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Phragmén-Lindelöf Alternative Results for a Class of Thermoelastic Plate

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Abstract: The spatial properties of solutions for a class of thermoelastic plate with biharmonic operator were studied. The energy method was used. We constructed an energy expression. A differential inequality which the energy expression was controlled by a second-order differential inequality is deduced. The *Phragmén-Lindelöf* alternative results of the solutions were obtained by solving the inequality. These results show that the Saint-Venant principle is also valid for the hyperbolic–hyperbolic coupling equations. Our results can been seen as a version of symmetry in inequality for studying the *Phragmén-Lindelöf* alternative results.

Keywords: thermoelastic plate; *Phragmén-Lindelöf* alternative; Saint-Venant principle; biharmonic equation

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1. Introduction

Saint Venant principle points out that for any equilibrium force system on an elastic body, if its action point is limited to a given ball, the displacement and stress generated by the equilibrium force system at any point where the distance from the load is far greater than the radius of the ball can be ignored. This principle is widely used in engineering mechanics in practice. Many papers in the literature dealt with the study of the Saint-Venant principle. For example, Horgan and Knowles [1] and Horgan [2,3] studied the Saint-Venant principle in different equations and different situations. The traditional characteristic of the Saint-Venant theorem is to derive the energy decay estimates of the solutions. Usually, these decays are exponential with the spatial distance from the finite end to the infinity. In order to have some understandings about the study of the Saint-Venant Principle, one could refer to the papers [4–9]. In recent years, the studies of Saint-Venant principle for hyperbolic or quasihyperbolic equations are abundant. Especially for the studies of the spatial behavior of viscoelasticity equations, we could see papers [10–13]. When the spatial variable tends to infinity, the solution is decreasing. In the research of solution spacial decay estimates, people often need to add the solutions must satisfy some constraints at infinity. Many scholars have begun to study the *Phragmén-Lindelöf* alternative results of solutions. The advantage of this situation is that there is no need to add constraints on the solutions at infinity. The classical *Phragmén-Lindelöf* theorem states that the solutions of the harmonic equation must grow exponentially or decay exponentially with distance from the finite end of the cylinder to infinity. Payne and Schaefer [14] extended the study from harmonic equation to biharmonic equation. They obtained the *Phragmén-Lindelöf* alternative results for biharmonic equation in three different regions. Literatures [15–18] studied the spatial behaviors of biharmonic equations by various methods. In particular, we can see that Liu and Lin [19] studied the spatial properties for time-dependent stokes equation. They transformed the equation to a biharmonic equation and obtained the *Phragmén-Lindelöf* results by using a second-order differential inequality. The abovementioned studies from the literature all consider a single equation. Recently, there some new results about the studies of the hyperbolic equations or biharmonic Equations have been published (see [20–25]). For studies of other equations using energy method, see [26–29].

The domain we consider in this paper is defined as follows:

$$\Omega_0 = \{(x_1, x_2)|x_1 > 0, 0 < x_2 < h\}, \tag{1}$$

with h is a given positive number. We now give the following notation:

$$L_z = \{(x_1, x_2)|x_1 = z \geq 0, 0 \leq x_2 \leq h\}. \tag{2}$$

In reference [30], the authors studied the coupled system of wave-plate type with thermal effect. They obtained the results of the analytic property and the exponential stability of the C_0-semigroup. The equations are as follows:

$$\begin{cases} \rho_1 u_{,tt} - \Delta u - \mu \Delta u_{,t} + \lambda \Delta vs. = 0, \\ \rho_2 v_{,tt} + \gamma \Delta^2 vs. + \lambda \Delta u + m \Delta \theta = 0, \\ \tau \theta_{,t} - k \Delta \theta - m \Delta v_{,t} = 0. \end{cases} \tag{3}$$

The model is used to represent the evolution process of a system which contains an elastic membrane and plate. The plate has an elastic force and a thermal effect (see [31]). Here u is the vertical deflection of the membrane and v is the vertical deflection of the plate. θ is the difference of temperature. The coefficients ρ_1, ρ_2, μ, λ, m,τ, γ, and k are nonnegative constants. Δ denotes the Laplace operator, and Δ^2 denotes the biharmonic operator.

In the present paper, we consider the case when $\tau = 0$. In this case, Equation (3) can be rewritten as:

$$\rho_1 u_{,tt} - \Delta u - \mu \Delta u_{,t} + \lambda \Delta vs. = 0, \tag{4}$$

$$\rho_2 v_{,tt} + \gamma \Delta^2 v + \lambda \Delta u - \frac{m^2}{k} \Delta v_{,t} = 0. \tag{5}$$

We give the following initial and boundary value conditions:

$$\begin{cases} v(x_1, 0, t) = u(x_1, 0, t) = u_{,2}(x_1, 0, t) = 0, x_1 > 0, t > 0, \\ v(x_1, h, t) = u(x_1, h, t) = u_{,2}(x_1, h, t) = 0, x_1 > 0, t > 0, \\ v(0, x_2, t) = g_1(x_2, t), 0 \leq x_2 \leq h, t > 0, \\ u(0, x_2, t) = g_2(x_2, t), 0 \leq x_2 \leq h, t > 0, \\ u_{,1}(0, x_2, t) = g_3(x_2, t), 0 \leq x_2 \leq h, t > 0, \\ v(x_1, x_2, 0) = u(x_1, x_2, 0) = u_{,t}(x_1, x_2, 0), 0 \leq x_2 \leq h, x_1 > 0, \end{cases} \tag{6}$$

where $g_i(x_2, t), i = 1, 2, 3$ are the given functions and meet the following compatibility conditions:

$$\begin{cases} g_1(0, t) = g_1(h, t) = g_{1,2}(0, t) = g_{1,2}(h, t) = 0, \\ g_2(0, t) = g_2(h, t) = g_{2,2}(0, t) = g_{2,2}(h, t) = 0, \\ g_3(0, t) = g_3(h, t) = g_{3,2}(0, t) = g_{3,2}(h, t) = 0, \\ g_1(x_2, 0) = g_2(x_2, 0) = g_3(x_2, 0) = 0. \end{cases} \tag{7}$$

We try to establish the *Phragmén-Lindelöf* alternative results for the solutions of the biharmonic Equations (4) and (5) under conditions (6) and (7). We firstly define an energy expression of the solutions, then we derive that the energy expression satisfies a second-order differential inequality, and finally we obtain the *Phragmén-Lindelöf* alternative results of the solutions by solving the second-order inequality. For the inequality is symmetry, we show the application of symmetry in mathematical inequalities in practice. Since the system is a hyperbolic–hyperbolic coupling system, how to define the appropriate energy function will be the greatest innovation in this paper. How to control the energy function will be the difficulty of this paper. No similar studies have been found on the spatial properties for the solutions of the biharmonic equations with hyperbolic–hyperbolic coupling equations using the second-order differential inequality. In this paper, we use the comma to represent partial differentiation. $,k$ denotes the differentiation with respect to the direction x_k, thus $u_{,\alpha}$ denotes $\frac{\partial u}{\partial x_\alpha}$, and $u_{,t}$ denotes $\frac{\partial u}{\partial t}$. The usual summation convection is employed with repeated Greek subscripts α summed from 1 to 2. Hence, $u_{\alpha,\alpha} = \sum_{\alpha=1}^{2} \frac{\partial u_\alpha}{\partial x_\alpha}$. The symbol $dA = dx_1 dx_2$.

2. Energy Expression $\Phi(z,t)$

In order to get the *Phragmén-Lindelöf* alternative results, we must define an energy expression for the solutions. This expression plays an important role in obtaining our results. The energy expression will be constructed by the following Lemmas.

Lemma 1. *Let u and v be classical solutions of problems (4)–(7), we define the a function $\varphi_1(z,t)$ as:*

$$\varphi_1(z,t) = \frac{\mu}{2}\int_0^t \int_{L_z} exp(-\omega\eta) u_{,\eta}^2 \mathrm{d}x_2 \mathrm{d}\eta + \lambda \int_0^t \int_{L_z} exp(-\omega\eta) u v_{,\eta} \mathrm{d}x_2 \mathrm{d}\eta. \tag{8}$$

$\varphi_1(z,t)$ can also be expressed as:

$$\begin{aligned}
\varphi_1(z,t) = & \frac{\omega\rho_1}{2}\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)(z-\xi)u_{,\eta}^2 \mathrm{d}A\mathrm{d}\eta \\
& + \frac{\rho_1}{2}\int_0^z\int_{L_\xi} exp(-\omega t)(z-\xi)u_{,t}^2 \mathrm{d}A \\
& + \frac{\omega}{2}\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)(z-\xi)u_{,\alpha}u_{,\alpha} \mathrm{d}A\mathrm{d}\eta \\
& + \frac{1}{2}\int_0^z\int_{L_\xi} exp(-\omega t)(z-\xi)u_{,\alpha}u_{,\alpha} \mathrm{d}A \\
& + \mu\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)(z-\xi)u_{,\alpha\eta}u_{,\alpha\eta} \mathrm{d}A\mathrm{d}\eta \\
& + \lambda\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)(z-\xi)u_{,\alpha}v_{,\alpha\eta} \mathrm{d}A\mathrm{d}\eta \\
& - \int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)u_{,\eta}u_{,1} \mathrm{d}A\mathrm{d}\eta \\
& + \lambda\omega\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)(z-\xi)u_{,\alpha}v_{,\alpha} \mathrm{d}A\mathrm{d}\eta \\
& + \lambda\int_0^z\int_{L_\xi} exp(-\omega t)(z-\xi)u_{,\alpha}v_{,\alpha} \mathrm{d}A \\
& + \lambda\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)u_{,1}v_{,\eta} \mathrm{d}A\mathrm{d}\eta \\
& + \lambda\omega\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)uv_{,1} \mathrm{d}A\mathrm{d}\eta + \lambda\int_0^z\int_{L_\xi} exp(-\omega t)uv_{,1} \mathrm{d}A + k_1(z,t),
\end{aligned} \tag{9}$$

where

$$\begin{aligned}
k_1(z,t) = & z\int_0^t\int_{L_0} exp(-\omega\eta)u_{,\eta}u_{,1} \mathrm{d}x_2\mathrm{d}\eta + \frac{\mu}{2}\int_0^t\int_{L_0} exp(-\omega\eta)u_{,\eta}^2 \mathrm{d}x_2\mathrm{d}\eta \\
& + z\mu\int_0^t\int_{L_0} exp(-\omega\eta)u_{,\eta}u_{,1\eta} \mathrm{d}x_2\mathrm{d}\eta + \lambda\int_0^t\int_{L_0} exp(-\omega\eta)uv_{,\eta} \mathrm{d}x_2\mathrm{d}\eta.
\end{aligned} \tag{10}$$

Proof. Multiplying both sides of Equation (4) by $exp(-\omega\eta)(z-\xi)u_{,\eta}$ and integrating, we obtain

$$
\begin{aligned}
0 &= \int_0^t \int_0^z \int_{L_\xi} exp(-\omega\eta)(z-\xi)u_{,\eta}(\rho_1 u_{,\eta\eta} - u_{,\alpha\alpha} - \mu u_{,\alpha\alpha\eta} \\
&+ \lambda v_{,\alpha\alpha})\mathrm{d}A\mathrm{d}\eta \\
&= \frac{\omega\rho_1}{2}\int_0^t \int_0^z \int_{L_\xi} exp(-\omega\eta)(z-\xi)u_{,\eta}^2\mathrm{d}A\mathrm{d}\eta \\
&+ \frac{\rho_1}{2}\int_0^z \int_{L_\xi} exp(-\omega t)(z-\xi)u_{,t}^2\mathrm{d}A \\
&+ \frac{\omega}{2}\int_0^t \int_0^z \int_{L_\xi} exp(-\omega\eta)(z-\xi)u_{,\alpha}u_{,\alpha}\mathrm{d}A\mathrm{d}\eta \\
&+ \frac{1}{2}\int_0^z \int_{L_\xi} exp(-\omega t)(z-\xi)u_{,\alpha}u_{,\alpha}\mathrm{d}A \\
&- \int_0^t \int_0^z \int_{L_\xi} exp(-\omega\eta)u_{,\eta}u_{,1}\mathrm{d}A\mathrm{d}\eta \\
&+ z\int_0^t \int_{L_0} exp(-\omega\eta)u_{,\eta}u_{,1}\mathrm{d}x_2\mathrm{d}\eta \\
&+ \mu\int_0^t \int_0^z \int_{L_\xi} exp(-\omega\eta)(z-\xi)u_{,\alpha\eta}u_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta \\
&- \frac{\mu}{2}\int_0^t \int_{L_z} exp(-\omega\eta)u_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta \\
&+ \frac{\mu}{2}\int_0^t \int_{L_0} exp(-\omega\eta)u_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta \\
&+ z\mu\int_0^t \int_{L_0} exp(-\omega\eta)u_{,\eta}u_{,1\eta}\mathrm{d}x_2\mathrm{d}\eta \\
&+ \lambda\int_0^t \int_0^z \int_{L_\xi} exp(-\omega\eta)(z-\xi)u_{,\alpha}v_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta \\
&+ \lambda\omega\int_0^t \int_0^z \int_{L_\xi} exp(-\omega\eta)(z-\xi)u_{,\alpha}v_{,\alpha}\mathrm{d}A\mathrm{d}\eta \\
&+ \lambda\int_0^z \int_{L_\xi} exp(-\omega t)(z-\xi)u_{,\alpha}v_{,\alpha}\mathrm{d}A + \lambda\int_0^t \int_0^z \int_{L_\xi} exp(-\omega\eta)u_{,1}v_{,\eta}\mathrm{d}A\mathrm{d}\eta \\
&- \lambda\int_0^t \int_{L_z} exp(-\omega\eta)uv_{,\eta}\mathrm{d}x_2\mathrm{d}\eta + \lambda\int_0^t \int_{L_0} exp(-\omega\eta)uv_{,\eta}\mathrm{d}x_2\mathrm{d}\eta \\
&+ \lambda\omega\int_0^t \int_0^z \int_{L_\xi} exp(-\omega\eta)uv_{,1}\mathrm{d}A\mathrm{d}\eta \\
&+ \lambda\int_0^z \int_{L_\xi} exp(-\omega t)uv_{,1}\mathrm{d}A.
\end{aligned}
\tag{11}
$$

By combining Equations (8) and (11), we can get (9). The proof of Lemma 1 is finished. □

Lemma 2. *We suggest u and v are the classical solutions of problems (4)–(7), and we define a function $\varphi_2(z,t)$ as:*

$$\begin{aligned}\varphi_2(z,t) = &\int_0^t \int_0^z \int_{L_\xi} exp(-\omega\eta)(z-\xi)(u_{,\alpha\alpha})^2 \mathrm{d}A\mathrm{d}\eta + \mu\omega \int_0^t \int_0^z \int_{L_\xi} exp(-\omega\eta)(z-\xi)(u_{,\alpha\alpha})^2 \mathrm{d}A\mathrm{d}\eta \\ &+ \mu \int_0^z \int_{L_\xi} exp(-\omega\eta)(z-\xi)(u_{,\alpha\alpha})^2 \mathrm{d}A - \rho_1 \int_0^t \int_0^z \int_{L_\xi} exp(-\omega\eta)(z-\xi) u_{,\beta\eta} u_{,\beta\eta} \mathrm{d}A\mathrm{d}\eta \\ &+ \frac{\rho_1}{2} \int_0^t \int_0^z \int_{L_\xi} exp(-\omega\eta) u_{,\eta}^2 \mathrm{d}A\mathrm{d}\eta - \omega\rho_1 \int_0^t \int_0^z \int_{L_\xi} exp(-\omega\eta)(z-\xi) u_{,\beta\beta} u_{,\eta} \mathrm{d}A\mathrm{d}\eta \\ &- \rho_1 \int_0^z \int_{L_\xi} exp(-\omega t)(z-\xi) u_{,\beta\beta} u_{,t} \mathrm{d}A - \lambda \int_0^t \int_0^z \int_{L_\xi} exp(-\omega\eta)(z-\xi) u_{,\beta\beta} v_{,\alpha\alpha} \mathrm{d}A\mathrm{d}\eta \\ &+ k_2(z,t),\end{aligned} \tag{12}$$

$\varphi_2(z,t)$ can also be expressed as:

$$\varphi_2(z,t) = -\frac{\rho_1}{2} \int_0^t \int_{L_z} exp(-\omega\eta) u_{,\eta}^2 \mathrm{d}x_2 \mathrm{d}\eta, \tag{13}$$

where

$$k_2(z,t) = \frac{\rho_1 z}{2} \int_0^t \int_{L_0} exp(-\omega\eta) u_{,\eta}^2 \mathrm{d}x_2 \mathrm{d}\eta - \rho_1 z \int_0^t \int_{L_0} exp(-\omega\eta) u_{,\eta} u_{,1\eta} \mathrm{d}x_2 \mathrm{d}\eta.$$

Proof. Multiplying both sides of Equation (4) by $exp(-\omega\eta)(z-\xi)u_{,\beta\beta}$ and integrating, we can obtain

$$\begin{aligned} 0 = &\int_0^t \int_0^z \int_{L_\xi} exp(-\omega\eta)(z-\xi) u_{,\beta\beta}(\rho_1 u_{,\eta\eta} - u_{,\alpha\alpha} - \mu u_{,\alpha\alpha\eta} + \lambda v_{,\alpha\alpha}) \mathrm{d}A\mathrm{d}\eta \\ = &\rho_1 \int_0^t \int_0^z \int_{L_\xi} exp(-\omega\eta)(z-\xi) u_{,\beta\eta} u_{,\beta\eta} \mathrm{d}A\mathrm{d}\eta \\ &- \rho_1 \int_0^t \int_0^z \int_{L_\xi} exp(-\omega\eta)(z-\xi) u_{,\eta} u_{,1\eta} \mathrm{d}A\mathrm{d}\eta \\ &+ \rho_1 z \int_0^t \int_{L_0} exp(-\omega\eta) u_{,\eta} u_{,1\eta} \mathrm{d}x_2 \mathrm{d}\eta \\ &+ \omega\rho_1 \int_0^t \int_0^z \int_{L_\xi} exp(-\omega\eta)(z-\xi) u_{,\beta\beta} u_{,\eta} \mathrm{d}A\mathrm{d}\eta \\ &+ \rho_1 \int_0^z \int_{L_\xi} exp(-\omega t)(z-\xi) u_{,\beta\beta} u_{,t} \mathrm{d}A \\ &- \int_0^t \int_0^z \int_{L_\xi} exp(-\omega\eta)(z-\xi)(u_{,\alpha\alpha})^2 \mathrm{d}A\mathrm{d}\eta \\ &- \mu\omega \int_0^t \int_0^z \int_{L_\xi} exp(-\omega\eta)(z-\xi)(u_{,\alpha\alpha})^2 \mathrm{d}A\mathrm{d}\eta \\ &- \mu \int_0^z \int_{L_\xi} exp(-\omega t)(z-\xi)(u_{,\alpha\alpha})^2 \mathrm{d}A \\ &+ \lambda \int_0^t \int_0^z \int_{L_\xi} exp(-\omega\eta)(z-\xi) u_{,\beta\beta} v_{,\alpha\alpha} \mathrm{d}A\mathrm{d}\eta. \end{aligned} \tag{14}$$

By combining Equations (12) and (14), we can get (13).
The proof of Lemma 2 is finished. □

Lemma 3. *We suggest u and v are classical solutions of problems (4)–(7), and we define a function $\varphi_3(z,t)$ as:*

$$
\begin{aligned}
\varphi_3(z,t) = &\frac{\omega\rho_2}{2}\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)(z-\xi)v_{,\eta}^2 \mathrm{d}A\mathrm{d}\eta \\
&+\frac{\rho_2}{2}\int_0^z\int_{L_\xi} exp(-\omega t)(z-\xi)v_{,t}^2 \mathrm{d}x_2\mathrm{d}\eta \\
&+\frac{m^2}{k}\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)(z-\xi)v_{,\alpha\eta}v_{,\alpha\eta} \mathrm{d}A\mathrm{d}\eta \\
&+\frac{\gamma\omega}{2}\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)(z-\xi)v_{,\alpha\beta}v_{,\alpha\beta} \mathrm{d}A\mathrm{d}\eta \\
&+\frac{\gamma}{2}\int_0^z\int_{L_\xi} exp(-\omega t)v_{,\alpha\beta}v_{,\alpha\beta} \mathrm{d}A \\
&-\lambda\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)(z-\xi)v_{,\alpha\eta}u_{,\alpha} \mathrm{d}A\mathrm{d}\eta \\
&+\lambda\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)v_{,\eta}u_{,1} \mathrm{d}A\mathrm{d}\eta \\
&-2\gamma\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha 1} \mathrm{d}A\mathrm{d}\eta \\
&+\frac{2\gamma\kappa\rho_2\omega}{m^2}\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)v_{,\eta}v_{,1} \mathrm{d}A\mathrm{d}\eta \\
&+\frac{2\gamma\kappa\rho_2}{m^2}\int_0^z\int_{L_\xi} exp(-\omega\eta)v_{,t}v_{,1} \mathrm{d}A \\
&-\frac{2\kappa\lambda\gamma}{m^2}\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)u_{,\alpha}v_{,1\alpha} \mathrm{d}A\mathrm{d}\eta + k_3(z,t).
\end{aligned}
\tag{15}
$$

$\varphi_3(z,t)$ can also be expressed as:

$$
\begin{aligned}
\varphi_3(z,t) = &\frac{m^2}{2k}\int_0^t\int_{L_z} exp(-\omega\eta)v_{,\eta}^2 \mathrm{d}x_2\mathrm{d}\eta \\
&+2\gamma\int_0^t\int_{L_z} exp(-\omega\eta)v_{,1\eta}v_{,1} \mathrm{d}x_2\mathrm{d}\eta \\
&-\frac{2k\lambda\gamma}{m^2}\int_0^t\int_{L_z} exp(-\omega\eta)u_{,1}v_{,11} \mathrm{d}x_2\mathrm{d}\eta \\
&-\frac{k\gamma^2}{m^2}\int_0^t\int_{L_z} exp(-\omega\eta)v_{,\alpha\beta}v_{,\alpha\beta} \mathrm{d}x_2\mathrm{d}\eta \\
&+\frac{2k\gamma^2}{m^2}\int_0^t\int_{L_z} exp(-\omega\eta)v_{,1\alpha}v_{,1\alpha} \mathrm{d}x_2\mathrm{d}\eta \\
&-\frac{2k\gamma^2}{m^2}\int_0^t\int_{L_z} exp(-\omega\eta)v_{,1}v_{,1\beta\beta} \mathrm{d}x_2\mathrm{d}\eta,
\end{aligned}
\tag{16}
$$

with

$$
\begin{aligned}
k_3(z,t) = & (\frac{m^2}{2k} + \frac{k\rho_2\gamma}{m^2}) \int_0^t \int_{L_0} exp(-\omega\eta) v_{,\eta}^2 \mathrm{d}x_2 \mathrm{d}\eta \\
& + \frac{m^2}{k} z \int_0^t \int_{L_0} exp(-\omega\eta) v_{,\eta} v_{,1\eta} \mathrm{d}x_2 \mathrm{d}\eta \\
& + \gamma z \int_0^t \int_{L_0} exp(-\omega\eta) v_{,\alpha\eta} v_{,\alpha 1} \mathrm{d}x_2 \mathrm{d}\eta \\
& - \gamma z \int_0^t \int_{L_0} exp(-\omega\eta) v_{,\eta} v_{,1\beta\beta} \mathrm{d}x_2 \mathrm{d}\eta \\
& + 2\gamma \int_0^t \int_{L_0} exp(-\omega\eta) v_{,1\eta} v_{,1} \mathrm{d}x_2 \mathrm{d}\eta \\
& - \frac{2k\lambda\gamma}{m^2} \int_0^t \int_{L_0} exp(-\omega\eta) u_{,1} v_{,11} \mathrm{d}x_2 \mathrm{d}\eta \\
& - \frac{k\gamma^2}{m^2} \int_0^t \int_{L_0} exp(-\omega\eta) v_{,\alpha\beta} v_{,\alpha\beta} \mathrm{d}x_2 \mathrm{d}\eta \\
& + \frac{2k\gamma^2}{m^2} \int_0^t \int_{L_0} exp(-\omega\eta) v_{,1\alpha} v_{,1\alpha} \mathrm{d}x_2 \mathrm{d}\eta \\
& - \frac{2k\gamma^2}{m^2} \int_0^t \int_{L_0} exp(-\omega\eta) v_{,1} v_{,1\beta\beta} \mathrm{d}x_2 \mathrm{d}\eta.
\end{aligned}
$$

Proof. Multiplying both sides of Equation (5) by $exp(-\omega\eta)(z-\xi)v_{,\eta}$ and integrating, we can obtain

$$
\begin{aligned}
0 = & \int_0^t \int_0^z \int_{L_\xi} exp(-\omega\eta)(z-\xi) v_{,\eta} (\rho_2 v_{,\eta\eta} + \gamma v_{,\alpha\alpha\beta\beta} + \lambda u_{,\alpha\alpha} - \frac{m^2}{k} v_{,\alpha\alpha\eta}) \mathrm{d}A \mathrm{d}\eta \\
= & \frac{\omega\rho_2}{2} \int_0^t \int_0^z \int_{L_\xi} exp(-\omega\eta)(z-\xi) v_{,\eta}^2 \mathrm{d}A \mathrm{d}\eta \\
& + \frac{\rho_2}{2} \int_0^z \int_{L_\xi} exp(-\omega t)(z-\xi) v_{,t}^2 \mathrm{d}A \\
& - \lambda \int_0^t \int_0^z \int_{L_\xi} exp(-\omega\eta)(z-\xi) v_{,\alpha\eta} u_{,\alpha} \mathrm{d}A \mathrm{d}\eta \\
& + \lambda \int_0^t \int_0^z \int_{L_\xi} exp(-\omega\eta) v_{,\eta} u_{,1} \mathrm{d}A \mathrm{d}\eta \\
& + \gamma \int_0^t \int_0^z \int_{L_\xi} exp(-\omega\eta)(z-\xi) v_{,\eta} v_{,\alpha\alpha\beta\beta} \mathrm{d}A \mathrm{d}\eta \\
& + \frac{m^2}{k} \int_0^t \int_0^z \int_{L_\xi} exp(-\omega\eta)(z-\xi) v_{,\alpha\eta} v_{,\alpha\eta} \mathrm{d}A \mathrm{d}\eta \\
& - \frac{m^2}{2k} \int_0^t \int_{L_z} exp(-\omega\eta) v_{,\eta}^2 \mathrm{d}x_2 \mathrm{d}\eta \\
& + \frac{m^2}{2k} \int_0^t \int_{L_0} exp(-\omega\eta) v_{,\eta}^2 \mathrm{d}x_2 \mathrm{d}\eta \\
& + \frac{m^2}{k} z \int_0^t \int_{L_0} exp(-\omega\eta) v_{,\eta} v_{,1\eta} \mathrm{d}x_2 \mathrm{d}\eta.
\end{aligned} \tag{17}
$$

Next, we begin to deal with the term $\gamma\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)(z-\xi)v_{,\eta}v_{,\alpha\alpha\beta\beta}\mathrm{d}A\mathrm{d}\eta$.

$$\begin{aligned}
&\gamma\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)(z-\xi)v_{,\eta}v_{,\alpha\alpha\beta\beta}\mathrm{d}A\mathrm{d}\eta\\
&=\frac{\gamma\omega}{2}\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)(z-\xi)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta\\
&+\frac{\gamma}{2}\int_0^z\int_{L_\xi} exp(-\omega\eta)(z-\xi)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A\\
&-2\gamma\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha 1}\mathrm{d}A\mathrm{d}\eta\\
&+\gamma z\int_0^t\int_{L_0} exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha 1}\mathrm{d}x_2\mathrm{d}\eta\\
&-\gamma z\int_0^t\int_{L_0} exp(-\omega\eta)v_{,\eta}v_{,1\beta\beta}\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned}\tag{18}$$

Now let us deal with $-2\gamma\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha 1}\mathrm{d}A\mathrm{d}\eta$.

$$\begin{aligned}
&-2\gamma\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha 1}\mathrm{d}A\mathrm{d}\eta\\
&=2\gamma\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)v_{,\alpha\alpha\eta}v_{,1}\mathrm{d}A\mathrm{d}\eta\\
&-2\gamma\int_0^t\int_{L_z} exp(-\omega\eta)v_{,1\eta}v_{,1}\mathrm{d}x_2\mathrm{d}\eta\\
&+2\gamma\int_0^t\int_{L_0} exp(-\omega\eta)v_{,1\eta}v_{,1}\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned}\tag{19}$$

Using the Equation (5), we can get

$$\begin{aligned}
&2\gamma\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)v_{,\alpha\alpha\eta}v_{,1}\mathrm{d}A\mathrm{d}\eta\\
&=2\gamma\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)(\frac{k}{m^2}\rho_2 v_{,\eta\eta}+\frac{k}{m^2}\gamma v_{,\alpha\alpha\beta\beta}+\frac{k}{m^2}\lambda u_{,\alpha\alpha})v_{,1}\mathrm{d}A\mathrm{d}\eta\\
&=-\frac{k\rho_2\gamma}{m^2}\int_0^t\int_{L_z} exp(-\omega\eta)v_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta+\frac{k\rho_2\gamma}{m^2}\int_0^t\int_{L_0} exp(-\omega\eta)v_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta\\
&+\frac{2k\gamma\rho_2\omega}{m^2}\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)v_{,\eta}v_{,1}\mathrm{d}A\mathrm{d}\eta+\frac{2k\gamma\rho_2}{m^2}\int_0^z\int_{L_\xi} exp(-\omega\eta)v_{,t}v_{,1}\mathrm{d}A\\
&+\frac{2k\gamma^2}{m^2}\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)v_{,1}v_{,\alpha\alpha\beta\beta}\mathrm{d}A\mathrm{d}\eta\\
&-\frac{2k\lambda\gamma}{m^2}\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)u_{,\alpha}v_{,1\alpha}\mathrm{d}A\mathrm{d}\eta\\
&+\frac{2k\lambda\gamma}{m^2}\int_0^t\int_{L_z} exp(-\omega\eta)u_{,1}v_{,11}\mathrm{d}x_2\mathrm{d}\eta\\
&-\frac{2k\lambda\gamma}{m^2}\int_0^t\int_{L_0} exp(-\omega\eta)u_{,1}v_{,11}\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned}\tag{20}$$

Now, let us deal with the term $\frac{2k\gamma^2}{m^2}\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)v_{,1}v_{,\alpha\alpha\beta\beta}\mathrm{d}A\mathrm{d}\eta$.

$$
\begin{aligned}
&\frac{2k\gamma^2}{m^2}\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)v_{,1}v_{,\alpha\alpha\beta\beta}\mathrm{d}A\mathrm{d}\eta \\
&= -\frac{2k\gamma^2}{m^2}\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)v_{,1\alpha}v_{,\alpha\beta\beta}\mathrm{d}A\mathrm{d}\eta + \frac{2k\gamma^2}{m^2}\int_0^t\int_{L_z} exp(-\omega\eta)v_{,1}v_{,1\beta\beta}\mathrm{d}x_2\mathrm{d}\eta \\
&- \frac{2k\gamma^2}{m^2}\int_0^t\int_{L_0} exp(-\omega\eta)v_{,1}v_{,1\beta\beta}\mathrm{d}x_2\mathrm{d}\eta \\
&= \frac{k\gamma^2}{m^2}\int_0^t\int_{L_z} exp(-\omega\eta)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}x_2\mathrm{d}\eta - \frac{k\gamma^2}{m^2}\int_0^t\int_{L_0} exp(-\omega\eta)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}x_2\mathrm{d}\eta \\
&- \frac{2k\gamma^2}{m^2}\int_0^t\int_{L_z} exp(-\omega\eta)v_{,1\alpha}v_{,1\alpha}\mathrm{d}x_2\mathrm{d}\eta + \frac{2k\gamma^2}{m^2}\int_0^t\int_{L_0} exp(-\omega\eta)v_{,1\alpha}v_{,1\alpha}\mathrm{d}x_2\mathrm{d}\eta \\
&+ \frac{2k\gamma^2}{m^2}\int_0^t\int_{L_z} exp(-\omega\eta)v_{,1}v_{,1\beta\beta}\mathrm{d}x_2\mathrm{d}\eta - \frac{2k\gamma^2}{m^2}\int_0^t\int_{L_0} exp(-\omega\eta)v_{,1}v_{,1\beta\beta}\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned} \tag{21}
$$

A combination of (15) and (17)–(21), we obtain

$$
\begin{aligned}
\varphi_3(z,t) = &\frac{m^2}{2k}\int_0^t\int_{L_z} exp(-\omega\eta)v_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta + 2\gamma\int_0^t\int_{L_z} exp(-\omega\eta)v_{,1\eta}v_{,1}\mathrm{d}x_2\mathrm{d}\eta \\
&- \frac{2k\lambda\gamma}{m^2}\int_0^t\int_{L_z} exp(-\omega\eta)u_{,1}v_{,11}\mathrm{d}x_2\mathrm{d}\eta - \frac{k\gamma^2}{m^2}\int_0^t\int_{L_z} exp(-\omega\eta)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}x_2\mathrm{d}\eta \\
&+ \frac{2k\gamma^2}{m^2}\int_0^t\int_{L_z} exp(-\omega\eta)v_{,1\alpha}v_{,1\alpha}\mathrm{d}x_2\mathrm{d}\eta - \frac{2k\gamma^2}{m^2}\int_0^t\int_{L_z} exp(-\omega\eta)v_{,1}v_{,1\beta\beta}\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned}
$$

The proof of Lemma 3 is finished. □

Lemma 4. *We suggest u and v are classical solutions of problems (4)–(7), and we define a function $\varphi_4(z,t)$ as :*

$$
\begin{aligned}
\varphi_4(z,t) = &-\frac{\rho_2}{2}\int_0^t\int_{L_z} exp(-\omega\eta)v_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta + \gamma\int_0^t\int_{L_z} exp(-\omega\eta)v_{,1\alpha\beta}v_{,\alpha\beta}\mathrm{d}x_2\mathrm{d}\eta \\
&- \frac{\gamma}{2}\int_0^t\int_{L_z} exp(-\omega\eta)v_{,1\alpha}v_{,1\alpha}\mathrm{d}x_2\mathrm{d}\eta + \frac{\gamma}{2}\int_0^t\int_{L_z} exp(-\omega\eta)v_{,11}^2\mathrm{d}x_2\mathrm{d}\eta \\
&- \frac{\gamma}{2}\int_0^t\int_{L_z} exp(-\omega\eta)v_{,12}^2\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned} \tag{22}
$$

Then, $\varphi_4(z,t)$ can also be expressed as:

$$
\begin{aligned}
\varphi_4(z,t) = &-\rho_2\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)(z-\xi)v_{,1\eta}^2\mathrm{d}A\mathrm{d}\eta + \frac{m^2}{2k}\int_0^z\int_{L_\xi} exp(-\omega t)(z-\xi)v_{,1\alpha}v_{,1\alpha}\mathrm{d}A \\
&+ \gamma\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)v_{,1\alpha\beta}v_{,1\alpha\beta}\mathrm{d}A\mathrm{d}\eta - \rho_2\omega\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)(z-\xi)v_{,11}v_{,\eta}\mathrm{d}A\mathrm{d}\eta \\
&- \rho_2\int_0^z\int_{L_\xi} exp(-\omega t)(z-\xi)v_{,11}v_{,t}\mathrm{d}A + \lambda\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)(z-\xi)v_{,11\alpha}u_{,\alpha}\mathrm{d}A\mathrm{d}\eta \\
&- \lambda\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)v_{,11}u_{,1}\mathrm{d}A\mathrm{d}\eta + \int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)v_{,12}v_{,2\eta}\mathrm{d}A\mathrm{d}\eta + k_4(z,t),
\end{aligned} \tag{23}
$$

with

$$
\begin{aligned}
k_4(z,t) = &-\frac{\rho_2}{2}\int_0^t\int_{L_0} exp(-\omega\eta)v_{,\eta}^2 \mathrm{d}x_2\mathrm{d}\eta \\
&-\rho_2 z\int_0^z\int_{L_0} exp(-\omega\eta)v_{,1\eta}v_{,\eta}\mathrm{d}x_2\mathrm{d}\eta \\
&+\frac{m^2}{k}z\int_0^t\int_{L_0} exp(-\omega\eta)v_{,12}v_{,2\eta}\mathrm{d}x_2\mathrm{d}\eta \\
&+\gamma\int_0^t\int_{L_0} exp(-\omega\eta)v_{,1\alpha\beta}v_{,\alpha\beta}\mathrm{d}x_2\mathrm{d}\eta \\
&-\frac{\gamma}{2}\int_0^t\int_{L_0} exp(-\omega\eta)v_{,1\alpha}v_{,1\alpha}\mathrm{d}x_2\mathrm{d}\eta \\
&-\gamma z\int_0^t\int_{L_0} exp(-\omega\eta)v_{,11\alpha}v_{,\alpha 1}\mathrm{d}x_2\mathrm{d}\eta \\
&+\frac{\gamma}{2}\int_0^t\int_{L_0} exp(-\omega\eta)v_{,11}^2\mathrm{d}x_2\mathrm{d}\eta \\
&-\frac{\gamma}{2}\int_0^t\int_{L_0} exp(-\omega\eta)v_{,12}^2\mathrm{d}x_2\mathrm{d}\eta \\
&+\gamma z\int_0^t\int_{L_0} exp(-\omega\eta)v_{,11}v_{,1\beta\beta}\mathrm{d}x_2\mathrm{d}\eta \\
&+\lambda z\int_0^t\int_{L_0} exp(-\omega\eta)v_{,11}u_{,1}\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned}
$$

Proof. Multiplying both sides of Equation (5) by $exp(-\omega\eta)(z-\xi)v_{,11}$ and integrating, we can obtain

$$
\begin{aligned}
0 = &\rho_2\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)(z-\xi)v_{,11}v_{,\eta\eta}\mathrm{d}A\mathrm{d}\eta \\
&+\gamma\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)(z-\xi)v_{,11}v_{,\alpha\alpha\beta\beta}\mathrm{d}A\mathrm{d}\eta \\
&+\lambda\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)(z-\xi)v_{,11}u_{,\alpha\alpha}\mathrm{d}A\mathrm{d}\eta \\
&-\frac{m^2}{k}\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)(z-\xi)v_{,11}v_{,\alpha\alpha\eta}\mathrm{d}A\mathrm{d}\eta.
\end{aligned}
\tag{24}
$$

Using the divergence theorem, the first term on the right of Equation (22) can be rewritten as

$$
\begin{aligned}
&\rho_2\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)(z-\xi)v_{,11}v_{,\eta\eta}\mathrm{d}A\mathrm{d}\eta \\
&=\rho_2\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)(z-\xi)v_{,1\eta}v_{,1\eta}\mathrm{d}A\mathrm{d}\eta-\frac{\rho_2}{2}\int_0^t\int_{L_z} exp(-\omega\eta)v_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta \\
&+\frac{\rho_2}{2}\int_0^t\int_{L_0} exp(-\omega\eta)v_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta+\rho_2 z\int_0^t\int_{L_0} exp(-\omega\eta)v_{,1\eta}v_{,\eta}\mathrm{d}x_2\mathrm{d}\eta \\
&+\rho_2\omega\int_0^t\int_0^z\int_{L_\xi} exp(-\omega\eta)(z-\xi)v_{,11}v_{,\eta}\mathrm{d}A\mathrm{d}\eta \\
&+\rho_2\int_0^z\int_{L_\xi} exp(-\omega t)(z-\xi)v_{,11}v_{,t}\mathrm{d}A.
\end{aligned}
\tag{25}
$$

Similarly, the second term on the right of Equation (22) can be rewritten as

$$\begin{aligned}
&\gamma \int_0^t \int_0^z \int_{L_\xi} exp(-\omega\eta)(z-\xi)v_{,11}v_{,\alpha\alpha\beta\beta}\mathrm{d}A\mathrm{d}\eta \\
&= -\gamma \int_0^t \int_0^z \int_{L_\xi} exp(-\omega\eta)v_{,1\alpha\beta}v_{,1\alpha\beta}\mathrm{d}A\mathrm{d}\eta + \gamma \int_0^t \int_{L_z} exp(-\omega\eta)v_{,1\alpha\beta}v_{,\alpha\beta}\mathrm{d}x_2\mathrm{d}\eta \\
&- \gamma \int_0^t \int_{L_0} exp(-\omega\eta)v_{,1\alpha\beta}v_{,\alpha\beta}\mathrm{d}x_2\mathrm{d}\eta - \frac{\gamma}{2}\int_0^t \int_{L_z} exp(-\omega\eta)v_{,1\alpha}v_{,1\alpha}\mathrm{d}x_2\mathrm{d}\eta \\
&+ \frac{\gamma}{2}\int_0^t \int_{L_0} exp(-\omega\eta)v_{,1\alpha}v_{,1\alpha}\mathrm{d}x_2\mathrm{d}\eta + \gamma z\int_0^t \int_{L_0} exp(-\omega\eta)v_{,11\alpha}v_{,\alpha 1}\mathrm{d}x_2\mathrm{d}\eta \\
&+ \frac{\gamma}{2}\int_0^t \int_{L_z} exp(-\omega\eta)v_{,11}^2\mathrm{d}x_2\mathrm{d}\eta - \frac{\gamma}{2}\int_0^t \int_{L_0} exp(-\omega\eta)v_{,11}^2\mathrm{d}x_2\mathrm{d}\eta \\
&- \frac{\gamma}{2}\int_0^t \int_{L_z} exp(-\omega\eta)v_{,12}^2\mathrm{d}x_2\mathrm{d}\eta + \frac{\gamma}{2}\int_0^t \int_{L_0} exp(-\omega\eta)v_{,12}^2\mathrm{d}x_2\mathrm{d}\eta \\
&- \gamma z\int_0^t \int_{L_0} exp(-\omega\eta)v_{,11}v_{,1\beta\beta}\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned} \tag{26}$$

The third term on the right of Equation (22) can be rewritten as

$$\begin{aligned}
&\lambda \int_0^t \int_0^z \int_{L_\xi} exp(-\omega\eta)(z-\xi)v_{,11}u_{,\alpha\alpha}\mathrm{d}A\mathrm{d}\eta \\
&= -\lambda \int_0^t \int_0^z \int_{L_\xi} exp(-\omega\eta)(z-\xi)v_{,11\alpha}u_{,\alpha}\mathrm{d}A\mathrm{d}\eta \\
&+ \lambda \int_0^t \int_0^z \int_{L_\xi} exp(-\omega\eta)v_{,11}u_{,1}\mathrm{d}A\mathrm{d}\eta \\
&- \lambda z \int_0^t \int_{L_0} exp(-\omega\eta)v_{,11}u_{,1}\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned} \tag{27}$$

The fourth term on the right of Equation (22) can be rewritten as

$$\begin{aligned}
&-\frac{m^2}{k}\int_0^t \int_0^z \int_{L_\xi} exp(-\omega\eta)(z-\xi)v_{,11}v_{,\alpha\alpha\eta}\mathrm{d}A\mathrm{d}\eta \\
&= -\frac{m^2}{2k}\int_0^z \int_{L_\xi} exp(-\omega t)(z-\xi)v_{,1\alpha}v_{,1\alpha}\mathrm{d}A \\
&- \frac{m^2}{k}z\int_0^t \int_{L_0} exp(-\omega\eta)(z-\xi)v_{,12}v_{,2\eta}\mathrm{d}x_2\mathrm{d}\eta \\
&- \int_0^t \int_0^z \int_{L_\xi} exp(-\omega\eta)v_{,12}v_{,2\eta}\mathrm{d}A\mathrm{d}\eta.
\end{aligned} \tag{28}$$

A combination of (24)–(28) gives (23). The proof of Lemma 4 is finished. □

Lemma 5. *We define new energy expressions $\varphi(z,t)$ and $\Phi(z,t)$ as follows:*

$$\varphi(z,t) = \varphi_1(z,t) + k_1\varphi_2(z,t) + \varphi_3(z,t) + k_2\varphi_4(z,t), \tag{29}$$

and

$$\Phi(z,t) = \int_0^t \varphi(z,s)ds. \tag{30}$$

The following second-order partial differential inequality holds

$$|\Phi(z,t)| \le k_3 \frac{\partial^2\Phi(z,t)}{\partial z^2}, \tag{31}$$

where k_1 k_2 and k_3 are positive constance to be defined later.

Proof. From (9), we can get

$$\begin{aligned}
\frac{\partial^2 \varphi_1(z,t)}{\partial z^2} = & \frac{\omega\rho_1}{2}\int_0^t\int_{L_z} exp(-\omega\eta)u_{,\eta}^2 \mathrm{d}x_2\mathrm{d}\eta \\
& + \frac{\rho_1}{2}\int_{L_z} exp(-\omega t)u_{,t}^2 \mathrm{d}x_2 \\
& + \frac{\omega}{2}\int_0^t\int_{L_z} exp(-\omega\eta)u_{,\alpha}u_{,\alpha}\mathrm{d}x_2\mathrm{d}\eta \\
& + \frac{1}{2}\int_{L_z} exp(-\omega t)u_{,\alpha}u_{,\alpha}\mathrm{d}x_2 \\
& - \lambda\int_0^t\int_{L_z} exp(-\omega\eta)u_{,1\eta}u_{,1}\mathrm{d}x_2\mathrm{d}\eta \\
& - \lambda\int_0^t\int_{L_z} exp(-\omega\eta)u_{,\eta}u_{,11}\mathrm{d}x_2\mathrm{d}\eta \\
& + \lambda\int_0^t\int_{L_z} exp(-\omega\eta)u_{,11}v_{,\eta}\mathrm{d}x_2\mathrm{d}\eta \\
& + \lambda\int_0^t\int_{L_z} exp(-\omega\eta)u_{,1}v_{,1\eta}\mathrm{d}x_2\mathrm{d}\eta \\
& + \lambda\omega\int_0^t\int_{L_z} exp(-\omega\eta)u_{,1}v_{,1}\mathrm{d}x_2\mathrm{d}\eta \\
& + \lambda\omega\int_0^t\int_{L_z} exp(-\omega\eta)uv_{,11}\mathrm{d}x_2\mathrm{d}\eta \\
& + \mu\int_0^t\int_{L_z} exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta \\
& + \lambda\int_0^t\int_{L_z} exp(-\omega\eta)u_{,\alpha}v_{,\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta \\
& + \lambda\int_{L_z} exp(-\omega t)u_{,1}v_{,1}\mathrm{d}x_2 \\
& + \lambda\int_{L_z} exp(-\omega t)uv_{,11}\mathrm{d}x_2 \\
& + \lambda\omega\int_0^t\int_{L_z} exp(-\omega\eta)u_{,\alpha}v_{,\alpha}\mathrm{d}x_2\mathrm{d}\eta \\
& + \lambda\int_{L_z} exp(-\omega t)u_{,\alpha}v_{,\alpha}\mathrm{d}x_2.
\end{aligned} \tag{32}$$

From (12), we can get

$$\begin{aligned}
\frac{\partial^2 \varphi_2(z,t)}{\partial z^2} = & \int_0^t\int_{L_z} exp(-\omega\eta)(u_{,\alpha\alpha})^2 \mathrm{d}x_2\mathrm{d}\eta \\
& + \mu\omega\int_0^t\int_{L_z} exp(-\omega\eta)(u_{,\alpha\alpha})^2 \mathrm{d}x_2\mathrm{d}\eta \\
& + \mu\int_{L_z} exp(-\omega\eta)(u_{,\alpha\alpha})^2 \mathrm{d}x_2 \\
& - \rho_1\int_0^t\int_{L_z} exp(-\omega\eta)u_{,\beta\eta}u_{,\beta\eta}\mathrm{d}x_2\mathrm{d}\eta \\
& + \rho_1\int_0^t\int_{L_z} exp(-\omega\eta)u_{,1\eta}u_{,1}\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned} \tag{33}$$

From (15), we can get

$$\begin{aligned}
\frac{\partial^2\varphi_3(z,t)}{\partial z^2} =& \frac{\omega\rho_2}{2}\int_0^t\int_{L_z} exp(-\omega\eta)v_{,\eta}^2 \mathrm{d}x_2\mathrm{d}\eta + \frac{\rho_2}{2}\int_{L_z} exp(-\omega t)v_{,t}^2\mathrm{d}x_2 \\
&+ \frac{m^2}{k}\int_0^t\int_{L_z} exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta + \frac{\gamma\omega}{2}\int_0^t\int_{L_z} exp(-\omega\eta)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}x_2\mathrm{d}\eta \\
&+ \frac{\gamma}{2}\int_{L_z} exp(-\omega t)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}x_2 - \lambda\int_0^t\int_{L_z} exp(-\omega\eta)v_{,\alpha\eta}u_{,\alpha}\mathrm{d}x_2\mathrm{d}\eta \\
&+ \lambda\int_0^t\int_{L_z} exp(-\omega\eta)v_{,1\eta}u_{,1}\mathrm{d}x_2\mathrm{d}\eta + \lambda\int_0^t\int_{L_z} exp(-\omega\eta)v_{,\eta}u_{,11}\mathrm{d}x_2\mathrm{d}\eta \\
&+ \frac{2\gamma\kappa\rho_2\omega}{m^2}\int_0^t\int_{L_z} exp(-\omega\eta)v_{,1\eta}v_{,1}\mathrm{d}x_2\mathrm{d}\eta + \frac{2\gamma\kappa\rho_2\omega}{m^2}\int_0^t\int_{L_z} exp(-\omega\eta)v_{,\eta}v_{,11}\mathrm{d}x_2\mathrm{d}\eta \\
&+ \frac{2\gamma\kappa\rho_2}{m^2}\int_{L_z} exp(-\omega t)v_{,1t}v_{,1}\mathrm{d}x_2 + \frac{2\gamma\kappa\rho_2}{m^2}\int_{L_z} exp(-\omega t)v_{,t}v_{,11}\mathrm{d}x_2 \\
&- \frac{2\kappa\lambda\gamma}{m^2}\int_0^t\int_{L_z} exp(-\omega\eta)u_{,1\alpha}v_{,1\alpha}\mathrm{d}x_2\mathrm{d}\eta - \frac{2\kappa\lambda\gamma}{m^2}\int_0^t\int_{L_z} exp(-\omega\eta)u_{,\alpha}v_{,11\alpha}\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned} \tag{34}$$

From (23), we can get

$$\begin{aligned}
\frac{\partial^2\varphi_4(z,t)}{\partial z^2} =& -\rho_2\int_0^t\int_{L_z} exp(-\omega\eta)v_{,1\eta}^2\mathrm{d}x_2\mathrm{d}\eta + \frac{m^2}{2k}\int_{L_z} exp(-\omega t)v_{,1\alpha}v_{,1\alpha}\mathrm{d}x_2 \\
&+ \gamma\int_0^t\int_{L_z} exp(-\omega\eta)v_{,1\alpha\beta}v_{,1\alpha\beta}\mathrm{d}x_2\mathrm{d}\eta - \rho_2\omega\int_0^t\int_{L_z} exp(-\omega\eta)v_{,11}v_{,\eta}\mathrm{d}x_2\mathrm{d}\eta \\
&+ \rho_2\int_{L_z} exp(-\omega\eta)v_{,11}v_{,t}\mathrm{d}x_2 + \lambda\int_0^t\int_{L_z} exp(-\omega\eta)v_{,11\alpha}u_{,\alpha}\mathrm{d}x_2\mathrm{d}\eta \\
&- \lambda\int_0^t\int_{L_z} exp(-\omega\eta)v_{,111}u_{,1}\mathrm{d}x_2\mathrm{d}\eta - \lambda\int_0^t\int_{L_z} exp(-\omega\eta)v_{,11}u_{,11}\mathrm{d}x_2\mathrm{d}\eta \\
&+ \int_0^t\int_{L_z} exp(-\omega\eta)v_{,112}v_{,2\eta}\mathrm{d}x_2\mathrm{d}\eta + \int_0^t\int_{L_z} exp(-\omega\eta)v_{,12}v_{,12\eta}\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned} \tag{35}$$

A combination of (8), (13), (21), (22) and (29) leads to

$$\begin{aligned}
\varphi(z,t) =& \frac{\mu}{2}\int_0^t\int_{L_z} exp(-\omega\eta)u_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta + \lambda\int_0^t\int_{L_z} exp(-\omega\eta)uv_{,\eta}\mathrm{d}x_2\mathrm{d}\eta \\
&- k_1\frac{\rho_1}{2}\int_0^t\int_{L_z} exp(-\omega\eta)u_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta - k_2\frac{\gamma}{2}\int_0^t\int_{L_z} exp(-\omega\eta)v_{,12}^2\mathrm{d}x_2\mathrm{d}\eta \\
&+ \frac{m^2}{2k}\int_0^t\int_{L_z} exp(-\omega\eta)v_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta + 2\gamma\int_0^t\int_{L_z} exp(-\omega\eta)v_{,1\eta}v_{,1}\mathrm{d}x_2\mathrm{d}\eta \\
&- \frac{2k\lambda\gamma}{m^2}\int_0^t\int_{L_z} exp(-\omega\eta)u_{,1}v_{,11}\mathrm{d}x_2\mathrm{d}\eta - \frac{k\gamma^2}{m^2}\int_0^t\int_{L_z} exp(-\omega\eta)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}x_2\mathrm{d}\eta \\
&+ \frac{2k\gamma^2}{m^2}\int_0^t\int_{L_z} exp(-\omega\eta)v_{,1\alpha}v_{,1\alpha}\mathrm{d}x_2\mathrm{d}\eta - \frac{2k\gamma^2}{m^2}\int_0^t\int_{L_z} exp(-\omega\eta)v_{,1}v_{,1\beta\beta}\mathrm{d}x_2\mathrm{d}\eta \\
&- k_2\frac{\rho_2}{2}\int_0^t\int_{L_z} exp(-\omega\eta)v_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta + k_2\gamma\int_0^t\int_{L_z} exp(-\omega\eta)v_{,1\alpha\beta}v_{,\alpha\beta}\mathrm{d}x_2\mathrm{d}\eta \\
&- k_2\frac{\gamma}{2}\int_0^t\int_{L_z} exp(-\omega\eta)v_{,1\alpha}v_{,1\alpha}\mathrm{d}x_2\mathrm{d}\eta + k_2\frac{\gamma}{2}\int_0^t\int_{L_z} exp(-\omega\eta)v_{,11}^2\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned} \tag{36}$$

Combining (32)–(36), we have

$$
\begin{aligned}
\frac{\partial^2\varphi(z,t)}{\partial z^2} =& \frac{\omega\rho_1}{2}\int_0^t\int_{L_z} exp(-\omega\eta)u_{,\eta}^2 \mathrm{d}x_2\mathrm{d}\eta + \frac{\rho_1}{2}\int_{L_z} exp(-\omega t)u_{,t}^2\mathrm{d}x_2 \\
&+ \frac{\omega}{2}\int_0^t\int_{L_z} exp(-\omega\eta)u_{,\alpha}u_{,\alpha}\mathrm{d}x_2\mathrm{d}\eta + \frac{1}{2}\int_{L_z} exp(-\omega t)u_{,\alpha}u_{,\alpha}\mathrm{d}x_2 \\
&+ u\int_0^t\int_{L_z} exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta + \lambda\omega\int_0^t\int_{L_z} exp(-\omega\eta)u_{,\alpha}v_{,\alpha}\mathrm{d}x_2\mathrm{d}\eta \\
&+ \lambda\int_{L_z} exp(-\omega t)u_{,\alpha}v_{,\alpha}\mathrm{d}x_2 + (\rho_1 k_1 - \lambda)\int_0^t\int_{L_z} exp(-\omega\eta)u_{,1\eta}u_{,1}\mathrm{d}x_2\mathrm{d}\eta \\
&- \lambda\int_0^t\int_{L_z} exp(-\omega\eta)u_{,\eta}u_{,11}\mathrm{d}x_2\mathrm{d}\eta + 2\lambda\int_0^t\int_{L_z} exp(-\omega\eta)u_{,11}v_{,\eta}\mathrm{d}x_2\mathrm{d}\eta \\
&+ 2\lambda\int_0^t\int_{L_z} exp(-\omega\eta)u_{,1}v_{,1\eta}\mathrm{d}x_2\mathrm{d}\eta + \lambda\omega\int_0^t\int_{L_z} exp(-\omega\eta)u_{,1}v_{,1}\mathrm{d}x_2\mathrm{d}\eta \\
&+ \lambda\omega\int_0^t\int_{L_z} exp(-\omega\eta)uv_{,11}\mathrm{d}x_2\mathrm{d}\eta + \lambda\int_{L_z} exp(-\omega t)u_{,1}v_{,1}\mathrm{d}x_2 \\
&+ \lambda\int_{L_z} exp(-\omega t)uv_{,11}\mathrm{d}x_2 + (1+\mu\omega)k_1\int_0^t\int_{L_z} exp(-\omega\eta)(u_{,\alpha\alpha})^2\mathrm{d}x_2\mathrm{d}\eta \\
&+ \mu k_1\int_{L_z} exp(-\omega t)(u_{,\alpha\alpha})^2\mathrm{d}x_2 - \rho_1 k_1\int_0^t\int_{L_z} exp(-\omega\eta)u_{,\beta\eta}u_{,\beta\eta}\mathrm{d}x_2\mathrm{d}\eta \\
&+ \frac{\rho_2\omega}{2}\int_0^t\int_{L_z} exp(-\omega\eta)v_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta + \frac{\rho_1}{2}\int_{L_z} exp(-\omega t)v_{,t}^2\mathrm{d}x_2 \\
&+ \frac{m^2}{k}\int_0^t\int_{L_z} exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta + \frac{\omega\gamma}{2}\int_0^t\int_{L_z} exp(-\omega t)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}x_2\mathrm{d}\eta \\
&+ \frac{\gamma}{2}\int_{L_z} exp(-\omega t)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}x_2 + \frac{2\gamma k\rho_2\omega}{m^2}\int_0^t\int_{L_z} exp(-\omega\eta)v_{,1\eta}v_{,1}\mathrm{d}x_2\mathrm{d}\eta \\
&+ (\frac{2\gamma k\rho_2\omega}{m^2} - \rho_2\omega k_2)\int_0^t\int_{L_z} exp(-\omega\eta)v_{,\eta}v_{,11}\mathrm{d}x_2\mathrm{d}\eta + \frac{2\gamma k\rho_2}{m^2}\int_{L_z} exp(-\omega t)v_{,1t}v_{,1}\mathrm{d}x_2 \\
&+ (\frac{2\gamma k\rho_2}{m^2} + \rho_2 k_2)\int_{L_z} exp(-\omega t)v_{,t}v_{,11}\mathrm{d}x_2 - \frac{2k\lambda\gamma}{m^2}\int_0^t\int_{L_z} exp(-\omega\eta)u_{,1\alpha}v_{,1\alpha}\mathrm{d}x_2\mathrm{d}\eta \\
&+ (\lambda k_2 - \frac{2k\lambda\gamma}{m^2})\int_0^t\int_{L_z} exp(-\omega\eta)u_{,\alpha}v_{,11\alpha}\mathrm{d}x_2\mathrm{d}\eta - \rho_2 k_2\int_0^t\int_{L_z} exp(-\omega\eta)v_{,1\eta}^2\mathrm{d}x_2\mathrm{d}\eta \\
&+ \frac{m^2}{2k}k_2\int_{L_z} exp(-\omega t)v_{,1\alpha}v_{,1\alpha}\mathrm{d}x_2 + \gamma k_2\int_0^t\int_{L_z} exp(-\omega\eta)v_{,1\alpha\beta}v_{,1\alpha\beta}\mathrm{d}x_2\mathrm{d}\eta \\
&- \lambda k_2\int_0^t\int_{L_z} exp(-\omega\eta)v_{,111}u_{,1}\mathrm{d}x_2\mathrm{d}\eta - \lambda k_2\int_0^t\int_{L_z} exp(-\omega\eta)v_{,11}u_{,11}\mathrm{d}x_2\mathrm{d}\eta \\
&+ k_2\int_0^t\int_{L_z} exp(-\omega\eta)v_{,112}v_{,2\eta}\mathrm{d}x_2\mathrm{d}\eta + k_2\int_0^t\int_{L_z} exp(-\omega\eta)v_{,12}u_{,12\eta}\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned}
\tag{37}
$$

Using the results (3.1)–(3.3) in [19], we have

$$\int_0^t\int_{L_z} exp(-\omega\eta)v_{,\alpha}v_{,\alpha}\mathrm{d}x_2\mathrm{d}\eta \le c_1\int_0^t\int_{L_z} exp(-\omega\eta)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}x_2\mathrm{d}\eta, \tag{38}$$

$$\int_{L_z} exp(-\omega t)v_{,\alpha}v_{,\alpha}\mathrm{d}x_2 \le c_1\int_{L_z} exp(-\omega t)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}x_2, \tag{39}$$

$$\int_0^t\int_{L_z} exp(-\omega\eta)u^2\mathrm{d}x_2\mathrm{d}\eta \le c_2\int_0^t\int_{L_z} exp(-\omega\eta)u_{,\alpha\alpha}u_{,\alpha\alpha}\mathrm{d}x_2\mathrm{d}\eta, \tag{40}$$

$$\int_{L_z} exp(-\omega t)u^2\mathrm{d}x_2 \le c_2\int_{L_z} exp(-\omega t)u_{,\alpha\alpha}u_{,\alpha\alpha}\mathrm{d}x_2, \tag{41}$$

$$\int_{L_z} exp(-\omega t)v_{,1}v_{,1}\mathrm{d}x_2 \leq c_3 \int_{L_z} exp(-\omega t)v_{,1\alpha}v_{,1\alpha}\mathrm{d}x_2\mathrm{d}\eta, \tag{42}$$

with c_1, c_2, and c_3 are positive constants .

Using the Schwarz inequality, and combining (37)–(42), we obtain

$$\begin{aligned}
\frac{\partial^2\varphi(z,t)}{\partial z^2} \geq & (\frac{\omega\rho_1}{2} - \frac{\lambda}{2}\varepsilon_4)\int_0^t\int_{L_z} exp(-\omega\eta)u_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta + \frac{\rho_1}{2}\int_{L_z} exp(-\omega t)u_{,t}^2\mathrm{d}x_2 \\
& + [\frac{\omega}{2} - \frac{\lambda\omega}{2}\varepsilon_1 - \frac{\rho_1 k_1 - \lambda}{2\varepsilon_3} - \lambda\varepsilon_6 - \frac{\lambda\omega}{2}\varepsilon_7 - \frac{k\lambda\gamma}{m^2}\varepsilon_{15} - (\lambda k_2 - \frac{2k\lambda\gamma}{m^2})\frac{\varepsilon_{16}}{2} \\
& - \frac{\lambda k_2}{2\varepsilon_{17}}]\int_0^t\int_{L_z} exp(-\omega\eta)u_{,\alpha}u_{,\alpha}\mathrm{d}x_2\mathrm{d}\eta + (\frac{1}{2} - \frac{\lambda}{2}\varepsilon_2 - \frac{\lambda}{2}\varepsilon_9)\int_{L_z} exp(-\omega t)u_{,\alpha}u_{,\alpha}\mathrm{d}x_2 \\
& + [\mu - \frac{\rho_1 k_1 - \lambda}{2}\varepsilon_3 - \rho_1 k_1]\int_0^t\int_{L_z} exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta \\
& + [(1+\mu\omega)k_1 - \frac{1}{2\varepsilon_4} - \lambda\varepsilon_5 - \frac{\lambda\omega c_2}{2}\varepsilon_8 - \frac{k\lambda\gamma}{m^2}\varepsilon_{14} - \frac{\lambda k_2}{2\varepsilon_{18}}]\int_0^t\int_{L_z} exp(-\omega\eta)(u_{,\alpha\eta})^2\mathrm{d}x_2\mathrm{d}\eta \\
& + (\mu k_1 - \frac{\lambda c_2}{2}\varepsilon_{10})\int_{L_z} exp(-\omega t)(u_{,\alpha\alpha})^2\mathrm{d}x_2 \\
& + [\frac{\rho_2\omega}{2} - \frac{\lambda}{\varepsilon_5} - \frac{1}{2}(\frac{2\gamma k\rho_2\omega}{m^2} - \rho_2\omega k_2)\varepsilon_{12}]\int_0^t\int_{L_z} exp(-\omega\eta)v_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta \\
& + [\frac{\rho_2}{2} - \frac{1}{2}(\frac{2\gamma k\rho_2}{m^2} + \rho_2 k_2)\varepsilon_{13}]\int_{L_z} exp(-\omega t)v_{,t}^2\mathrm{d}x_2 \\
& + [\frac{m^2}{k} - \frac{\lambda}{\varepsilon_6} - \frac{\gamma k\rho_2\omega}{m^2}\varepsilon_{11} - \frac{k_2}{2\varepsilon_{19}} - \rho_2 k_2]\int_0^t\int_{L_z} exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta \\
& + [\frac{\omega\gamma}{2} - \frac{\lambda\omega c_1}{2\varepsilon_1} - \frac{\lambda\omega c_1}{2\varepsilon_7} - \frac{\lambda\omega}{2\varepsilon_8} - \frac{\gamma k\rho_2\omega c_1}{m^2\varepsilon_{11}} - \frac{1}{2}(\frac{2\gamma k\rho_2\omega}{m^2} - \rho_2\omega k_2)\frac{1}{\varepsilon_{12}} - \frac{k\lambda\gamma}{m^2}\frac{1}{\varepsilon_{14}} \\
& - \frac{\lambda k_2}{2}\varepsilon_{18}]\int_0^t\int_{L_z} exp(-\omega\eta)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}x_2\mathrm{d}\eta + [\frac{\gamma}{2} - \frac{\lambda c_1}{2\varepsilon_2}]\int_{L_z} exp(-\omega t)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}x_2 \\
& + [\frac{m^2}{2k}k_2 - \frac{\lambda c_3}{2\varepsilon_9} - \frac{\lambda}{2\varepsilon_{10}} - \frac{1}{2\varepsilon_{13}}(\frac{2\gamma k\rho_2}{m^2} + \rho_2 k_2)]\int_{L_z} exp(-\omega t)v_{,1\alpha}v_{,1\alpha}\mathrm{d}x_2 \\
& + [\gamma k_2 - \frac{k\lambda\gamma}{m^2\varepsilon_{15}} - (\lambda k_2 - \frac{2k\lambda\gamma}{m^2})\frac{1}{2\varepsilon_{16}} - \frac{\lambda k_2}{2}\varepsilon_{17} - \frac{k_2}{2}\varepsilon_{19}]\int_0^t\int_{L_z} exp(-\omega\eta)v_{,1\alpha\beta}v_{,1\alpha\beta}\mathrm{d}x_2\mathrm{d}\eta \\
& + \frac{k_2}{2}\int_{L_z} exp(-\omega t)v_{,12}v_{,12}\mathrm{d}x_2 + \frac{2\gamma k\rho_2}{m^2}\int_{L_z} exp(-\omega t)v_{,1t}v_{,1}\mathrm{d}x_2,
\end{aligned} \tag{43}$$

where $\varepsilon_i, (i = 1, 2, \cdots, 20)$ are arbitrary positive constants .

In (43), if we choose $\varepsilon_1 = \varepsilon_2 = \varepsilon_7 = \varepsilon_9 = \frac{1}{4\lambda}, \varepsilon_3 = 2, \varepsilon_4 = \frac{\omega\rho_1}{2\lambda}, \varepsilon_5 = \frac{\omega\rho_2}{8\lambda}, \varepsilon_6 = \lambda, \varepsilon_8 = 1$, $\varepsilon_{10} = \frac{\mu k_1}{\lambda c_2}, \varepsilon_{11} = \varepsilon_{12} = \gamma, \varepsilon_{13} = \frac{1}{2}(\frac{2\gamma k\rho_2}{m^2} + \rho_2 k_2)^{-1}\rho_1, \varepsilon_{14} = \frac{\gamma}{\omega}, \varepsilon_{15} = \frac{k\lambda\gamma}{m^2}, \varepsilon_{16} = \frac{2\lambda}{\gamma}, \varepsilon_{17} = \frac{\gamma}{2\lambda}$, $\varepsilon_{18} = \omega, \varepsilon_{19} = \frac{\gamma}{2}, k_1 = \frac{1}{2\rho_1}(\frac{\mu}{2} + \lambda), k_2 = \frac{4}{\gamma}; m \geq \max\{\sqrt{\frac{4k}{k_2}[2\lambda^2 c_3 + \frac{\lambda^2 c_2}{2\mu k_1} + \frac{(2\gamma k\rho_2 + \rho_2 k_2)^2}{\rho_1}]}, 1, 8\gamma^2 k, \sqrt{2k(1 + \gamma^2 k\rho_2\omega + \frac{4}{\gamma^2} + \frac{4\rho_2}{\gamma})}\}, \omega \geq \max\{2k_1\rho_1 + 16\lambda^2 + 8k^2\lambda^2\gamma^2, 1\}, \mu \geq \sqrt{\frac{8\lambda}{\omega^2} + \rho_1\rho_2 + 4\rho_1\lambda c_1 + 8\rho_1 k\lambda\gamma^2}, \gamma \geq \max\{16\lambda^2 c_1 + 10\lambda + 4k\rho_2 c_1 + 4k\rho_2 + 4k\lambda, 1\}$, we can get

$$
\begin{aligned}
\frac{\partial^2\varphi(z,t)}{\partial z^2} \geq & \frac{\omega\rho_1}{4}\int_0^t\int_{L_z} exp(-\omega\eta)u_{,\eta}^2 \mathrm{d}x_2\mathrm{d}\eta + \frac{\rho_1}{2}\int_{L_z} exp(-\omega t)u_{,t}^2 \mathrm{d}x_2 \\
& + \frac{\omega}{8}\int_0^t\int_{L_z} exp(-\omega\eta)u_{,\alpha}u_{,\alpha}\mathrm{d}x_2\mathrm{d}\eta + \frac{1}{4}\int_{L_z} exp(-\omega t)u_{,\alpha}u_{,\alpha}\mathrm{d}x_2 \\
& + \frac{\mu}{2}\int_0^t\int_{L_z} exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta + \frac{\mu\omega k_1}{2}\int_0^t\int_{L_z} exp(-\omega\eta)(u_{,\alpha\alpha})^2\mathrm{d}x_2\mathrm{d}\eta \\
& + \frac{\mu k_1}{2}\int_{L_z} exp(-\omega t)(u_{,\alpha\alpha})^2\mathrm{d}x_2 + \frac{\rho_2\omega}{4}\int_0^t\int_{L_z} exp(-\omega\eta)v_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta \\
& + \frac{\rho_2}{4}\int_{L_z} exp(-\omega t)v_{,t}^2\mathrm{d}x_2 + \frac{m^2}{2k}\int_0^t\int_{L_z} exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta \\
& + \frac{\omega\gamma}{4}\int_0^t\int_{L_z} exp(-\omega\eta)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}x_2\mathrm{d}\eta + \frac{\gamma}{4}\int_{L_z} exp(-\omega t)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}x_2 \\
& + \frac{m^2}{4k}k_2\int_{L_z} exp(-\omega t)v_{,1\alpha}v_{,1\alpha}\mathrm{d}x_2 + \int_0^t\int_{L_z} exp(-\omega\eta)v_{,1\alpha\beta}v_{,1\alpha\beta}\mathrm{d}x_2\mathrm{d}\eta \\
& + \frac{k_2}{2}\int_{L_z} exp(-\omega t)v_{,12}v_{,12}\mathrm{d}x_2 + \frac{2\gamma k\rho_2}{m^2}\int_{L_z} exp(-\omega t)v_{,1t}v_{,1}\mathrm{d}x_2 \\
& = E(z,t).
\end{aligned}
\tag{44}
$$

We now define a new function $F(z,t)$ as

$$
F(z,t) = \int_0^t E(z,s)ds. \tag{45}
$$

We get

$$
\begin{aligned}
F(z,t) = & \frac{\omega\rho_1}{4}\int_0^t\int_0^s\int_{L_z} exp(-\omega\eta)u_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta ds + \frac{\rho_1}{2}\int_0^t\int_{L_z} exp(-\omega s)u_{,s}^2\mathrm{d}x_2 ds \\
& + \frac{\omega}{8}\int_0^t\int_0^s\int_{L_z} exp(-\omega\eta)u_{,\alpha}u_{,\alpha}\mathrm{d}x_2\mathrm{d}\eta ds + \frac{1}{4}\int_0^t\int_{L_z} exp(-\omega s)u_{,\alpha}u_{,\alpha}\mathrm{d}x_2 ds \\
& + \frac{\mu}{2}\int_0^t\int_0^s\int_{L_z} exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta ds + \frac{\mu\omega k_1}{2}\int_0^t\int_0^s\int_{L_z} exp(-\omega\eta)(u_{,\alpha\alpha})^2\mathrm{d}x_2\mathrm{d}\eta ds \\
& + \frac{\mu k_1}{2}\int_0^t\int_{L_z} exp(-\omega s)(u_{,\alpha\alpha})^2\mathrm{d}x_2 ds + \frac{\rho_2\omega}{4}\int_0^t\int_0^s\int_{L_z} exp(-\omega\eta)v_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta ds \\
& + \frac{\rho_2}{4}\int_0^t\int_{L_z} exp(-\omega s)v_{,t}^2\mathrm{d}x_2 ds + \frac{m^2}{2k}\int_0^t\int_0^s\int_{L_z} exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta ds \\
& + \frac{\omega\gamma}{4}\int_0^t\int_0^s\int_{L_z} exp(-\omega\eta)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}x_2\mathrm{d}\eta ds + \frac{\gamma}{4}\int_0^t\int_{L_z} exp(-\omega s)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}x_2 ds \\
& + \frac{m^2}{4k}k_2\int_0^t\int_{L_z} exp(-\omega s)v_{,1\alpha}v_{,1\alpha}\mathrm{d}x_2 ds + \int_0^t\int_0^s\int_{L_z} exp(-\omega\eta)v_{,1\alpha\beta}v_{,1\alpha\beta}\mathrm{d}x_2\mathrm{d}\eta ds \\
& + \frac{k_2}{2}\int_0^t\int_{L_z} exp(-\omega s)v_{,12}^2\mathrm{d}x_2 ds + \frac{\omega\gamma k\rho_2}{m^2}\int_0^t\int_{L_z} exp(-\omega s)v_{,1}^2\mathrm{d}x_2 ds \\
& + \frac{\gamma k\rho_2}{m^2}\int_{L_z} exp(-\omega t)v_{,1}^2\mathrm{d}x_2.
\end{aligned}
\tag{46}
$$

We can easily get

$$F(z,t) \geq 0. \tag{47}$$

Following the same procedures as (37)–(44), we obtain

$$\frac{\partial^2 \varphi(z,t)}{\partial z^2} \leq \frac{3}{2} E(z,t). \tag{48}$$

Inserting (30) into (48), we have

$$\frac{\partial^2 \Phi(z,t)}{\partial z^2} \leq \frac{3}{2} F(z,t). \tag{49}$$

We can also get

$$\frac{\partial^2 \Phi(z,t)}{\partial z^2} \geq F(z,t). \tag{50}$$

Combining (47) and (50), we obtain

$$\frac{\partial^2 \Phi(z,t)}{\partial z^2} = \frac{\partial^2 \int_0^t \varphi(z,s)ds}{\partial z^2} \geq 0. \tag{51}$$

Combining (36), (44) and (51), using the Schwarz's inequality, we can obtain

$$|\Phi(z,t)| \leq k_3 \frac{\partial^2 \Phi(z,t)}{\partial z^2},$$

where k_3 is a computable positive constant. The proof of Lemma 5 is finished. □

3. *Phragmén-Lindelöf* Alternative Results

Based on Lemmas 1–5, we can get the following Lemmas:

Lemma 6. *We suggest u and v are classical solutions of problems* (4)–(7) *in the semi-infinite strip* Ω_0 *defined by* (1)*, if there exists a* $z_0 \geq 0$ *such that* $\frac{\partial \Phi(z_0,t)}{\partial z} > 0$*, then the following inequality holds:*

$$\lim_{z \to \infty} e^{-k_4 z} G(z,t) \geq c_1(t), \tag{52}$$

where $c_1(t) = \frac{2}{3}[\frac{\partial}{\partial z}\Phi(z_1,t) + k_4 \Phi(z_1,t)]e^{-k_4 z_1}$*,* $G(z,t)$*will be defined in* (61)*.*

Proof. Since $\frac{\partial^2 \Phi(z_0,t)}{\partial z^2} \geq 0$ for all $z \geq 0$, we can get $\frac{\partial \Phi(z,t)}{\partial z} > 0$ for all $z \geq z_0$.

We know the fact $\Phi(z,t) \geq \Phi(z_0,t) + \frac{\partial \Phi(z_0,t)}{\partial z}(z - z_0)$ for all $z \geq z_0$.

If we let $z \to +\infty$, we can obtain $\Phi(z,t) > 0$.

So, wo have the following results:

There exists a $z_1 > z_0$ such that $\frac{\partial \Phi(z_1,t)}{\partial z} > 0$ and $\Phi(z_1,t) > 0$.

From (31), we can get

$$\frac{\partial^2 \Phi(z,t)}{\partial z^2} - k_4^2 \Phi(z,t) \geq 0, \tag{53}$$

with $k_4 = \sqrt{\frac{1}{k_3}}$.

Equation (53) can be rewritten as

$$\frac{\partial}{\partial z}\left(e^{-k_4 z}\left[\frac{\partial}{\partial z}\Phi(z,t)+k_4\Phi(z,t)\right]\right)\geq 0, \tag{54}$$

or

$$\frac{\partial}{\partial z}\left(e^{k_4 z}\left[\frac{\partial}{\partial z}\Phi(z,t)-k_4\Phi(z,t)\right]\right)\geq 0. \tag{55}$$

Integrating (54) and (55), we obtain

$$\frac{\partial}{\partial z}\Phi(z,t)+k_4\Phi(z,t)\geq [\frac{\partial}{\partial z}\Phi(z_1,t)+k_4\Phi(z_1,t)]e^{k_4(z-z_1)}, \tag{56}$$

or

$$\frac{\partial}{\partial z}\Phi(z,t)-k_4\Phi(z,t)\geq [\frac{\partial}{\partial z}\Phi(z_1,t)-k_4\Phi(z_1,t)]e^{-k_4(z-z_1)}, \tag{57}$$

for all $z \geq z_1$.

Combining (56) and (57), we have

$$\begin{aligned}\frac{\partial}{\partial z}\Phi(z,t)\geq \frac{\partial}{\partial z}\Phi(z_1,t)\frac{e^{k_4(z-z_1)}+e^{-k_4(z-z_1)}}{2}\\ +k_4\Phi(z_1,t)\frac{e^{k_4(z-z_1)}-e^{-k_4(z-z_1)}}{2}.\end{aligned} \tag{58}$$

Integrating (49) from z_1 to z, we obtain

$$\int_{z_1}^{z}\frac{\partial^2\Phi(z,t)}{\partial z^2}d\xi \leq \frac{3}{2}\int_{z_1}^{z}F(\xi,t)d\xi. \tag{59}$$

Inserting (59) into (58), we have

$$\begin{aligned}\frac{3}{2}\int_{z_1}^{z}F(\xi,t)d\xi \geq \frac{\partial}{\partial z}\Phi(z_1,t)\left[\frac{e^{k_4(z-z_1)}+e^{-k_4(z-z_1)}}{2}-1\right]\\ +k_4\Phi(z_1,t)\frac{e^{k_4(z-z_1)}-e^{-k_4(z-z_1)}}{2}.\end{aligned} \tag{60}$$

If we define

$$G(z,t)=\int_{z_1}^{z}F(\xi,t)d\xi, \tag{61}$$

Combining (46) and (61), we have

$$\begin{aligned}
G(z,t) = & \frac{\omega\rho_1}{4}\int_0^t\int_0^s\int_{z_1}^z\int_{L_\xi} exp(-\omega\eta)u_{,\eta}^2 \mathrm{d}A\mathrm{d}\eta ds \\
& + \frac{\rho_1}{2}\int_0^t\int_{z_1}^z\int_{L_\xi} exp(-\omega s)u_{,s}^2 \mathrm{d}Ads \\
& + \frac{\omega}{8}\int_0^t\int_0^s\int_{z_1}^z\int_{L_\xi} exp(-\omega\eta)u_{,\alpha}u_{,\alpha} \mathrm{d}A\mathrm{d}\eta ds \\
& + \frac{1}{4}\int_0^t\int_{z_1}^z\int_{L_\xi} exp(-\omega s)u_{,\alpha}u_{,\alpha} \mathrm{d}Ads \\
& + \frac{\mu}{2}\int_0^t\int_0^s\int_{z_1}^z\int_{L_\xi} exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta} \mathrm{d}A\mathrm{d}\eta ds \\
& + \frac{\mu\omega k_1}{2}\int_0^t\int_0^s\int_{z_1}^z\int_{L_\xi} exp(-\omega\eta)(u_{,\alpha\alpha})^2 \mathrm{d}A\mathrm{d}\eta ds \\
& + \frac{\mu k_1}{2}\int_0^t\int_{z_1}^z\int_{L_\xi} exp(-\omega s)(u_{,\alpha\alpha})^2 \mathrm{d}Ads \\
& + \frac{\rho_2\omega}{4}\int_0^t\int_0^s\int_{z_1}^z\int_{L_\xi} exp(-\omega\eta)v_{,\eta}^2 \mathrm{d}A\mathrm{d}\eta ds \\
& + \frac{\rho_2}{4}\int_0^t\int_{z_1}^z\int_{L_\xi} exp(-\omega s)v_{,t}^2 \mathrm{d}Ads \\
& + \frac{m^2}{2k}\int_0^t\int_0^s\int_{z_1}^z\int_{L_\xi} exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha\eta} \mathrm{d}A\mathrm{d}\eta ds \\
& + \frac{\omega\gamma}{4}\int_0^t\int_0^s\int_{z_1}^z\int_{L_\xi} exp(-\omega\eta)v_{,\alpha\beta}v_{,\alpha\beta} \mathrm{d}A\mathrm{d}\eta ds \\
& + \frac{\gamma}{4}\int_0^t\int_{z_1}^z\int_{L_\xi} exp(-\omega s)v_{,\alpha\beta}v_{,\alpha\beta} \mathrm{d}Ads \\
& + \frac{m^2}{4k}k_2\int_0^t\int_{z_1}^z\int_{L_\xi} exp(-\omega s)v_{,1\alpha}v_{,1\alpha} \mathrm{d}Ads \\
& + \int_0^t\int_0^s\int_{z_1}^z\int_{L_\xi} exp(-\omega\eta)v_{,1\alpha\beta}v_{,1\alpha\beta} \mathrm{d}A\mathrm{d}\eta ds \\
& + \frac{k_2}{2}\int_0^t\int_{z_1}^z\int_{L_\xi} exp(-\omega s)v_{,12}^2 \mathrm{d}Ads \\
& + \frac{\omega\gamma k\rho_2}{m^2}\int_0^t\int_{z_1}^z\int_{L_\xi} exp(-\omega s)v_{,1}^2 \mathrm{d}Ads \\
& + \frac{\gamma k\rho_2}{m^2}\int_{z_1}^z\int_{L_\xi} exp(-\omega t)v_{,1}^2 \mathrm{d}A.
\end{aligned} \tag{62}$$

we obtain

$$\lim_{z\to\infty} e^{-k_4 z}G(z,t) \geq c_1(t),$$

with $c_1(t) = \frac{2}{3}[\frac{\partial}{\partial z}\Phi(z_1,t) + k_4\Phi(z_1,t)]e^{-k_4 z_1}$. □

Lemma 7. *We suggest u and v are classical solutions of problems (4)–(7) in the semi-infinite strip Ω_0 defined by (1). If $\frac{\partial\Phi(z,t)}{\partial z} \leq 0$ for all $z \geq 0$, then the following inequality holds:*

$$H(z,t) \leq c_2(t)e^{-k_4 z}. \tag{63}$$

where $c_2(t) = -\frac{\partial}{\partial z}\Phi(0,t) + k_4\Phi(0,t)]$, $H(z,t)$will be defined in (66).

Proof. If we suggest there exists a $z_0 > 0$, such that $\Phi(z_0,t) < 0$. Since $\frac{\partial\Phi(z,t)}{\partial z} \leq 0$ for all $z \geq 0$, we can get $\Phi(z,t) \leq \Phi(z_0,t) < 0$ for all $z \geq z_0$. From (31), we have $\frac{\partial\Phi(z,t)}{\partial z} - \frac{\partial\Phi(z_0,t)}{\partial z} = \frac{\partial^2\Phi(\xi,t)}{\partial z}(z-z_0) \geq -\frac{1}{k_3}\Phi(\xi,t)(z-z_0)$, with $z_0 < \xi < z$. let $z \to \infty$, we have $\frac{\partial\Phi(z,t)}{\partial z} > 0$. Which gives a contradiction to $\frac{\partial\Phi(z,t)}{\partial z} \leq 0$ for all $z \geq 0$. So we can conclude $\Phi(z,t) \geq 0$ for all $z \geq 0$.

Integrating (53) from 0 to z, we obtain

$$-\frac{\partial\Phi(z,t)}{\partial z}+k_4\Phi(z,t)\leq c_2(t)e^{-k_4 z}, \tag{64}$$

with $c_2(t)=-\frac{\partial}{\partial z}\Phi(0,t)+k_4\Phi(0,t)]$.

Since $\Phi(z,t)\geq 0$ for for all $z\geq 0$, we have

$$-\frac{\partial\Phi(z,t)}{\partial z}\leq c_2(t)e^{-k_4 z}. \tag{65}$$

From (64), we can get the results $\Phi(z,t)$ and $-\frac{\partial\Phi(z,t)}{\partial z}$ tend to 0 as $z\to\infty$. We thus have

$$\begin{aligned}-\frac{\partial\Phi(z,t)}{\partial z}&=\int_z^\infty\frac{\partial^2\Phi(\xi,t)}{\partial z^2}d\xi\\&\geq\int_z^\infty F(\xi,t)d\xi\\&=H(z,t).\end{aligned} \tag{66}$$

Combining (46) and (66), we have

$$\begin{aligned}H(z,t)=&\frac{\omega\rho_1}{4}\int_0^t\int_0^s\int_z^\infty\int_{L_\xi}exp(-\omega\eta)u_{,\eta}^2\mathrm{d}A\mathrm{d}\eta ds\\&+\frac{\rho_1}{2}\int_0^t\int_z^\infty\int_{L_\xi}exp(-\omega s)u_{,s}^2\mathrm{d}Ads\\&+\frac{\omega}{8}\int_0^t\int_0^s\int_z^\infty\int_{L_\xi}exp(-\omega\eta)u_{,\alpha}u_{,\alpha}\mathrm{d}A\mathrm{d}\eta ds\\&+\frac{1}{4}\int_0^t\int_z^\infty\int_{L_\xi}exp(-\omega s)u_{,\alpha}u_{,\alpha}\mathrm{d}x_2ds\\&+\frac{\mu}{2}\int_0^t\int_0^s\int_z^\infty\int_{L_\xi}exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta ds\\&+\frac{\mu\omega k_1}{2}\int_0^t\int_0^s\int_z^\infty\int_{L_\xi}exp(-\omega\eta)(u_{,\alpha\alpha})^2\mathrm{d}A\mathrm{d}\eta ds\\&+\frac{\mu k_1}{2}\int_0^t\int_z^\infty\int_{L_\xi}exp(-\omega s)(u_{,\alpha\alpha})^2\mathrm{d}Ads\\&+\frac{\rho_2\omega}{4}\int_0^t\int_0^s\int_z^\infty\int_{L_\xi}exp(-\omega\eta)v_{,\eta}^2\mathrm{d}A\mathrm{d}\eta ds\\&+\frac{\rho_2}{4}\int_0^t\int_z^\infty\int_{L_\xi}exp(-\omega s)v_{,t}^2\mathrm{d}Ads\\&+\frac{m^2}{2k}\int_0^t\int_0^s\int_z^\infty\int_{L_\xi}exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta ds\\&+\frac{\omega\gamma}{4}\int_0^t\int_0^s\int_z^\infty\int_{L_\xi}exp(-\omega\eta)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta ds\\&+\frac{\gamma}{4}\int_0^t\int_z^\infty\int_{L_\xi}exp(-\omega s)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}Ads\\&+\frac{m^2}{4k}k_2\int_0^t\int_z^\infty\int_{L_\xi}exp(-\omega s)v_{,1\alpha}v_{,1\alpha}\mathrm{d}Ads\\&+\int_0^t\int_0^s\int_z^\infty\int_{L_\xi}exp(-\omega\eta)v_{,1\alpha\beta}v_{,1\alpha\beta}\mathrm{d}A\mathrm{d}\eta ds\\&+\frac{k_2}{2}\int_0^t\int_z^\infty\int_{L_\xi}exp(-\omega s)v_{,12}^2\mathrm{d}Ads\\&+\frac{\omega\gamma k\rho_2}{m^2}\int_0^t\int_z^\infty\int_{L_\xi}exp(-\omega s)v_{,1}^2\mathrm{d}Ads\\&+\frac{\gamma k\rho_2}{m^2}\int_z^\infty\int_{L_\xi}exp(-\omega t)v_{,1}^2\mathrm{d}A.\end{aligned} \tag{67}$$

Inserting (66) into (65), we obtain

$$H(z,t) \leq c_2(t)e^{-k_4 z}.$$

□

Based on Lemmas 6 and 7, we can get the following theorem.

Theorem 1. *We suggest u and v are classical solutions of problems* (4)–(7) *in the semi-infinite strip* Ω_0 *defined by* (1)*, then either inequality*

$$\lim_{z\to\infty} e^{-k_4 z} G(z,t) \geq c_1(t)$$

holds or

$$H(z,t) \leq c_2(t)e^{-k_4 z}$$

holds.

Theorem 1 shows that either the energy expression $G(z,t)$ grows exponentially or the energy expression $H(z,t)$ decays exponentially.

4. Conclusions

In this paper, we studied the spatial properties of solutions for a class of thermoelastic plate with biharmonic operator in a semi-infinite cylinder in R^2. The *Phragmén-Lindelöf* alternative results were obtained based on a second-order inequality. Our method is also valid for the hyperbolic–parabolic coupling equations. We can only deal with the linear equations. For the case of nonlinear equations, it is difficult to study the spatial properties. The results of these future studies will be of great interest to researchers in our field.

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