Article

Geometry of $\alpha$-Cosymplectic Metric as $\ast$-Conformal $\eta$-Ricci–Yamabe Solitons Admitting Quarter-Symmetric Metric Connection

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Abstract: The outline of this research article is to initiate the development of a $\ast$-conformal $\eta$-Ricci–Yamabe soliton in $\alpha$-Cosymplectic manifolds according to the quarter-symmetric metric connection. Here, we have established some curvature properties of $\alpha$-Cosymplectic manifolds in regard to the quarter-symmetric metric connection. Further, the attributes of the soliton when the manifold gratifies a quarter-symmetric metric connection have been displayed in this article. Later, we picked up the Laplace equation from $\ast$-conformal $\eta$-Ricci–Yamabe soliton equation when the potential vector field $\xi$ of the soliton is of gradient type, admitting quarter-symmetric metric connection. Next, we evolved the nature of the soliton when the vector field’s conformal killing reveals a quarter-symmetric metric connection. We show an example of a 5-dimensional $\alpha$-cosymplectic metric as a $\ast$-conformal $\eta$-Ricci–Yamabe soliton acknowledges quarter-symmetric metric connection to prove our results.

Keywords: Ricci–Yamabe soliton; $\ast$-conformal $\eta$-Ricci–Yamabe soliton; conformal killing vector field; $\alpha$-cosymplectic manifolds

1. Background and Motivations

Many mathematicians know that the concept of Ricci flow, which is an evolution equation for metrics on a Riemannian manifold, was introduced by R. S. Hamilton [1] in the year 1982. We call a self-similar solution of the Ricci flow [1], ref. [2] a Ricci soliton [3] if it moves only depending on one parameter family of scaling and diffeomorphism. After the introduction of Ricci flow, for the sake of constructing Yamabe metrics on compact Riemannian manifolds, Hamilton [3] developed the idea of Yamabe flow.

The Ricci flow is equivalent to the Yamabe flow in two-dimension [1]. However, the situation is different when the dimension bigger than two. The Yamabe flow preserves the conformal class of the metric, but in general the Ricci flow does not.

Since the introduction of these geometric flows, the respective solitons and their generalizations, etc., have been the center of attention of many geometers viz. [1,3–31] who have provided new approaches to understand the geometry of different kinds of the Riemannian manifold. Recently, Sarkar et al. [32] studied $\ast$-conformal $\eta$-Ricci soliton and $\ast$-conformal Ricci soliton from the viewpoint of contact geometry and obtained some beautiful results.

In 2019, S. Güler et al. [21] introduced a new geometry flow, which is a scalar combination of the Ricci and Yamabe flow under the Ricci–Yamabe map. The new flow is also known as the Ricci–Yamabe flow for type $(\rho, q)$.
Suppose that $M$ is a Riemannian manifold of dimension $n$ and $T^2_2(M)$ is a linear space of its symmetric tensor fields for $(0,2)$-type and $\text{Riem}(M) \subsetneq T^2_2(M)$ is infinite space of its Riemannian metrics. In the paper [21], the authors have stated the below definition:

**Definition 1.** [21] Suppose that a Riemannian flow on $M$ is a smooth map:

$$g : I \subseteq \mathbb{R} \rightarrow \text{Riem}(M),$$

here, $I$ is an open interval. We can also call it a time-dependent (or non-stationary) Riemannian metric.

**Definition 2.** [21] We call the map $R_Y(\rho, q, g) : I \rightarrow T^2_2(M)$ which is defined by:

$$R_Y(\rho, q, g) := 2\rho S(t) + qr(t)g(t) + \frac{\partial}{\partial t}g(t),$$

is $(\rho, q)$-Ricci–Yamabe map of $(M^n, g)$, where $\rho, q$ are some scalars. If $R_Y(\rho, q, g) \equiv 0$, after that, call $g(\cdot)$ an $(\rho, q)$-Ricci–Yamabe flow.

In [21], the authors characterized that the $(\rho, q)$-Ricci–Yamabe flow is said to be

- Ricci flow [1] if $\rho = 1, q = 0$;
- Yamabe flow [3] if $\rho = 0, q = 1$;

We call a soliton for the Ricci–Yamabe flow the Ricci–Yamabe solitons [29] if it moves depending on one parameter group of diffeomorphism and scaling. We say that the metric of the Riemannian manifold $(M^n, g)$, $n > 2$ admit $(\rho, q)$-Ricci–Yamabe soliton or simply Ricci–Yamabe soliton (RYS) $(g, V, \Lambda, \rho, q)$ if it satisfies the equation

$$\mathcal{L}_Vg + 2\rho S + [2\Lambda - qr]g = 0, \quad (1)$$

here, $S$ is the Ricci tensor, $\Lambda, \rho, q$ are the real scalars, $\mathcal{L}_Vg$ denotes the Lie derivative of the metric $g$ along $V$, $r$ is the scalar curvature.

Moreover, we say that the Ricci–Yamabe soliton is shrinking, expanding, or steady, depending on whether the $\Lambda$ is positive, negative, or zero, respectively. If $\Lambda, \rho, q$ become smooth functions, after that, we would call (1) to almost be a Ricci–Yamabe soliton.

In 2020, Mohd. Danish Siddiqi et al. [29] introduced a new generalization of the Ricci–Yamabe soliton, namely the $\eta$-Ricci–Yamabe soliton, which is

$$2\mu \eta \otimes \eta + \mathcal{L}_Vg + 2\rho S + [2\Lambda - qr]g = 0, \quad (2)$$

where $\eta$ is a 1-form on $M$ and $\mu$ is a constant.

Very recently [32], S.R. et al. studied the conformal Ricci–Yamabe soliton and defined it as:

$$\mathcal{L}_Vg + 2\rho S + \left[2\Lambda - \left(p + \frac{2}{n}\right) - qr\right]g = 0, \quad (3)$$

where, $p$ is a scalar non-dynamical field (time dependent scalar field), $r$ is the scalar curvature, $\mathcal{L}_Vg$ denotes the Lie derivative of the metric $g$ along the vector field $V$, $n$ is the dimension of the manifold, $S$ is the Ricci tensor, and $(\Lambda, \alpha, \beta)$ are real scalars. We call the conformal Ricci–Yamabe soliton as either being steady, shrinking, or expanding, depending on $\Lambda$ being either zero, negative, or positive. If the vector field $V$ is a gradient type, that is $V = \text{grad}(f)$, for $f$ as a smooth function of $M$, then we call the Equation (3) a conformal gradient Ricci–Yamabe soliton.
One can also get the notion of conformal $\eta$-Ricci–Yamabe soliton from [32], and it is defined as:

$$\mathcal{L}_V g + 2\mu \eta \otimes \eta + 2pS + \left[2\lambda - \left(p + \frac{2}{n}\right) - qr\right]g = 0,$$

where $S$ is the Ricci tensor, $r$ is the scalar curvature, $p$ is a scalar non-dynamical field (time dependent scalar field), $n$ is the dimension of the manifold, $(\lambda, \alpha, \beta, \mu)$ are real scalars, and $\mathcal{L}_V g$ denotes the Lie derivative of the metric $g$ along the vector field $V$. We say that the conformal $\eta$-Ricci–Yamabe soliton is steady, shrinking, or expanding if $\lambda$ is zero, negative, or positive.

Recently, in the paper [15], S.D. and S.R. give the definition of $\ast \cdot \eta$-Ricci soliton, a generalization of $\eta$-Ricci soliton, which can be given as,

$$\mathcal{L}_\xi g + 2\mu \eta \otimes \eta + 2\Lambda g + 2S^* = 0, \quad (5)$$

where $S^*$ is $\ast$-Ricci tensor, which was given by Tachibana [33] on almost Hermitian manifolds and then developed by Hamada [22] on real hypersurfaces of non-flat complex space forms.

Very recently, in 2019, S.R. et al. [32] defined a $\ast \cdot \eta$-conformal $\eta$-Ricci soliton as,

$$\mathcal{L}_\xi g + 2S^* + \left[2\lambda - \left(p + \frac{2}{n}\right)ight]g + 2\mu \eta \otimes \eta = 0. \quad (6)$$

Motivated by the above generalizations, we now give the definition of a $\ast \cdot \eta$-conformal $\eta$-Ricci–Yamabe soliton as:

**Definition 3.** We say that the Riemannian manifold $(M, g)$ of dimension $n$ admit $\ast \cdot \eta$-conformal $\eta$-Ricci–Yamabe soliton, if

$$\mathcal{L}_\xi g + 2\mu \eta \otimes \eta + 2\Lambda g + 2S^* + \left[2\Lambda - \left(p + \frac{2}{n}\right) - qr^*\right]g = 0,$$

where $\ast$-scalar curvature is $r^* = \text{Tr}(S^*)$.

Moreover, in the above equation, if the vector field $\xi$ is the gradient of a smooth function $f$ (denoted by $Df$, where $D$ denotes the gradient operator) then the Equation (7) is called a gradient $\ast \cdot \eta$-Ricci–Yamabe soliton and it is defined as:

$$\text{Hess} f + \mu \eta \otimes \eta + \rho S^* + \left[\Lambda - \frac{1}{2} \left(p + \frac{2}{n}\right) - qr^*\right]g = 0,$$

where, $\text{Hess} f$ is the Hessian of the smooth function $f$.

On the other hand, we call a linear connection $\tilde{\nabla}$ on an $n$-dimensional Riemannian manifold $(M, g)$ as a quarter-symmetric connection $[16,19]$, if its torsion tensor of the connection $\tilde{\nabla}$,

$$T(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y]$$

satisfies,

$$T(X, Y) = -\eta(X)\phi Y + \eta(Y)\phi X,$$

where, $\eta$ is differentiable 1-form, $\phi$ is $(1, 1)$ tensor field.

Furthermore, if $\tilde{\nabla}$ satisfies,

$$(\tilde{\nabla}_X g)(Y, Z) = 0,$$

for all of vector fields $X, Z, Y$ on $(M, g)$, after that, we call it a quarter-symmetric metric connection $[16,19]$.

If $\phi X$ is replaced by $X$, after that, we call the connection a semi-symmetric metric connection $[34]$. 

Based on the above facts and discussions in the research of contact geometry, a natural question arises:

The question is that if there are almost contact metric manifolds, that metrics are $*$-conformal $\eta$-Ricci–Yamabe soliton?

In some next sections, we explain that the answer to the question is affirmative. This paper is organized as below: In Section 2, we talk about some basic formulas of contact geometry and curvature properties of $\alpha$-cosymplectic manifolds. In Section 3, we demonstrate some features of $n$-dimensional $\alpha$-cosymplectic manifold in regard to quarter-symmetric metric connection. For Section 4, we allocate with $*$-conformal $\eta$-Ricci–Yamabe soliton admitting quarter-symmetric metric connection in $\alpha$-cosymplectic manifolds. In this section, we develop the relationship between soliton constants $\Lambda$ and $\mu$. Section 5 is also committed to construct the Laplace equation from $\eta$-Ricci–Yamabe soliton equation in terms of quarter-symmetric metric connection when the potential vector field $\zeta$ of the soliton is of a gradient type. Further, we have also put up some applications on Laplace equation and proved some theorems on the harmonic aspect of $*$-conformal $\eta$-Ricci–Yamabe soliton in $n$-dimensional $\alpha$-cosymplectic manifold with a quarter-symmetric metric connection in Sections 5 and 6, respectively. We next contemplated the potential vector field $V$ of the soliton as a conformal killing vector field to characterize the vector field to accessorize the nature of this soliton on this manifold with a quarter-symmetric metric connection. In Section 7, we have set up an example to adorn an alive $*$-conformal $\eta$-Ricci–Yamabe soliton on a 5-dimensional $\alpha$-cosymplectic manifold, which acknowledges the quarter-symmetric metric connection.

2. Background Materials and Preliminaries

In this part, we give some basic notions and background materials which we need in this paper. We consider a $(2n+1)$-dimensional connected almost by contact metric manifold $M$. The $M$ with an almost contact metric structure $(\varphi, \xi, \eta, g)$, here $g$ is the compatible Riemannian metric, $\eta$ is 1-form, $\xi$ is the vector field, $\varphi$ is $(1, 1)$ the tensor field, such that [10],

$$\varphi \xi = 0, \eta \circ \varphi = 0, \quad (11)$$

$$\varphi^2 (X) = -X + \eta (X) \xi, \eta (\xi) = 1,$$

$$-g(\varphi X, Y) = g(X, \varphi Y), \quad (12)$$

$$g(\varphi X, \varphi Y) = -\eta (X) \eta (Y) + g(X, Y), \quad (13)$$

$$g(X, \xi) = \eta (X) \quad \text{(14)}$$

for all of vector fields $Y, X \in \chi(M)$, here, $\chi(M)$ denotes the collection of all of smooth vector fields of $M$.

On this manifold $M$, the 2-form $\Phi$ is defined as

$$\Phi(X, Y) = g(\varphi X, Y) \quad (15)$$

for all of vector fields $Y, X \in \chi(M)$.

Suppose that $(M, \varphi, \xi, \eta, g)$ is an almost contact metric manifold, we call it an almost cosymplectic [20,23] if $d\Phi = 0$, $d\eta = 0$, here, $d$ is to represent the exterior differential operator.

Suppose that $(M, \varphi, \xi, \eta, g)$ is an almost contact manifold, we call it normal if the Nijenhuis torsion,

$$N_\varphi(X, Y) = 2d\eta(X, Y)\xi + \varphi^2 [X, Y]$$

$$-\varphi [X, \varphi Y] + [\varphi X, \varphi Y - \varphi [\varphi X, Y]]$$

vanishes for any of vector fields $Y$ and $X$. We call the normal almost cosymplectic manifold as a cosymplectic manifold [20,23].
Suppose that \( M \) is an almost contact metric manifold, if \( \alpha \) is a non-zero real constant, \( d\Phi = 2\alpha \eta \wedge \Phi, d\eta = 0 \), then we call \( M \) as an almost \( \alpha \)-Kenmotsu.

In 2005, a combination of almost \( \alpha \)-Kenmotsu and almost cosymplectic manifolds was developed by Kim and Pak [24], into an almost \( \alpha \)-cosymplectic manifold, and \( \alpha \) as a scalar. An almost \( \alpha \)-cosymplectic manifold can be defined as [23,35],

\[
d\Phi = 2\alpha \eta \wedge \Phi, d\eta = 0
\]

for any real number \( \alpha \).

We call the normal almost \( \alpha \)-cosymplectic manifold as the \( \alpha \)-cosymplectic manifold. There are two different kinds for \( \alpha \)-cosymplectic depending on different conditions. If \( \alpha = 0 \), then the \( \alpha \)-cosymplectic manifold is cosymplectic. If \( \alpha \neq 0 \), for \( \alpha \in \mathbb{R} \), then the \( \alpha \)-cosymplectic manifold is \( \alpha \)-Kenmotsu [23].

In an \( \alpha \)-cosymplectic manifold, we have [23,35]

\[
(\nabla_X \phi)Y = a(g(\phi X, Y)\xi, \eta(Y)\phi X).
\]

(16)

Suppose that \( M \) is an \( n \)-dimensional \( \alpha \)-cosymplectic manifold. Then, according to (16), we obtain

\[
\nabla_X \xi = -\alpha \phi^2 X = a[\eta(X)\xi + X],
\]

(17)

here \( \nabla \) is the Levi-Civita connection associated with \( g \).

On an \( n \)-dimensional \( \alpha \)-cosymplectic manifold \( M \), we have the below relationship [23]

\[
S(X, \xi) = -a^2(-1 + n)\eta(X)
\]

(18)

\[
R(\xi, X)\xi = a^2[-\eta(X)\xi + X],
\]

(19)

\[
R(X, Y)\xi = a^2[-\eta(Y)X + \eta(X)Y],
\]

(20)

\[
\eta(R(X, Y)Z) = a^2[-\eta(X)g(Y, Z) + \eta(Y)g(X, Z)],
\]

(21)

\[
R(\xi, X)Y = a^2[-\eta(X)\xi + \eta(Y)X],
\]

(22)

for all of vector fields \( X, Z, Y \) on \( M \), where \( S \) is the Ricci tensor of \( M \), and \( R \) is the Riemannian curvature tensor.

In paper [23], Lemma 2.2, the authors give the proof of the \( \star \)-Ricci tensor on an \( n \)-dimensional \( \alpha \)-cosymplectic manifold as:

\[
S^\star(Y, Z) = a^2\eta(Y)\eta(Z) + a^2(n - 2)g(Y, Z) + S(Y, Z)
\]

(23)

for any vector field \( Z, Y \) on \( M \), here \( S^\star \) is \( \star \)-Ricci tensor for type \((0, 2)\) on \( M \), \( S \) is Ricci tensor for type \((0, 2)\) on \( M \).

Taking \( Z = Y = e_i \), here \( e_i \)'s is the orthonormal basis of \( T_p(M) \) for \( i = 1, 2, ...n \), after that, we obtain that

\[
r^\star = a^2(n^2 - 2n + 1) + r,
\]

(24)

here, \( r^\star = \text{Tr}(S^\star) \) is \( \star \)-scalar curvature, and \( r \) is scalar curvature.

3. On an \( n \)-Dimensional \( \alpha \)-Cosymplectic Manifold in Terms of Quarter-Symmetric Metric Connection

Suppose that \( \tilde{\nabla} \) is a linear connection, \( \nabla \) is a Levi-Civita connection of an almost contact metric manifold \( M \) such that

\[
\tilde{\nabla}_X Y = H(X, Y) + \nabla_X Y,
\]

(25)
here, \( H \) is the tensor for type \((1,1)\). For \( \tilde{\nabla} \) to be a quarter-symmetric metric connection on \( M \), after that, we get \([19]\)

\[
H(X,Y) = \frac{1}{2} T'(Y,X) + \frac{1}{2} T(X,Y) + \frac{1}{2} T'(X,Y),
\]  

(26)

here

\[
g(T(Z,X),Y) = g(T'(X,Y),Z).
\]  

(27)

Now according to equation (10), (27), we acquire

\[
T'(X,Y) = -\eta(X)\phi Y + g(\phi Y, X)\xi.
\]  

(28)

Now, we use the equations (9), (28) and (26) to yield

\[
-\eta(X)\phi Y = H(X,Y).
\]  

(29)

Therefore, \( \tilde{\nabla} \) in a \( \alpha \)-cosymplectic manifold is defined by

\[
\tilde{\nabla}XY = -\eta(X)\phi Y + \nabla XY.
\]  

(30)

Suppose that \( \tilde{R} \) are curvature tensors of \( \tilde{\nabla} \) of a \( \alpha \)-cosymplectic manifold, \( R \) is curvature tensors of \( \nabla \) of a \( \alpha \)-cosymplectic manifold.

After that, using (30), (17), we infer

\[
\tilde{R}(X,Y)Z = -\eta(Y)(\nabla_X\phi)Z + \eta(X)(\nabla_Y\phi)Z + R(X,Y)Z.
\]  

(31)

At present, we fetch the above equation to the identity (16) to arrive

\[
\tilde{R}(X,Y)Z = a\eta(Y)\eta(Z)\phi X - a\eta(X)\eta(Z)\phi Y - a\eta(Y)g(\phi X, Z)\xi + a\eta(X)g(\phi Y, Z) + R(X,Y)Z.
\]  

(32)

Moreover, we could give a nice relationship between the curvature tensor of \( M \) in regard to \( \tilde{\nabla} \) and \( \nabla \). The relation is shown in the Equation (32).

We take an inner product of (32) with \( W \) to yield

\[
\tilde{S}(Y,Z) = a\eta(Y)\eta(Z)g(\phi Y, Z) - a\eta(Y)\eta(W)g(\phi Y, W) - a\eta(Y)\eta(W)g(\phi Y, Z) + a\eta(X)\eta(W)g(\phi Y, Z) + R(X,Y, Z, W).
\]  

(33)

Taking (33) over \( W, X \), we achieve

\[
\tilde{S}(Y,Z) = a\eta(Y)\eta(Z)g(\phi Y, Z) + S(Y, Z).
\]  

(34)

here, \( \tilde{S} \) are the Ricci tensors of \( \tilde{\nabla} \), \( S \) are the Ricci tensors of \( \nabla \).

Therefore, it is not symmetric for the Ricci tensor of the quarter-symmetric metric connection in \( \alpha \)-cosymplectic manifold.

Once again, we contract (34) over \( Y, Z \) to find

\[
\tilde{r} = r,
\]  

(35)

here, \( \tilde{r} \) is the scalar curvatures of \( \tilde{\nabla} \), \( r \) is the scalar curvatures of \( \nabla \).
Therefore, we obtain the following theorem:

**Theorem 1.** We consider a $\alpha$-cosymplectic manifold $M$ with $\tilde{\nabla}$, then

(i) $\tilde{r} = r$,

(ii) $\tilde{S}$ is not symmetric,

(iii) $\tilde{S}(Y, \zeta) = -\alpha^2(n-1)\eta(Y) = S(Y, \zeta)$,

(iv) $\tilde{R}(Y, X, Z, W) + \tilde{R}(X, Y, Z, W) = 0$,

(v) $\tilde{S}$ is given by (34),

(vi) $\tilde{R}$ is given by (32). (vii) $\tilde{R}(X, Y, Z, W) + \tilde{R}(X, Y, W, Z) = 0$,

### 4. Main Results

In this part, we present our main results in the paper. Suppose that $M$ is an $n$-dimensional $\alpha$-cosymplectic manifold, which admits a $\ast$-conformal $\eta$-Ricci–Yamabe soliton. Then from (7), we acquire

$$2\mu\eta(Y)\eta(Z) + (\tilde{L}_Y g)(Y, Z) + 2\rho S(Y, Z) + \left[2\Lambda - \left(p + \frac{2}{n} \right) - q\rho \right]g(Y, Z) = 0$$

(36)

for any vector fields $Z, Y$ on $M$.

Now, we fetch the above equation into the identities (23) and (24) to arrive

$$(\tilde{L}_Y g)(Y, Z) + 2\rho S(Y, Z) + \left[2\Lambda - \left(p + \frac{2}{n} \right) + 2\alpha^2\rho(n-2) - qr \right. \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qandaq; \left.qr \right]g(Y, Z) + \left[2\mu + 2\alpha^2\rho \right]g(Y)g(Z) = 0.$$ \hspace{1cm} (37)

We consider $\tilde{L}_V$ is the Lie derivative along $V$ in regard to the quarter-symmetric metric connection $\nabla$.

Now, if we consider $M$ that is the $n$-dimensional $\alpha$-cosymplectic manifold that admits a $\ast$-conformal $\eta$-Ricci–Yamabe soliton in regard to $\tilde{\nabla}$, then from (37), we achieve

$$[2\mu + 2\alpha^2\rho]g(Y)g(Z) + (\tilde{L}_Y g)(Y, Z) + 2\rho S(Y, Z) + \left[2\Lambda - \left(p + \frac{2}{n} \right) + 2\alpha^2\rho(n-2) - qr \right. \qquad \qquad \qquad \qquad \qandaq; \left.qr \right]g(Y, Z) + \left[2\mu + 2\alpha^2\rho \right]g(Y)g(Z) = 0.$$ \hspace{1cm} (38)

From the viewpoint of (30), we infer

$$(\tilde{L}_V g)(X, Y) = g(\tilde{\nabla}_X V, Y) + g(X, \tilde{\nabla}_Y V)$$

$$(\tilde{L}_V g)(X, Y) - \eta(Y)g(X, \phi V) - \eta(X)g(\phi V, Y).$$

(39)

Then, we use the identity (11) to get

$$(\tilde{L}_Y g)(X, Y) = (\tilde{L}_X g)(X, Y)$$

(40)

for any of vector fields $Y, X$ on $M$.

From (40), (34) and (35), (38) becomes

$$\left[2\rho S(Y, Z) + \tilde{L}_Y g)(Y, Z) + \left[2\Lambda - \left(p + \frac{2}{n} \right) + 2\alpha^2\rho(n-2) - qr \right. \qquad \qquad \qandaq; \left.qr \right]g(Y, Z) + \left[2\mu + 2\alpha^2\rho \right]g(Y)g(Z) \right]g(Y, Z) = 0.$$ \hspace{1cm} (41)

Now in an $n$-dimensional $\alpha$-cosymplectic manifold, according to (17), we acquire

$$(\tilde{L}_Y g)(Y, Z) = g(Y, \nabla_Z \xi) + g(\nabla_Y \xi, Z)$$

$$= 2\alpha[-\eta(Y)g(Z) + g(Y, Z)].$$

(42)
We put this value in (41) and then take $Z = \xi$ to obtain
\[
2\rho S(Y, \xi) + \left[2\Lambda - \left(p + \frac{2}{n}\right) + 2\alpha^2\rho(n - 2) - qr - q\alpha^2(n - 1)^2\right]\eta(Y)
\]
\[+ \left[2\mu + 2\alpha^2\rho\right]\eta(Y) = 0. \tag{43}
\]
Using (18), we achieve
\[
\Lambda + \mu = \frac{1}{2} \left(p + \frac{2}{n}\right) + \frac{qr}{2} + \frac{q\alpha^2(n - 1)^2}{2}. \tag{44}
\]
Therefore, we could obtain the following theorem.

**Theorem 2.** If the metric of an n-dimensional $\alpha$-cosymplectic manifold admits a $*$-conformal $\eta$-Ricci–Yamabe soliton in regard to $\tilde{\nabla}$, then the soliton constants $\Lambda$ and $\mu$ are related by the Equation (44).

Again, using (37), (41) takes the form
\[
\alpha\rho g(\phi Y, Z) = 0. \tag{45}
\]
for any vector fields $Z, Y$ on $M$.

Then, $\alpha = 0$, since in a $*$-conformal $\eta$-Ricci–Yamabe soliton $\rho$ could not become zero. Therefore, for $\alpha = 0$, then $M$ reduces to an $n$-dimensional cosymplectic manifold.

This gives,

**Corollary 1.** If the metric of an n-dimensional $\alpha$-cosymplectic manifold admits a $*$-conformal $\eta$-Ricci–Yamabe soliton in regard to the Levi-Civita connection $\nabla$. Then the metric admits the soliton is regard to a quarter-symmetric metric connection $\tilde{\nabla}$ if and only if the manifold becomes an $n$-dimensional cosymplectic manifold.

We take $Z = Y = e_i$, here $e_i$'s are the orthonormal basis of $T_p(M)$ for $i = 1, 2, ..., n$ to infer
\[
div\xi + \rho r + n \left[\Lambda - \frac{1}{2} \left(p + \frac{2}{n}\right) + \alpha^2\rho(n - 2) - \frac{qr}{2} - \frac{q\alpha^2(n^2 - 2n + 1)}{2}\right]
\]
\[+ \mu + \alpha^2\rho = 0, \tag{46}
\]
where $div\xi$ is the divergence of the vector field $\xi$.

Replacing the value of $\mu$ from (44) in the above equation, $\Lambda$ takes the form
\[
\Lambda = \frac{1}{2} \left(\frac{2}{n} + p\right) + \frac{qr}{2} - \frac{div\xi + \rho r}{n - 1} + \frac{q\alpha^2(n^2 - 2n + 1)}{2} - \alpha^2\rho(n - 1). \tag{47}
\]
In view of (47), (44) becomes
\[
\mu = \alpha^2\rho(n - 1) + \frac{div\xi + \rho r}{n - 1}. \tag{48}
\]
Hence, we can state

**Corollary 2.** If the metric of an n-dimensional $\alpha$-cosymplectic manifold admits a $*$-conformal $\eta$-Ricci–Yamabe soliton in regard to $\tilde{\nabla}$. Then the soliton constants $\Lambda$ and $\mu$ takes the form of
\[
\frac{qr}{2} + \frac{1}{2} \left(\frac{2}{n} + p\right) + \frac{q\alpha^2(n^2 - 2n + 1)}{2} - \alpha^2\rho(n - 1) - \frac{div\xi + \rho r}{n - 1},
\]
and $\alpha^2\rho(n - 1) + \frac{div\xi + \rho r}{n - 1}$,
respectively, where $\text{div}\xi$ is the divergence of $\xi$.

If $\xi = \text{grad}(f)$, where $\text{grad}(f)$ is the gradient of the smooth function $f$, after that, from (46), we have the following:

**Theorem 3.** Suppose that an $n$-dimensional $\ast$-cosymplectic manifold admits a $\ast$-conformal $\eta$-Ricci–Yamabe soliton in regard to $\nabla$. If the vector field $\xi$, associated with the soliton, is of the form $\text{grad}(f)$, where $f$ is a smooth function. After that, Laplacian equation satisfied by $f$ becomes:

\[
\Delta(f) = -n\left[\Lambda - \frac{1}{2}\left(p + \frac{2}{n}\right) + \alpha^2 \rho(n - 2) - \frac{qr}{2} - \frac{q\alpha^2(n - 1)^2}{2}\right] - \rho(\alpha^2 + r) - \mu. \tag{49}
\]

**5. Some Applications**

As an application, we obtain the following results for the $\ast$-conformal $\eta$-Ricci soliton, $\ast$-conformal $\eta$-Yamabe soliton, and $\ast$-conformal $\eta$-Einstein soliton ($\rho = 1, q = 0, \rho = 0, q = 1$, and $\rho = 1, q = -1$ (cf. [1,3,11])).

**Corollary 3.** Suppose that the $n$-dimensional $\ast$-cosymplectic manifold admits a $\ast$-conformal $\eta$-Yamabe soliton in regard to $\nabla$. If the vector field $\xi$, associated with the soliton, is the form $\text{grad}(f)$, and $f$ is a smooth function. After that, Laplacian equation satisfied by $f$ becomes:

\[
\Delta(f) = -n\left[\Lambda - \frac{1}{2}\left(p + \frac{2}{n}\right) + \alpha^2(n - 2)\right] - \mu - (\alpha^2 + r). \tag{50}
\]

**Corollary 4.** Suppose that the $n$-dimensional $\ast$-cosymplectic manifold admits a $\ast$-conformal $\eta$-Yamabe soliton in regard to $\nabla$. If the vector field $\xi$, associated with the soliton, is the form $\text{grad}(f)$, and $f$ is a smooth function. After that, Laplacian equation satisfied by $f$ becomes:

\[
\Delta(f) = -n\left[\Lambda - \frac{1}{2}\left(p + \frac{2}{n}\right) - \frac{r}{2} - \frac{\alpha^2(n - 1)^2}{2}\right] - \mu. \tag{51}
\]

**Corollary 5.** Suppose that the $n$-dimensional $\ast$-cosymplectic manifold admits a $\ast$-conformal $\eta$-Einstein soliton in regard to $\nabla$. If the vector field $\xi$, associated with the soliton, is the form $\text{grad}(f)$, where $f$ is a smooth function. After that, the Laplacian equation satisfied by $f$ becomes:

\[
\Delta(f) = -n\left[\Lambda - \frac{1}{2}\left(p + \frac{2}{n}\right) + \alpha^2(n - 2) + \frac{r}{2} + \frac{\alpha^2(n^2 - 2n + 1)}{2}\right] - \mu - (\alpha^2 + r). \tag{52}
\]

**6. Harmonic Aspect of $\ast$-Conformal $\eta$-Ricci–Yamabe Soliton on $n$-Dimensional $\ast$-Cosymplectic Manifold Admitting Quarter-Symmetric Metric Connection**

This section is based on the fact that a function $f : M \to \mathbb{R}$, we say that it is harmonic, if $\Delta f = 0$, here $\Delta$ is the Laplacian operator on $M$ [36]. Therefore, let us consider the fact that the $\xi$ is the gradient of a harmonic function $f$, after that, from Theorem 3, we can give the following conclusions:

**Theorem 4.** Suppose that $M$ is the $n$-dimensional $\ast$-cosymplectic manifold that admits a $\ast$-conformal $\eta$-Ricci–Yamabe soliton in regard to $\nabla$ and $\xi$, associated with the soliton, is of the form $\text{grad}(f)$ if $f$ is a harmonic function on $M$. After that, the $\ast$-conformal $\eta$-Ricci–Yamabe soliton is expanding, steady, and shrinking:

1. \(\frac{q}{2}\left(r + \alpha^2(n - 1)^2\right) + \frac{1}{2}\left(p + \frac{2}{n}\right) > \left\{\frac{\mu + p\alpha^2 + r}{n}\right\} + \alpha^2 \rho(n - 2)\},\)
2. \(\frac{q}{2}\left(r + \alpha^2(n - 1)^2\right) + \frac{1}{2}\left(p + \frac{2}{n}\right) = \left\{\frac{\mu + p\alpha^2 + r}{n}\right\} + \alpha^2 \rho(n - 2)\} \) and
3. \(\frac{q}{2}\left(r + \alpha^2(n - 1)^2\right) + \frac{1}{2}\left(p + \frac{2}{n}\right) < \left\{\frac{\mu + p\alpha^2 + r}{n}\right\} + \alpha^2 \rho(n - 2)\} \), respectively.
Proof. From Equation (49), we can easily obtain the desired result. \( \square \)

**Theorem 5.** Suppose that \( M \) is the \( n \)-dimensional \( \ast \)-cosymplectic manifold that admits a \( \ast \)-conformal \( \eta \)-Ricci soliton in regard to \( \nabla \) and the vector field \( \xi \), associated with the soliton is of the form \( \nabla f \) if \( f \) is a harmonic function on \( M \). Then, the \( \ast \)-conformal \( \eta \)-Ricci soliton is shrinking.

**Proof.** Using the Equation (50), we can easily get the required result. \( \square \)

**Theorem 6.** Let \( M \) be the \( n \)-dimensional \( \ast \)-cosymplectic manifold that admits a \( \ast \)-conformal \( \eta \)-Yamabe soliton in regard to \( \nabla \) and \( \xi \), associated with the soliton is of the form \( \nabla f \) if \( f \) is a harmonic function on \( M \). After that, the \( \ast \)-conformal \( \eta \)-Yamabe soliton is expanding, steady, and shrinking:

1. \( \frac{1}{2} \left\{ \left( p + \frac{3}{2} \right) + r + \alpha^2(n^2 - 2n + 1) \right\} > \frac{\mu}{n} \),
2. \( \frac{1}{2} \left\{ \left( p + \frac{3}{2} \right) + r + \alpha^2(n^2 - 2n + 1) \right\} = \frac{\mu}{n} \) and
3. \( \frac{1}{2} \left\{ \left( p + \frac{3}{2} \right) + r + \alpha^2(n^2 - 2n + 1) \right\} < \frac{\mu}{n} \), respectively.

**Proof.** From Equation (51), we can easily obtain our necessary result. \( \square \)

**Theorem 7.** Suppose that \( M \) is the \( n \)-dimensional \( \ast \)-cosymplectic manifold that admits a \( \ast \)-conformal \( \eta \)-Einstein soliton in regard to \( \nabla \) and \( \xi \), associated with the soliton is of the form \( \nabla f \) if \( f \) is a harmonic function on \( M \). Then the \( \ast \)-conformal \( \eta \)-Einstein soliton is also shrinking.

**Proof.** We use Equation (52) to arrive our findings. \( \square \)

**Definition 4.** A vector field \( V \) is said to be a conformal Killing vector field if and only if the following relation holds:

\[
(\mathcal{L}_V g)(Y, Z) = 2\theta g(Y, Z),
\]

(53)

where \( \theta \) is some function of the co-ordinates (conformal scalar).

Moreover, if \( \theta \) is not constant, we say that the conformal killing vector field \( V \) is proper. Additionally, when \( \theta \) is constant, we say that \( V \) is a homothetic vector field and when the constant \( \theta \) becomes non zero, we say that \( V \) is proper homothetic vector field. If \( \theta = 0 \) in the above equation, then we call \( V \) the Killing vector field.

We consider \( M \) that is the \( n \)-dimensional \( \ast \)-cosymplectic manifold and admitting a \( \ast \)-conformal \( \eta \)-Ricci–Yamabe soliton, here, \( \xi = V \).

After that, using (7), (23), and (24), we obtain that

\[
(\mathcal{L}_V g)(Y, Z) + 2\rho S(Y, Z) + \left[ 2\Lambda - \left( p + \frac{2}{n} \right) + 2\alpha^2 \rho(n - 2) - qr \right.
\]

\[
-qa^2(n^2 + 1 - 2n) \bigg] g(Y, Z) + 2\mu \eta(Z) \eta(Y) + 2\alpha^2 \rho \eta(Z) \eta(Y) = 0
\]

(54)

Now, if \( M \), which is the \( n \)-dimensional \( \ast \)-cosymplectic manifold that admits a \( \ast \)-conformal \( \eta \)-Ricci–Yamabe soliton \( (g, V, \Lambda, \rho, q) \) in regard to \( \tilde{\nabla} \), after that, according to (54), we acquire

\[
(\tilde{\nabla}_V g)(Y, Z) + 2\rho \tilde{S}(Y, Z) + \left[ 2\Lambda - \left( p + \frac{2}{n} \right) + 2\alpha^2 \rho(-2 + n) - q\tilde{r} \right.
\]

\[
-qa^2(n^2 + 1 - 2n) \bigg] g(Y, Z) + 2\mu \eta(Z) \eta(Y) + 2\alpha^2 \rho \eta(Z) \eta(Y) = 0
\]

(55)
In the view of (34), (35), and (39), the above equation provides:

\[ -\eta(Y)g(\phi V, Z) + (\xi g)(Y, Z) - \eta(Z)g(Y, \phi V) + 2\varrho [S(Y, Z) + 2\varphi g(\phi Y, Z)] \] 
\[ + \left[ 2\Lambda - \left( \frac{2}{n} + p \right) + 2\alpha^2 \rho(n - 2) - qr - q\alpha^2(n^2 + 1 - 2n) \right] g(Y, Z) \]
\[ 2\mu \eta(Z)\eta(Y) + 2\alpha^2 \rho \eta(Z)\eta(Y) + 2\varrho S(Y, Z) + 2\varphi g(\phi Y, Z) \]
\[ - \eta(Z)g(Y, \phi V) - \eta(Y)g(\phi V, Z) = 0. \]

We use the Equation (53) into the identity (56) to arrive

\[ \left[ 2\theta + 2\Lambda - \left( \frac{2}{n} + p \right) + 2\mu - qr - q\alpha^2(n^2 - 2n + 1) \right] \eta(Y), \] 

which can be written as

\[ \phi V = \left[ 2\theta + 2\Lambda - \left( \frac{2}{n} + p \right) + 2\mu - qr - q\alpha^2(n - 1)^2 \right] \xi. \]

**Theorem 8.** Suppose that the n-dimensional \( \alpha \)-cosymplectic manifold that admits a *-conformal \( \eta \)-Ricci–Yamabe soliton \((g, V, \Lambda, \mu, \rho, q)\) in regard to \( \nabla \). If \( V \) is a conformal killing vector field, then (58) holds.

**Remark 1.** For particular values of \( \rho \) and \( q \), we can also obtain the similar results such as Theorem 8 for *-conformal \( \eta \)-Ricci soliton, and for *-conformal \( \eta \)-Yamabe soliton, and for *-conformal \( \eta \)-Einstein soliton, respectively.

7. Example of a 5-Dimensional \( \alpha \)-Cosymplectic Metric as a *-Conformal \( \eta \)-Ricci–Yamabe Soliton in Regard to Quarter-Symmetric Metric Connection

Suppose that \( M = \{ (x, y, z, u, v) \in \mathbb{R}^5 \} \) is the 5-dimensional manifold, and \((x, y, z, u, v)\) are standard coordinates in \( \mathbb{R}^5 \).

Suppose that \( e_2, e_3, e_4, e_5, e_3 \) is linearly independent frame field for \( M \), the formulas are in the following:

\[ e_2 = e^{\alpha\beta} \frac{\partial}{\partial y}, \quad e_1 = e^{\alpha\beta} \frac{\partial}{\partial x}, \quad e_3 = e^{\alpha\beta} \frac{\partial}{\partial z}, \quad e_5 = -\frac{\partial}{\partial u}, \quad e_4 = e^{\alpha\beta} \frac{\partial}{\partial u}. \]

Suppose that \( g \) is Riemannian metric defined by the following:

\[ g(e_i, e_j) = 0, i, j = 1, 2, 3, 4, 5, i \neq j, \]
\[ g(e_1, e_1) = 1, g(e_3, e_3) = 1, g(e_4, e_4) = 1, g(e_5, e_5) = 1, g(e_2, e_2) = 1. \]

Suppose that \( \eta \) is a 1-form that is \( \eta(Z) = g(Z, e_5) \) for any \( Z \in \chi(M) \), \( \phi \) is (1, 1)-tensor field given by:

\[ -e_2 = \phi e_1, \quad e_1 = \phi e_2, \quad -e_4 = \phi e_3, \quad e_3 = \phi e_4, \quad 0 = \phi e_5. \]

where \( \chi(M) \) is the set of differentiable vector fields for \( M \).
Moreover, taking the linearity of $g, \phi$, for any $U, Z \in \chi(M)$, we obtain that,

\[
g(\phi Z, \phi U) = -\eta(Z)\eta(U) + g(Z, U),
\]
\[
\phi^2(Z) = -Z + \eta(Z)e_5,
\]
\[
\eta(e_5) = 1.
\]

Therefore, $e_5 = \xi$. Suppose that $\nabla$ is the Levi-Civita connection in regard to $g$. Then, we obtain that

\[
[e_2, e_5] = ae_2, [e_1, e_5] = ae_1, [e_3, e_5] = ae_3
\]
\[
[e_4, e_5] = ae_4, \text{ and for others } i, j, [e_i, e_j] = 0.
\]

$\nabla$ of the metric $g$ is defined by

\[
2g(\nabla_X Y, Z) = -Zg(X, Y) + Yg(Z, X)
\]
\[
+ g(Z, [X, Y] - g(X, [Y, Z]) - g(Y, [X, Z])) + Xg(Y, Z),
\]

this formula is known as Koszul’s formula.

Taking Koszul’s formula, we obtain that

\[
\nabla_{e_1} e_5 = ae_1, \nabla_{e_1} e_4 = 0, \nabla_{e_1} e_3 = 0,
\]
\[
\nabla_{e_2} e_5 = ae_2, \nabla_{e_2} e_4 = 0, \nabla_{e_2} e_3 = 0,
\]
\[
\nabla_{e_3} e_5 = ae_3, \nabla_{e_3} e_4 = 0, \nabla_{e_3} e_3 = -ae_5,
\]
\[
\nabla_{e_4} e_5 = ae_4, \nabla_{e_4} e_4 = -ae_5, \nabla_{e_4} e_3 = 0,
\]
\[
\nabla_{e_5} e_5 = 0, \nabla_{e_5} e_4 = 0, \nabla_{e_5} e_3 = 0,
\]

It is easy to know that for $e_5 = \xi$, the manifold satisfies,

\[
(\nabla_X \phi) Y = a[g(\phi X, Y)\xi - \eta(Y)\phi X],
\]
\[
\nabla_X \xi = a[-\eta(X)\xi + X].
\]

Therefore, $M$ is the $\alpha$-cosymplectic manifold.

Furthermore, $R$ is defined by,

\[
R(X, Y)Z = -\nabla_{[X,Y]}Z - \nabla_Y \nabla_X Z + \nabla_X \nabla_Y Z.
\]

Thus,

\[
R(e_1, e_4)e_4 = -a^2 e_1, R(e_1, e_5)e_5 = -a^2 e_1, R(e_1, e_2)e_2 - a^2 e_1, R(e_1, e_3)e_3 = -a^2 e_1,
\]
Then, taking (30), we acquire
\[ R(e_5, e_5) e_5 = a^2 e_3, R(e_2, e_3) e_2 = a^2 e_3, R(e_1, e_2) e_1 = a^2 e_2, R(e_1, e_3) e_1 = a^2 e_3, \]
\[ R(e_2, e_5) e_5 = -a^2 e_2, R(e_3, e_4) e_4 = -a^2 e_3, R(e_2, e_4) e_4 = -a^2 e_2, R(e_2, e_3) e_3 = -a^2 e_2, \]
\[ R(e_4, e_5) e_5 = a^2 e_5, R(e_3, e_5) e_3 = a^2 e_5, R((e_1, e_5) e_1 = a^2 e_5, R(e_2, e_5) e_2 = a^2 e_5, \]
\[ R(e_5, e_4) e_5 = a^2 e_4, R(e_1, e_4) e_1 = a^2 e_4, R(e_2, e_4) e_2 = a^2 e_4, R(e_3, e_4) e_3 = a^2 e_4. \]

From the above relations, we have,
\[ S(e_1, e_1) = -4a^2, S(e_4, e_4) = -4a^2, S(e_3, e_3) = -4a^2, S(e_5, e_5) = -4a^2, S(e_2, e_2) = -4a^2. \]

Hence \( r = -20a^2 \).

Let \( \nabla \) be a quarter-symmetric metric connection on \( M \).

Then, taking (30), we acquire
\[ \begin{align*}
\nabla' e_1 e_5 &= ae_1, \nabla' e_1 e_3 = -ae_3, \\
\nabla' e_5 e_5 &= 0, \nabla' e_3 e_2 = 0, \nabla' e_1 e_4 = 0, \\
\nabla' e_2 e_5 &= ae_2, \nabla' e_2 e_2 = -ae_3, \\
\nabla' e_2 e_1 &= 0, \nabla' e_2 e_3 = 0, \nabla' e_4 e_4 = 0, \\
\nabla' e_3 e_5 &= ae_3, \nabla' e_3 e_3 = -ae_5, \\
\nabla' e_3 e_1 &= 0, \nabla' e_3 e_4 = 0, \nabla' e_5 e_2 = 0, \\
\nabla' e_4 e_5 &= ae_4, \nabla' e_4 e_4 = -ae_5, \\
\nabla' e_4 e_2 &= 0, \nabla' e_4 e_1 = 0, \nabla' e_5 e_3 = 0, \\
\nabla' e_5 e_1 &= e_2, \nabla' e_5 e_4 = -e_3, \nabla' e_5 e_3 = e_4, \\
\n\nabla' e_5 e_2 &= -e_1, \nabla' e_5 e_5 = 0.
\end{align*} \]

Then
\[ \begin{align*}
R(e_1, e_5) e_5 &= -a^2 e_1 - ae_2, R(e_1, e_2) e_2 = -a^2 e_1, \\
R(e_1, e_4) e_4 &= -a^2 e_1, R(e_1, e_3) e_3 = -a^2 e_1, \\
R(e_5, e_3) e_5 &= a^2 e_3 + ae_4, R(e_1, e_2) e_1 = a^2 e_2, \\
R(e_2, e_3) e_2 &= a^2 e_3, R(e_1, e_3) e_1 = a^2 e_3, \\
R(e_5, e_4) e_4 &= -a^2 e_5, R(e_1, e_4) e_5 = -a^2 e_2 + ae_1, \\
R(e_2, e_3) e_3 &= -a^2 e_2, R(e_2, e_4) e_4 = -a^2 e_2, \\
R(e_4, e_5) e_4 &= a^2 e_5, R(e_2, e_5) e_2 = a^2 e_5, \\
R(e_3, e_3) e_3 &= a^2 e_5, R((e_1, e_5) e_1 = a^2 e_5.
\end{align*} \]
\[ \tilde{R}(e_5, e_4) e_5 = \alpha^2 e_4 - \alpha e_3, \tilde{R}(e_1, e_4) e_1 = \alpha^2 e_4, \]
\[ \tilde{R}(e_3, e_4) e_3 = \alpha^2 e_4, \tilde{R}(e_2, e_4) e_2 = \alpha^2 e_4. \]

The above expressions of \( \tilde{R} \) satisfy our result (32).

From the above relations, we find
\[ \tilde{S}(e_2, e_2) = -4\alpha^2, \tilde{S}(e_4, e_4) = -4\alpha^2, \tilde{S}(e_1, e_1) = -4\alpha^2, \]
\[ \tilde{S}(e_5, e_5) = -4\alpha^2, \tilde{S}(e_3, e_3) = -4\alpha^2, \]
which satisfies our result (34).

Hence
\[ \tilde{r} = -20\alpha^2 = r, \]
which satisfies our result (35).

Now, we set \( X = Y = e_5 \) into identity (40) and use (43) to yield
\[ (\tilde{\mathcal{E}} \xi g)(e_5, e_5) = 0. \]

Then, we take \( X = Y = e_5 \) into identity (38) and use (60)–(62) to get
\[ -4\alpha^2 p + \left[ \Lambda - \frac{1}{2} \left( p + \frac{2}{n} \right) + 3\alpha^2 p + 2\alpha^2 q \right] + \left[ \mu + \alpha^2 \rho \right] = 0, \]
which provides
\[ \Lambda + \mu = -2\alpha^2 q + \frac{1}{2} \left( p + \frac{2}{n} \right). \]

Hence the above equation proves that, \( \Lambda \) and \( \mu \) satisfies our result (44) for \( n = 5 \) and \( g \) gives a \( \ast \)-conformal and \( \eta \)-conformal Ricci–Yamabe soliton on the 5-dimensional \( \alpha \)-cosymplectic manifold, admitting a quarter-symmetric metric connection \( \tilde{\nabla} \).

8. Conclusions and Remarks

The study of a \( \ast \)-conformal \( \eta \)-Ricci–Yamabe soliton on Riemannian manifolds and pseudo-Riemannian manifolds is a great importance in the area of differential geometry, especially in Riemannian geometry and in special relativistic physics as well. Basically, Ricci–Yamabe flow is the most prominent flagship of modern physics. Here we have characterized the \( \alpha \)-cosymplectic manifold, which admits the \( \ast \)-conformal \( \eta \)-Ricci–Yamabe soliton, in terms of a quarter-symmetric metric connection. The \( \ast \)-conformal \( \eta \)-Ricci–Yamabe soliton is a new notion not only in the area of differentiable manifold but in the area of mathematical physics, general relativity and quantum cosmology, quantum gravity, and black hole theory as well. It deals geometric and physical applications with relativistic viscous fluid spacetime, admitting heat flux and stress, dark and dust fluid general relativistic spacetime, and radiation era in general relativistic spacetime. When a manifold is endowed with a geometric structure, we have more opportunities to explore its geometric properties. Affine geometry, Riemannian geometry, contact geometry, Kähler geometry, CR geometry, or Finsler geometry and so on are all only a few examples of such differential geometric structures. Several theoretical and practical applications have been obtained over the years \([25,32,35,37–64]\). On the other hand, the theory of submanifolds represents an important field in differential geometry, especially when the ambient manifold carries geometric structures. The connection between the intrinsic geometry of the submanifold with its extrinsic geometry has been extensively developed in recent decades. Follow this, there are some topics that arise from this article that require further research.

(i) Is Theorem 2 true if we do not assume a quarter-symmetric connection on a \( \alpha \)-cosymplectic manifold?
(ii) Is Theorem 3 true without considering the potential vector field of the soliton is of gradient type?
(iii) If the conformal vector field is not killing, then is Theorem 8 true?
(iv) What are the results of our paper in true Trans-Sasakian manifolds, Co-Kähler manifold, or para-contact geometry?

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