_j}) \right)}, s_{\left(\prod_{j=1}^{k} \eta(\bar{a}_{i_j}) \right)}, \\
& s_{\left(\prod_{j=1}^{k} \varphi(\bar{a}_{i_j}) \right)} \Big], \Big(\prod_{j=1}^{k} \big(\mu(\bar{a}_{i_j}) \big), 1 - \prod_{j=1}^{k} \big(1 - \nu(\bar{a}_{i_j}) \big) \Big) \Big\rangle
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{1 \le i_1 < \cdots < i_k \le n} (\prod_{j=1}^{k} \bar{a}_{i_j}) = & \Big\langle \Big[s_{\sum_{1 \le i_1 < \cdots < i_k \le n} \left(\prod_{j=1}^{k} \sigma(\bar{a}_{i_j}) \right)}, s_{\sum_{1 \le i_1 < \cdots < i_k \le n} \left(\prod_{j=1}^{k} \zeta(\bar{a}_{i_j}) \right)}, \\
& s_{\sum_{1 \le i_1 < \cdots < i_k \le n} \left(\prod_{j=1}^{k} \eta(\bar{a}_{i_j}) \right)}, s_{\sum_{1 \le i_1 < \cdots < i_k \le n} \left(\prod_{j=1}^{k} \varphi(\bar{a}_{i_j}) \right)} \Big], \\
& \Big(1 - \prod_{1 \le i_1 < \cdots < i_k \le n} \Big(1 - \prod_{j=1}^{k} \big(\mu(\bar{a}_{i_j}) \big) \Big), \\
& \prod_{1 \le i_1 < \cdots < i_k \le n} \Big(1 - \prod_{j=1}^{k} \big(1 - \nu(\bar{a}_{i_j}) \big) \Big) \Big) \Big\rangle
\end{aligned}
$$

then,

$$
\begin{aligned}
\frac{\sum_{1 \le i_1 < \cdots < i_k \le n} (\prod_{j=1}^{k} \bar{a}_{i_j})}{C_n^k} = & \Big\langle \Big[s_{\frac{\sum_{1 \le i_1 < \cdots < i_k \le n} \left(\prod_{j=1}^{k} \sigma(\bar{a}_{i_j}) \right)}{C_n^k}}, s_{\frac{\sum_{1 \le i_1 < \cdots < i_k \le n} \left(\prod_{j=1}^{k} \zeta(\bar{a}_{i_j}) \right)}{C_n^k}}, \\
& s_{\frac{\sum_{1 \le i_1 < \cdots < i_k \le n} \left(\prod_{j=1}^{k} \eta(\bar{a}_{i_j}) \right)}{C_n^k}}, s_{\frac{\sum_{1 \le i_1 < \cdots < i_k \le n} \left(\prod_{j=1}^{k} \varphi(\bar{a}_{i_j}) \right)}{C_n^k}} \Big], \\
& \Big(1 - \Big(\prod_{1 \le i_1 < \cdots < i_k \le n} \Big(1 - \prod_{j=1}^{k} \big(\mu(\bar{a}_{i_j}) \big) \Big) \Big)^{\frac{1}{C_n^k}}, \\
& \Big(\prod_{1 \le i_1 < \cdots < i_k \le n} \Big(1 - \prod_{j=1}^{k} \big(1 - \nu(\bar{a}_{i_j}) \big) \Big) \Big)^{\frac{1}{C_n^k}} \Big) \Big\rangle
\end{aligned}
$$

Therefore,

$$TIFLMSM^{(k)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) = \left(\frac{\sum_{\substack{1 \le i_1 < \ldots \\ < i_k \le n}} \prod_{j=1}^{k} \bar{a}_{i_j}}{C_n^k} \right)^{\frac{1}{k}}$$

$$= \left\langle \left[s_{\left(\frac{\sum_{1 \le i_1 < \ldots < i_k \le n} \prod_{j=1}^{k} \sigma(\bar{a}_{i_j})}{C_n^k} \right)^{\frac{1}{k}}}, s_{\left(\frac{\sum_{1 \le i_1 < \ldots < i_k \le n} \prod_{j=1}^{k} \zeta(\bar{a}_{i_j})}{C_n^k} \right)^{\frac{1}{k}}}, s_{\left(\frac{\sum_{1 \le i_1 < \ldots < i_k \le n} \prod_{j=1}^{k} \eta(\bar{a}_{i_j})}{C_n^k} \right)^{\frac{1}{k}}}, s_{\left(\frac{\sum_{1 \le i_1 < \ldots < i_k \le n} \prod_{j=1}^{k} \varphi(\bar{a}_{i_j})}{C_n^k} \right)^{\frac{1}{k}}} \right], \right.$$

$$\left(\left(1 - \Big(\prod_{1 \le i_1 < \cdots < i_k \le n} (1 - \prod_{j=1}^{k} \mu(\bar{a}_{i_j})) \Big)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}}, \right.$$

$$\left. \left. 1 - \left(1 - \Big(\prod_{1 \le i_1 < \cdots < i_k \le n} \big(1 - \prod_{j=1}^{k} (1 - \nu(\bar{a}_{i_j}))\big) \Big)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \right) \right\rangle$$

□

Property 1. *Let* $\bar{a}_i = \langle [s_{\sigma(\bar{a}_i)}, s_{\zeta(\bar{a}_i)}, s_{\eta(\bar{a}_i)}, s_{\varphi(\bar{a}_i)}], (\mu(\bar{a}_i), \nu(\bar{a}_i)) \rangle (i = 1, 2, \ldots, n)$ *and* $\bar{b}_i = \langle [s_{\sigma(\bar{b}_i)}, s_{\zeta(\bar{b}_i)}, s_{\eta(\bar{b}_i)}, s_{\varphi(\bar{b}_i)}], (\mu(\bar{b}_2), \nu(\bar{b}_i)) \rangle (i = 1, 2, \ldots, n)$ *be collections of TIFLNs, then the following properties of* TIFLMSM$^{(K)}$ *are shown.*

(1) Idempotency. If $\bar{a}_i = \bar{a} = \langle [s_{\sigma(\bar{a})}, s_{\zeta(\bar{a})}, s_{\eta(\bar{a})}, s_{\varphi(\bar{a})}], (\mu(\bar{a}), \nu(\bar{a})) \rangle$ *for each i, then* $TIFL\ MSM^{(K)}(\bar{a}, \bar{a}, \ldots, \bar{a}) = a = \langle [s_{\sigma(\bar{a})}, s_{\zeta(\bar{a})}, s_{\eta(\bar{a})}, s_{\varphi(\bar{a})}], (\mu(\bar{a}), \nu(\bar{a})) \rangle$.

(2) Commutativity. If $\bar{a}_i$ *is a permutation of* $\bar{b}_i$ *for each i, then* $TIFLMSM^{(K)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) = TIFLMSM^{(K)}(\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_n)$.

(3) Monotonicity. If $\sigma(\bar{a}_i) \le \sigma(\bar{b}_i), \zeta(\bar{a}_i) \le \zeta(\bar{b}_i), \eta(\bar{a}_i) \le \eta(\bar{b}_i), \varphi(\bar{a}_i) \le \varphi(\bar{b}_i), \mu(\bar{a}_i) \le \mu(\bar{b}_i)$ *and* $\nu(\bar{a}_i) \ge \nu(\bar{b}_i)$ *for all i* $(i = 1, 2, \ldots, n)$*, then* $\bar{a}_i \le \bar{b}_i$ *and* $TIFLMSM^{(K)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) \le TIFLMSM^{(K)}(\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_n)$.

(4) Boundedness. $min(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) \le TIFLMSM^{(K)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) \le max(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n)$.

Proof.

1. Since each $\bar{a}_i = \bar{a} = \langle [s_{\sigma(\bar{a})}, s_{\zeta(\bar{a})}, s_{\eta(\bar{a})}, s_{\varphi(\bar{a})}], (\mu(\bar{a}), \nu(\bar{a})) \rangle$, so

$$TIFLMSM^{(k)}(\bar{a}, \bar{a}, \ldots, \bar{a}) = \left\langle \left[s_{\left(\frac{C_n^k (\sigma(\bar{a}))^k}{C_n^k} \right)^{\frac{1}{k}}}, s_{\left(\frac{C_n^k (\zeta(\bar{a}))^k}{C_n^k} \right)^{\frac{1}{k}}}, s_{\left(\frac{C_n^k (\eta(\bar{a}))^k}{C_n^k} \right)^{\frac{1}{k}}}, s_{\left(\frac{C_n^k (\varphi(\bar{a}))^k}{C_n^k} \right)^{\frac{1}{k}}} \right], \right.$$

$$\left(\left(1 - \big((1 - (\mu(\bar{a}))^k)^{C_n^k} \big)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}}, \right.$$

$$\left. \left. 1 - \left(1 - \Big(\big(1 - (1 - \nu(\bar{a}))^k\big)^{C_n^k} \Big)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \right) \right\rangle$$

$$= \langle [s_{\sigma(\bar{a})}, s_{\zeta(\bar{a})}, s_{\eta(\bar{a})}, s_{\varphi(\bar{a})}], (\mu(\bar{a}), \nu(\bar{a})) \rangle = \bar{a}$$

2. The Commutativity of TIFLMSM$^{(K)}$ is easy to prove; therefore, the proof is omitted.
3. If $\sigma(\bar{a}_i) \le \sigma(\bar{b}_i), \zeta(\bar{a}_i) \le \zeta(\bar{b}_i), \eta(\bar{a}_i) \le \eta(\bar{b}_i), \varphi(\bar{a}_i) \le \varphi(\bar{b}_i), \mu(\bar{a}_i) \le \mu(\bar{b}_i)$ and $\nu(\bar{a}_i) \ge \nu(\bar{b}_i), \bar{a}_i = \bar{a} = \langle [s_{\sigma(\bar{a})}, s_{\zeta(\bar{a})}, s_{\eta(\bar{a})}, s_{\varphi(\bar{a})}], (\mu(\bar{a}), \nu(\bar{a})) \rangle$ and $\bar{b}_i = \bar{b} = \langle [s_{\sigma(\bar{b})}, s_{\zeta(\bar{b})}, s_{\eta(\bar{b})}, s_{\varphi(\bar{b})}], (\mu(\bar{b}), \nu(\bar{b})) \rangle$ for all i $(i = 1, 2, \ldots, n)$. According to Idempotency, $TIFLMSM^{(K)}(\bar{a}, \bar{a}, \ldots, \bar{a}) = \langle [s_{\sigma(\bar{a})}, s_{\zeta(\bar{a})}, s_{\eta(\bar{a})}, s_{\varphi(\bar{a})}], (\mu(\bar{a}), \nu(\bar{a})) \rangle$ and $TIFLMSM^{(K)}(\bar{b}, \bar{b}, \ldots, \bar{b}) = \langle [s_{\sigma(\bar{b})}, s_{\zeta(\bar{b})}, s_{\eta(\bar{b})}, s_{\varphi(\bar{b})}], (\mu(\bar{b}), \nu(\bar{b})) \rangle$.Therefore, we can derive the following result:

$$TIFLMSM^{(K)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) \le TIFLMSM^{(K)}(\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_n)$$

4. According to the idempotency, suppose that $min(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) = \bar{a}_{min} = TIFLMSM^{(K)}(\bar{a}_{min}, \bar{a}_{min}, \ldots, \bar{a}_{min})$, and $max(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) = \bar{a}_{max} = TIFLMSM^{(K)}(\bar{a}_{max}, \bar{a}_{max}, \ldots, \bar{a}_{max})$. According to the Monotonicity, if $\bar{a}_{min} \leq \bar{a}_i$ and $\bar{a}_{max} \geq \bar{a}_i (i = 1, 2, \ldots, n)$, then we can obtain $\bar{a}_{min} = TIFLMSM^{(K)}(\bar{a}_{min}, \bar{a}_{min}, \ldots, \bar{a}_{min}) \leq TIFLMSM^{(K)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n)$ and $TIFLMSM^{(K)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) \leq \bar{a}_{max} = TIFLMSM^{(K)}(\bar{a}_{max}, \bar{a}_{max}, \ldots, \bar{a}_{max})$. Therefore, we can obtain the conclusion as follows.

$$min(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) \leq TIFLMSM^{(K)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) \leq max(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n)$$

□

Next, some particular cases of $TIFLMSM^{(K)}$ operator with regard to parameter k are discussed.

(1) When k = 1, we can obtain the formula as follows.

$$TIFLMSM^{(1)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) = \frac{\bigoplus_{i=1}^{n} \bar{a}_i}{C_n^1} = \Big\langle \Big[s_{\frac{\sum_{i=1}^{n} \sigma(\bar{a}_i)}{C_n^1}}, s_{\frac{\sum_{i=1}^{n} \zeta(\bar{a}_i)}{C_n^1}}, s_{\frac{\sum_{i=1}^{n} \eta(\bar{a}_i)}{C_n^1}}, s_{\frac{\sum_{i=1}^{n} \varphi(\bar{a}_i)}{C_n^1}} \Big], \Big(\Big(1 - (\prod_{1 \leq i \leq n} (1 - \mu(\bar{a}_i)))^{\frac{1}{n}} \Big), \Big(\prod_{1 \leq i \leq n} (\nu(\bar{a}_i)) \Big)^{\frac{1}{n}} \Big) \Big\rangle \tag{27}$$

When k = 1, the TIFLMSM operator degenerates to the trapezoid intuitionistic fuzzy linguistic average (TIFLA) operator.

(2) When k = 2, we can obtain formula as follows.

$$\begin{aligned} TIFLMSM^{(2)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) &= \Big(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \bar{a}_i \bar{a}_j \Big)^{\frac{1}{2}} \\ &= \Big\langle \Big[s_{\left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \sigma(\bar{a}_i)\sigma(\bar{a}_j) \right)^{\frac{1}{2}}}, s_{\left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \zeta(\bar{a}_i)\zeta(\bar{a}_j) \right)^{\frac{1}{2}}}, \\ &\quad s_{\left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \eta(\bar{a}_i)\eta(\bar{a}_j) \right)^{\frac{1}{2}}}, s_{\left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \varphi(\bar{a}_i)\varphi(\bar{a}_j) \right)^{\frac{1}{2}}} \Big], \\ &\quad \Big(\Big(1 - (\prod_{\substack{i,j=1 \\ i \neq j}}^{n} (1 - \mu(\bar{a}_i)\mu(\bar{a}_j)))^{\frac{1}{n(n-1)}} \Big)^{\frac{1}{2}}, \\ &\quad 1 - \Big(1 - \Big(\prod_{\substack{i,j=1 \\ i \neq j}}^{n} \Big(1 - (1 - \nu(\bar{a}_i))(1 - \nu(\bar{a}_j)) \Big) \Big)^{\frac{1}{n(n-1)}} \Big)^{\frac{1}{2}} \Big) \Big\rangle \end{aligned} \tag{28}$$

When k = 2, the TIFLMSM operator degenerates to the trapezoid intuitionistic fuzzy linguistic Bonferroni mean (TIFLBM) operator (p = 1, q = 1).

(3) When $k = n$, we can obtain the formula as follows.

$$TIFLMSM^{(n)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) = \Big(\prod_{i=1}^{n} \bar{a}_i \Big)^{\frac{1}{n}} = \Big\langle \Big[s_{\left(\prod_{i=1}^{n} \sigma(\bar{a}_i) \right)^{\frac{1}{n}}}, s_{\left(\prod_{i=1}^{n} \zeta(\bar{a}_i) \right)^{\frac{1}{n}}}, s_{\left(\prod_{i=1}^{n} \eta(\bar{a}_i) \right)^{\frac{1}{n}}}, s_{\left(\prod_{i=1}^{n} \varphi(\bar{a}_i) \right)^{\frac{1}{n}}} \Big], \Big(\Big(\prod_{i=1}^{n} \mu(\bar{a}_i) \Big)^{\frac{1}{n}}, 1 - \Big(\prod_{i=1}^{n} \Big(1 - \nu(\bar{a}_i) \Big) \Big)^{\frac{1}{n}} \Big) \Big\rangle \tag{29}$$

When $k = n$, the TIFLMSM operator degenerates to the trapezoid intuitionistic fuzzy linguistic geometric (TIFLG) operator.

Definition 10. *Let* $\bar{a}_i = \langle [s_{\sigma(x_i)}, s_{\zeta(x_i)}, s_{\eta(x_i)}, s_{\varphi(x_i)}], (\mu(x_i), \nu(x_i)) \rangle (i = 1, 2, \ldots, n)$ *be a set of the TIFLNs, then the TILFMSM operator:* $\Omega^n \rightarrow \Omega$ *is given as follows.*

$$TILFGMSM^{(k,p_1,p_2,\ldots,p_k)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) = \left(\frac{\sum_{\substack{1 \le i_1 < \ldots \\ < i_k \le n}} \prod_{j=1}^{k} \bar{a}_{i_j}^{p_j}}{C_n^k} \right)^{\frac{1}{p_1+p_2+\cdots+p_k}}, \tag{30}$$

where Ω *is a collection of TIFLNs and* $k = 1, 2, \ldots, n$.

According to the operational laws of TIFLNs, we can obtain the following result of the TIFLGMSM operator below.

Theorem 3. *Let* $\bar{a}_i = \langle [s_{\sigma(a_i)}, s_{\zeta(a_i)}, s_{\eta(a_i)}, s_{\varphi(a_i)}], (\mu(a_i), \nu(a_i)) \rangle (i = 1, 2, \ldots, n)$ *be a set of the TIFLNs, then the aggregated result from Definition 11 is still a TIFLN.*

$$\begin{aligned} TIFLGMSM^{(k,p_1,p_2,\ldots,p_k)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) &= \left(\frac{\sum_{\substack{1 \le i_1 < \ldots \\ < i_k \le n}} \prod_{j=1}^{k} \bar{a}_{i_j}^{p_j}}{C_n^k} \right)^{\frac{1}{p_1+p_2+\cdots+p_k}} \\ = \Bigg\langle \Bigg[& s_{\left(\frac{\sum_{\substack{1 \le i_1 < \ldots \\ < i_k \le n}} \prod_{j=1}^{k} (\sigma(\bar{a}_{i_j}))^{p_j}}{C_n^k} \right)^{\frac{1}{p_1+p_2+\cdots+p_k}}}, s_{\left(\frac{\sum_{\substack{1 \le i_1 < \ldots \\ < i_k \le n}} \prod_{j=1}^{k} (\zeta(\bar{a}_{i_j}))^{p_j}}{C_n^k} \right)^{\frac{1}{p_1+p_2+\cdots+p_k}}}, \\ & s_{\left(\frac{\sum_{\substack{1 \le i_1 < \ldots \\ < i_k \le n}} \prod_{j=1}^{k} (\eta(\bar{a}_{i_j}))^{p_j}}{C_n^k} \right)^{\frac{1}{p_1+p_2+\cdots+p_k}}}, s_{\left(\frac{\sum_{\substack{1 \le i_1 < \ldots \\ < i_k \le n}} \prod_{j=1}^{k} (\varphi(\bar{a}_{i_j}))^{p_j}}{C_n^k} \right)^{\frac{1}{p_1+p_2+\cdots+p_k}}} \Bigg], \\ & \left(\left(1 - \left(\prod_{1 \le i_1 < \cdots < i_k \le n} \left(1 - \prod_{j=1}^{k} (\mu(\bar{a}_{i_j}))^{p_j} \right) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{p_1+p_2+\cdots+p_k}}, \right. \\ & \left. 1 - \left(1 - \left(\prod_{1 \le i_1 < \cdots < i_k \le n} \left(1 - \prod_{j=1}^{k} (1 - \nu(\bar{a}_{i_j}))^{p_j} \right) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{p_1+p_2+\cdots+p_k}} \right) \Bigg\rangle \end{aligned} \tag{31}$$

The proof with respect to the TIFLGMSM operator is similar with Theorem 2, so it is omitted here.

Property 2. *Let* $\bar{a}_i = \langle [s_{\sigma(\bar{a}_i)}, s_{\zeta(\bar{a}_i)}, s_{\eta(\bar{a}_i)}, s_{\varphi(\bar{a}_i)}], (\mu(\bar{a}_i), \nu(\bar{a}_i)) \rangle (i = 1, 2, \ldots, n)$ *and* $\bar{b}_i = \langle [s_{\sigma(\bar{b}_i)}, s_{\zeta(\bar{b}_i)}, s_{\eta(\bar{b}_i)}, s_{\varphi(\bar{b}_i)}], (\mu(\bar{b}_2), \nu(\bar{b}_i)) \rangle (i = 1, 2, \ldots, n)$ *be collections of TIFLNs, then the following properties of* $TIFLGMSM^{(k,p_1,p_2,\ldots,p_k)}$ *are shown.*

(1) *Idempotency. If* $\bar{a}_i = \bar{a} = \langle [s_{\sigma(\bar{a})}, s_{\zeta(\bar{a})}, s_{\eta(\bar{a})}, s_{\varphi(\bar{a})}], (\mu(\bar{a}), \nu(\bar{a})) \rangle$ *for each i, then* $TIFLGMSM^{(k,p_1,p_2,\ldots,p_k)}(\bar{a}, \bar{a}, \ldots, \bar{a}) = a = \langle [s_{\sigma(\bar{a})}, s_{\zeta(\bar{a})}, s_{\eta(\bar{a})}, s_{\varphi(\bar{a})}], (\mu(\bar{a}), \nu(\bar{a})) \rangle$.

(2) *Commutativity. If* $\bar{a}_i$ *is a permutation of* $\bar{b}_i$ *for each i, then* $TIFLGMSM^{(k,p_1,p_2,\ldots,p_k)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) = TIFLGMSM^{(k,p_1,p_2,\ldots,p_k)}(\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_n)$.

(3) *Monotonicity. If* $\sigma(\bar{a}_i) \le \sigma(\bar{b}_i), \zeta(\bar{a}_i) \le \zeta(\bar{b}_i), \eta(\bar{a}_i) \le \eta(\bar{b}_i), \varphi(\bar{a}_i) \le \varphi(\bar{b}_i), \mu(\bar{a}_i) \le \mu(\bar{b}_i)$ *and* $\nu(\bar{a}_i) \ge \nu(\bar{b}_i)$ *for all i* $(i = 1, 2, \ldots, n)$, *then* $\bar{a}_i \le \bar{b}_i$ *and* $TIFLGMSM^{(k,p_1,p_2,\ldots,p_k)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) \le TIFLGMSM^{(k,p_1,p_2,\ldots,p_k)}(\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_n)$.

(4) *Boundedness.* $min(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) \le TIFLGMSM^{(k,p_1,p_2,\ldots,p_k)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) \le max(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n)$.

The proof of the Property 2 with respect to the TIFLGMSM operator is similar to Property 1, so it is omitted here.

Next, some particular cases of the $TIFLGMSM^{(k,p_1,p_2,\ldots,p_k)}$ operator with regard to parameter k are discussed.

(1) When $k = 1$, we can obtain formula as follows.

$$
\begin{aligned}
&TIFLGMSM^{(1,p_1)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) = \Big(\frac{\bigoplus_{i=1}^{n} \bar{a}_i}{C_n^1}\Big)^{\frac{1}{p_1}} \\
&= \Big\langle \Big[s_{(\frac{\sum_{i=1}^{n}\sigma(\bar{a}_i)}{C_n^1})^{\frac{1}{p_1}}}, s_{(\frac{\sum_{i=1}^{n}\zeta(\bar{a}_i)}{C_n^1})^{\frac{1}{p_1}}}, s_{(\frac{\sum_{i=1}^{n}\eta(\bar{a}_i)}{C_n^1})^{\frac{1}{p_1}}}, s_{(\frac{\sum_{i=1}^{n}\varphi(\bar{a}_i)}{C_n^1})^{\frac{1}{p_1}}} \Big], \\
&\Big(\Big(1-(\prod_{1\le i\le n}(1-(\mu(\bar{a}_i))^{p_1}))^{\frac{1}{n}}\Big)^{\frac{1}{p_1}}, 1-\Big(1-(\prod_{1\le i\le n}\big(1-(1-\nu(\bar{a}_i))^{p_1}\big))^{\frac{1}{n}}\Big)^{\frac{1}{p_1}}\Big)\Big\rangle
\end{aligned} \tag{32}
$$

(2) When k = 2, we can obtain formula as follows.

$$
\begin{aligned}
&TIFLGMSM^{(2,p_1,p_2)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) = \Big(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \bar{a}_i^{p_1}\bar{a}_j^{p_2}\Big)^{\frac{1}{p_1+p_2}} \\
&= \Big\langle \Big[s_{\Big(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}(\sigma(\bar{a}_i))^{p_1}(\sigma(\bar{a}_j))^{p_2}\Big)^{\frac{1}{p_1+p_2}}}, s_{\Big(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}(\zeta(\bar{a}_i))^{p_1}(\zeta(\bar{a}_j))^{p_2}\Big)^{\frac{1}{p_1+p_2}}}, \\
&s_{\Big(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}(\eta(\bar{a}_i))^{p_1}(\eta(\bar{a}_j))^{p_2}\Big)^{\frac{1}{p_1+p_2}}}, s_{\Big(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}(\varphi(\bar{a}_i))^{p_1}(\varphi(\bar{a}_j))^{p_2}\Big)^{\frac{1}{p_1+p_2}}} \Big], \\
&\Big(\Big(1-(\prod_{\substack{i,j=1\\ i\neq j}}^{n}(1-(\mu(\bar{a}_i))^{p_1}(\mu(\bar{a}_j))^{p_2}))^{\frac{1}{n(n-1)}}\Big)^{\frac{1}{p_1+p_2}}, \\
&1-\Big(1-\Big(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\Big(1-(1-\nu(\bar{a}_i))^{p_1}(1-\nu(\bar{a}_j))^{p_2}\Big)\Big)^{\frac{1}{n(n-1)}}\Big)^{\frac{1}{p_1+p_2}}\Big)\Big\rangle
\end{aligned} \tag{33}
$$

When k = 2, the TIFLGMSM operator degenerates to the trapezoid intuitionistic fuzzy linguistic Bonferroni mean (TIFLBM) operator with arguments p_1, p_2.

(3) When $k = n$, we can obtain formula as follows.

$$
\begin{aligned}
&TIFLGMSM^{(n,p_1,p_2,\ldots,p_n)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) = \Big(\prod_{i=1}^{n}(\bar{a}_i)^{p_i}\Big)^{\frac{1}{p_1+p_2+\cdots+p_n}} \\
&= \Big\langle \Big[s_{\Big(\prod_{i=1}^{n}(\sigma(\bar{a}_i))^{p_i}\Big)^{\frac{1}{p_1+p_2+\cdots+p_n}}}, s_{\Big(\prod_{i=1}^{n}(\zeta(\bar{a}_i))^{p_i}\Big)^{\frac{1}{p_1+p_2+\cdots+p_n}}}, \\
&s_{\Big(\prod_{i=1}^{n}(\eta(\bar{a}_i))^{p_i}\Big)^{\frac{1}{p_1+p_2+\cdots+p_n}}}, s_{\Big(\prod_{i=1}^{n}(\varphi(\bar{a}_i))^{p_i}\Big)^{\frac{1}{p_1+p_2+\cdots+p_n}}} \Big], \\
&\Big(\Big(\prod_{i=1}^{n}(\mu(\bar{a}_i))^{p_i}\Big)^{\frac{1}{p_1+p_2+\cdots+p_n}}, 1-\Big(\prod_{i=1}^{n}\Big(1-\nu(\bar{a}_i)\Big)^{p_i}\Big)^{\frac{1}{p_1+p_2+\cdots+p_n}}\Big)\Big\rangle
\end{aligned} \tag{34}
$$

3.2. The TIFLWMSM and TIFLWGMSM Operators

Definition 11. *Let* $\bar{a}_i = \langle [s_{\sigma(x_i)}, s_{\zeta(x_i)}, s_{\eta(x_i)}, s_{\varphi(x_i)}], (\mu(x_i), \nu(x_i))\rangle (i = 1, 2, \ldots, n)$ *be a set of the TIFLNs, then the TIFLWMSM operator:* $\Omega^n \to \Omega$ *is given as follows.*

$$
TIFLWMSM^{(k)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) = \Big(\frac{\sum_{\substack{1\le i_1<\ldots\\ <i_k\le n}} \prod_{j=1}^{k}(nw_{i_j}\bar{a}_{i_j})}{C_n^k}\Big)^{\frac{1}{k}}, \tag{35}
$$

where Ω *is a collection of TIFLNs and* $k = 1, 2, \ldots, n$.

According to the calculation laws of TIFLNs, we can obtain the following result of the TIFLWMSM operator as follows.

Theorem 4. *Let* $\bar{a}_i = \langle [s_{\sigma(a_i)}, s_{\zeta(a_i)}, s_{\eta(a_i)}, s_{\varphi(a_i)}], (\mu(a_i), \nu(a_i)) \rangle (i = 1, 2, \ldots, n)$ *be a set of the TIFLNs, then the aggregated result from Definition 12 is still a TIFLN.*

$$\begin{aligned} TIFLWMSM^{(k)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) &= \left(\frac{\sum_{\substack{1 \le i_1 < \ldots \\ < i_k \le n}} \prod_{j=1}^{k} (nw_{i_j} \bar{a}_{i_j})}{C_n^k} \right)^{\frac{1}{k}} \\ &= \Bigg\langle \Bigg[s_{\left(\frac{\sum_{\substack{1 \le i_1 < \ldots \\ < i_k \le n}} \prod_{j=1}^{k} (nw_{i_j} \sigma(\bar{a}_{i_j}))}{C_n^k} \right)^{\frac{1}{k}}}, s_{\left(\frac{\sum_{\substack{1 \le i_1 < \ldots \\ < i_k \le n}} \prod_{j=1}^{k} (nw_{i_j} \zeta(\bar{a}_{i_j}))}{C_n^k} \right)^{\frac{1}{k}}}, \\ & s_{\left(\frac{\sum_{\substack{1 \le i_1 < \ldots \\ < i_k \le n}} \prod_{j=1}^{k} (nw_{i_j} \eta(\bar{a}_{i_j}))}{C_n^k} \right)^{\frac{1}{k}}}, s_{\left(\frac{\sum_{\substack{1 \le i_1 < \ldots \\ < i_k \le n}} \prod_{j=1}^{k} (nw_{i_j} \varphi(\bar{a}_{i_j}))}{C_n^k} \right)^{\frac{1}{k}}} \Bigg], \\ & \Bigg(\Bigg(1 - \Bigg(\prod_{1 \le i_1 < \cdots < i_k \le n} \Bigg(1 - \prod_{j=1}^{k} \left(1 - \left(1 - \mu(\bar{a}_{i_j}) \right)^{nw_{i_j}} \right) \Bigg) \Bigg)^{\frac{1}{C_n^k}} \Bigg)^{\frac{1}{k}}, \\ & 1 - \Bigg(1 - \Bigg(\prod_{1 \le i_1 < \cdots < i_k \le n} \Bigg(1 - \prod_{j=1}^{k} \left(1 - \left(\nu(\bar{a}_{i_j}) \right)^{nw_{i_j}} \right) \Bigg) \Bigg)^{\frac{1}{C_n^k}} \Bigg)^{\frac{1}{k}} \Bigg) \Bigg\rangle \end{aligned} \tag{36}$$

The proof with respect to the TIFLWMSM operator is similar to Theorem 2, so it is omitted here.

Property 3. *Let* $\bar{a}_i = \langle [s_{\sigma(\bar{a}_i)}, s_{\zeta(\bar{a}_i)}, s_{\eta(\bar{a}_i)}, s_{\varphi(\bar{a}_i)}], (\mu(\bar{a}_i), \nu(\bar{a}_i)) \rangle (i = 1, 2, \ldots, n)$ *and* $\bar{b}_i = \langle [s_{\sigma(\bar{b}_i)}, s_{\zeta(\bar{b}_i)}, s_{\eta(\bar{b}_i)}, s_{\varphi(\bar{b}_i)}], (\mu(\bar{b}_2), \nu(\bar{b}_i)) \rangle (i = 1, 2, \ldots, n)$ *be collections of TIFLNs, then the following properties of* $\text{TIFLWMSM}^{(k)}$ *are shown.*

(1) Reducibility. When $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$, *then* $TIFLWMSM^{(k)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) = TIFLMSM^{(k)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n)$

(2) Monotonicity. If $\sigma(\bar{a}_i) \le \sigma(\bar{b}_i), \zeta(\bar{a}_i) \le \zeta(\bar{b}_i), \eta(\bar{a}_i) \le \eta(\bar{b}_i), \varphi(\bar{a}_i) \le \varphi(\bar{b}_i), \mu(\bar{a}_i) \le \mu(\bar{b}_i)$ *and* $\nu(\bar{a}_i) \ge \nu(\bar{b}_i)$ *for all* i $(i = 1, 2, \ldots, n)$, *then* $\bar{a}_i \le \bar{b}_i$ *and* $TIFLWMSM^{(k)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) \le TIFLWMSM^{(k)}(\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_n)$.

(3) Boundedness. $min(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) \le TIFLWMSM^{(k)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) \le max(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n)$.

Proof. Let $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$, then

$$\begin{aligned} TIFLWMSM^{(k)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) &= \left(\frac{\sum_{\substack{1 \le i_1 < \ldots \\ < i_k \le n}} \prod_{j=1}^{k} (\bar{a}_{i_j})}{C_n^k} \right)^{\frac{1}{k}} \\ &= \Bigg\langle \Bigg[s_{\left(\frac{\sum_{\substack{1 \le i_1 < \ldots \\ < i_k \le n}} \prod_{j=1}^{k} (\sigma(\bar{a}_{i_j}))}{C_n^k} \right)^{\frac{1}{k}}}, s_{\left(\frac{\sum_{\substack{1 \le i_1 < \ldots \\ < i_k \le n}} \prod_{j=1}^{k} (\zeta(\bar{a}_{i_j}))}{C_n^k} \right)^{\frac{1}{k}}}, \\ & s_{\left(\frac{\sum_{\substack{1 \le i_1 < \ldots \\ < i_k \le n}} \prod_{j=1}^{k} (\eta(\bar{a}_{i_j}))}{C_n^k} \right)^{\frac{1}{k}}}, s_{\left(\frac{\sum_{\substack{1 \le i_1 < \ldots \\ < i_k \le n}} \prod_{j=1}^{k} (\varphi(\bar{a}_{i_j}))}{C_n^k} \right)^{\frac{1}{k}}} \Bigg], \\ & \Bigg(\Bigg(1 - \Bigg(\prod_{1 \le i_1 < \cdots < i_k \le n} \Bigg(1 - \prod_{j=1}^{k} \left(\mu(\bar{a}_{i_j}) \right) \Bigg) \Bigg)^{\frac{1}{C_n^k}} \Bigg)^{\frac{1}{k}}, \\ & 1 - \Bigg(1 - \Bigg(\prod_{1 \le i_1 < \cdots < i_k \le n} \Bigg(1 - \prod_{j=1}^{k} \left(1 - \nu(\bar{a}_{i_j}) \right) \Bigg) \Bigg)^{\frac{1}{C_n^k}} \Bigg)^{\frac{1}{k}} \Bigg) \Bigg\rangle \\ &= TIFLMSM^{(k)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) \end{aligned}$$

□

The proof of Monotonicity and Boundedness for the TIFLWMSM operator is similar to Property 1, so the proof is omitted here.

Next, some particular cases of the $TIFLWMSM^{(k)}$ operator with regard to parameter k are discussed.

(1) When $k = 1$, we can obtain formula as follows.

$$TIFLWMSM^{(1)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) = \Big(\bigoplus_{i=1}^{n} w_i \bar{a}_i\Big)$$
$$= \Big\langle \Big[s_{(\sum_{i=1}^{n} w_i \sigma(\bar{a}_i))}, s_{(\sum_{i=1}^{n} w_i \zeta(\bar{a}_i))}, s_{(\sum_{i=1}^{n} w_i \eta(\bar{a}_i))}, s_{(\sum_{i=1}^{n} w_i \varphi(\bar{a}_i))} \Big], \Big(\Big(1 - \prod_{1 \le i \le n} (1 - \mu(\bar{a}_i))^{w_i}\Big), \prod_{1 \le i \le n} \Big(\nu(\bar{a}_i)\Big)^{w_i} \Big) \Big\rangle \tag{37}$$

(2) When $k = 2$, we can obtain formula as follows.

$$TIFLWMSM^{(2)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) = \Bigg(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} (nw_i \bar{a}_i) \otimes (nw_j \bar{a}_j)\Bigg)^{\frac{1}{2}}$$
$$= \Bigg\langle \Bigg[s_{\Big(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} (nw_i \sigma(\bar{a}_i)) \otimes (nw_j \sigma(\bar{a}_j))\Big)^{\frac{1}{2}}}, s_{\Big(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} (nw_i \zeta(\bar{a}_i)) \otimes (nw_j \zeta(\bar{a}_j))\Big)^{\frac{1}{2}}},$$
$$s_{\Big(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} (nw_i \eta(\bar{a}_i)) \otimes (nw_j \eta(\bar{a}_j))\Big)^{\frac{1}{2}}}, s_{\Big(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} (nw_i \varphi(\bar{a}_i)) \otimes (nw_j \varphi(\bar{a}_j))\Big)^{\frac{1}{2}}} \Bigg],$$
$$\Bigg(\Bigg(1 - \Big(\prod_{\substack{i,j=1 \\ i \neq j}}^{n} 1 - \big(1 - (1 - \mu(\bar{a}_i))^{w_i}\big) \otimes \big(1 - (1 - \mu(\bar{a}_j))^{w_j}\big) \Big)^{\frac{1}{n(n-1)}} \Bigg)^{\frac{1}{2}},$$
$$1 - \Bigg(1 - \Big(\prod_{\substack{i,j=1 \\ i \neq j}}^{n} \Big(1 - (1 - (\nu(\bar{a}_i))^{w_i}) \otimes (1 - (\nu(\bar{a}_j))^{w_j})\Big) \Big)^{\frac{1}{n(n-1)}} \Bigg)^{\frac{1}{2}} \Bigg) \Bigg\rangle \tag{38}$$

When $k = 2$, the TIFLWMSM operator degenerates to the trapezoid intuitionistic fuzzy linguistic weighted Bonferroni mean (TIFLWBM) operator ($p = 1$, $q = 1$).

(3) When $k = n$, we can obtain formula as follows.

$$TIFLWMSM^{(n)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) = \Big(\prod_{i=1}^{n} (nw_i \bar{a}_i)\Big)^{\frac{1}{n}}$$
$$= \Big\langle \Big[s_{\big(\prod_{i=1}^{n} (nw_i \sigma(\bar{a}_i))\big)^{\frac{1}{n}}}, s_{\big(\prod_{i=1}^{n} (nw_i \zeta(\bar{a}_i))\big)^{\frac{1}{n}}}, s_{\big(\prod_{i=1}^{n} (nw_i \eta(\bar{a}_i))\big)^{\frac{1}{n}}}, s_{\big(\prod_{i=1}^{n} (nw_i \varphi(\bar{a}_i))\big)^{\frac{1}{n}}} \Big],$$
$$\Big(\Big(\prod_{i=1}^{n} (1 - (1 - \mu(\bar{a}_i))^{nw_i}) \Big)^{\frac{1}{n}}, 1 - \Big(\prod_{i=1}^{n} \Big(1 - (\nu(\bar{a}_i))^{nw_i}\Big) \Big)^{\frac{1}{n}} \Big) \Big\rangle \tag{39}$$

Definition 12. *Let $\bar{a}_i = \langle [s_{\sigma(x_i)}, s_{\zeta(x_i)}, s_{\eta(x_i)}, s_{\varphi(x_i)}], (\mu(x_i), \nu(x_i)) \rangle (i = 1, 2, \ldots, n)$ be a set of the TIFLNs, then the TIFLGWMSM operator: $\Omega^n \to \Omega$ is given as follows.*

$$TIFLGWMSM^{(k, p_1, p_2, \ldots, p_k)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) = \left(\frac{\sum_{\substack{1 \le i_1 < \ldots \\ < i_k \le n}} \prod_{j=1}^{k} (nw_{i_j} \bar{a}_{i_j})^{p_j}}{C_n^k} \right)^{\frac{1}{p_1 + p_2 + \cdots + p_k}}, \tag{40}$$

where Ω is a collection of TIFLNs and $k = 1, 2, \ldots, n$.

According to the calculation laws of TIFLNs, we can obtain the following result of TIFLGWMSM operator as follows.

Theorem 5. *Let $\bar{a}_i = \langle [s_{\sigma(a_i)}, s_{\zeta(a_i)}, s_{\eta(a_i)}, s_{\varphi(a_i)}], (\mu(a_i), \nu(a_i)) \rangle (i = 1, 2, \ldots, n)$ be a set of the TIFLNs, then the aggregated result from Definition 12 is still a TIFLN.*

$$TIFLGWMSM^{(k,p_1,p_2,\ldots,p_k)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) = \left(\frac{\sum_{\substack{1 \le i_1 < \ldots \\ < i_k \le n}} \prod_{j=1}^{k} (nw_{i_j} \bar{a}_{i_j})^{p_j}}{C_n^k} \right)^{\frac{1}{p_1+p_2+\cdots+p_k}}$$
$$= \Bigg\langle \Bigg[s_{\left(\frac{\sum_{\substack{1 \le i_1 < \ldots \\ < i_k \le n}} \prod_{j=1}^{k} (nw_{i_j} \sigma(\bar{a}_{i_j}))^{p_j}}{C_n^k} \right)^{\frac{1}{p_1+p_2+\cdots+p_k}}}, s_{\left(\frac{\sum_{\substack{1 \le i_1 < \ldots \\ < i_k \le n}} \prod_{j=1}^{k} (nw_{i_j} \zeta(\bar{a}_{i_j}))^{p_j}}{C_n^k} \right)^{\frac{1}{p_1+p_2+\cdots+p_k}}},$$
$$s_{\left(\frac{\sum_{\substack{1 \le i_1 < \ldots \\ < i_k \le n}} \prod_{j=1}^{k} (nw_{i_j} \eta(\bar{a}_{i_j}))^{p_j}}{C_n^k} \right)^{\frac{1}{p_1+p_2+\cdots+p_k}}}, s_{\left(\frac{\sum_{\substack{1 \le i_1 < \ldots \\ < i_k \le n}} \prod_{j=1}^{k} (nw_{i_j} \varphi(\bar{a}_{i_j}))^{p_j}}{C_n^k} \right)^{\frac{1}{p_1+p_2+\cdots+p_k}}} \Bigg], \tag{41}$$
$$\left(\left(1 - \left(\prod_{1 \le i_1 < \cdots < i_k \le n} \left(1 - \prod_{j=1}^{k} \left(1 - \left(1 - \mu(\bar{a}_{i_j}) \right)^{nw_{i_j}} \right)^{p_j} \right) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{p_1+p_2+\cdots+p_k}},$$
$$1 - \left(1 - \left(\prod_{1 \le i_1 < \cdots < i_k \le n} \left(1 - \prod_{j=1}^{k} \left(1 - \left(\nu(\bar{a}_{i_j}) \right)^{nw_{i_j}} \right)^{p_j} \right) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{p_1+p_2+\cdots+p_k}} \right) \Bigg\rangle$$

The proof with respect to the TIFLGWMSM operator is similar to Theorem 2, so it is omitted here.

Property 4. *Let $\bar{a}_i = \langle [s_{\sigma(\bar{a}_i)}, s_{\zeta(\bar{a}_i)}, s_{\eta(\bar{a}_i)}, s_{\varphi(\bar{a}_i)}], (\mu(\bar{a}_i), \nu(\bar{a}_i)) \rangle (i = 1, 2, \ldots, n)$ and $\bar{b}_i = \langle [s_{\sigma(\bar{b}_i)}, s_{\zeta(\bar{b}_i)}, s_{\eta(\bar{b}_i)}, s_{\varphi(\bar{b}_i)}], (\mu(\bar{b}_2), \nu(\bar{b}_i)) \rangle (i = 1, 2, \ldots, n)$ be collections of TIFLNs, the following properties of $TIFLGWMSM^{(k,p_1,p_2,\ldots,p_k)}$ are shown.*

(1) *Reducibility. When $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$, then $TIFLGWMSM^{(k,p_1,p_2,\ldots,p_k)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) = TIFLGMSM^{(k,p_1,p_2,\ldots,p_k)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n)$*

(2) *Monotonicity. If $\sigma(\bar{a}_i) \le \sigma(\bar{b}_i), \zeta(\bar{a}_i) \le \zeta(\bar{b}_i), \eta(\bar{a}_i) \le \eta(\bar{b}_i), \varphi(\bar{a}_i) \le \varphi(\bar{b}_i), \mu(\bar{a}_i) \le \mu(\bar{b}_i)$ and $\nu(\bar{a}_i) \ge \nu(\bar{b}_i)$ for all i $(i = 1, 2, \ldots, n)$, then $\bar{a}_i \le \bar{b}_i$ and $TIFLGWMSM^{(k,p_1,p_2,\ldots,p_k)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) \le TIFLGWMSM^{(k,p_1,p_2,\ldots,p_k)}(\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_n)$.*

(3) *Boundedness. $min(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) \le TIFLGWMSM^{(k,p_1,p_2,\ldots,p_k)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) \le max(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n)$.*

The proof of property for the TIFLGWMSM operator is similar to Property 3; therefore, it is ommitted here.

Next, some particular cases of the $TIFLGWMSM^{(k,p_1,p_2,\ldots,p_k)}$ operator with regard to parameter k are discussed.

(1) When $k = 1$, we can obtain formula as follows.

$$TIFLGWMSM^{(1,p_1)}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) = \left(\frac{\bigoplus_{i=1}^{n} (nw_i \bar{a}_i)^{p_1}}{C_n^1} \right)^{\frac{1}{p_1}}$$
$$= \Bigg\langle \Bigg[s_{\left(\frac{\sum_{i=1}^{n} (nw_i \sigma(\bar{a}_i))^{p_1}}{C_n^1} \right)^{\frac{1}{p_1}}}, s_{\left(\frac{\sum_{i=1}^{n} (nw_i \zeta(\bar{a}_i))^{p_1}}{C_n^1} \right)^{\frac{1}{p_1}}}, s_{\left(\frac{\sum_{i=1}^{n} (nw_i \eta(\bar{a}_i))^{p_1}}{C_n^1} \right)^{\frac{1}{p_1}}},$$
$$s_{\left(\frac{\sum_{i=1}^{n} (nw_i \varphi(\bar{a}_i))^{p_1}}{C_n^1} \right)^{\frac{1}{p_1}}} \Bigg], \left(\left(1 - \left(\prod_{1 \le i \le n} \left(1 - \left(1 - \left(1 - \mu(\bar{a}_i) \right)^{nw_i} \right)^{p_1} \right) \right)^{\frac{1}{n}} \right)^{\frac{1}{p_1}}, \tag{42}$$
$$1 - \left(1 - \left(\prod_{1 \le i \le n} \left(1 - \left(1 - \left(\nu(\bar{a}_i) \right)^{nw_i} \right)^{p_1} \right) \right)^{\frac{1}{n}} \right)^{\frac{1}{p_1}} \right) \Bigg\rangle$$

(2) When $k = 2$, we can obtain formula as follows.

$$TIFLGWMSM^{(2,p_1,p_2)}(\bar{a}_1,\bar{a}_2,\ldots,\bar{a}_n)=\Big(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}(nw_i\bar{a}_i)^{p_i}\otimes(nw_j\bar{a}_j)^{p_j}\Big)^{\frac{1}{p_1+p_2}}$$

$$=\Big\langle\Big[s_{\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}(nw_i\sigma(\bar{a}_i))^{p_i}\otimes(nw_j\sigma(\bar{a}_j))^{p_j}\right)^{\frac{1}{p_1+p_2}}},s_{\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}(nw_i\zeta(\bar{a}_i))^{p_i}\otimes(nw_j\zeta(\bar{a}_j))^{p_j}\right)^{\frac{1}{p_1+p_2}}},$$
$$s_{\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}(nw_i\eta(\bar{a}_i))^{p_i}\otimes(nw_j\eta(\bar{a}_j))^{p_j}\right)^{\frac{1}{p_1+p_2}}},s_{\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}(nw_i\varphi(\bar{a}_i))^{p_i}\otimes(nw_j\varphi(\bar{a}_j))^{p_j}\right)^{\frac{1}{p_1+p_2}}}\Big],\tag{43}$$
$$\Big(\Big(1-\big(\prod_{\substack{i,j=1\\ i\neq j}}^{n}1-\big(1-(1-\mu(\bar{a}_i))^{nw_i}\big)^{p_i}\otimes\big(1-(1-\mu(\bar{a}_j))^{nw_j}\big)^{p_j}\big)^{\frac{1}{n(n-1)}}\Big)^{\frac{1}{p_1+p_2}},$$
$$1-\Big(1-\Big(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\Big(1-(1-(\nu(\bar{a}_i))^{nw_i})^{p_i}\otimes(1-(\nu(\bar{a}_j))^{nw_j})^{p_j}\Big)\Big)^{\frac{1}{n(n-1)}}\Big)^{\frac{1}{p_1+p_2}}\Big)\Big\rangle$$

(3) When $k = n$, we can obtain formula as follows.

$$TIFLGWMSM^{(n,p_1,p_2,\ldots,p_n)}(\bar{a}_1,\bar{a}_2,\ldots,\bar{a}_n)=\Big(\prod_{i=1}^{n}(nw_i\bar{a}_i)^{p_i}\Big)^{\frac{1}{p_1+p_2+\cdots+p_n}}$$
$$=\Big\langle\Big[s_{\left(\prod_{i=1}^{n}(nw_i\sigma(\bar{a}_i))^{p_i}\right)^{\frac{1}{p_1+p_2+\cdots+p_n}}},s_{\left(\prod_{i=1}^{n}(nw_i\zeta(\bar{a}_i))^{p_i}\right)^{\frac{1}{p_1+p_2+\cdots+p_n}}},$$
$$s_{\left(\prod_{i=1}^{n}(nw_i\eta(\bar{a}_i))^{p_i}\right)^{\frac{1}{p_1+p_2+\cdots+p_n}}},s_{\left(\prod_{i=1}^{n}(nw_i\varphi(\bar{a}_i))^{p_i}\right)^{\frac{1}{p_1+p_2+\cdots+p_n}}}\Big],\tag{44}$$
$$\Big(\Big(\prod_{i=1}^{n}(1-(1-\mu(\bar{a}_i))^{nw_i})^{p_i}\Big)^{\frac{1}{p_1+p_2+\cdots+p_n}},$$
$$1-\Big(\prod_{i=1}^{n}\Big(1-\big(\nu(\bar{a}_i)\big)^{nw_i}\Big)^{p_i}\Big)^{\frac{1}{p_1+p_2+\cdots+p_n}}\Big)\Big\rangle$$

4. MAGDM Method Based on the TIFLWMSM Operator and TIFLWGMSM Operator

For a MAGDM issue under trapezoid intuitionistic fuzzy linguistic environment: Suppose $A=\{A_1,A_2,\ldots,A_m\}$ is a collection of alternatives; $C=\{C_1,C_2,\ldots,C_n\}$ is the set of attributes; and $W=\{w_1,w_2,\ldots,w_n\}$ is the weight vector of the attributes $C_j(j=1,2,\ldots,n)$, where $w_j\geq 0(j=1,2,\ldots,n),\sum_{j=1}^{n}w_j=1$. Let $D=\{D_1,D_2,\ldots,D_d\}$ be the set of decision makers, and $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_d)^T$ be the weight vector of decision makers $D_e(e=1,2,\ldots,d)$, where $\lambda_e\geq 0,\sum_{e=1}^{d}\lambda_e=1$. Suppose $A^e=[\bar{a}_{ij}^e]_{m\times n}$ are the decision matrices, where $\bar{a}_{ij}^e=\langle[s_{\sigma(\bar{a}_i)},s_{\zeta(\bar{a}_i)},s_{\eta(\bar{a}_i)},s_{\varphi(\bar{a}_i)}],(\mu(\bar{a}_i),\nu(\bar{a}_i))\rangle$ takes the form of the TIFLVs given by the decision maker D_e for an alternative A_i with respect to attribute C_j. Finally, we will obtain the ranking of alternatives.

The steps are given below:

Step 1. Utilize the following TIFLWMSM operator or the TIFLWGMSM operator to aggregate all decision matrices $A^e(e=1,2,3,4)$ into one decision matrix M.

$$\bar{a}_{ij} = TIFLWMSM(\bar{a}_{ij}^1, \bar{a}_{ij}^2, \ldots, \bar{a}_{ij}^d) = \left(\frac{\sum_{1\le i_1<\cdots<i_k\le d}\prod_{j=1}^k (d\lambda_{i_j}\bar{a}_{i_j})}{C_d^k}\right)^{\frac{1}{k}}$$
$$= \Big\langle \Big[s_{\left(\frac{\sum_{1\le i_1<\cdots<i_k\le d}\prod_{j=1}^k (d\lambda_{i_j}\sigma(\bar{a}_{i_j}))}{C_d^k}\right)^{\frac{1}{k}}}, s_{\left(\frac{\sum_{1\le i_1<\cdots<i_k\le d}\prod_{j=1}^k (d\lambda_{i_j}\zeta(\bar{a}_{i_j}))}{C_d^k}\right)^{\frac{1}{k}}},$$
$$s_{\left(\frac{\sum_{1\le i_1<\cdots<i_k\le d}\prod_{j=1}^k (d\lambda_{i_j}\eta(\bar{a}_{i_j}))}{C_d^k}\right)^{\frac{1}{k}}}, s_{\left(\frac{\sum_{1\le i_1<\cdots<i_k\le d}\prod_{j=1}^k (d\lambda_{i_j}\varphi(\bar{a}_{i_j}))}{C_d^k}\right)^{\frac{1}{k}}} \Big], \tag{45}$$
$$\left(\left(1-\left(\prod_{1\le i_1<\cdots<i_k\le d}\left(1-\prod_{j=1}^k\left(1-\left(1-\mu(\bar{a}_{i_j})\right)^{d\lambda_{i_j}}\right)\right)\right)^{\frac{1}{C_d^k}}\right)^{\frac{1}{k}},\right.$$
$$\left.1-\left(1-\left(\prod_{1\le i_1<\cdots<i_k\le d}\left(1-\prod_{j=1}^k\left(1-\left(\nu(\bar{a}_{i_j})\right)^{d\lambda_{i_j}}\right)\right)\right)^{\frac{1}{C_d^k}}\right)^{\frac{1}{k}}\right)\Big\rangle$$

or

$$\bar{a}_{ij} = TIFLGWMSM(\bar{a}_{ij}^1, \bar{a}_{ij}^2, \ldots, \bar{a}_{ij}^d) = \left(\frac{\sum_{1\le i_1<\cdots<i_k\le d}\prod_{j=1}^k (d\lambda_{i_j}\bar{a}_{i_j})^{p_j}}{C_d^k}\right)^{\frac{1}{p_1+p_2+\cdots+p_k}}$$
$$= \Big\langle \Big[s_{\left(\frac{\sum_{1\le i_1<\cdots<i_k\le d}\prod_{j=1}^k (d\lambda_{i_j}\sigma(\bar{a}_{i_j}))^{p_j}}{C_d^k}\right)^{\frac{1}{p_1+p_2+\cdots+p_k}}}, s_{\left(\frac{\sum_{1\le i_1<\cdots<i_k\le d}\prod_{j=1}^k (d\lambda_{i_j}\zeta(\bar{a}_{i_j}))^{p_j}}{C_d^k}\right)^{\frac{1}{p_1+p_2+\cdots+p_k}}},$$
$$s_{\left(\frac{\sum_{1\le i_1<\cdots<i_k\le d}\prod_{j=1}^k (d\lambda_{i_j}\eta(\bar{a}_{i_j}))^{p_j}}{C_d^k}\right)^{\frac{1}{p_1+p_2+\cdots+p_k}}}, s_{\left(\frac{\sum_{1\le i_1<\cdots<i_k\le d}\prod_{j=1}^k (d\lambda_{i_j}\varphi(\bar{a}_{i_j}))^{p_j}}{C_d^k}\right)^{\frac{1}{p_1+p_2+\cdots+p_k}}} \Big], \tag{46}$$
$$\left(\left(1-\left(\prod_{1\le i_1<\cdots<i_k\le d}\left(1-\prod_{j=1}^k\left(1-\left(1-\mu(\bar{a}_{i_j})\right)^{d\lambda_{i_j}}\right)^{p_j}\right)\right)^{\frac{1}{C_d^k}}\right)^{\frac{1}{p_1+p_2+\cdots+p_k}},\right.$$
$$\left.1-\left(1-\left(\prod_{1\le i_1<\cdots<i_k\le d}\left(1-\prod_{j=1}^k\left(1-\left(\nu(\bar{a}_{i_j})\right)^{d\lambda_{i_j}}\right)^{p_j}\right)\right)^{\frac{1}{C_d^k}}\right)^{\frac{1}{p_1+p_2+\cdots+p_k}}\right)\Big\rangle$$

where λ denotes the weight of the decision maker.

Step 2. Utilize the following TIFLWMSM operator or the TIFLWGMSM operator to aggregate all attribute values $\bar{a}_{ij}(j = 1, 2, \ldots, n)$ and obtain a comprehensive value $\bar{a}_i$.

$$\bar{a}_{i} = TIFLWMSM(\bar{a}_{i1}, \bar{a}_{i2}, \ldots, \bar{a}_{in}) = \left(\frac{\sum_{1\le i_1<\cdots<i_k\le n}\prod_{j=1}^k (nw_{i_j}\bar{a}_{i_j})}{C_n^k}\right)^{\frac{1}{k}}$$
$$= \Big\langle \Big[s_{\left(\frac{\sum_{1\le i_1<\cdots<i_k\le n}\prod_{j=1}^k (nw_{i_j}\sigma(\bar{a}_{i_j}))}{C_n^k}\right)^{\frac{1}{k}}}, s_{\left(\frac{\sum_{1\le i_1<\cdots<i_k\le n}\prod_{j=1}^k (nw_{i_j}\zeta(\bar{a}_{i_j}))}{C_n^k}\right)^{\frac{1}{k}}},$$
$$s_{\left(\frac{\sum_{1\le i_1<\cdots<i_k\le n}\prod_{j=1}^k (nw_{i_j}\eta(\bar{a}_{i_j}))}{C_n^k}\right)^{\frac{1}{k}}}, s_{\left(\frac{\sum_{1\le i_1<\cdots<i_k\le n}\prod_{j=1}^k (nw_{i_j}\varphi(\bar{a}_{i_j}))}{C_n^k}\right)^{\frac{1}{k}}} \Big], \tag{47}$$
$$\left(\left(1-\left(\prod_{1\le i_1<\cdots<i_k\le n}\left(1-\prod_{j=1}^k\left(1-\left(1-\mu(\bar{a}_{i_j})\right)^{nw_{i_j}}\right)\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}},\right.$$
$$\left.1-\left(1-\left(\prod_{1\le i_1<\cdots<i_k\le n}\left(1-\prod_{j=1}^k\left(1-\left(\nu(\bar{a}_{i_j})\right)^{nw_{i_j}}\right)\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}\right)\Big\rangle$$

or

$$\bar{a}_i = TIFLWGMSM(\bar{a}_{i1}, \bar{a}_{i2}, \ldots, \bar{a}_{in}) = \left(\frac{\sum_{1\le i_1<\cdots<i_k\le n}\prod_{j=1}^{k}(nw_{i_j}\bar{a}_{i_j})^{p_j}}{C_n^k}\right)^{\frac{1}{p_1+p_2+\cdots+p_k}}$$
$$= \Big\langle \Big[s_{\left(\frac{\sum_{1\le i_1<\cdots<i_k\le n}\prod_{j=1}^{k}(nw_{i_j}\sigma(\bar{a}_{i_j}))^{p_j}}{C_n^k}\right)^{\frac{1}{p_1+p_2+\cdots+p_k}}}, s_{\left(\frac{\sum_{1\le i_1<\cdots<i_k\le n}\prod_{j=1}^{k}(nw_{i_j}\zeta(\bar{a}_{i_j}))^{p_j}}{C_n^k}\right)^{\frac{1}{p_1+p_2+\cdots+p_k}}},$$
$$s_{\left(\frac{\sum_{1\le i_1<\cdots<i_k\le n}\prod_{j=1}^{k}(nw_{i_j}\eta(\bar{a}_{i_j}))^{p_j}}{C_n^k}\right)^{\frac{1}{p_1+p_2+\cdots+p_k}}}, s_{\left(\frac{\sum_{1\le i_1<\cdots<i_k\le n}\prod_{j=1}^{k}(nw_{i_j}\varphi(\bar{a}_{i_j}))^{p_j}}{C_n^k}\right)^{\frac{1}{p_1+p_2+\cdots+p_k}}} \Big], \tag{48}$$
$$\left(\left(1-\left(\prod_{1\le i_1<\cdots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-(1-\mu(\bar{a}_{i_j}))^{nw_{i_j}}\right)^{p_j}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{p_1+p_2+\cdots+p_k}},\right.$$
$$\left.1-\left(1-\left(\prod_{1\le i_1<\cdots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-(\nu(\bar{a}_{i_j}))^{nw_{i_j}}\right)^{p_j}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{p_1+p_2+\cdots+p_k}}\right)\Big\rangle$$

where w denotes the weight of the attribute.

Step 3. Calculate the expected value $S(\bar{a}_i)$ of $\bar{a}_i$ with Equation(21).

Step 4. Rank $\bar{a}_i (i = 1, 2, \ldots, m)$ according to the size of the expected value.

Step 5. End.

5. A Numerical Example

In this section, we will illustrate the application of the proposed method with an instance adapted from [20]. An investment company wishes to invest in one of the following five alternative companies $A_i (i = 1, 2, 3, 4, 5)$, including (1) A_1 is a insurance company; (2) A_2 is a beverage company; (3) A_3 is an airline company; (4) A_4 is a motor company; (5) A_5 is an advertisement company. The investment company needs to make decisions based on the following four attributes (whose weight vector is $w = (0.3, 0.4, 0.2, 0.1)$): (1) C_1 denotes the risk index; (2) C_2 denotes the growth index; (3) C_3 denotes the social–political impact index; (4) C_4 denotes the environmental impact index. The decision makers $D_d (d = 1, 2, 3, 4)$ (assume that their weight vector is $\lambda = (0.25, 0.20, 0.3, 0.25)$) evaluate the five companies $A_i (i = 1, 2, 3, 4, 5)$ according to the attributes $C_j (j = 1, 2, 3, 4)$ mentioned above. The decision makers can evaluate through LTS $S = (s_0, s_1, s_2, s_3, s_4, s_5, s_6)$, and the following Tables 1–4 represent decision matrices $A^e = [\bar{a}^e_{ij}]_{(5\times4)} (e = 1, 2, 3, 4)$, respectively.

Table 1. Decision matrices A^1.

	C_1	C_2	C_3	C_4
A_1	$\langle c[s_2, s_3, s_4, s_5], (0.6, 0.3)\rangle$	$\langle c[s_2, s_4, s_5, s_6], (0.7, 0.3)\rangle$	$\langle c[s_1, s_3, s_4, s_5], (0.5, 0.4)\rangle$	$\langle c[s_2, s_3, s_5, s_6], (0.7, 0.2)\rangle$
A_2	$\langle c[s_1, s_2, s_4, s_5], (0.7, 0.3)\rangle$	$\langle c[s_2, s_3, s_5, s_6], (0.6, 0.3)\rangle$	$\langle c[s_0, s_1, s_4, s_5], (0.7, 0.1)\rangle$	$\langle c[s_2, s_3, s_4, s_5], (0.6, 0.3)\rangle$
A_3	$\langle c[s_0, s_2, s_4, s_5], (0.7, 0.2)\rangle$	$\langle c[s_1, s_2, s_4, s_5], (0.5, 0.3)\rangle$	$\langle c[s_1, s_2, s_3, s_5], (0.8, 0.1)\rangle$	$\langle c[s_2, s_4, s_5, s_6], (0.7, 0.2)\rangle$
A_4	$\langle c[s_1, s_2, s_3, s_5], (0.5, 0.2)\rangle$	$\langle c[s_1, s_3, s_4, s_5], (0.7, 0.1)\rangle$	$\langle c[s_2, s_3, s_4, s_5], (0.6, 0.3)\rangle$	$\langle c[s_3, s_4, s_5, s_6], (0.5, 0.4)\rangle$
A_5	$\langle c[s_2, s_3, s_5, s_6], (0.6, 0.2)\rangle$	$\langle c[s_2, s_3, s_4, s_5], (0.7, 0.3)\rangle$	$\langle c[s_0, s_1, s_5, s_6], (0.5, 0.4)\rangle$	$\langle c[s_1, s_2, s_4, s_5], (0.7, 0.1)\rangle$

Table 2. Decision matrices A^2.

	C_1	C_2	C_3	C_4
A_1	$\langle c[s_0, s_1, s_5, s_6], (0.7, 0.2)\rangle$	$\langle c[s_1, s_2, s_4, s_5], (0.6, 0.4)\rangle$	$\langle c[s_2, s_3, s_4, s_5], (0.5, 0.5)\rangle$	$\langle c[s_3, s_4, s_5, s_6], (0.4, 0.2)\rangle$
A_2	$\langle c[s_2, s_3, s_5, s_6], (0.5, 0.3)\rangle$	$\langle c[s_3, s_4, s_5, s_6], (0.5, 0.4)\rangle$	$\langle c[s_1, s_2, s_3, s_5], (0.8, 0.1)\rangle$	$\langle c[s_1, s_3, s_4, s_5], (0.5, 0.5)\rangle$
A_3	$\langle c[s_0, s_1, s_4, s_5], (0.6, 0.2)\rangle$	$\langle c[s_0, s_1, s_5, s_6], (0.7, 0.2)\rangle$	$\langle c[s_3, s_4, s_5, s_6], (0.6, 0.3)\rangle$	$\langle c[s_1, s_2, s_4, s_5], (0.7, 0.2)\rangle$
A_4	$\langle c[s_1, s_3, s_4, s_5], (0.6, 0.4)\rangle$	$\langle c[s_2, s_4, s_5, s_6], (0.8, 0.2)\rangle$	$\langle c[s_2, s_3, s_5, s_6], (0.6, 0.2)\rangle$	$\langle c[s_0, s_2, s_4, s_5], (0.6, 0.3)\rangle$
A_5	$\langle c[s_2, s_4, s_5, s_6], (0.7, 0.3)\rangle$	$\langle c[s_2, s_3, s_4, s_5], (0.6, 0.3)\rangle$	$\langle c[s_1, s_2, s_4, s_5], (0.7, 0.1)\rangle$	$\langle c[s_2, s_3, s_4, s_5], (0.6, 0.4)\rangle$

Table 3. Decision matrices A^3.

	C_1	C_2	C_3	C_4
A_1	$\langle c[s_0, s_1, s_4, s_6], (0.7, 0.1)\rangle$	$\langle c[s_3, s_4, s_5, s_6], (0.6, 0.1)\rangle$	$\langle c[s_1, s_2, s_4, s_5], (0.7, 0.2)\rangle$	$\langle c[s_0, s_1, s_4, s_5], (0.6, 0.2)\rangle$
A_2	$\langle c[s_2, s_4, s_5, s_6], (0.8, 0.2)\rangle$	$\langle c[s_2, s_3, s_5, s_6], (0.6, 0.2)\rangle$	$\langle c[s_0, s_2, s_4, s_6], (0.6, 0.3)\rangle$	$\langle c[s_1, s_3, s_4, s_5], (0.6, 0.1)\rangle$
A_3	$\langle c[s_2, s_3, s_4, s_5], (0.6, 0.3)\rangle$	$\langle c[s_1, s_2, s_4, s_6], (0.7, 0.1)\rangle$	$\langle c[s_2, s_3, s_4, s_5], (0.6, 0.4)\rangle$	$\langle c[s_2, s_4, s_5, s_6], (0.7, 0.3)\rangle$
A_4	$\langle c[s_1, s_2, s_4, s_5], (0.6, 0.2)\rangle$	$\langle c[s_2, s_3, s_4, s_5], (0.5, 0.5)\rangle$	$\langle c[s_3, s_4, s_5, s_6], (0.5, 0.2)\rangle$	$\langle c[s_0, s_1, s_3, s_4], (0.7, 0.2)\rangle$
A_5	$\langle c[s_3, s_4, s_5, s_6], (0.5, 0.4)\rangle$	$\langle c[s_1, s_2, s_3, s_5], (0.8, 0.1)\rangle$	$\langle c[s_1, s_3, s_4, s_5], (0.5, 0.5)\rangle$	$\langle c[s_2, s_3, s_5, s_6], (0.5, 0.3)\rangle$

Table 4. Decision matrices A^4.

	C_1	C_2	C_3	C_4
A_1	$\langle c[s_1, s_2, s_4, s_5], (0.7, 0.2)\rangle$	$\langle c[s_3, s_4, s_5, s_6], (0.6, 0.3)\rangle$	$\langle c[s_0, s_1, s_4, s_5], (0.6, 0.2)\rangle$	$\langle c[s_0, s_1, s_5, s_6], (0.7, 0.2)\rangle$
A_2	$\langle c[s_2, s_3, s_4, s_6], (0.6, 0.2)\rangle$	$\langle c[s_1, s_2, s_4, s_5], (0.7, 0.1)\rangle$	$\langle c[s_2, s_4, s_5, s_6], (0.7, 0.3)\rangle$	$\langle c[s_2, s_3, s_4, s_5], (0.6, 0.3)\rangle$
A_3	$\langle c[s_0, s_2, s_4, s_5], (0.6, 0.3)\rangle$	$\langle c[s_2, s_3, s_5, s_6], (0.6, 0.2)\rangle$	$\langle c[s_1, s_3, s_4, s_5], (0.5, 0.4)\rangle$	$\langle c[s_2, s_4, s_5, s_6], (0.8, 0.2)\rangle$
A_4	$\langle c[s_3, s_4, s_5, s_6], (0.7, 0.2)\rangle$	$\langle c[s_2, s_3, s_4, s_5], (0.5, 0.5)\rangle$	$\langle c[s_0, s_1, s_5, s_6], (0.7, 0.2)\rangle$	$\langle c[s_1, s_2, s_3, s_5], (0.6, 0.4)\rangle$
A_5	$\langle c[s_1, s_3, s_4, s_5], (0.5, 0.5)\rangle$	$\langle c[s_1, s_2, s_3, s_5], (0.8, 0.1)\rangle$	$\langle c[s_2, s_3, s_5, s_6], (0.5, 0.3)\rangle$	$\langle c[s_3, s_4, s_5, s_6], (0.5, 0.4)\rangle$

5.1. The Evaluation Steps by the TIFLWMSM Operator

The following specific steps are shown:

1. Firstly, we can aggregate all decision matrices $A^e(e = 1, 2, 3, 4)$ into one decision matrix using the TIFLWMSM operator (let k = 2) and the calculation process of $\bar{a}_{11}$ as follows:

$$
\begin{aligned}
\bar{a}_{11} =& TIFLWMSM(\bar{a}_{11}^1, \bar{a}_{11}^2, \bar{a}_{11}^3, \bar{a}_{11}^4) \\
=& \Big\langle \Big[s_{\left(\frac{\sum_{1\le i_1<i_2\le 4}\prod_{j=1}^{2}(4\lambda_{i_j}\sigma(\bar{a}_{i_j}))}{C_4^2}\right)^{\frac{1}{2}}}, s_{\left(\frac{\sum_{1\le i_1<i_2\le 4}\prod_{j=1}^{2}(4\lambda_{i_j}\zeta(\bar{a}_{i_j}))}{C_4^2}\right)^{\frac{1}{2}}}, \\
& s_{\left(\frac{\sum_{1\le i_1<i_2\le 4}\prod_{j=1}^{2}(4\lambda_{i_j}\eta(\bar{a}_{i_j}))}{C_4^2}\right)^{\frac{1}{2}}}, s_{\left(\frac{\sum_{1\le i_1<i_2\le 4}\prod_{j=1}^{2}(4\lambda_{i_j}\varphi(\bar{a}_{i_j}))}{C_4^2}\right)^{\frac{1}{2}}} \Big], \\
& \left(\left(1 - \left(\prod_{1\le i_1<i_2\le 4} \left(1 - \prod_{j=1}^{2} \left(1 - \left(1 - \mu(\bar{a}_{i_j})\right)^{4\lambda_{i_j}}\right)\right)\right)^{\frac{1}{C_4^2}}\right)^{\frac{1}{2}}, \right. \\
& \left. 1 - \left(1 - \left(\prod_{1\le i_1<i_2\le n} \left(1 - \prod_{j=1}^{2} \left(1 - \left(\nu(\bar{a}_{i_j})\right)^{4\lambda_{i_j}}\right)\right)\right)^{\frac{1}{C_4^2}}\right)^{\frac{1}{2}} \right) \Big\rangle
\end{aligned}
$$

$$=\Big\langle\Big[s_{\left(\frac{16(0.25\times 2\times 0.2\times 0)+16(0.25\times 2\times 0.3\times 0)+\cdots+16(0.3\times 0\times 0.25\times 1)}{6}\right)^{\frac{1}{2}}},$$
$$s_{\left(\frac{16(0.25\times 3\times 0.2\times 1)+16(0.25\times 3\times 0.3\times 1)+\cdots+16(0.3\times 1\times 0.25\times 2)}{6}\right)^{\frac{1}{2}}},$$
$$s_{\left(\frac{16(0.25\times 4\times 0.2\times 5)+16(0.25\times 4\times 0.3\times 4)+\cdots+16(0.3\times 4\times 0.25\times 4)}{6}\right)^{\frac{1}{2}}},$$
$$s_{\left(\frac{16(0.25\times 5\times 0.2\times 6)+16(0.25\times 5\times 0.3\times 6)+\cdots+16(0.3\times 6\times 0.25\times 5)}{6}\right)^{\frac{1}{2}}}\Big],$$
$$\bigg(\Big(1-\Big(\big(1-(1-(1-0.6)^{4\times 0.25})\times(1-(1-0.7)^{4\times 0.2})\big)$$
$$\times\cdots\times\big(1-(1-(1-0.7)^{4\times 0.3})\times(1-(1-0.7)^{4\times 0.25})\big)\Big)^{\frac{1}{6}}\Big)^{\frac{1}{2}},$$
$$1-\Big(1-\Big(\big(1-(1-0.3^{4\times 0.25})\times(1-0.2^{4\times 0.2})\big)$$
$$\times\cdots\times\big(1-(1-0.1^{4\times 0.3})\times(1-0.2^{4\times 0.25})\big)\Big)^{\frac{1}{6}}\Big)^{\frac{1}{2}}\bigg)\Big\rangle$$
$$=\langle[s_{0.58},s_{1.68},s_{4.20},s_{5.47}],(0.6713,0.2058)\rangle$$

According to the above calculation method, we can aggregate all the results into the following decision matrix:

	C_1	C_2	C_3	C_4
A_1	$\langle c[s_{0.58},s_{1.68},s_{4.20},s_{5.47}],(0.6713,0.2058)\rangle$	$\langle c[s_{2.27},s_{3.53},s_{4.76},s_{5.76}],(0.6221,0.2803)\rangle$	$\langle c[s_{0.89},s_{2.16},s_{3.99},s_{4.98}],(0.5721,0.3223)\rangle$	$\langle c[s_{0.89},s_{2.02},s_{4.69},s_{5.69}],(0.6037,0.2041)\rangle$
A_2	$\langle c[s_{1.72},s_{2.99},s_{4.47},s_{5.72}],(0.6488,0.2536)\rangle$	$\langle c[s_{1.92},s_{2.93},s_{4.73},s_{5.72}],(0.5991,0.2459)\rangle$	$\langle c[s_{0.52},s_{2.15},s_{4.01},s_{5.51}],(0.6978,0.1952)\rangle$	$\langle c[s_{1.47},s_{2.99},s_{3.99},s_{4.98}],(0.5734,0.301)\rangle$
A_3	$\langle c[s_{0},s_{2.02},s_{3.99},s_{4.98}],(0.6221,0.2524)\rangle$	$\langle c[s_{0.97},s_{2.00},s_{4.44},s_{5.72}],(0.6211,0.2058)\rangle$	$\langle c[s_{1.65},s_{2.93},s_{3.93},s_{5.19}],(0.6226,0.3016)\rangle$	$\langle c[s_{1.77},s_{3.53},s_{4.76},s_{5.76}],(0.7221,0.2275)\rangle$
A_4	$\langle c[s_{1.41},s_{2.66},s_{3.97},s_{5.23}],(0.5970,0.2487)\rangle$	$\langle c[s_{1.72},s_{3.20},s_{4.20},s_{5.19}],(0.6230,0.3236)\rangle$	$\langle c[s_{1.64},s_{2.69},s_{4.73},s_{5.72}],(0.5961,0.2286)\rangle$	$\langle c[s_{0.71},s_{2.11},s_{3.67},s_{4.93}],(0.5959,0.3307)\rangle$
A_5	$\langle c[s_{1.97},s_{3.47},s_{4.73},s_{5.72}],(0.5708,0.3523)\rangle$	$\langle c[s_{1.43},s_{2.44},s_{3.44},s_{4.98}],(0.7243,0.1989)\rangle$	$\langle c[s_{0.91},s_{2.22},s_{4.48},s_{5.48}],(0.5456,0.3217)\rangle$	$\langle c[s_{1.95},s_{2.96},s_{4.51},s_{5.51}],(0.5708,0.2986)\rangle$

2. To acquire the collective comprehensive values by the TIFLWMSM operator (let k = 2), the calculation process of $\bar{a}_1$ is as follows:

$$\bar{a}_1=TIFLWMSM(\bar{a}_{11},\bar{a}_{12},\bar{a}_{13},\bar{a}_{14})$$
$$=\Big\langle\Big[s_{\left(\frac{\sum_{1\le i_1<i_2\le 4}\prod_{j=1}^{2}(4w_{i_j}\sigma(\bar{a}_{i_j}))}{C_4^2}\right)^{\frac{1}{2}}},s_{\left(\frac{\sum_{1\le i_1<i_2\le 4}\prod_{j=1}^{2}(4w_{i_j}\zeta(\bar{a}_{i_j}))}{C_4^2}\right)^{\frac{1}{2}}},$$
$$s_{\left(\frac{\sum_{1\le i_1<i_2\le 4}\prod_{j=1}^{2}(4w_{i_j}\eta(\bar{a}_{i_j}))}{C_4^2}\right)^{\frac{1}{2}}},s_{\left(\frac{\sum_{1\le i_1<i_2\le 4}\prod_{j=1}^{2}(4w_{i_j}\varphi(\bar{a}_{i_j}))}{C_4^2}\right)^{\frac{1}{2}}}\Big],$$
$$\bigg(\Big(1-\Big(\prod_{1\le i_1<i_2\le 4}\Big(1-\prod_{j=1}^{2}\big(1-(1-\mu(\bar{a}_{i_j}))^{4w_{i_j}}\big)\Big)\Big)^{\frac{1}{C_4^2}}\Big)^{\frac{1}{2}},$$
$$1-\Big(1-\Big(\prod_{1\le i_1<i_2\le n}\Big(1-\prod_{j=1}^{2}\big(1-(\nu(\bar{a}_{i_j}))^{4w_{i_j}}\big)\Big)\Big)^{\frac{1}{C_4^2}}\Big)^{\frac{1}{2}}\bigg)\Big\rangle$$

$$=\Big\langle \Big[s_{\left(\frac{16(0.3\times0.58\times0.4\times2.27)+16(0.3\times0.58\times0.2\times0.89)+\cdots+16(0.2\times0.89\times0.1\times0.89)}{6}\right)^{\frac{1}{2}}},$$

$$s_{\left(\frac{16(0.3\times1.68\times0.4\times3.53)+16(0.3\times1.68\times0.2\times2.16)+\cdots+16(0.2\times2.16\times0.1\times2.02)}{6}\right)^{\frac{1}{2}}},$$

$$s_{\left(\frac{16(0.3\times4.20\times0.4\times4.76)+16(0.3\times4.20\times0.2\times3.99)+\cdots+16(0.2\times3.99\times0.1\times4.69)}{6}\right)^{\frac{1}{2}}},$$

$$s_{\left(\frac{16(0.3\times5.47\times0.4\times5.76)+16(0.3\times5.47\times0.2\times4.98)+\cdots+16(0.2\times4.98\times0.1\times5.69)}{6}\right)^{\frac{1}{2}}}\Big],$$

$$\bigg(\Big(1-\Big(\Big(1-\big(1-(1-0.6713)^{4\times0.3}\big)\times\big(1-(1-0.6221)^{4\times0.4}\big)\Big)$$

$$\times\cdots\times\Big(1-\big(1-(1-0.5721)^{4\times0.2}\big)\times\big(1-(1-0.6037)^{4\times0.1}\big)\Big)\Big)^{\frac{1}{6}}\Big)^{\frac{1}{2}},$$

$$1-\Big(1-\Big(\Big(1-\big(1-0.2058^{4\times0.3}\big)\times\big(1-0.2803^{4\times0.4}\big)\Big)$$

$$\times\cdots\times\Big(1-\big(1-0.3223^{4\times0.2}\big)\times\big(1-0.2041^{4\times0.1}\big)\Big)\Big)^{\frac{1}{6}}\Big)^{\frac{1}{2}}\bigg)\Big\rangle$$

$$=\langle[s_{1.11},s_{2.32},s_{4.25},s_{5.30}],(0.5856,0.2938)\rangle$$

According to the above calculation method, we can obtain the following collective comprehensive values of alternatives A_2, A_3, A_4, A_5.

$$\bar{a}_2=\langle[s_{1.40},s_{2.68},s_{4.24},s_{5.39}],(0.6017,0.2828)\rangle$$
$$\bar{a}_3=\langle[s_{0.82},s_{2.32},s_{4.08},s_{5.19}],(0.6067,0.2910)\rangle$$
$$\bar{a}_4=\langle[s_{1.42},s_{2.68},s_{4.04},s_{5.11}],(0.5700,0.3145)\rangle$$
$$\bar{a}_5=\langle[s_{1.47},s_{2.65},s_{4.04},s_{5.20}],(0.5670,0.3385)\rangle$$

3. Calculate the value $S(\bar{a}_i)$ of $\bar{a}_i$ ($i=1,2,3,4$) by Equation (21). The calculation process of $S(\bar{a}_1)$ is as follows:

$$S(\bar{a}_1)=s_{\frac{(\sigma(\tilde{\alpha})+\zeta(\tilde{\alpha})+\eta(\tilde{\alpha})+\varphi(\tilde{\alpha}))\times(1+\mu(\tilde{\alpha})-\nu(\tilde{\alpha}))}{8}}$$
$$=s_{\frac{(1.11+2.32+4.25+5.30)\times(1+0.5856-0.2938)}{8}}$$
$$=s_{2.096}$$

According to the above calculation method, we can obtain the following expected values of alternatives A_2, A_3, A_4, A_5.

$$S(\bar{a}_2)=s_{2.260}\quad S(\bar{a}_3)=s_{2.041}\quad S(\bar{a}_4)=s_{2.079}\quad S(\bar{a}_5)=s_{2.052}$$

4. We can obtain the following results according to the above expected values.

$$A_2>A_1>A_4>A_5>A_3$$

The best choice is A_2.

5. End.

5.2. The Evaluation Steps by the TIFLWGMSM Operator

The following specific steps are shown:

1. Firstly, we will utilize the TIFLWGMSM operator (let k = 2, $p_1=1,p_2=2$); the calculation process is similar to the step 1 of the TIFLWMSM operator, so it is omitted. Therefore, we can obtain:

	C_1	C_2	C_3	C_4
A_1	$\langle c[s_{0.69}, s_{1.66}, s_{4.23}, s_{5.59}]$, $(0.6843, 0.1899)\rangle$	$\langle c[s_{2.51}, s_{3.71}, s_{4.89}, s_{5.90}]$, $(0.6182, 0.2611)\rangle$	$\langle c[s_{1.02}, s_{2.11}, s_{4.05}, s_{5.06}]$, $(0.5942, 0.2899)\rangle$	$\langle c[s_{1.24}, s_{1.99}, s_{4.72}, s_{5.74}]$, $(0.6176, 0.1998)\rangle$
A_2	$\langle c[s_{1.84}, s_{3.20}, s_{4.58}, s_{5.90}]$, $(0.6615, 0.2372)\rangle$	$\langle c[s_{1.94}, s_{2.93}, s_{4.75}, s_{5.76}]$, $(0.6178, 0.2150)\rangle$	$\langle c[s_{0.81}, s_{2.53}, s_{4.22}, s_{5.73}]$, $(0.6964, 0.2126)\rangle$	$\langle c[s_{1.53}, s_{3.04}, s_{4.05}, s_{5.06}]$, $(0.5823, 0.2785)\rangle$
A_3	$\langle c[s_0, s_{2.19}, s_{4.05}, s_{5.06}]$, $(0.6182, 0.2594)\rangle$	$\langle c[s_{1.20}, s_{2.20}, s_{4.56}, s_{5.90}]$, $(0.6353, 0.1899)\rangle$	$\langle c[s_{1.73}, s_{3.03}, s_{4.05}, s_{5.24}]$, $(0.6037, 0.3277)\rangle$	$\langle c[s_{1.85}, s_{3.71}, s_{4.89}, s_{5.90}]$, $(0.7362, 0.2262)\rangle$
A_4	$\langle c[s_{1.71}, s_{2.90}, s_{4.21}, s_{5.40}]$, $(0.6196, 0.2375)\rangle$	$\langle c[s_{1.84}, s_{3.21}, s_{4.23}, s_{5.24}]$, $(0.6067, 0.3578)\rangle$	$\langle c[s_{2.05}, s_{2.94}, s_{4.89}, s_{5.90}]$, $(0.6090, 0.2164)\rangle$	$\langle c[s_{0.79}, s_{1.96}, s_{3.53}, s_{4.88}]$, $(0.6139, 0.3169)\rangle$
A_5	$\langle c[s_{2.14}, s_{3.57}, s_{4.75}, s_{5.76}]$, $(0.5624, 0.3728)\rangle$	$\langle c[s_{1.36}, s_{2.37}, s_{3.38}, s_{5.06}]$, $(0.7446, 0.3728)\rangle$	$\langle c[s_{1.15}, s_{2.48}, s_{4.55}, s_{5.56}]$, $(0.5449, 0.3186)\rangle$	$\langle c[s_{2.18}, s_{3.20}, s_{4.72}, s_{5.73}]$, $(0.5553, 0.3167)\rangle$

2. To acquire the collective comprehensive values by TIFLWGMSM (let $k = 2$, $p_1 = 1$, $p_2 = 2$), the calculation process is similar to step 2 of TIFLWMSM operator, so it is omitted. Thus, we can obtain:

$$\begin{aligned} \bar{a}_1 &= \langle [s_{1.44}, s_{2.54}, s_{4.38}, s_{5.45}], (0.5785, 0.2971)\rangle \\ \bar{a}_2 &= \langle [s_{1.62}, s_{2.82}.s_{4.40}, s_{5.56}], (0.6, 0.2844)\rangle \\ \bar{a}_3 &= \langle [s_{0.97}, s_{2.44}, s_{4.20}, s_{5.36}], (0.5924, 0.3083)\rangle \\ \bar{a}_4 &= \langle [s_{1.75}, s_{2.90}, s_{4.20}, s_{5.24}], (0.5652, 0.3303)\rangle \\ \bar{a}_5 &= \langle [s_{1.51}, s_{2.64}, s_{3.94}, s_{5.22}], (0.5664, 0.3461)\rangle \end{aligned}$$

3. Calculate the value $S(\bar{a}_i)$ of $\bar{a}_i$ $(i = 1, 2, 3, 4)$ by Equation (21); the calculation process is similar to the step 3 of TIFLWMSM operator, so it is omitted. Thus, we can obtain:

$$S(\bar{a}_1) = s_{2.212} \quad S(\bar{a}_2) = s_{2.368} \quad S(\bar{a}_3) = s_{2.082} \quad S(\bar{a}_4) = s_{2.175} \quad S(\bar{a}_5) = s_{2.030}$$

4. We can obtain the following results according to the above expected values.

$$A_2 > A_1 > A_4 > A_3 > A_5$$

   The best choice is A_2.
5. End.

5.3. Comparative Analysis and Discussion

In this part, we firstly compare the two methods presented in this paper with other methods, which include Chu et al.'s [20] proposed TIFLWBM and TIFLWGBM operators and the ITrFLWA and ITrFLOWA operators developed by Ju et al. [15]. The results of the comparison are shown in Table 5. From Table 5, we originally find that the best alternative in all the methods is A_2, which proves that the methods proposed in this paper are valid and applicable. In addition, the sorting results are slightly different in this paper and other methods because the two methods proposed in [20] can only capture the interrelationships between two input arguments, while the two methods from this paper can consider interrelations among multi-input parameters. Additionally, the methods developed by [15] ignore the correlations between arguments, which leads to a different sorting result from the two methods in this paper. Moreover, the ranking results of the two methods are slightly different in this paper because different values of p_1 and p_2 in the $TIFLWGMSM$ operator result in different sorting.

Table 5. Compared with other methods.

Methods	Operator	Ranking
Methods in this paper	$TIFLWMSM^{(k)}(k=2)$ $TIFLWGMSM^{(k,p_1,p_2)}(k=2,p_1=1,p_2=2)$	$A_2 > A_1 > A_4 > A_5 > A_3$ $A_2 > A_1 > A_4 > A_3 > A_5$
Method in [20]	TIFLWBM TIFLWGBM	$A_2 > A_1 > A_3 > A_5 > A_4$ $A_2 > A_5 > A_4 > A_3 > A_1$
Method in [15]	ITrFLWA ITrFLOWA	$A_2 > A_1 > A_5 > A_4 > A_3$ $A_2 > A_1 > A_5 > A_4 > A_3$

Furthermore, we take different values for P_1 and P_2 when $k = 2$ in the $TIFLWGMSM^{(k,p_1,p_2)}$ operator, whose comparison results are shown in Table 6. From Table 6, we find that the ordering results are obviously different and inconsistent with a practical situation when $P_1 = 0$ or $P_2 = 0$. The reason for the above situation is that the relationship between the input parameters is not considered. Therefore, the value of P_1 or P_2 must be assigned to real numbers in actual applications. Moreover, we can also know the best choice is A_2, followed by A_1, and the worst alternative is A_3 or A_5 when P_1 and P_2 are not equal to zero. Finally, alternatives A_3, A_4 and A_5 are easily influenced when values of P_1 and P_2 change.

Table 6. Comparison results when $k = 2$ in the $TIFLWGMSM^{(k,p_1,p_2)}$ operator.

P_1	P_2	$S(\bar{a}_i)$	Ranking
0	1	$S(\bar{a}_1) = 1.596, S(\bar{a}_2) = 1.679, S(\bar{a}_3) = 1.698$ $S(\bar{a}_4) = 1.394, S(\bar{a}_5) = 1.6$	$A_3 > A_2 > A_5 > A_1 > A_4$
1	0	$S(\bar{a}_1) = 3.323, S(\bar{a}_2) = 3.448, S(\bar{a}_3) = 2.867$ $S(\bar{a}_4) = 3.138, S(\bar{a}_5) = 3.487$	$A_5 > A_2 > A_1 > A_4 > A_3$
1	1	$S(\bar{a}_1) = 2.096, S(\bar{a}_2) = 2.26, S(\bar{a}_3) = 2.041$ $S(\bar{a}_4) = 2.079, S(\bar{a}_5) = 2.052$	$A_2 > A_1 > A_4 > A_5 > A_3$
1	2	$S(\bar{a}_1) = 2.212, S(\bar{a}_2) = 2.368, S(\bar{a}_3) = 2.082$ $S(\bar{a}_4) = 2.175, S(\bar{a}_5) = 2.03$	$A_2 > A_1 > A_4 > A_3 > A_5$
1	3	$S(\bar{a}_1) = 2.554, S(\bar{a}_2) = 2.653, S(\bar{a}_3) = 2.284$ $S(\bar{a}_4) = 2.392, S(\bar{a}_5) = 2.194$	$A_2 > A_1 > A_4 > A_3 > A_5$
2	1	$S(\bar{a}_1) = 2.560, S(\bar{a}_2) = 2.679, S(\bar{a}_3) = 2.378$ $S(\bar{a}_4) = 2.428, S(\bar{a}_5) = 2.493$	$A_2 > A_1 > A_5 > A_4 > A_3$
2	2	$S(\bar{a}_1) = 2.441, S(\bar{a}_2) = 2.631, S(\bar{a}_3) = 2.290$ $S(\bar{a}_4) = 2.383, S(\bar{a}_5) = 2.317$	$A_2 > A_1 > A_4 > A_5 > A_3$
2	3	$S(\bar{a}_1) = 2.608, S(\bar{a}_2) = 2.816, S(\bar{a}_3) = 2.381$ $S(\bar{a}_4) = 2.525, S(\bar{a}_5) = 2.375$	$A_2 > A_1 > A_4 > A_3 > A_5$
3	1	$S(\bar{a}_1) = 2.954, S(\bar{a}_2) = 2.993, S(\bar{a}_3) = 2.631$ $S(\bar{a}_4) = 2.689, S(\bar{a}_5) = 2.813$	$A_2 > A_1 > A_5 > A_4 > A_3$
3	2	$S(\bar{a}_1) = 2.696, S(\bar{a}_2) = 2.841, S(\bar{a}_3) = 2.475$ $S(\bar{a}_4) = 2.555, S(\bar{a}_5) = 2.562$	$A_2 > A_1 > A_5 > A_4 > A_3$
3	3	$S(\bar{a}_1) = 2.735, S(\bar{a}_2) = 2.941, S(\bar{a}_3) = 2.496$ $S(\bar{a}_4) = 2.623, S(\bar{a}_5) = 2.552$	$A_2 > A_1 > A_4 > A_5 > A_3$

By the above analysis, we can learn that the two methods presented in this paper are adaptable and valid compared to other existing methods for solving the problems. In addition, selecting the suitable arguments is of vital significance in coping with MAGDM problems. Thus, the methods proposed in this paper are more advisable to solve the MAGDM problems.

6. Conclusions

The MSM operator can reflect the interrelationship between input arguments. In addition, TIFLNs can clearly express uncertain information. Therefore, in this paper, we extend the MSM operator to deal with TIFL information and put forward a trapezoid intuitionistic fuzzy linguistic Maclaurin symmetric mean (TIFLMSM) operator and a trapezoid intuitionistic fuzzy linguistic generalized Maclaurin symmetric mean (TIFLGMSM) operator. In view of the input parameters with different importance, which have a significant impact on the final decision results, the trapezoid intuitionistic fuzzy linguistic weighted Maclaurin symmetric mean (TIFLWMSM) operator and the trapezoid intuitionistic fuzzy linguistic weighted generalized Maclaurin symmetric mean (TIFLWGMSM) operator are proposed. Then, two methods to solve the MAGDM problems in which the attribute values are depicted in a trapezoid intuitionistic fuzzy linguistic environment are proposed. We also verify the practicality and reliability of the proposed methods by an illustrative example. In the end, a comparison with other existing methods proves the effectiveness and usability of our proposed methods.

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References

1. Zadeh, L.A. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–353, http://doi.org/10.1016/S0019-9958(65)90241-X.
2. Atanassov, K.T. Intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1986**, *20*, 87–96, http://doi.org/10.1016/S0165-0114(86)80034-3.
3. Atanassov, K.T. More on intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1989**, *33*, 37–45, http://doi.org/10.1016/0165-0114(89)90215-7.
4. Fan, Y.; Xiao, F. Two dimensional intuitionistic fuzzy sets. *Eng. Appl. Artif. Intell.* **2020**, *95*, 103882, http://doi.org/10.1016/j.engappai.2020.103882.
5. Zadeh, L.A. The concept of a linguistic variable and its application to approximate reasoning. *Inf. Sci.* **1975**, *8*, 199–249, http://doi.org/10.1016/0020-0255(75)90036-5.
6. Xu, Z.S. Uncertain linguistic aggregation operators based approach to multiple attribute group decision making under uncertain linguistic environment. *Inf. Sci.* **2004**, *168*, 171–184, http://doi.org/10.1016/j.ins.2004.02.003.
7. Chai, J.; Xian, S.; Lu, S. Z-uncertain probabilistic linguistic variables and its application in emergency decision making for treatment of COVID-19 patients. *Int. J. Intell. Syst.* **2021**, *36*, 362–402, http://doi.org/10.1002/int.22303.
8. Herrera, F.; Martínez, L. A 2-tuple fuzzy linguistic representation model for computing with words. *IEEE Trans. Fuzzy Syst.* **2000**, *8*, 746–752, http://doi.org/10.1109/91.890332.
9. Ju, Y.; Liu, X.; Yang, S. Trapezoid fuzzy 2-tuple linguistic aggregation operators and their applications to multiple attribute decision making. *J. Intell. Fuzzy Syst.* **2014**, *27*, 1219–1232, http://doi.org/10.3233/IFS-131087.
10. Xu, Z.S. An Approach Based on Similarity Measure to Multiple Attribute Decision Making with trapezoid Fuzzy Linguistic Variables. In Proceedings of the International Conference on Fuzzy Systems and Knowledge Discovery, Changsha, China, 27–29 August 2005; pp. 110–117.
11. Wang, J.Q.; Li, J.J. The multi-criteria group decision making method based on multi-granularity intuitionistic two semantics. *Sci. Technol. Inf.* **2009**, *33*, 8–9.
12. Liu, P.; Jin, F. Methods for aggregating intuitionistic uncertain linguistic variables and their application to group decision making. *Inf. Sci.* **2012**, *205*, 58–71, http://doi.org/10.1016/j.ins.2012.04.014.
13. Liu, P. Some geometric aggregation operators based on interval intuitionistic uncertain linguistic variables and their application to group decision making. *Appl. Math. Model.* **2013**, *37*, 2430–2444, http://doi.org/10.1016/j.apm.2012.05.032.
14. Atanassov, K.T.; Gargov, G. Interval-valued intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1989**, *31*, 343–349, http://doi.org/10.1016/0165-0114(89)90205-4.
15. Ju, Y.B.; Yang, S.H.; Wang, A.H. Some aggregation operators with intuitionistic trapezoid fuzzy linguistic information and their applications to multi-attribute group decision making. *Appl. Math. Inf. Sci.* **2014**, *8*, 2427–2436, http://doi.org/10.12785/amis/080538.
16. Bonferroni, C. Sulle medie multiple di potenze. *Boll. Mater. Ital.* **1950**, *5*, 267–270.

17. Wei, G.W.; Gao, H.; Wei, Y. Some q-Rung Orthopair Fuzzy Heronian Mean Operators in Multiple Attribute Decision Making. *Int. J. Intell. Syst.* **2018**, *33*, 1426–1458, http://doi.org/10.1002/int.21985.
18. Chen, Y.; Liu, P. Multi-attribute decision-making approach based on intuitionistic trapezoidal fuzzy generalized heronian OWA operator. *J. Intell. Fuzzy Syst.* **2014**, *27*, 1381–1392, http://doi.org/10.3233/IFS-131105.
19. Maclaurin, C. A second letter to Martin Foulkes, Esq.: Concerning the roots of equations, with the demonstrations of other rules of algebra. *Philos. Trans. R. Soc. Lond. Ser. A* **1729**, *36*, 59–96.
20. Chu, Y.; Liu, P.; Li, H. Multi-attribute group decision making method based on some trapezoid intuitionistic fuzzy linguistic Bonferroni mean aggregation operators. *J. Intell. Fuzzy Syst*. **2019**, *36*, 3869–3889, http://doi.org/ 10.3233/JIFS-181045.
21. Beliakov, G.; Pradera, A.; Calvo, T. Aggregation functions: A guide for practitioners. *Stud. Fuzziness Soft Comput.* **2010**, *2010*, 139–141.
22. Liu, P. Multiple Attribute Group Decision Making Method Based on Interval-Valued Intuitionistic Fuzzy Power Heronian Aggregation Operators. *Comput. Ind. Eng.* **2017**, *108*, 199–212, http://doi.org/10.1016/j.cie.2017.04.033.
23. Yang, Z.; Li, X.; Garg, H.; Qi, M. A Cognitive Information-Based Decision-Making Algorithm Using Interval-Valued q-Rung Picture Fuzzy Numbers and Heronian Mean Operators. *Cogn. Comput.* **2021**, *13*, 357–380, http://doi.org/10.1007/s12559-020-09811-8.
24. Ayub, S.; Abdullah, S.; Ghani, F.; Qiyas, M.; Khan, M.Y. Cubic fuzzy Heronian mean Dombi aggregation operators and their application on multi-attribute decision-making problem. *Soft Comput.* **2021**, *25*, 4175–4189, http://doi.org/10.1007/s00500-020-05512-4.
25. Detemple, D.; Robertson, J. On generalized symmetric means of two variables. *Angew. Chem.* **1979**, *47*, 4638–4660.
26. Qin, J.; Liu, X. An Approach to Intuitionistic Fuzzy Multiple Attribute Decision Making Based on Maclaurin Symmetric Mean Operators. *J. Intell. Fuzzy Syst.* **2014**, *27*, 2177–2190, http://doi.org/10.3233/IFS-141182.
27. Qin, J.; Liu, X. Approaches to uncertain linguistic multiple attribute decision making based on dual Maclaurin symmetric mean. *J. Intell. Fuzzy Syst.* **2015**, *29*, 171–186, http://doi.org/10.3233/IFS-151584.
28. Liu, P.; Zhang, X. Some Maclaurin Symmetric Mean Operators for Single-Valued Trapezoidal Neutrosophic Numbers and Their Applications to Group Decision Making. *Int. J. Fuzzy Syst.* **2017**, *20*, 45–61, http://doi.org/10.1007/s40815-017-0335-9.
29. Geng, Y.S.; Wang, X.G.; Li, X.M.; Yu, K. Some Interval Neutrosophic Linguistic Maclaurin Symmetric Mean Operators and Their Application in Multiple Attribute Decision Making. *Symmetry* **2018**, *10*, 127, http://doi.org/10.3390/sym10040127.
30. Ali, Z.; Mahmood, T. Maclaurin symmetric mean operators and their applications in the environment of complex q-rung orthopair fuzzy sets. *Comput. Appl. Math.* **2020**, *39*, 1–27, http://doi.org/10.1007/s40314-020-01145-3.
31. Wei, G.W. Uncertain linguistic hybrid geometric mean operator and its Application to group decision making under uncertain linguistic environment. *Int. J. Uncertain. Fuzz. Knowl. Based Syst.* **2009**, *17*, 251–267, http://doi.org/10.1142/S021848850900584X.
32. Xu, Z.S. An approach based on the uncertain LOWG and induced uncertain LOWG operators to group decision making with uncertain multiplicative linguistic preference relations. *Decis. Support Syst.* **2006** , *41*, 488–499, http://doi.org/10.1016/j.dss.2004.08.011.
33. Xu, Z.S. Induced uncertain linguistic OWA operators applied to group decision making. *Inf. Fusion* **2006**, *7*, 231–238, http://doi.org/10.1016/j.inffus.2004.06.005.
34. Liu, P.D. Some Heronian mean operators with 2-tuple linguistic information and their application to multiple attribute group decision making. *Technol. Econ. Dev. Econ.* **2015**, *21*, 797–814, http://doi.org/10.3846/20294913.2015.1055614.
35. Xu, Z.S. Goal programming models for multiple attribute decision making under linguistic setting. *J. Manag. Sci. China* **2006**, *9*, 9–17.
36. Liu, P.; Su, Y. The Extended TOPSIS Based on Trapezoid Fuzzy Linguistic Variables. *J. Converg. Inf. Technol.* **2010**, *5*, 38–53.
37. Liu, Z.C.; Yong, Y. Models for comprehensive evaluating modeling of investment project risk with trapezoid fuzzy linguistic information. *J. Intell. Fuzzy Syst.* **2015**, *28*, 151–156, http://doi.org/10.3233/IFS-141284.
38. Wang, J.Q.; Yang, Y.; Li, L. Multi-criteria decision-making method based on single-valued neutrosophic linguistic Maclaurin symmetric mean operators. *Neural Comput. Appl.* **2018**, *30*, 1529–1547, http://doi.org/10.1007/s00521-016-2747-0.