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On Some Laws of Large Numbers for Uncertain Random Variables

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Abstract: Baoding Liu created uncertainty theory to describe the information represented by human language. In turn, Yuhan Liu founded chance theory for modelling phenomena where both uncertainty and randomness are present. The first theory involves an uncertain measure and variable, whereas the second one introduces the notions of a chance measure and an uncertain random variable. Laws of large numbers (LLNs) are important theorems within both theories. In this paper, we prove a law of large numbers (LLN) for uncertain random variables being continuous functions of pairwise independent, identically distributed random variables and regular, independent, identically distributed uncertain variables, which is a generalisation of a previously proved version of LLN, where the independence of random variables was assumed. Moreover, we prove the Marcinkiewicz–Zygmund type LLN in the case of uncertain random variables. The proved version of the Marcinkiewicz–Zygmund type theorem reflects the difference between probability and chance theory. Furthermore, we obtain the Chow type LLN for delayed sums of uncertain random variables and formulate counterparts of the last two theorems for uncertain variables. Finally, we provide illustrative examples of applications of the proved theorems. All the proved theorems can be applied for uncertain random variables being functions of symmetrically or asymmetrically distributed random variables, and symmetrical or asymmetrical uncertain variables. Furthermore, in some special cases, under the assumption of symmetry of the random and uncertain variables, the limits in the first and the third theorem have forms of symmetrical uncertain variables.

Keywords: law of large numbers; Chow theorem; Marcinkiewicz–Zygmund theorem; uncertain random variables

Citation: Nowak, P.; Hryniewicz, O. On Some Laws of Large Numbers for Uncertain Random Variables. *Symmetry* **2021**, *13*, 2258. https://doi.org/10.3390/sym13122258

Academic Editors: Jian Zhou, Ke Wang and Yuanyuan Liu

Received: 9 November 2021
Accepted: 23 November 2021
Published: 27 November 2021

1. Introduction

The classical probability theory was developed by Kolmogorov. It is a field of mathematics that is used to address practical problems in which randomness naturally occurs. Among other things, the authors would like to highlight its applications in finance, where the notion of a stochastic process, an important part of probability theory, plays a substantial role. Apart from the Kolmogorov theory, other models of randomness have been proposed, including the boolean algebraic probability theory and MV-algebraic probability theory (see [1] and references therein).

Fuzzy sets were introduced in [2] to model non-random imprecision. However, since they can have various interpretations in practice, some generalisations of fuzzy sets have been proposed, such as Atanassov's intuitionistic fuzzy sets (IFSs) (see [3] and references therein), interval-valued fuzzy sets (IVFs), defined in [4], or grey sets (see [5]).

Fuzzy random variables are used for the description of complex phenomena, which are both random and imprecise. Definitions of fuzzy random variables, e.g., proposed by Kwakernaak [6] or Puri and Ralescu [7], correspond to their different conceptions. For an interesting discussion on this issue, we refer the reader to [8]. The MV-algebraic probability theory, where observables and states are counterparts of random variables and probability measures, has also been applied for intuitionistic fuzzy events and interval-valued fuzzy events, defined as IFSs and IVFs, respectively. MV-algebraic central limit theorems were considered in [9], whereas their counterparts for intuitionistic fuzzy events and interval-valued fuzzy events were proved in [10,11], respectively. In turn, the approach proposed in [12] was devoted to the martingale convergence theorem in the intuitionistic fuzzy framework.

As described in [13], some authors have noted that human uncertainty is not adequately described by fuzziness. This fact was one of the reasons why Baoding Liu created uncertainty theory. Within this theory (see [14]), an uncertain measure of an event indicates the belief degree that this event will occur. The properties of uncertain measures, including normality, self-duality, monotonicity, countable subadditivity, and fulfilling product axiom (see [15]), have made the uncertainty theory a part of mathematics, different from classical probability. Moreover, this theory has provided the notion of an uncertain variable, describing uncertain quantities. Chance theory was proposed by Yuhan Liu in [13], who introduced the notion of a chance measure, integrating both probability and uncertain measures, and the concept of an uncertain random variable to deal with appearance simultaneously uncertain and random phenomena.

The laws of large numbers (LLNs) take an important place among theoretical issues concerning random and fuzzy random variables. The literature devoted to LLNs within probability theory is very rich. For classical versions of these theorems, we refer the reader to [16]. Many papers have been devoted to LLNs for fuzzy random variables too. We only mention a few of them. Basic versions of the law of large numbers were proved in [17] for fuzzy random variables defined by Kwakernaak and in [18] in the case where the definition proposed by Puri and Ralescu was used. Marcinkiewicz type and Chung type LLNs were considered in [19,20], respectively. Finally, the paper [21] was devoted to a weak LLN for linearly negative quadrant dependent fuzzy random variables. LLNs have also been considered for sequences of observables. Basic versions of LLNs in the MV-algebraic framework were discussed in [22], and some of their generalisations were proved in [23]. A weak law of large numbers for intuitionistic fuzzy events was considered in [24]. Moreover, LLNs were obtained in [25,26] for M-observables and IF-observables, respectively. Finally, the paper [27] concerned the interval-valued fuzzy case. In [28], LLN was proved for uncertain random variables being functions of independent, identically distributed random variables and independent, identically distributed regular uncertain variables, in the version corresponding to the classical Kolmogorov Theorem. The LLN mentioned above was further generalised to cases where random and uncertain variables are independent, random variables are identically distributed, and uncertain variables are not identically distributed (see [29]). In turn, the cases where random and uncertain variables are independent, but both are not identically distributed, were considered in [30,31]. Finally, in [32], LLNs were studied for sequences of uncertain random variables described by dependent random variables and independent uncertain variables.

In this paper, we prove an LLN for uncertain random variables being functions of pairwise, independent, identically distributed random variables and independent, identically distributed regular uncertain variables. Thus, the obtained theorem is a generalisation of the LLN from [28]. Moreover, we prove the Marcinkiewicz–Zygmund and the Chow type LLNs for sequences of uncertain random variables. We also present counterparts of the last two theorems for uncertain variables. Finally, we provide illustrative examples of applications of the proved theorems. Although the proving technique in the case of the first and the third theorems is similar to the one used in [28], the mentioned theorems are essential generalisations of the LLN from that paper. As described in Remarks 2 and 5, in some special cases, under the assumption of symmetry of the random and uncertain variables, the limits in the first and the third theorem also have symmetrical forms.

The rest of the paper is organised as follows. In Section 2, we give a brief exposition of basic elements of uncertainty and chance theory, including the notions of uncertain variables and uncertain random variables. Section 3 is dedicated to the generalisation of the LLN from [28]. The Marcinkiewicz–Zygmund type LLN within the theory of uncertain random variables is proved in Section 4. Section 5 is devoted to the Chow LLN for delayed sums of uncertain random variables. A short conclusion is presented in Section 6. Finally, classical probabilistic versions of the proved theorems are presented in Appendix A.

2. Preliminaries

This section is dedicated to basic definitions and theorems concerning uncertain variables and uncertain random variables.

2.1. Uncertainty Space and Uncertain Variable

We denote by $\mathbb{R}$, $\mathcal{B}(\mathbb{R})$, and $\mathbb{N}$ the set of real numbers, the σ-algebra of Borel subsets of $\mathbb{R}$, and the set of positive integers, respectively.

Definition 1 (see [14])**.** *Let Γ be a nonempty set and $\mathcal{L}$ be a σ-algebra of subsets of Γ. We call a set function $\mathcal{M} : \mathcal{L} \to [0,1]$ an uncertain measure and $(\Gamma, \mathcal{L}, \mathcal{M})$ an uncertainty space if $\mathcal{M}$ satisfies the following conditions:*
Axiom 1 (normality axiom): $\mathcal{M}\{\Gamma\} = 1$;
Axiom 2 (duality axiom): $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1,\ \Lambda \in \mathcal{L}$;
Axiom 3 (subadditivity axiom): for each sequence of events $\Lambda_1, \Lambda_2, \ldots$, the following inequality holds:

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

Definition 2 (see [15])**.** *Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k),\ k = 1, 2, \ldots$, be a sequence of uncertainty spaces. We call an uncertain measure $\mathcal{M}$ on the product σ-algebra a product uncertain measure if*

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

for arbitrary $\Lambda_k \in \mathcal{L}_k,\ k = 1, 2, \ldots$

Definition 3 (see [14])**.** *A function $\tau : \Gamma \to \mathbb{R}$ is called an uncertain variable if it is measurable, i.e., if*

$$\{\tau \in B\} = \{\gamma \in \Gamma | \tau(\gamma) \in B\} \in \mathcal{L}$$

for each $B \in \mathcal{B}(\mathbb{R})$. Its uncertainty distribution is a function given by

$$\Psi(x) = \mathcal{M}\{\tau \le x\},\ x \in \mathbb{R}.$$

If its inverse function $\Psi^{-1}(\alpha)$ exists and is unique for each $\alpha \in (0,1)$, then Ψ and τ are called a regular uncertainty distribution and regular uncertain variable, respectively.

Definition 4 (see [15])**.** *Let $\tau_1, \tau_2, \ldots, \tau_n$ be uncertain variables on an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. They are said to be independent if*

$$\mathcal{M}\left\{\bigcap_{i=1}^{n} (\tau_i \in B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\tau_i \in B_i\}$$

for arbitrary $B_i \in \mathcal{B}(\mathbb{R}),\ i = 1, 2, \ldots, n.$

2.2. Chance Space and Uncertain Random Variable

Definition 5 (see [13])**.** *Let* $(\Gamma, \mathcal{L}, \mathcal{M})$ *be an uncertainty space and* $(\Omega, \mathcal{A}, \mathbb{P})$ *be a probability space. A chance space is the space of the form:*

$$(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \mathbb{P}) = (\Gamma \times \Omega, \mathcal{L} \times \mathcal{A}, \mathcal{M} \times \mathbb{P}).$$

If $\Theta \in \mathcal{L} \times \mathcal{A}$ *(i.e.,* Θ *is an uncertain random event), then the chance measure* Ch *of* Θ *is given by*

$$\mathrm{Ch}\{\Theta\} = \int_0^1 \mathbb{P}\{\omega \in \Omega | \, \mathcal{M}\{\gamma \in \Gamma | (\gamma, \omega) \in \Theta\} \geq r\} dr. \quad (1)$$

It was proved in [13] that Ch satisfies normality, duality, and monotonicity, i.e.,

$\mathrm{Ch}\{\Gamma \times \Omega\} = 1,$

$\mathrm{Ch}\{\Theta\} + \mathrm{Ch}\{\Theta^c\} = 1, \ \Theta \in \mathcal{L} \times \mathcal{A},$

$\mathrm{Ch}\{\Theta_1\} \leq \mathrm{Ch}\{\Theta_2\}$ for events Θ_1, Θ_2, such that $\Theta_1 \subseteq \Theta_2$.

Moreover, the author of [33] proved that Ch is subadditive.

Definition 6 (see [13])**.** *A function* ξ *from the chance space* $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \mathbb{P})$ *to* $\mathbb{R}$ *is called an uncertain random variable if it is measurable, i.e., for each* $B \in \mathcal{B}(\mathbb{R})$

$$\{\xi \in B\} = \{(\gamma, \omega) | \xi(\gamma, \omega) \in B\} \in \mathcal{L} \times \mathcal{A}.$$

Its chance distribution Ξ *is defined as follows:*

$$\Xi(x) = \mathrm{Ch}\{\xi \leq x\}, \ x \in \mathbb{R}.$$

Definition 7 (see [34])**.** *An uncertain random sequence* $\{\xi_n\}_{n=1}^{\infty}$ *is said to be convergent to an uncertain random variable* ξ *in measure if*

$$\lim_{n \to \infty} \mathrm{Ch}\{(\gamma, \omega) \in \mathcal{L} \times \mathcal{A} \, || \xi_n(\gamma, \omega) - \xi(\gamma, \omega)| \geq \varepsilon\} = 0$$

for each $\varepsilon > 0$.

Definition 8 (see [13])**.** *An uncertain random sequence* $\{\xi_n\}_{n=1}^{\infty}$ *with chance distributions* $\{\Xi_n\}_{n=1}^{\infty}$ *is said to be convergent to an uncertain random variable* ξ *with chance distribution* Ξ *in distribution if* $\lim_{n \to \infty} \Xi_n(x) = \Xi(x)$ *for each* $x \in \mathbb{R}$ *at which* $\Xi(x)$ *is continuous.*

The following theorem from [35] shows the relationship between both types of convergence.

Theorem 1. *If an uncertain random sequence* $\{\xi_n\}_{n=1}^{\infty}$ *converges in measure to an uncertain random variable* ξ*, then it converges in distribution to* ξ*.*

3. A Generalised Version of LLN

Let $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \mathbb{P})$ be a chance space.

Let $\{\eta_i\}_{i=1}^{\infty}$ be a sequence of random variables and $\{\tau_i\}_{i=1}^{\infty}$ be a sequence of uncertain variables. We denote by C^I the class of continuous functions $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $f(x, y)$ is strictly increasing with respect to y for each $x \in \mathbb{R}$, by C^D the class of continuous functions $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $f(x, y)$ is strictly decreasing with respect to y for each $x \in \mathbb{R}$, by $\bar{C}^I$ the class of continuous functions $g : \mathbb{R} \to \mathbb{R}$ strictly increasing, and by $\bar{C}^D$ the class of continuous functions $g : \mathbb{R} \to \mathbb{R}$ strictly decreasing. Let

$$S_n = \sum_{i=1}^{n} f(\eta_i, \tau_i),\ n \in \mathbb{N}, f \in C^I \cup C^D,$$
$$S_n(y) = \sum_{i=1}^{n} f(\eta_i, y),\ y \in \mathbb{R},\ n \in \mathbb{N}, f \in C^I \cup C^D,$$
$$\bar{S}_n = \sum_{i=1}^{n} g(\tau_i),\ n \in \mathbb{N}, g \in \bar{C}^I \cup \bar{C}^D.$$

We use the symbol $\mathbb{E}$ to denote the expected value with respect to probability measure $\mathbb{P}$.

Definition 9. *Random variables $\{\eta_i\}_{i=1}^{\infty}$ are said to be pairwise independent if for each $i, j \in \mathbb{N}$, $i \neq j$, the random variables η_i and η_j are independent.*

Clearly, independence of random variables implies their pairwise independence.

Remark 1. *Let Φ be a cumulative distribution function, τ be a regular uncertain variable with uncertainty distribution Ψ, and $F(y) = \int_{-\infty}^{\infty} f(x, y) d\Phi(x)$, $y \in \mathbb{R}$. Then from considerations in [28], it follows that for $z \in (\inf\{F(y)|y \in \mathbb{R}\}, \sup\{F(y)|y \in \mathbb{R}\})$ the function*

$$z \mapsto \mathcal{M}\left\{\int_{-\infty}^{\infty} f(x, \tau) d\Phi(x) \leq z\right\} = \begin{cases} \Psi(F^{-1}(z)) & if \quad f \in C^I, \\ 1 - \Psi(F^{-1}(z)) & if \quad f \in C^D \end{cases} \tag{2}$$

is continuous and strictly increasing.

Theorem 2. *Let random variables $\{\eta_i\}_{i=1}^{\infty}$ be pairwise independent, identically distributed with cumulative distribution function Φ and uncertain variables $\{\tau_i\}_{i=1}^{\infty}$ be regular, independent, identically distributed with uncertainty distribution Ψ. Assume that $f \in C^I \cup C^D$ and $\mathbb{E}\,|f(\eta_1, y)| < \infty$, $y \in \mathbb{R}$. Then*

$$\lim_{n \to \infty} \frac{S_n}{n} = \int_{-\infty}^{\infty} f(x, \tau_1) d\Phi(x) \tag{3}$$

in the sense of convergence in distribution.

Proof. In the proof, we use the approach from [28].

Assume that $f \in C^I$. Fix $z \in \mathbb{R}$. Let $\varepsilon > 0$. By Theorem A1 in Appendix A, there exists $N_1 \in \mathbb{N}$ such that for each $n \geq N_1$

$$\mathbb{P}\left\{\frac{S_n(F^{-1}(z - \varepsilon))}{n} \leq z\right\} \geq 1 - \varepsilon, \tag{4}$$

where F is defined as in Remark 1 for the cumulative distribution function Φ of the random variables $\{\eta_i\}_{i=1}^{\infty}$.
Let $n \geq N_1$.

$$\begin{aligned}
\mathrm{Ch}\left\{\frac{S_n}{n} \leq z\right\} &= \int_0^1 \mathbb{P}\left\{\mathcal{M}\left\{\frac{S_n}{n} \leq z\right\} \geq r\right\} dr \\
&\geq \int_0^1 \mathbb{P}\left\{\left\{\frac{S_n(F^{-1}(z-\varepsilon))}{n} \leq z\right\} \cap \left\{\mathcal{M}\left\{\frac{S_n}{n} \leq z\right\} \geq r\right\}\right\} dr \\
&\geq \int_0^1 \mathbb{P}\left\{\left\{\frac{S_n(F^{-1}(z-\varepsilon))}{n} \leq z\right\} \cap \left\{\mathcal{M}\left\{\frac{S_n}{n} \leq \frac{S_n(F^{-1}(z-\varepsilon))}{n}\right\} \geq r\right\}\right\} dr \\
&= \int_0^1 \mathbb{P}\left\{\left\{\frac{S_n(F^{-1}(z-\varepsilon))}{n} \leq z\right\}\right. \\
&\quad \left.\cap \left\{\mathcal{M}\left\{\sum_{i=1}^{n} f(\eta_i, \tau_i) \leq \sum_{i=1}^{n} f(\eta_i, F^{-1}(z-\varepsilon))\right\} \geq r\right\}\right\} dr \\
&\geq \int_0^1 \mathbb{P}\left\{\left\{\frac{S_n(F^{-1}(z-\varepsilon))}{n} \leq z\right\}\right. \\
&\quad \left.\cap \left\{\mathcal{M}\left\{\bigcap_{i=1}^{n}\left\{f(\eta_i, \tau_i) \leq f(\eta_i, F^{-1}(z-\varepsilon))\right\}\right\} \geq r\right\}\right\} dr.
\end{aligned} \tag{5}$$

$\{\tau_i\}_{i=1}^{\infty}$ are independent and $f(x,y)$ is a strictly increasing function of y for each x. Thus, applying (4) and (5) we obtain

$$\begin{aligned}
\mathrm{Ch}\left\{\frac{S_n}{n} \leq z\right\} &\geq \int_0^1 \mathbb{P}\left\{\left\{\frac{S_n(F^{-1}(z-\varepsilon))}{n} \leq z\right\} \cap \left\{\bigwedge_{i=1}^{n} \mathcal{M}\left\{\tau_i \leq F^{-1}(z-\varepsilon)\right\} \geq r\right\}\right\} dr \\
&= \int_0^1 \mathbb{P}\left\{\left\{\frac{S_n(F^{-1}(z-\varepsilon))}{n} \leq z\right\} \cap \left\{\bigwedge_{i=1}^{n} \Psi(F^{-1}(z-\varepsilon)) \geq r\right\}\right\} dr \\
&\geq (1-\varepsilon)\Psi(F^{-1}(z-\varepsilon)).
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \mathrm{Ch}\left\{\frac{S_n}{n} \leq z\right\} \geq \Psi(F^{-1}(z)). \tag{6}$$

In a similar way, let $\varepsilon > 0$. Theorem A1 in Appendix A implies that there exists $N_2 \in \mathbb{N}$ such that for each $n \geq N_2$

$$\mathbb{P}\left\{\frac{S_n(F^{-1}(z+\varepsilon))}{n} > z\right\} \geq 1-\varepsilon. \tag{7}$$

Let $n \geq N_2$.

$$\begin{aligned}
\mathrm{Ch}\left\{\frac{S_n}{n} > z\right\} &= \int_0^1 \mathbb{P}\left\{\mathcal{M}\left\{\frac{S_n}{n} > z\right\} \geq r\right\} dr \\
&\geq \int_0^1 \mathbb{P}\left\{\left\{\frac{S_n(F^{-1}(z+\varepsilon))}{n} > z\right\} \cap \left\{\mathcal{M}\left\{\frac{S_n}{n} > z\right\} \geq r\right\}\right\} dr \\
&\geq \int_0^1 \mathbb{P}\left\{\left\{\frac{S_n(F^{-1}(z+\varepsilon))}{n} > z\right\} \cap \left\{\mathcal{M}\left\{\frac{S_n}{n} > \frac{S_n(F^{-1}(z+\varepsilon))}{n}\right\} \geq r\right\}\right\} dr \\
&= \int_0^1 \mathbb{P}\left\{\left\{\frac{S_n(F^{-1}(z+\varepsilon))}{n} > z\right\}\right. \\
&\quad \left.\cap \left\{\mathcal{M}\left\{\sum_{i=1}^n f(\eta_i, \tau_i) > \sum_{i=1}^n f(\eta_i, F^{-1}(z+\varepsilon))\right\} \geq r\right\}\right\} dr \\
&\geq \int_0^1 \mathbb{P}\left\{\left\{\frac{S_n(F^{-1}(z+\varepsilon))}{n} > z\right\}\right. \\
&\quad \left.\cap \left\{\mathcal{M}\left\{\bigcap_{i=1}^n \left\{f(\eta_i, \tau_i) > f(\eta_i, F^{-1}(z+\varepsilon))\right\}\right\} \geq r\right\}\right\} dr. \qquad (8)
\end{aligned}$$

From independence of $\{\tau_i\}_{i=1}^{\infty}$, the assumption that $f(x, y)$ is a strictly increasing function of y for each x, (7), and (8) it follows that

$$\begin{aligned}
\mathrm{Ch}\left\{\frac{S_n}{n} > z\right\} &\geq \int_0^1 \mathbb{P}\left\{\left\{\frac{S_n(F^{-1}(z+\varepsilon))}{n} > z\right\} \cap \left\{\bigwedge_{i=1}^n \mathcal{M}\left\{\tau_i > F^{-1}(z+\varepsilon)\right\} \geq r\right\}\right\} dr \\
&= \int_0^1 \mathbb{P}\left\{\left\{\frac{S_n(F^{-1}(z+\varepsilon))}{n} > z\right\} \cap \left\{\bigwedge_{i=1}^n (1 - \Psi(F^{-1}(z+\varepsilon))) \geq r\right\}\right\} dr \\
&\geq (1-\varepsilon)(1 - \Psi(F^{-1}(z+\varepsilon))). \qquad (9)
\end{aligned}$$

By (9) and the duality property of Ch,

$$\mathrm{Ch}\left\{\frac{S_n}{n} \leq z\right\} \leq 1 - (1-\varepsilon)(1 - \Psi(F^{-1}(z+\varepsilon))) \qquad (10)$$

for each $n \geq N_2$.

(10) implies

$$\lim_{n\to\infty} \mathrm{Ch}\left\{\frac{S_n}{n} \leq z\right\} \leq \Psi(F^{-1}(z)). \qquad (11)$$

Thus, by (6) and (11),

$$\lim_{n\to\infty} \mathrm{Ch}\left\{\frac{S_n}{n} \leq z\right\} = \Psi(F^{-1}(z)) = \mathcal{M}\left\{\int_{-\infty}^{\infty} f(x, \tau_1) d\Phi(x) \leq z\right\}.$$

If $f \in C^D$, then $-f \in C^I$. Proceeding like in the previous part of the proof, we get

$$\lim_{n\to\infty} \mathrm{Ch}\left\{-\frac{S_n}{n} < -z\right\} = \mathcal{M}\left\{-\int_{-\infty}^{\infty} f(x, \tau_1) d\Phi(x) < -z\right\},$$

being equivalent to

$$\lim_{n\to\infty} \mathrm{Ch}\left\{\frac{S_n}{n} > z\right\} = \mathcal{M}\left\{\int_{\infty}^{\infty} f(x, \tau_1) d\Phi(x) > z\right\}.$$

From the duality properties of Ch and $\mathcal{M}$ we obtain

$$\lim_{n\to\infty} \mathrm{Ch}\left\{\frac{S_n}{n} \le z\right\} = \mathcal{M}\left\{\int_{-\infty}^{\infty} f(x,\tau_1)d\Phi(x) \le z\right\}.$$

□

The LLN proposed in [28] follows from Theorem 2, since independent random variables are pairwise independent.

Example 1. *We consider a sequence of independent Rademacher random variables* $\{\epsilon_i\}_{i=-1}^{\infty}$*, i.e., random variables with distribution* $\mathbb{P}\{\epsilon_i = -1\} = \mathbb{P}\{\epsilon_i = 1\} = \frac{1}{2}$*,* $i \in \mathbb{N} \cup \{-1, 0\}$*, and a sequence of independent, identically distributed indicator functions* $\mathbb{1}_{A_i}$*,* $i \in \mathbb{N}$*, such that* $\mathbb{P}\{A_i\} = \alpha$*,* $0 < \alpha < 1$*,* $i \in \mathbb{N}$*. We assume that both sequences are independent. Let* $\{\eta_i\}_{i=1}^{\infty}$ *be also Rademacher distributed random variables, given by*

$$\eta_i = \mathbb{1}_{A_i}\epsilon_i + \mathbb{1}_{A_i^c}\epsilon_{i-1}\epsilon_{i-2},\ i \in \mathbb{N}.$$

Moreover, let uncertain variables $\{\tau_i\}_{i=1}^{\infty}$ *be independent, normally distributed with uncertainty distribution*

$$\Psi(x) = \left(1 + \exp\left(\frac{-\pi x}{\sqrt{3}}\right)\right)^{-1},\ x \in \mathbb{R}.$$

Then $\{\eta_i\}_{i=1}^{\infty}$ *are pairwise independent, but dependent (see [36] for the proof). Therefore, the LLN proposed in [28] can not be applied to the sequence of uncertain random variables* $\{\eta_i + \tau_i\}_{i=1}^{\infty}$*. However, from Theorem 2 it follows that*

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n}(\eta_i + \tau_i)}{n} = \tau_1$$

in the sense of convergence in distribution.

The above example is an illustration of the following remark.

Remark 2. *If random variables* $\{\eta_i\}_{i=1}^{\infty}$ *and uncertain variables* $\{\tau_i\}_{i=1}^{\infty}$ *satisfy the assumptions of Theorem 2,* $\{\eta_i\}_{i=1}^{\infty}$ *are symmetrically distributed, i.e.,* η_i *and* $-\eta_i$ *have the same distribution for* $i \in \mathbb{N}$*,* $\{\tau_i\}_{i=1}^{\infty}$ *are symmetrical, i.e.,* $\Psi(x) + \Psi(-x) = 1$*,* $x \in \mathbb{R}$ *(see, e.g., [37]), and* $f(x,y) = x + y$ *or* $f(x,y) = x - y$*, then* $\lim_{n\to\infty} \frac{S_n}{n}$ *has the form of the symmetrical uncertain variable* τ_1 *or* $-\tau_1$*, respectively.*

Remark 3. *Remark 1 and Formula (3) imply that for* $z \in (\inf\{F(y)|y \in \mathbb{R}\}, \sup\{F(y)|y \in \mathbb{R}\})$

$$\begin{aligned}\psi(z) &:= \lim_{n\to\infty} \mathrm{Ch}\left\{\frac{S_n}{n} \le z\right\} = \mathcal{M}\left\{\int_{-\infty}^{\infty} f(x,\tau_1)d\Phi(x) \le z\right\} \\ &= \begin{cases} \Psi(F^{-1}(z)) & if \quad f \in C^I, \\ 1 - \Psi(F^{-1}(z)) & if \quad f \in C^D \end{cases}\end{aligned} \tag{12}$$

is a continuous, strictly increasing function.

4. The Marcinkiewicz–Zygmund Type LLN for Independent Uncertain Random Variables

We will use the notation from the previous section. This section is devoted to the problem of convergence of the sequence

$$\frac{S_n - nc}{n^{1/p}} \text{ as } n \to \infty, \text{ where } p > 0 \text{ and } c \in \mathbb{R}. \tag{13}$$

At the beginning, we consider convergence of the sequence given in formula (13) in measure for $p \in (0,1)$.

Theorem 3. *Let random variables $\{\eta_i\}_{i=1}^{\infty}$ be independent, identically distributed with cumulative distribution function Φ and uncertain variables $\{\tau_i\}_{i=1}^{\infty}$ be independent, regular with uncertainty distributions $\{\Psi_i\}_{i=1}^{\infty}$ such that*

$$\lim_{y\to\infty} \bar{\Psi}(y) = 1, \ \lim_{y\to-\infty} \bar{\bar{\Psi}}(y) = 1, \tag{14}$$

where

$$\bar{\Psi}(y) = \lim_{n\to\infty} \bigwedge_{i=1}^{n} \Psi_i(y), \ \bar{\bar{\Psi}}(y) = \lim_{n\to\infty} \bigwedge_{i=1}^{n} (1 - \Psi_i(y)),$$

$y \in \mathbb{R}$. Let $p \in (0,1)$. If $f \in C^I \cup C^D$ and $\mathbb{E}\,|f(\eta_1, y)|^p < \infty$, $y \in \mathbb{R}$, then for arbitrary $c \in \mathbb{R}$

$$\lim_{n\to\infty} \frac{S_n - nc}{n^{1/p}} = 0$$

in the sense of convergence in measure.

Proof. Fix $\varepsilon > 0$. Let $0 < \varepsilon_1, \varepsilon_2, \varepsilon_1', \varepsilon_1'', \varepsilon_2', \varepsilon_2'' < 1$ be such that

$$(1-\varepsilon_1) < (1-\varepsilon_1')(1-\varepsilon_1''), \ (1-\varepsilon_2) < (1-\varepsilon_2')(1-\varepsilon_2'').$$

By Theorem A2 in Appendix A, there exist $N_1, N_2 \in \mathbb{N}, y_1, y_2 \in \mathbb{R}$ such that

$$\mathbb{P}\left\{ \frac{S_n(y_1) - nc}{n^{1/p}} < \varepsilon \right\} > 1 - \varepsilon_1', \bigwedge_{i=1}^{n} \Psi_i(y_1) \ge \bar{\Psi}(y_1) > 1 - \varepsilon_1'', \ n \ge N_1, \tag{15}$$

$$\mathbb{P}\left\{ \frac{S_n(y_2) - nc}{n^{1/p}} > -\varepsilon \right\} > 1 - \varepsilon_2', \bigwedge_{i=1}^{n} (1 - \Psi_i(y_2)) \ge \bar{\bar{\Psi}}(y_2) > 1 - \varepsilon_2'', \ n \ge \mathbb{N}_2\,. \tag{16}$$

Assume that $f \in C^I$. Let $n \ge N_1$.

$$\begin{aligned}
\mathrm{Ch}\left\{ \frac{S_n - nc}{n^{1/p}} < \varepsilon \right\} &= \int_0^1 \mathbb{P}\left\{ \mathcal{M}\left\{ \frac{S_n - nc}{n^{1/p}} < \varepsilon \right\} \ge r \right\} dr \\
&\ge \int_0^1 \mathbb{P}\left\{ \left\{ \frac{S_n(y_1) - nc}{n^{1/p}} < \varepsilon \right\} \ \cap \left\{ \mathcal{M}\left\{ \frac{S_n - nc}{n^{1/p}} < \varepsilon \right\} \ge r \right\} \right\} dr \\
&\ge \int_0^1 \mathbb{P}\left\{ \left\{ \frac{S_n(y_1) - nc}{n^{1/p}} < \varepsilon \right\} \ \cap \left\{ \mathcal{M}\left\{ \frac{S_n}{n^{1/p}} < \frac{S_n(y_1)}{n^{1/p}} \right\} \ge r \right\} \right\} dr \\
&= \int_0^1 \mathbb{P}\left\{ \left\{ \frac{S_n(y_1) - nc}{n^{1/p}} < \varepsilon \right\} \right. \\
&\quad \left. \cap \left\{ \mathcal{M}\left\{ \sum_{i=1}^{n} f(\eta_i, \tau_i) < \sum_{i=1}^{n} f(\eta_i, y_1) \right\} \ge r \right\} \right\} dr \\
&\ge \int_0^1 \mathbb{P}\left\{ \left\{ \frac{S_n(y_1) - nc}{n^{1/p}} < \varepsilon \right\} \right. \\
&\quad \left. \cap \left\{ \mathcal{M}\left\{ \bigcap_{i=1}^{n} \{ f(\eta_i, \tau_i) < f(\eta_i, y_1) \} \right\} \ge r \right\} \right\} dr.
\end{aligned} \tag{17}$$

By independence and regularity of $\{\tau_i\}_{i=1}^{\infty}$, the assumption that $f(x,y)$ is a strictly increasing function of y for each x, (15), and (17), it follows that

$$\begin{aligned}
&\mathrm{Ch}\left\{\frac{S_n - nc}{n^{1/p}} < \varepsilon\right\} \\
&\geq \int_0^1 \mathbb{P}\left\{\left\{\frac{S_n(y_1) - nc}{n^{1/p}} < \varepsilon\right\} \cap \left\{\bigwedge_{i=1}^{n} \mathcal{M}\{\tau_i < y_1\} \geq r\right\}\right\} dr \\
&\geq \int_0^1 \mathbb{P}\left\{\left\{\frac{S_n(y_1) - nc}{n^{1/p}} < \varepsilon\right\} \cap \left\{\bigwedge_{i=1}^{n} \Psi_i(y_1) \geq r\right\}\right\} dr \\
&= \mathbb{P}\left\{\frac{S_n(y_1) - nc}{n^{1/p}} < \varepsilon\right\} \bigwedge_{i=1}^{n} \Psi_i(y_1) > (1-\varepsilon_1')(1-\varepsilon_1'') > 1 - \varepsilon_1.
\end{aligned} \tag{18}$$

Thus,

$$\lim_{n\to\infty} \mathrm{Ch}\left\{\frac{S_n - nc}{n^{1/p}} < \varepsilon\right\} = 1. \tag{19}$$

Similarly, for $n \geq N_2$

$$\begin{aligned}
&\mathrm{Ch}\left\{\frac{S_n - nc}{n^{1/p}} > -\varepsilon\right\} = \int_0^1 \mathbb{P}\left\{\mathcal{M}\left\{\frac{S_n - nc}{n^{1/p}} > -\varepsilon\right\} \geq r\right\} dr \\
&\geq \int_0^1 \mathbb{P}\left\{\left\{\frac{S_n(y_2) - nc}{n^{1/p}} > -\varepsilon\right\} \cap \left\{\mathcal{M}\left\{\frac{S_n - nc}{n^{1/p}} > -\varepsilon\right\} \geq r\right\}\right\} dr \\
&\geq \int_0^1 \mathbb{P}\left\{\left\{\frac{S_n(y_2) - nc}{n^{1/p}} > -\varepsilon\right\} \cap \left\{\mathcal{M}\left\{\frac{S_n}{n^{1/p}} > \frac{S_n(y_2)}{n^{1/p}}\right\} \geq r\right\}\right\} dr \\
&= \int_0^1 \mathbb{P}\left\{\left\{\frac{S_n(y_2) - nc}{n^{1/p}} > -\varepsilon\right\}\right. \\
&\quad \left.\cap \left\{\mathcal{M}\left\{\sum_{i=1}^{n} f(\eta_i, \tau_i) > \sum_{i=1}^{n} f(\eta_i, y_2)\right\} \geq r\right\}\right\} dr \\
&\geq \int_0^1 \mathbb{P}\left\{\left\{\frac{S_n(y_2) - nc}{n^{1/p}} > -\varepsilon\right\}\right. \\
&\quad \left.\cap \left\{\mathcal{M}\left\{\bigcap_{i=1}^{n} \{f(\eta_i, \tau_i) > f(\eta_i, y_2)\}\right\} \geq r\right\}\right\} dr.
\end{aligned} \tag{20}$$

$\{\tau_i\}_{i=1}^{\infty}$ are independent and regular, $f(x,y)$ is a strictly increasing function of y for each x. Therefore, (16) and (20) imply

$$\begin{aligned}
&\mathrm{Ch}\left(\frac{S_n - nc}{n^{1/p}} > -\varepsilon\right) \\
&\geq \int_0^1 \mathbb{P}\left\{\left\{\frac{S_n(y_2) - nc}{n^{1/p}} > -\varepsilon\right\} \cap \left\{\bigwedge_{i=1}^{n} \mathcal{M}\{\tau_i > y_2\} \geq r\right\}\right\} dr \\
&= \int_0^1 \mathbb{P}\left\{\left\{\frac{S_n(y_2) - nc}{n^{1/p}} > -\varepsilon\right\} \cap \left\{\bigwedge_{i=1}^{n} (1 - \Psi_i(y_2)) \geq r\right\}\right\} dr \\
&= \mathbb{P}\left\{\frac{S_n(y_2) - nc}{n^{1/p}} > -\varepsilon\right\} \bigwedge_{i=1}^{n} (1 - \Psi_i(y_2)) \\
&> (1-\varepsilon_2')(1-\varepsilon_2'') > 1 - \varepsilon_2.
\end{aligned} \tag{21}$$

Thus,

$$\lim_{n\to\infty} \mathrm{Ch}\left\{ \frac{S_n - nc}{n^{1/p}} > -\varepsilon \right\} = 1. \tag{22}$$

From (19), (22), and properties of Ch it follows that

$$\begin{aligned}
&\lim_{n\to\infty} \mathrm{Ch}\left\{ \left| \frac{S_n - nc}{n^{1/p}} \right| \geq \varepsilon \right\} \\
&= \lim_{n\to\infty} \mathrm{Ch}\left\{ \left\{ \frac{S_n - nc}{n^{1/p}} \leq -\varepsilon \right\} \cup \left\{ \frac{S_n - nc}{n^{1/p}} \geq \varepsilon \right\} \right\} \\
&\leq \lim_{n\to\infty} \mathrm{Ch}\left\{ \frac{S_n - nc}{n^{1/p}} \leq -\varepsilon \right\} + \lim_{n\to\infty} \mathrm{Ch}\left\{ \frac{S_n - nc}{n^{1/p}} \geq \varepsilon \right\} \\
&= 1 - \lim_{n\to\infty} \mathrm{Ch}\left\{ \frac{S_n - nc}{n^{1/p}} > -\varepsilon \right\} + 1 - \lim_{n\to\infty} \mathrm{Ch}\left\{ \frac{S_n - nc}{n^{1/p}} < \varepsilon \right\} = 0.
\end{aligned} \tag{23}$$

If $f \in C^D$, then $-f \in C^I$. Therefore, we can apply (23) for $-f$ and $-c$. Thus,

$$\begin{aligned}
&\lim_{n\to\infty} \mathrm{Ch}\left\{ \left| \frac{\sum_{i=1}^{n} -f(\eta_i, \tau_i) + nc}{n^{1/p}} \right| \geq \varepsilon \right\} \\
&= \lim_{n\to\infty} \mathrm{Ch}\left\{ \left| \frac{\sum_{i=1}^{n} f(\eta_i, \tau_i) - nc}{n^{1/p}} \right| \geq \varepsilon \right\} = 0.
\end{aligned} \tag{24}$$

□

The following corollary concerns the case of strictly monotone functions of uncertain variables.

Corollary 1. *Let uncertain variables $\{\tau_i\}_{i=1}^{\infty}$ be independent, regular with uncertainty distributions $\{\Psi_i\}_{i=1}^{\infty}$ satisfying (14), and $p \in (0,1)$. If $g \in \bar{C}^I \cup \bar{C}^D$, then for arbitrary $c \in \mathbb{R}$ $\lim_{n\to\infty} \frac{\bar{S}_n - nc}{n^{1/p}} = 0$ in the sense of convergence in measure.*

The convergence (13) in distribution for $p > 0$ is established by our next theorem.

Theorem 4. *Let random variables $\{\eta_i\}_{i=1}^{\infty}$ be identically distributed with cumulative distribution function Φ and uncertain variables $\{\tau_i\}_{i=1}^{\infty}$ be regular, independent, identically distributed with uncertainty distribution Ψ. Let $f \in C^I \cup C^D$.*
(i) If $\{\eta_i\}_{i=1}^{\infty}$ are independent, $p \in (0,1)$, $\mathbb{E}\,|f(\eta_1, y)|^p < \infty$, $y \in \mathbb{R}$, and $c \in \mathbb{R}$, then

$$\lim_{n\to\infty} \frac{S_n - nc}{n^{1/p}} = 0$$

in the sense of convergence in distribution.
(ii) If $\{\eta_i\}_{i=1}^{\infty}$ are pairwise independent, $p = 1$, $\mathbb{E}\,|f(\eta_1, y)| < \infty$, $y \in \mathbb{R}$, and $c \in \mathbb{R}$, then

$$\lim_{n\to\infty} \frac{S_n - nc}{n^{1/p}} = \lim_{n\to\infty} \frac{S_n}{n} - c = \int_{-\infty}^{\infty} f(x, \tau_1) d\Phi(x) - c$$

in the sense of convergence in distribution.
(iii) If $\{\eta_i\}_{i=1}^{\infty}$ are pairwise independent, $p > 1$, $\mathbb{E}\,|f(\eta_1, y)| < \infty$, $y \in \mathbb{R}$, and $c \in \{u \in \mathbb{R} \,|\, \mathcal{M}\left\{ \int_{-\infty}^{\infty} f(x, \tau_1) d\Phi(x) \leq u \right\} \in (0,1)\}$, then

$$\lim_{n\to\infty} \frac{S_n - nc}{n^{1/p}} = m_{f,c}$$

in the sense of convergence in distribution, where $m_{f,c}$ is an uncertain variable with constant uncertainty distribution equal to

$$\mathcal{M}\left\{\int_{-\infty}^{\infty} f(x,\tau_1)d\Phi(x) \le c\right\}.$$

Proof. Case (i) We use Theorem 3. Clearly, $\bar{\Psi} = \Psi$, $\bar{\bar{\Psi}} = 1 - \Psi$. Therefore, $\lim_{y\to\infty} \bar{\Psi}(y) = 1$ and $\lim_{y\to-\infty} \bar{\bar{\Psi}}(y) = 1$. Thus, the assumptions of Theorem 3 are satisfied and application of Theorem 1 finishes the proof.

Case (ii) Let $z \in \mathbb{R}$. By Theorem 2,

$$\lim_{n\to\infty} \mathrm{Ch}\left\{\frac{S_n - nc}{n^{1/p}} \le z\right\} = \lim_{n\to\infty} \mathrm{Ch}\left\{\frac{S_n}{n} \le z + c\right\}$$
$$= \mathcal{M}\{\mathbb{E} f(\eta_1,\tau_1) \le z + c\} = \mathcal{M}\{\mathbb{E} f(\eta_1,\tau_1) - c \le z\}. \tag{25}$$

This finishes the proof.

Case (iii) Fix $z \in \mathbb{R}$. For each $\delta > 0$ there exists $N \in \mathbb{N}$ such that $\frac{|z|}{n^{1-1/p}} < \delta$ for each $n \ge N$. If $z \ge 0$, then

$$\mathrm{Ch}\left\{\frac{S_n}{n} \le c\right\} \le \mathrm{Ch}\left\{\frac{S_n - nc}{n^{1/p}} \le z\right\} = \mathrm{Ch}\left\{\frac{S_n}{n} \le c + \frac{z}{n^{1-1/p}}\right\}$$
$$\le \mathrm{Ch}\left\{\frac{S_n}{n} \le c + \delta\right\}, n \ge N.$$

By Theorem 2,

$$\lim_{n\to\infty} \mathrm{Ch}\left\{\frac{S_n}{n} \le c\right\} = \mathcal{M}\left\{\int_{-\infty}^{\infty} f(x,\tau_1)d\Phi(x) \le c\right\} = \psi(c), \tag{26}$$

$$\lim_{n\to\infty} \mathrm{Ch}\left\{\frac{S_n}{n} \le c + \delta\right\} = \mathcal{M}\left\{\int_{-\infty}^{\infty} f(x,\tau_1)d\Phi(x) \le c + \delta\right\} = \psi(c+\delta). \tag{27}$$

From (26) and (27) it follows that

$$\psi(c) \le \liminf_{n\to\infty} \mathrm{Ch}\left\{\frac{S_n - nc}{n^{1/p}} \le z\right\} \le \limsup_{n\to\infty} \mathrm{Ch}\left\{\frac{S_n - nc}{n^{1/p}} \le z\right\} \le \psi(c+\delta). \tag{28}$$

Since $\delta > 0$ is arbitrary, (28) and continuity of ψ imply

$$\liminf_{n\to\infty} \mathrm{Ch}\left\{\frac{S_n - nc}{n^{1/p}} \le z\right\} = \limsup_{n\to\infty} \mathrm{Ch}\left\{\frac{S_n - nc}{n^{1/p}} \le z\right\}$$
$$= \lim_{n\to\infty} \mathrm{Ch}\left\{\frac{S_n - nc}{n^{1/p}} \le z\right\} = \psi(c) = \mathcal{M}\left\{\int_{-\infty}^{\infty} f(x,\tau_1)d\Phi(x) \le c\right\}. \tag{29}$$

If $z < 0$, then

$$\mathrm{Ch}\left\{\frac{S_n}{n} \le c - \delta\right\} \le \mathrm{Ch}\left\{\frac{S_n - nc}{n^{1/p}} \le z\right\} = \mathrm{Ch}\left\{\frac{S_n}{n} \le c + \frac{z}{n^{1-1/p}}\right\}$$
$$\le \mathrm{Ch}\left\{\frac{S_n}{n} \le c\right\},\ n \ge N.$$

By Theorem 2,

$$\lim_{n\to\infty} \mathrm{Ch}\left\{\frac{S_n}{n} \le c - \delta\right\} = \mathcal{M}\left\{\int_{-\infty}^{\infty} f(x,\tau_1)d\Phi(x) \le c - \delta\right\} = \psi(c-\delta). \tag{30}$$

From (26) and (30)

$$\psi(c-\delta) \leq \liminf_{n\to\infty} \mathrm{Ch}\left\{ \frac{S_n - nc}{n^{1/p}} \leq z \right\}$$
$$\leq \limsup_{n\to\infty} \mathrm{Ch}\left\{ \frac{S_n - nc}{n^{1/p}} \leq z \right\} \leq \psi(c). \quad (31)$$

$\delta > 0$ is arbitrary. Therefore, by (31) and continuity of ψ

$$\liminf_{n\to\infty} \mathrm{Ch}\left\{ \frac{S_n - nc}{n^{1/p}} \leq z \right\} = \limsup_{n\to\infty} \mathrm{Ch}\left\{ \frac{S_n - nc}{n^{1/p}} \leq z \right\}$$
$$= \lim_{n\to\infty} \mathrm{Ch}\left\{ \frac{S_n - nc}{n^{1/p}} \leq z \right\} = \psi(c) = \mathcal{M}\left\{ \int_{-\infty}^{\infty} f(x,\tau_1) d\Phi(x) \leq c \right\}. \quad (32)$$

□

Remark 4. *Case (iii) of the above theorem reflects the difference between chance theory and probability theory, where such convergence for random variables generally does not hold for $p > 2$.*

Example 2. *Let random variables $\{\eta_i\}_{i=1}^{\infty}$ be independent, identically distributed with cumulative distribution function Φ. Let uncertain variables $\{\tau_i\}_{i=1}^{\infty}$ be regular, independent, identically distributed with uncertainty distribution Ψ. Then from Theorem 4 it follows that*
(i) If $p \in (0,1)$, $\mathbb{E}\,|\eta_1|^p < \infty$, and $c \in \mathbb{R}$, then

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n}(\eta_i + \tau_i) - nc}{n^{1/p}} = 0, \quad \lim_{n\to\infty} \frac{\sum_{i=1}^{n}(\eta_i - \tau_i) - nc}{n^{1/p}} = 0$$

in the sense of convergence in distribution.
(ii) If $p = 1$, $\mathbb{E}\,|\eta_1| < \infty$, and $c \in \mathbb{R}$, then

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n}(\eta_i + \tau_i) - nc}{n^{1/p}} = \lim_{n\to\infty} \frac{\sum_{i=1}^{n}(\eta_i + \tau_i)}{n} - c$$
$$= \tau_1 + \mathbb{E}\,\eta_1 - c,$$
$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n}(\eta_i - \tau_i) - nc}{n^{1/p}} = \lim_{n\to\infty} \frac{\sum_{i=1}^{n}(\eta_i - \tau_i)}{n} - c$$
$$= -\tau_1 + \mathbb{E}\,\eta_1 - c$$

in the sense of convergence in distribution.
(iii) If $p > 1$ and $\mathbb{E}\,|\eta_1| < \infty$, then

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n}(\eta_i + \tau_i) - nc}{n^{1/p}} = m_1, \; c \in \{u \in \mathbb{R}\,|\,\Psi(u - \mathbb{E}\,\eta_1) \in (0,1)\},$$
$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n}(\eta_i - \tau_i) - nc}{n^{1/p}} = m_2, \; c \in \{u \in \mathbb{R}\,|\,\Psi(\mathbb{E}\,\eta_1 - u) \in (0,1)\},$$

in the sense of convergence in distribution, where m_1 and m_2 are uncertain variables with constant uncertainty distributions equal to $\Psi(c - \mathbb{E}\,\eta_1)$ and $1 - \Psi(\mathbb{E}\,\eta_1 - c)$, respectively.

Example 3. *Let the assumptions from Example 2 be fulfilled, η_1 have the discrete distribution*

$$\mathbb{P}\left\{ \eta_1 = 2^{5i/8} \right\} = 1/2^i, i \in \mathbb{N},$$

and τ_1 have the linear distribution

$$\Psi(x) = \begin{cases} 0 & \text{if} \quad x \leq 0; \\ x & \text{if} \quad 0 < x \leq 1; \\ 1 & \text{if} \quad x > 1. \end{cases}$$

$\mathbb{E}\,|\eta_1|^p < \infty$ *if and only if* $p < 8/5$. *Moreover,* $\mathbb{E}\,\eta_1 = \frac{2^{-3/8}}{1-2^{-3/8}} = \frac{1}{2^{3/8}-1}$. *Therefore,*

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n(\eta_i + \tau_i) - nc}{n^{1/p}} = 0, \ \lim_{n\to\infty} \frac{\sum_{i=1}^n(\eta_i - \tau_i) - nc}{n^{1/p}} = 0, \ p \in (0,1), \ c \in \mathbb{R},$$
$$\lim_{n\to\infty} \frac{\sum_{i=1}^n(\eta_i + \tau_i) - nc}{n} = \tau_1 + \frac{1}{2^{3/8}-1} - c,$$

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n(\eta_i - \tau_i)}{n} - c = -\tau_1 + \frac{1}{2^{3/8}-1} - c, \ c \in \mathbb{R},$$
$$\lim_{n\to\infty} \frac{\sum_{i=1}^n(\eta_i + \tau_i) - nc}{n^{1/p}} = m_1,$$
$$p > 1, \ c \in \left(\frac{1}{2^{3/8}-1}, \frac{1}{2^{3/8}-1} + 1\right),$$
$$\lim_{n\to\infty} \frac{\sum_{i=1}^n(\eta_i - \tau_i) - nc}{n^{1/p}} = m_2,$$
$$p > 1, \ c \in \left(\frac{1}{2^{3/8}-1} - 1, \frac{1}{2^{3/8}-1}\right),$$

in the sense of convergence in distribution, where m_1 *and* m_2 *are the uncertain variables with constant uncertainty distributions equal to* $c - \frac{1}{2^{3/8}-1}$ *and* $c - \frac{1}{2^{3/8}-1} + 1$, *respectively.*

In the case of strictly monotone functions of uncertain variables, we have the following corollary.

Corollary 2. *Let uncertain variables* $\{\tau_i\}_{i=1}^\infty$ *be regular, independent, identically distributed with uncertainty distribution* Ψ. *Let* $g \in \bar{\mathcal{C}}^I \cup \bar{\mathcal{C}}^D$.
(i) If $p \in (0,1)$ *and* $c \in \mathbb{R}$, *then*

$$\lim_{n\to\infty} \frac{\bar{S}_n - nc}{n^{1/p}} = 0$$

in the sense of convergence in distribution.
(ii) If $p = 1$ *and* $c \in \mathbb{R}$, *then*

$$\lim_{n\to\infty} \frac{\bar{S}_n - nc}{n^{1/p}} = \lim_{n\to\infty} \frac{\bar{S}_n}{n} - c = g(\tau_1) - c$$

in the sense of convergence in distribution.
(iii) If $p > 1$ *and* $c \in \{u \in \mathbb{R} \,|\, \mathcal{M}\{g(\tau_1) \le u\} \in (0,1)\}$, *then*

$$\lim_{n\to\infty} \frac{\bar{S}_n - nc}{n^{1/p}} = m_{g,c}$$

in the sense of convergence in distribution, where $m_{g,c}$ *is an uncertain variable with constant uncertainty distribution equal to* $\mathcal{M}\{g(\tau_1) \le c\}$.

Example 4. *Let uncertain variables* $\{\tau_i\}_{i=1}^\infty$ *be regular, independent, identically distributed with uncertainty distribution* Ψ *from Example 3. We apply Corollary 2.*
(i) If $p \in (0,1)$ *and* $c \in \mathbb{R}$, *then*

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n \tau_i^3 - nc}{n^{1/p}} = 0 \ \text{ and } \ \lim_{n\to\infty} \frac{\sum_{i=1}^n(-\tau_i^3) - nc}{n^{1/p}} = 0$$

in the sense of convergence in distribution.

(ii) If $p = 1$ and $c \in \mathbb{R}$, then

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} \tau_i^3 - nc}{n^{1/p}} = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} \tau_i^3}{n} - c = \tau_1^3 - c$$
$$\text{and} \quad \lim_{n\to\infty} \frac{\sum_{i=1}^{n} (-\tau_i^3) - nc}{n^{1/p}} = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} (-\tau_i^3)}{n} - c = -\tau_1^3 - c$$

in the sense of convergence in distribution.
(iii) If $p > 1$, then

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} \tau_i^3 - nc}{n^{1/p}} = m_1,\ c \in (0,1),$$
$$\text{and} \quad \lim_{n\to\infty} \frac{\sum_{i=1}^{n} (-\tau_i^3) - nc}{n^{1/p}} = m_2,\ c \in (-1,0),$$

in the sense of convergence in distribution, where m_1 and m_2 are uncertain variables with constant uncertainty distributions equal to $c^{1/3}$ and $c^{1/3} + 1$, respectively.

5. The Chow LLN for Delayed Sums of Uncertain Random Variables

We will use the notation from the previous sections. Let additionally,

$$T_{n,n+k} = \sum_{i=n+1}^{n+k} f(\eta_i, \tau_i),\ n \in \mathbb{N} \cup \{0\}, k \in \mathbb{N},\ f \in C^I \cup C^D,$$
$$T_{n,n+k}(y) = \sum_{i=n+1}^{n+k} f(\eta_i, y),\ y \in \mathbb{R},\ n \in \mathbb{N} \cup \{0\},\ k \in \mathbb{N}, f \in C^I \cup C^D,$$
$$\bar{T}_{n,n+k} = \sum_{i=n+1}^{n+k} g(\tau_i),\ n \in \mathbb{N} \cup \{0\}, k \in \mathbb{N},\ g \in \bar{C}^I \cup \bar{C}^D.$$

We will prove the LLN for uncertain random variables, which corresponds to the Chow law for delayed sums of random variables. In contradistinction to Theorem 2 and the LLN proved in [28], where for each $n \in \mathbb{N}$ the scaled sums $S_n = \sum_{i=1}^{n} f(\eta_i, \tau_i)$ were considered, the following Theorem 5 concerns the scaled sums of the form

$$T_{n,n+n^\alpha} = \sum_{i=n+1}^{n+n^\alpha} f(\eta_i, \tau_i)$$

for a fixed number $\alpha \in (0,1)$.

Theorem 5. *Let $\alpha \in (0,1)$. Let random variables $\{\eta_i\}_{i=1}^{\infty}$ be independent, identically distributed with cumulative distribution function Φ. Let uncertain variables $\{\tau_i\}_{i=1}^{\infty}$ be regular, independent, identically distributed with uncertainty distribution Ψ. Assume that $f \in C^I \cup C^D$ and $\mathbb{E}\,|f(\eta_1, y)|^{1/\alpha} < \infty$, $y \in \mathbb{R}$. Then*

$$\lim_{n\to\infty} \frac{T_{n,n+n^\alpha}}{n^\alpha} = \int_{-\infty}^{\infty} f(x, \tau_1) d\Phi(x) \tag{33}$$

in the sense of convergence in distribution.

Proof. In the following proof of this theorem, we use the general approach proposed in [28] with several important modifications.

Assume that $f \in C^I$. Fix $z \in \mathbb{R}$. Let $\varepsilon > 0$. By Corollary A1 in Appendix A, there exists $N_1 \in \mathbb{N}$ such that for each $n \geq N_1$

$$\mathbb{P}\left\{ \frac{T_{n,n+n^\alpha}(F^{-1}(z-\varepsilon))}{n^\alpha} \leq z \right\} \geq 1 - \varepsilon. \tag{34}$$

Let $n \geq N_1$.

$$
\begin{aligned}
\mathrm{Ch}\left\{\frac{T_{n,n+n^\alpha}}{n^\alpha} \leq z\right\} &= \int_0^1 \mathbb{P}\left\{\mathcal{M}\left\{\frac{T_{n,n+n^\alpha}}{n^\alpha} \leq z\right\} \geq r\right\}dr \\
&\geq \int_0^1 \mathbb{P}\left\{\left\{\frac{T_{n,n+n^\alpha}(F^{-1}(z-\varepsilon))}{n^\alpha} \leq z\right\}\right. \\
&\left.\cap\left\{\mathcal{M}\left\{\frac{T_{n,n+n^\alpha}}{n^\alpha} \leq z\right\} \geq r\right\}\right\}dr \\
&\geq \int_0^1 \mathbb{P}\left\{\left\{\frac{T_{n,n+n^\alpha}(F^{-1}(z-\varepsilon))}{n^\alpha} \leq z\right\}\right. \\
&\left.\cap\left\{\mathcal{M}\left\{\frac{T_{n,n+n^\alpha}}{n^\alpha} \leq \frac{T_{n,n+n^\alpha}(F^{-1}(z-\varepsilon))}{n^\alpha}\right\} \geq r\right\}\right\}dr \\
&= \int_0^1 \mathbb{P}\left\{\left\{\frac{T_{n,n+n^\alpha}(F^{-1}(z-\varepsilon))}{n^\alpha} \leq z\right\}\right. \\
&\left.\cap\left\{\mathcal{M}\left\{\sum_{i=n+1}^{n+n^\alpha} f(\eta_i,\tau_i) \leq \sum_{i=n+1}^{n+n^\alpha} f(\eta_i, F^{-1}(z-\varepsilon))\right\} \geq r\right\}\right\}dr \\
&\geq \int_0^1 \mathbb{P}\left\{\left\{\frac{T_{n,n+n^\alpha}(F^{-1}(z-\varepsilon))}{n^\alpha} \leq z\right\}\right. \\
&\left.\cap\left\{\mathcal{M}\left\{\bigcap_{i=n+1}^{n+n^\alpha}\left\{f(\eta_i,\tau_i) \leq f(\eta_i, F^{-1}(z-\varepsilon))\right\}\right\} \geq r\right\}\right\}dr.
\end{aligned}
\tag{35}
$$

Independence of $\{\tau_i\}_{i=1}^\infty$, the assumption that $f(x,y)$ is a strictly increasing function of y for each x, (34), and (35) imply that

$$
\begin{aligned}
&\mathrm{Ch}\left\{\frac{T_{n,n+n^\alpha}}{n^\alpha} \leq z\right\} \\
&\geq \int_0^1 \mathbb{P}\left\{\left\{\frac{T_{n,n+n^\alpha}(F^{-1}(z-\varepsilon))}{n^\alpha} \leq z\right\}\right. \\
&\left.\cap\left\{\bigwedge_{i=n+1}^{n+n^\alpha} \mathcal{M}\left\{\tau_i \leq F^{-1}(z-\varepsilon)\right\} \geq r\right\}\right\}dr \\
&= \int_0^1 \mathbb{P}\left\{\left\{\frac{T_{n,n+n^\alpha}(F^{-1}(z-\varepsilon))}{n^\alpha} \leq z\right\}\right. \\
&\left.\cap\left\{\bigwedge_{i=n+1}^{n+n^\alpha} \Psi(F^{-1}(z-\varepsilon)) \geq r\right\}\right\}dr \geq (1-\varepsilon)\Psi(F^{-1}(z-\varepsilon)).
\end{aligned}
$$

Thus,

$$
\lim_{n\to\infty} \mathrm{Ch}\left\{\frac{T_{n,n+n^\alpha}}{n^\alpha} \leq z\right\} \geq \Psi(F^{-1}(z)). \tag{36}
$$

Similarly, let $\varepsilon > 0$. By Corollary A1 in Appendix A, there exists $N_2 \in \mathbb{N}$ such that for each $n \geq N_2$

$$
\mathbb{P}\left\{\frac{T_{n,n+n^\alpha}(F^{-1}(z+\varepsilon))}{n^\alpha} > z\right\} \geq 1-\varepsilon. \tag{37}
$$

Let $n \geq N_2$.

$$\begin{aligned}
\mathrm{Ch}\left\{\frac{T_{n,n+n^\alpha}}{n^\alpha} > z\right\} &= \int_0^1 \mathbb{P}\left\{\mathcal{M}\left\{\frac{T_{n,n+n^\alpha}}{n^\alpha} > z\right\} \geq r\right\}dr \\
&\geq \int_0^1 \mathbb{P}\left\{\left\{\frac{T_{n,n+n^\alpha}(F^{-1}(z+\varepsilon))}{n^\alpha} > z\right\} \cap \left\{\mathcal{M}\left\{\frac{T_{n,n+n^\alpha}}{n^\alpha} > z\right\} \geq r\right\}\right\}dr \\
&\geq \int_0^1 \mathbb{P}\left\{\left\{\frac{T_{n,n+n^\alpha}(F^{-1}(z+\varepsilon))}{n^\alpha} > z\right\}\right. \\
&\quad \left.\cap\left\{\mathcal{M}\left\{\frac{T_{n,n+n^\alpha}}{n^\alpha} > \frac{T_{n,n+n^\alpha}(F^{-1}(z+\varepsilon))}{n^\alpha}\right\} \geq r\right\}\right\}dr \\
&= \int_0^1 \mathbb{P}\left\{\left\{\frac{T_{n,n+n^\alpha}(F^{-1}(z+\varepsilon))}{n^\alpha} > z\right\}\right. \\
&\quad \left.\cap\left\{\mathcal{M}\left\{\sum_{i=n+1}^{n+n^\alpha} f(\eta_i,\tau_i) > \sum_{i=n+1}^{n+n^\alpha} f(\eta_i, F^{-1}(z+\varepsilon))\right\} \geq r\right\}\right\}dr \\
&\geq \int_0^1 \mathbb{P}\left\{\left\{\frac{T_{n,n+n^\alpha}(F^{-1}(z+\varepsilon))}{n^\alpha} > z\right\}\right. \\
&\quad \left.\cap\left\{\mathcal{M}\left\{\bigcap_{i=n+1}^{n+n^\alpha}\left\{f(\eta_i,\tau_i) > f(\eta_i, F^{-1}(z+\varepsilon))\right\}\right\} \geq r\right\}\right\}dr. \qquad (38)
\end{aligned}$$

By independence of $\{\tau_i\}_{i=1}^\infty$, the assumption that $f(x,y)$ is a strictly increasing function of y for each x, (37), and (38), we obtain

$$\begin{aligned}
&\mathrm{Ch}\left\{\frac{T_{n,n+n^\alpha}}{n^\alpha} > z\right\} \\
&\geq \int_0^1 \mathbb{P}\left\{\left\{\frac{T_{n,n+n^\alpha}(F^{-1}(z+\varepsilon))}{n^\alpha} > z\right\}\right. \\
&\quad \left.\cap\left\{\bigwedge_{i=n+1}^{n+n^\alpha} \mathcal{M}\left\{\tau_i > F^{-1}(z+\varepsilon)\right\} \geq r\right\}\right\}dr \\
&= \int_0^1 \mathbb{P}\left\{\left\{\frac{T_{n,n+n^\alpha}(F^{-1}(z+\varepsilon))}{n^\alpha} > z\right\}\right. \\
&\quad \left.\cap\left\{\bigwedge_{i=n+1}^{n+n^\alpha} (1-\Psi(F^{-1}(z+\varepsilon))) \geq r\right\}\right\}dr \\
&\geq (1-\varepsilon)(1-\Psi(F^{-1}(z+\varepsilon))). \qquad (39)
\end{aligned}$$

By (39) and the duality property of Ch,

$$\mathrm{Ch}\left\{\frac{T_{n,n+n^\alpha}}{n^\alpha} \leq z\right\} \leq 1-(1-\varepsilon)(1-\Psi(F^{-1}(z+\varepsilon))) \qquad (40)$$

for each $n \geq N_2$. From (40) it follows that

$$\lim_{n\to\infty} \mathrm{Ch}\left\{\frac{T_{n,n+n^\alpha}}{n^\alpha} \leq z\right\} \leq \Psi(F^{-1}(z)). \qquad (41)$$

Therefore, by (36) and (41)

$$\lim_{n\to\infty} \mathrm{Ch}\left\{\frac{T_{n,n+n^\alpha}}{n^\alpha} \leq z\right\} = \Psi(F^{-1}(z)) = \mathcal{M}\left\{\int_{-\infty}^{\infty} f(x,\tau_1)d\Phi(x) \leq z\right\}.$$

If $f \in C^D$, then $-f \in C^I$. Thus, proceeding similarly as in the previous part of the proof, we obtain

$$\lim_{n\to\infty} \mathrm{Ch}\left\{-\frac{T_{n,n+n^\alpha}}{n^\alpha} < -z\right\} = \mathcal{M}\left\{-\int_{-\infty}^{\infty} f(x,\tau_1)d\Phi(x) < -z\right\},$$

which is equivalent to

$$\lim_{n\to\infty} \mathrm{Ch}\left\{\frac{T_{n,n+n^\alpha}}{n^\alpha} > z\right\} = \mathcal{M}\left\{\int_{\infty}^{\infty} f(x,\tau_1)d\Phi(x) > z\right\}.$$

The duality properties of Ch and $\mathcal{M}$ imply

$$\lim_{n\to\infty} \mathrm{Ch}\left\{\frac{T_{n,n+n^\alpha}}{n^\alpha} \le z\right\} = \mathcal{M}\left\{\int_{-\infty}^{\infty} f(x,\tau_1)d\Phi(x) \le z\right\}.$$

□

Remark 5. *If random variables $\{\eta_i\}_{i=1}^\infty$ and uncertain variables $\{\tau_i\}_{i=1}^\infty$ satisfy the assumptions of Theorem 5, $\{\eta_i\}_{i=1}^\infty$ are symmetrically distributed, $\{\tau_i\}_{i=1}^\infty$ are symmetrical, and $f(x,y) = x+y$ or $f(x,y) = x-y$, then $\lim_{n\to\infty} \frac{T_{n,n+n^\alpha}}{n^\alpha}$ has the form of the symmetrical uncertain variable τ_1 or $-\tau_1$, respectively.*

In the special case of strictly monotone functions of independent uncertain variables, we obtain.

Corollary 3. *Let $\alpha \in (0,1)$. Let uncertain variables $\{\tau_i\}_{i=1}^\infty$ be regular, independent, identically distributed with uncertainty distribution Ψ. Assume that $g \in \bar{C}^I \cup \bar{C}^D$. Then*

$$\lim_{n\to\infty} \frac{\bar{T}_{n,n+n^\alpha}}{n^\alpha} = g(\tau_1)$$

in the sense of convergence in distribution.

Example 5. *Let $\alpha = 2/3$. Let random variables $\{\eta_i\}_{i=1}^\infty$ and uncertain variables $\{\tau_i\}_{i=1}^\infty$ satisfy the assumptions from Example 3. Then*

$$\mathbb{E}(\eta_1 + y) = y + \frac{1}{2^{3/8}-1} \text{ and } \mathbb{E}\,|\eta_1 + y|^{3/2} < \infty,\ y \in \mathbb{R}.$$

Therefore, by Theorem 5 and Corollary 3

$$\lim_{n\to\infty} \frac{\sum_{i=n+1}^{n+n^{2/3}}(\eta_i + \tau_i)}{n^{2/3}} = \tau_1 + \frac{1}{2^{3/8}-1} \text{ and } \lim_{n\to\infty} \frac{\sum_{i=n+1}^{n+n^{2/3}} \tau_i}{n^{2/3}} = \tau_1$$

in the sense of convergence in distribution.

6. Conclusions

This paper is devoted to the development of the theory of uncertain random variables and uncertain variables. We have considered uncertain random variables in the form of continuous functions of a random variable and an uncertain variable. It has been additionally assumed that the functions are strictly monotone with respect to their second arguments. The sequences of uncertain random variables of this form have been defined for a fixed function satisfying the above conditions and given sequences of pairwise independent or independent, identically distributed random variables and regular, independent, identically distributed uncertain variables. This paper's main contribution is the formulation and proof of the generalisation of LLN from [28], the Marcinkiewicz–Zygmund type LNN, and the Chow type LLN for such defined sequences of uncertain random variables. The first theorem has the form corresponding to the classical Etemadi

Theorem. Within the second theorem, under the assumption that for each fixed value of the second argument, the p-th moment of the considered function of random variables exists, $p \in (0,1]$, the convergence in distribution of scaled sums of the uncertain random variables has been proved. Moreover, their convergence in measure has been obtained in the case of $p \in (0,1)$, where the assumption of the same distribution of the regular uncertain variables has been weakened. In turn, the third theorem has established the convergence in distribution of delayed, scaled sums of uncertain random variables. Furthermore, we have formulated counterparts of the last two theorems for regular uncertain variables. Finally, we have provided illustrative examples of applications of the proved theorems. Our future research plans concern the formulation and proof of other versions of LLN, in particular, for dependent uncertain random variables. One possible direction for further research is a generalisation of Theorem 5, corresponding to the Chow law for delayed sums of random variables, to the case where uncertain random variables are functions of pairwise independent, identically distributed random variables and regular, independent, identically distributed uncertain variables. This generalisation requires having first proved a version of the Chow law for random variables that are pairwise independent.

Author Contributions: This paper was prepared using the contributions of both authors. P.N. mainly contributed in developing of mathematical model. O.H. contributed in the interpretation of the obtained results. All authors have read and approved the final manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

IFSs	intuitionistic fuzzy sets
IVFs	interval-valued fuzzy sets
LLN	Law of Large Numbers
LLNs	Laws of Large Numbers
$\mathbb{P}$-a.s.	$\mathbb{P}$-almost surely
SLLN	Strong Law of Large Numbers

Appendix A. Classical Versions of the Etemadi SLLN, the Marcinkiewicz–Zygmund SLLN and the Chow SLLN for Delayed Sums

Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of real-valued random variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and let $s_0 = 0$, $s_n = \sum_{i=1}^{n} X_i$, $t_{n,n+k} = \sum_{i=n+1}^{n+k} X_i$ for each $n \in \mathbb{N} \cup \{0\}$, $k \in \mathbb{N}$.

The following generalisations of the Kolmogorov SLLN (Strong Law of Large Numbers) hold (see [38,39]):

Theorem A1 (Etemadi SLLN)**.** *If the random variables $\{X_i\}_{i=1}^{\infty}$ are pairwise independent, identically distributed and $\mathbb{E}\,|X_1| < \infty$, then*

$$\lim_{n\to\infty} \frac{s_n}{n} = \mathbb{E}\, X_1 \ \mathbb{P}\text{-almost surely}(\mathbb{P}\text{-a.s.})$$

Theorem A2 (Marcinkiewicz–Zygmund SLLN)**.** *Let random variables $\{X_i\}_{i=1}^{\infty}$ be independent and identically distributed. If $\mathbb{E}\,|X_1|^p < \infty$ for $p \in (0,2)$, then*

$$\lim_{n\to\infty} \frac{s_n - nc}{n^{1/p}} = 0 \ \mathbb{P}\text{-a.s.},$$

where $c = 0$ *if* $0 < p < 1$ *and* $c = \mathbb{E}\,X_1$ *if* $1 \le p < 2$.

The next theorem (see [40]) concerns delayed sums of random variables.

Theorem A3 (Chow SLLN for delayed sums)**.** *Let random variables* $\{X_i\}_{i=1}^{\infty}$ *be independent and identically distributed. Let* $\alpha \in (0,1)$*. Then*

$$\lim_{n\to\infty} \frac{t_{n,n+n^\alpha}}{n^\alpha} = 0 \;\mathbb{P}\text{-a.s.} \;\Leftrightarrow\; \mathbb{E}\,|X_1|^{1/\alpha} < \infty \text{ and } \mathbb{E}\,X_1 = 0.$$

Let $\alpha \in (0,1)$ and $\mathbb{E}\,|X_1|^{1/\alpha} < \infty$. Then $\{Y_i = X_i - \mathbb{E}\,X_i\}_{i=1}^{\infty}$ satisfies the assumptions of Theorem A3. Therefore,

$$\lim_{n\to\infty} \frac{\sum_{i=n+1}^{n+n^\alpha} Y_i}{n^\alpha} = 0 \;\mathbb{P}\text{-a.s.}$$

This implies that $\mathbb{P}$-a.s.

$$\lim_{n\to\infty} \frac{\sum_{i=n+1}^{n+n^\alpha} X_i - [n^\alpha]\,\mathbb{E}\,X_1}{n^\alpha} = 0,$$

where $[n^\alpha]$ is the integer part of n^α. Since $\lim_{n\to\infty} \frac{[n^\alpha]\,\mathbb{E}\,X_1}{n^\alpha} = \mathbb{E}\,X_1$, we obtain:

Corollary A1. *Let* $\alpha \in (0,1)$*. If* $\mathbb{E}\,|X_1|^{1/\alpha} < \infty$*, then*

$$\lim_{n\to\infty} \frac{t_{n,n+n^\alpha}}{n^\alpha} = \mathbb{E}\,X_1 \;\mathbb{P}\text{-a.s.}$$

Table A1. Notation used in the paper.

Notation	Description
$\mathbb{R}, \mathbb{N}$	The sets of real numbers and positive integers, respectively
$\mathcal{B}(\mathbb{R})$	The σ-algebra of Borel subsets of $\mathbb{R}$
$(\Gamma, \mathcal{L}, \mathcal{M})$	An uncertainty space, where Γ is a nonempty set, $\mathcal{L}$ is a σ-algebra of subsets of Γ, and $\mathcal{M} : \mathcal{L} \to [0,1]$ is an uncertain measure
$\Lambda, \Lambda_i, \Lambda_k$	Events, where $i, k \in \mathbb{N}$
$(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k),\ k = 1, 2, \ldots$	A sequence of uncertainty spaces
τ, τ_i	Uncertain variables, where $i \in \mathbb{N}$
Ψ	An uncertainty distribution
$(\Omega, \mathcal{A}, \mathbb{P})$	A probability space, where Ω is a nonempty set, $\mathcal{A}$ is a σ-algebra of subsets of Ω, and $\mathbb{P} : \mathcal{A} \to [0,1]$ is a probability measure
$(\Gamma \times \Omega, \mathcal{L} \times \mathcal{A}, \mathcal{M} \times \mathbb{P})$	A chance space $(\Gamma \times \Omega, \mathcal{L} \times \mathcal{A}, \mathcal{M} \times \mathbb{P}) = (\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \mathbb{P})$, where $(\Gamma, \mathcal{L}, \mathcal{M})$ is an uncertainty space and $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space
$\Theta, \Theta_1, \Theta_2$	Uncertain random events
Ch	A chance measure
ξ	An uncertain random variable
Ξ	A chance distribution
$\{\xi_n\}_{n=1}^{\infty}$	An uncertain random sequence
$\{\Xi_n\}_{n=1}^{\infty}$	A sequence of chance distributions
$\{\eta_i\}_{i=1}^{\infty}, \{\tau_i\}_{i=1}^{\infty}$	Sequences of random and uncertain variables, respectively
$C^I, C^D, \bar{C}^I, \bar{C}^D$	Classes of functions (see Section 3)

Table A1. *Cont.*

Notation	Description
S_n	$S_n = \sum_{i=1}^{n} f(\eta_i, \tau_i), n \in \mathbb{N}, f \in C^I \cup C^D$, where $\{\eta_i\}_{i=1}^{\infty}$, $\{\tau_i\}_{i=1}^{\infty}$ are described above
$S_n(y)$	$S_n(y) = \sum_{i=1}^{n} f(\eta_i, y), y \in \mathbb{R}, n \in \mathbb{N}, f \in C^I \cup C^D$, where $\{\eta_i\}_{i=1}^{\infty}$ is described above
$\bar{S}_n$	$\bar{S}_n = \sum_{i=1}^{n} g(\tau_i), n \in \mathbb{N}, g \in \bar{C}^I \cup \bar{C}^D$, where $\{\tau_i\}_{i=1}^{\infty}$ is described above
$\mathbb{E}$	The expected value with respect to probability measure $\mathbb{P}$
Φ	A cumulative distribution function
F, ψ	The functions $F(y) = \int_{-\infty}^{\infty} f(x, y) d\Phi(x),\ y \in \mathbb{R}$, $\psi(z) = \mathcal{M}\left\{\int_{-\infty}^{\infty} f(x, \tau_1) d\Phi(x) \leq z\right\}$, $z \in (\inf\{F(y) \| y \in \mathbb{R}\}, \sup\{F(y) \| y \in \mathbb{R}\})$, where Φ is described above, $f \in C^I \cup C^D$, and τ_1 is a regular uncertain variable
$\{\epsilon_i\}_{i=-1}^{\infty}$	A sequence of independent Rademacher random variables
$\mathbb{1}_{A_i}, i \in \mathbb{N}$	A sequence independent, identically distributed indicator functions, such that $\mathbb{P}\{A_i\} = \alpha, 0 < \alpha < 1$
$\{\Psi_i\}_{i=1}^{\infty}$	A sequence of uncertainty distributions
$\bar{\Psi}, \bar{\bar{\Psi}}$	The functions $\bar{\Psi}(y) = \lim_{n\to\infty} \bigwedge_{i=1}^{n} \Psi_i(y)$, $\bar{\bar{\Psi}}(y) = \lim_{n\to\infty} \bigwedge_{i=1}^{n} (1 - \Psi_i(y)), y \in \mathbb{R}$, where $\{\Psi_i\}_{i=1}^{\infty}$ is described above
$m_{f,c}, m_{g,c}, m_1, m_2$	Uncertain variables with constant uncertainty distributions (see Section 4)
$T_{n,n+k}$	$T_{n,n+k} = \sum_{i=n+1}^{n+k} f(\eta_i, \tau_i), n \in \mathbb{N} \cup \{0\}, k \in \mathbb{N}, f \in C^I \cup C^D$, where $\{\eta_i\}_{i=1}^{\infty}$, $\{\tau_i\}_{i=1}^{\infty}$ are described above
$T_{n,n+k}(y)$	$T_{n,n+k}(y) = \sum_{i=n+1}^{n+k} f(\eta_i, y), y \in \mathbb{R}, n \in \mathbb{N} \cup \{0\}, k \in \mathbb{N}$, $f \in C^I \cup C^D$, where $\{\eta_i\}_{i=1}^{\infty}$ is described above
$\bar{T}_{n,n+k}$	$\bar{T}_{n,n+k} = \sum_{i=n+1}^{n+k} g(\tau_i), n \in \mathbb{N} \cup \{0\}, k \in \mathbb{N}, g \in \bar{C}^I \cup \bar{C}^D$, where $\{\tau_i\}_{i=1}^{\infty}$ is described above
$\{X_i\}_{i=1}^{\infty}$	A sequence of random variables
s_n	$s_0 = 0, s_n = \sum_{i=1}^{n} X_i$ for each $n \in \mathbb{N}$, where $\{X_i\}_{i=1}^{\infty}$ is described above
$t_{n,n+k}$	$t_{n,n+k} = \sum_{i=n+1}^{n+k} X_i$ for each $n \in \mathbb{N} \cup \{0\}, k \in \mathbb{N}$, where $\{X_i\}_{i=1}^{\infty}$ is described above
$[n^\alpha]$	The integer part of n^α, where $n \in \mathbb{N}, \alpha \in (0, 1)$

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