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Dynamical Symmetries of the 2D Newtonian Free Fall Problem Revisited

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Abstract: Among the few exactly solvable problems in theoretical physics, the 2D (two-dimensional) Newtonian free fall problem in Euclidean space is perhaps the least known as compared to the harmonic oscillator or the Kepler–Coulomb problems. The aim of this article is to revisit this problem at the classical level as well as the quantum level, with a focus on its dynamical symmetries. We show how these dynamical symmetries arise as a special limit of the dynamical symmetries of the Kepler–Coulomb problem, and how a connection to the quartic anharmonic oscillator problem, a long-standing unsolved problem in quantum mechanics, can be established. To this end, we construct the Hilbert space of states with free boundary conditions as a space of square integrable functions that have a special functional integral representation. In this functional space, the free fall dynamical symmetry algebra is shown to be isomorphic to the so-called Klink's algebra of the quantum quartic anharmonic oscillator problem. Furthermore, this connection entails a remarkable integral identity for the quantum quartic anharmonic oscillator eigenfunctions, which implies that these eigenfunctions are in fact *zonal* functions of an underlying symmetry group representation. Thus, an appropriate representation theory for the 2D Newtonian free fall quantum symmetry group may potentially open the way to exactly solving the difficult quantization problem of the quartic anharmonic oscillator. Finally, the initial value problem of the acoustic Klein–Gordon equation for wave propagation in a sound duct with a varying circular section is solved as an illustration of the techniques developed here.

Keywords: super-integrability; dynamical symmetry algebras; integral transforms

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1. Introduction

In an elementary freshman physics course, the problem of particle motion under constant gravitational acceleration on the surface of the earth in Euclidean space, usually considered as one dimensional, is frequently dubbed as a free fall problem [1,2]. This is not the more general relativistic problem in which an inertial particle subject to no force moves along a space-time geodesic [3].

To the best knowledge of the author, up to now, no comprehensive report on the dynamical symmetries of this 2D Newtonian free fall problem has been found in the literature besides the seminal work of T. Iwai and S. G. Rew [4]. As the one-dimensional free fall problem, which has been thoroughly treated classically and quantum mechanically, displays limited interesting features, we introduce an extra space dimension to allow for new specific features to emerge. This is how a body of dynamical symmetries comes about, and how a connection to the quartic anharmonic oscillator problem is generated, which is one of the most challenging theoretical problems as it is generally thought of as a non-trivial quantum field theory in zero space dimension. These issues were not considered in [4].

This paper is divided into two main sections and a short section on an application.

Section 2 is devoted to the classical physics of the 2D Newtonian free fall problem. For the convenience of the reader, some basic concepts on the symmetries of a system are recalled in the framework of Hamiltonian mechanics before a derivation of the dynamical symmetry algebra is undertaken along the lines of [4]. Then, we prove a theorem which states that the dynamical symmetry algebra of the free fall problem may be obtained as a special limit of the the dynamical symmetry algebra of the Kepler–Coulomb problem, which concerns the dynamics of a particle moving in the inverse separation distance potential field. Both problems are known to be super-integrable, and now they are shown to be connected at the classical level. Hence, one may access the free fall dynamical symmetry algebra from the well-known Kepler–Coulomb dynamical symmetry algebra. In addition, this connection goes even further when a higher order integral of motion in the free fall problem is shown to be an outgrowth of a Kepler–Coulomb integral of motion at zero coupling constant. Furthermore, a passage to parabolic coordinates reveals a connection between the free fall problem and the quartic anharmonic oscillator.

In Section 3, we apply the Schrödinger quantization to the system, which is in fact not the quantization scheme adopted by T. Iwai and S. G. Rew in [4]. For this, we introduced a Hilbert space of square integrable wave functions that verify free boundary conditions in $\mathbb{R}^2$ and are suitable for obtaining an image representation of the free fall dynamical symmetry algebra as Klink's so-called one-variable algebra of the quartic anharmonic oscillator. Thus, the Schrödinger quantization does naturally lead to the quartic anharmonic oscillator besides the passage to parabolic coordinates. This is due to a particular integral representation of the free fall wave functions, which takes the form of an Airy-Fourier transform. Finally, by performing a passage to parabolic coordinates, one obtains an integral identity for the eigenfunctions of the quartic anharmonic oscillator problem, which turns out to be the characteristic integral equation for *zonal* functions in the representation of some underlying group. As the only natural group arising from the 2D free fall problem is its own dynamical symmetry group, we suspect that a comprehensive development of its representation theory would reveal that the eigenfunctions of the quartic anharmonic oscillator problem are just the corresponding zonal functions of this group. We defer this tantalizing investigation to a future work. In short, the present results on quantizing the 2D Newtonian free fall dynamical symmetry algebra appear as a necessary intermediate step in the search for an exact solution to the quantum quartic anharmonic oscillator problem.

The last section, Section 4, uses the functional techniques of Section 3 to solve the initial problem of sound wave propagation in a cylindrical duct with a particular varying circular section.

The paper ends with a short conclusion and perspectives on possible further research.

2. The Classical Two-Dimensional Free Fall Problem

2.1. Generalities on Classical Symmetries, Canonical Structure, and Integrals of Motion

As generally admitted, the classical state of a non-moving physical system is described by a set of functions in a coordinate system. Under a coordinate transformation, these functions may take different forms. But if they remain *invariant*, the system is said to admit a *symmetry*.

Now, for a physical system in motion, its states are specified by functions of time t and dynamical coordinates in *phase space*. If under a transformation of dynamical coordinates its state functions remain invariant, we say that the system admits a *dynamical symmetry*. The set of symmetry transformations may have a group structure, which is called *dynamical symmetry group*. Such a dynamical symmetry group (or equivalently, the algebra of its generators) is of highest interest in the search for solutions of its equation of motion (The term *dynamical symmetry* was coined by A. O. Barut [5]. Earlier, such symmetry was known as *hidden* or *accidental* [6]).

In this paper, we are concerned with the dynamical symmetries of a system in the framework of Hamilton's *canonical* formalism of mechanics, with the time-independent Hamiltonian function H. We now recall some useful main points of this framework.

- A system with n degrees of freedom is described by $2n$ canonical variables (q_i, p_i) with $i = 1, \ldots, n$ in phase space, verifying the following fundamental Poisson bracket commutation relations:

$$\{q_i, q_j\} = \{p_i, p_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij}, \tag{1}$$

where the Poisson bracket between two functions in phase space ϕ and ψ is defined by the following:

$$\{\phi, \psi\} = \sum_{i=1}^{n} \left(\frac{\partial \phi}{\partial q_i} \frac{\partial \psi}{\partial p_i} - \frac{\partial \phi}{\partial p_i} \frac{\partial \psi}{\partial q_i} \right),$$

and the canonical equations of motion:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \tag{2}$$

the solutions of which depend on the $2n$ initial conditions $(q_i(t), p_i(t))|_{t=0} = (q_i^0, p_i^0)$.

Consequently a dynamical symmetry exists if it originates from a *canonical* coordinate transformation in phase space, leaving *both* the Hamiltonian H and the fundamental Poisson bracket relations invariant. In differential geometry, this is called a *symplectic* structure. Non-trivial dynamical symmetry exists only for $n > 1$.

- Canonical transformations (see Chapters 10–12 of [7]) are parts of a wider class of coordinate transforms in phase space called *contact transformations*. A differentiable mapping $(q_1, \ldots, q_n, p_1, \ldots, p_n) \to (Q_1, \ldots, Q_n, P_1, \ldots, P_n)$ is called a contact transformation if the differential form $\sum_{i=1}^{n}(P_i dQ_i - p_i dq_i)$ is the exact differential dW of a function $W(q_1, \ldots, q_n, p_1, \ldots, p_n)$.

This can be equivalently expressed by requiring the fundamental Poisson brackets in the variables $(Q_1, \ldots, Q_n, P_1, \ldots, P_n)$ to be valid:

$$\begin{aligned} \{Q_i, Q_j\} = \{P_i, P_j\} &= 0, \quad (i, j = 1, \ldots, n), \\ \{Q_i, P_j\} &= 0, \quad (i, j = 1, \ldots, n; i < j, i > j), \\ \{Q_i, P_i\} &= 1, \quad (i = 1, \ldots, n). \end{aligned} \tag{3}$$

The Poisson bracket of two functions (ϕ, ψ) is *invariant* under the contact transformations $\{\phi, \psi\}_{(Q,P)} = \{\phi, \psi\}_{(q,p)}$. The lower index (q, p) (resp. (Q, P)) refers to the variables in the partial derivatives of the respective Poisson brackets. A contact transformation $(q_1, \ldots, q_n, p_1, \ldots, p_n) \to (Q_1, \ldots, Q_n, P_1, \ldots, P_n)$, which *preserves* the equations of motion of the system is a *canonical* transformation if $\int \sum_{i=1}^{n} P_i dQ_i$ is an *integral invariant* of the system (*Jacobi's theorem*). Consequently, a canonical transformation implements a dynamical symmetry.

An *infinitesimal* canonical transformation of parameter τ has the following form:

$$Q_i = q_i + \frac{\partial F_\tau}{\partial p_i} \Delta\tau, \quad P_i = p_i - \frac{\partial F_\tau}{\partial q_i} \Delta\tau, \tag{4}$$

where F_τ is an arbitrary function of $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ and $\Delta\tau$ is a small increment of the parameter τ. F_τ is called the *the generating function* of the contact transformation. The variation Δf of an arbitrary function f in phase space under an infinitesimal contact transformation of generator F_τ is given by $\Delta f = \{f, F_\tau\}\Delta\tau$. F_τ can be an *integral of the motion*, a function which takes a constant value and does not explicitly depend on time t, see [7]. A system of integrals of motion F_i is said to be in involution when $\{F_i, F_j\} = 0$ for all $(i, j = 1, \ldots, m)$. If F and F' are two integrals of the motion, their Poisson bracket is also an integral of the motion (*Poisson's theorem*). The set of all F_j forms a *Lie* algebra with respect to the Poisson bracket.

The determination of all canonical transformations for a dynamical system is at the core of finding the dynamical symmetries of this system.

- For a system with an n degree of freedom, with a time-independent Hamiltonian H, total energy is *conserved* and $H = E$. If there exists n functionally *independent* integrals of motion F_i, where $i = 1, \ldots, n$ in involution (or $\{F_i, F_j\} = 0$ for all (i, j)), the system is said to be Liouville *integrable* and its solution can be given up to *quadratures*. If there are further m functionally independent integrals of motion F_m, where $m = 1, \ldots, (n-1)$, the system is called *super-integrable* [8]. For $m = 1$ (resp. $m = (n-1)$) it is called *minimal* super-integrable (resp. *maximal* super-integrable). These m extra integrals of motion usually build a Lie algebra. Maximal super-integrable systems are also known as *exactly solvable* systems, and their properties can be derived algebraically. Each integral of motion F_i may originate from Noether's conservation law or from a coordinate variable separation. The Kepler–Coulomb (or inverse distance potential) and the isotropic harmonic oscillator problems are known to be super-integrable systems in two dimensions [9].

2.2. Statement of the Classical Two-Dimensional Free Fall Problem

As it is widely known, the motion of a particle under constant force in two-dimensional space occurs along parabolas with a symmetry axis parallel to the direction of the constant force. To simplify the writing, the particle is assumed to have a unit mass and moves in a two-dimensional configuration space $(v, u) \in \mathbb{R}^2$, with Ov being the horizontal axis and the gravitational constant set is also equal to one. Let $(\dot{v}, \dot{u})$ be the time derivatives of the Cartesian coordinates (v, u). From the Lagrangian $L = \left(\frac{1}{2}(\dot{u}^2 + \dot{v}^2) - u\right)$, one gets the conjugate momenta and the Hamiltonian:

$$p_u = \frac{\partial L}{\partial \dot{u}} = \dot{u}, \quad p_v = \frac{\partial L}{\partial \dot{v}} = \dot{v}, \quad \text{and} \quad H = \frac{1}{2}p_v^2 + \left(\frac{1}{2}p_u^2 + u\right). \tag{5}$$

Note that H is the sum of the v-free motion part and of the u-free fall part.

2.3. Dynamical Symmetries

As in two dimensions, the maximal number of possible symmetries is three. Iwai and Rew [4] were the first to obtain dynamical symmetries by considering linear inhomogeneous transformations $S(\rho, \sigma, \tau)$ in the phase space dependent on three parameters (ρ, σ, τ):

$$\begin{aligned} V &= v + \tau p_v + \sigma p_u + \rho, & P_v &= p_v - \sigma, \\ U &= u + \sigma p_v - \tau p_u - \frac{1}{2}(\sigma^2 + \tau^2), & P_u &= p_u + \tau. \end{aligned} \tag{6}$$

Proposition 1. *$S(\rho, \sigma, \tau)$ is a canonical transformation.*

Proof. By the simple substitution of the (V, U, P_v, P_u) in terms of the (v, u, p_v, p_u), as given by Equation (6), it appears that the Hamiltonian function form is invariant:

$$H = \frac{1}{2}(P_u^2 + P_v^2) + U = \frac{1}{2}(p_u^2 + p_v^2) + u. \tag{7}$$

Moreover, using the expression of the Poisson bracket and Equation (6), one can also check the following:

$$\begin{aligned} \{V, U\} = \{P_v, P_u\} &= 0, \\ \{V, P_u\} = \{U, P_v\} &= 0, \\ \{V, P_v\} = \{U, p_u\} &= 1, \quad , \end{aligned} \tag{8}$$

Hence, $S(\rho, \sigma, \tau)$ is a *bona fide* canonical transformation. □

Proposition 2. *For all triplet $(\rho, \sigma, \tau) \in \mathbb{R}^3$, the canonical transforms form a group—the dynamical symmetry group of the free fall problem in two dimensions.*

Proof. *Composition law*. Let another canonical transform $S(\rho',\sigma',\tau')$ with parameters (ρ',σ',τ') act on the previous one $S(\rho,\sigma,\tau)$ and compute the resulting product $S(\rho',\sigma',\tau')$ $S(\rho,\sigma,\tau)$, which is expressed by the set of new parameters $(\overline{V},\overline{U},\overline{P}_v,\overline{P}_v)$:

$$\begin{aligned}\overline{V} &= V+\tau' P_v+\sigma' P_u+\rho', \\ \overline{U} &= U+\sigma' P_v-\tau' P_u-\frac{1}{2}(\sigma'^2+\tau'^2), \\ \overline{P}_v &= P_v-\sigma', \\ \overline{P}_u &= P_u+\tau'. \end{aligned} \tag{9}$$

Then, the substitution of Equation (6) into Equation (9) yields the following equations:

$$\begin{aligned}\overline{V} &= (v+\tau p_v+\sigma p_u+\rho)+\tau'(p_v-\sigma)+\sigma'(p_u+\tau)+\rho' \\ &= v+(\tau+\tau')p_v+(\sigma+\sigma')p_u+(\rho+\rho')-(\sigma\tau'-\sigma'\tau), \\ \overline{U} &= (u+\sigma p_v-\tau p_u-\frac{1}{2}(\sigma^2+\tau^2))+\sigma'(p_v-\sigma)-\tau'(p_u+\tau)-\frac{1}{2}(\sigma'^2+\tau'^2) \\ &= u+(\sigma+\sigma')p_v-(\tau+\tau')p_u-\frac{1}{2}(\sigma+\sigma')^2-\frac{1}{2}(\tau+\tau')^2, \\ \overline{P}_v &= (p_{v'}-\sigma)-\sigma'=p_v-(\sigma+\sigma'), \\ \overline{P}_u &= (p_{u'}+\tau')+\tau'=p_u+(\tau+\tau'). \end{aligned} \tag{10}$$

Hence, we conclude that the composition of two operations is as follows:

$$S(\rho',\sigma',\tau')\ \cdot\ S(\rho,\sigma,\tau)=S((\rho+\rho')-(\sigma\tau'-\sigma'\tau),\sigma+\sigma',\tau+\tau'). \tag{11}$$

Because of the extra term $(\sigma\tau'-\sigma'\tau)$, the previous formula does not correspond to the additive structure of a group operation with respect to the parameters (ρ,σ,τ). But since the v variable is a *cyclic* variable, this term is physically *irrelevant ´*; in this way, the group structure is *physically* restored.

Iwai and Rew proposed to represent each element of the group with fifth-order, upper-triangular matrices, as can be checked by the matrix multiplication rule:

$$\begin{pmatrix} 1 & 0 & \tau & \sigma & \rho \\ 0 & 1 & \sigma & -\tau & -\frac{1}{2}(\sigma^2+\tau^2) \\ 0 & 0 & 1 & 0 & -\sigma \\ 0 & 0 & 0 & 1 & \tau \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- The *inverse transform* $S^{-1}(\rho,\sigma,\tau)$ is clearly given by $S(-\rho,-\sigma,-\tau)$ since the extra term $(\sigma\tau'-\sigma'\tau)$ vanishes at $(\sigma+\sigma')=0$ and $(\tau+\tau')=0$.

-The *identity* matrix is obviously the neutral group element. □

2.4. Infinitesimal Iwai–Rew Canonical Linear Transforms and Integrals of Motion

The infinitesimal form of the Iwai–Rew canonical linear transformations is written in the form of Equation (8), with infinitesimal $(\Delta\rho,\Delta\sigma,\Delta\tau)$ as:

$$\begin{aligned} V &= v+p_v\Delta\tau+p_u\Delta\sigma+\Delta\rho+\ldots, \\ U &= u+p_v\Delta\sigma-p_u\Delta\tau+\ldots, \\ P_v &= p_v-\Delta\sigma+\ldots, \\ P_u &= p_u+\Delta\tau+\ldots. \end{aligned} \tag{12}$$

Proposition 3. *The infinitesimal transformations in phase space defined by Equation (12) are canonical transformations: H and the Poisson brackets* $\{V, P_v\}, \{V, P_u\}, \{V, U\}, \{U, P_u\}, \{U, P_v\}, \{P_u, P_v\}$ *are invariant.*

Proof. By working out the Poisson brackets using Equation (12), they will appear to remain in the canonical form of Equation (8), while the form of the Hamiltonian in the variables (V, U, P_v, P_u) is the same as the form in variables (v, u, p_v, p_u). □

We now determine the three generating functions $(F_\rho, F_\sigma, F_\tau)$ from the three subgroups of canonical transformations, respectively defined by the following equations:

$$\begin{aligned} F_\rho &: \quad V = v + \Delta\rho, U = u, P_v = p_v, P_u = p_u, \\ F_\sigma &: \quad V = v + p_u\Delta\sigma, U = u + p_v\Delta\sigma, P_v = p_v - \Delta\sigma, P_u = p_u, \\ F_\tau &: \quad V = v + p_v\Delta\tau, U = u - p_u\Delta\tau, P_v = p_v, P_u = p_u + \Delta\tau. \end{aligned} \tag{13}$$

Then, partial derivatives of $(F_\rho, F_\sigma, F_\tau)$ with respect to (v, u, p_v, p_u) can be deduced as follows:

$$\begin{aligned} F_\rho &: \quad \frac{\partial F_\rho}{\partial v} = 0, \frac{\partial F_\rho}{\partial u} = 0, \frac{\partial F_\rho}{\partial p_v} = 1, \frac{\partial F_\rho}{\partial p_u} = 0, \\ F_\sigma &: \quad \frac{\partial F_\sigma}{\partial v} = 1, \frac{\partial F_\sigma}{\partial u} = 0, \frac{\partial F_\sigma}{\partial p_v} = p_u, \frac{\partial F_\sigma}{\partial p_u} = p_v \\ F_\tau &: \quad \frac{\partial F_\sigma}{\partial v} = 1, \frac{\partial F_\sigma}{\partial u} = -1, \frac{\partial F_\sigma}{\partial p_v} = p_v, \frac{\partial F_\sigma}{\partial p_u} = -p_u. \end{aligned} \tag{14}$$

Therefore, we obtain the exact differentials $(dF_\rho, dF_\sigma, dF_\tau)$ in terms of (dv, du, dp_v, dp_u) (Schwarz's theorem is trivially verified for all pairs of variables). Their integration yields the sought integrals of motion:

$$F_\rho = p_v, \quad F_\sigma = (p_v p_u + v), \quad F_\tau = \frac{1}{2}p_v^2 - \left(\frac{1}{2}p_u^2 + u\right). \tag{15}$$

Proposition 4. *The free fall system in two dimensions is* super-integrable *since its has three integrals of the motion:*

$$\{F_\rho, H\} = \{F_\sigma, H\} = \{F_\tau, H\} = 0. \tag{16}$$

Proof. Compute the three Poisson brackets and observe that they are zero. □

Proposition 5. *The three generating functions* $(F_\rho, F_\sigma, F_\tau)$ *build a Lie algebra structure, the dynamical symmetry algebra* $\mathfrak{A}_{FF}$ *of the two-dimensional free fall with respect to the Poisson bracket:*

$$\{F_\rho, F_\sigma\} = -I, \quad \{F_\sigma, F_\tau\} = 2F_\rho, \quad \{F_\tau, F_\rho\} = 0. \tag{17}$$

Proof. Work out the Poisson brackets and use the expressions of the $(F_\rho, F_\sigma, F_\tau)$ from Equation (15). They are identical to those of Iwai–Rew [4]. If we make the substitutions $F_\rho \to F_3, \quad F_\sigma \to F_2, \quad F_\tau \to F_1$ in our Equation (15), we recover the Iwai–Rew commutation relations given by their Equation (2.21). □

Remark 1. *Observe that the classical trajectory data (two initial position coordinates, two initial momentum coordinates) can be used to compute the values of* $\widehat{H}, \widehat{F}_\rho, \widehat{F}_\sigma, \widehat{F}_\tau$ *and vice versa.*

Proposition 6. *Origins of the integrals of motion*

(a) F_ρ *is due to translational invariance of H in the Ov direction;*
(b) F_σ *is due the separability of H in a* $\pi/4$*-rotated coordinate system;*
(c) F_τ *is due to the manifest separability of H in v and u variables.*

Proof. (a) For this case, the proof is trivial because v is a *cyclic* variable in H.

(b) Consider the change of variables $\xi = (u+v), \quad \zeta = (u-v)$, from which one deduces $u = \frac{1}{2}(\xi+\zeta), v = \frac{1}{2}(\xi-\zeta)$. Then, since $p_v = \dot{v}, \quad p_u = \dot{u}$, we have $p_v = \frac{1}{2}(p_\xi + p_\zeta), \quad p_u = \frac{1}{2}(p_\xi - p_\zeta)$. Substitution into the expressions of H and F_σ yields the following equation:

$$H = \frac{1}{2}\left\{\left(\frac{1}{2}p_\xi^2 + \xi\right) + \left(\frac{1}{2}p_\zeta^2 + \zeta\right)\right\}, \text{ and } F_\sigma = \frac{1}{2}\left\{\left(\frac{1}{2}p_\xi^2 + \xi\right) - \left(\frac{1}{2}p_\zeta^2 + \zeta\right)\right\}. \quad (18)$$

where H and F_σ are both separable in the new coordinate system obtained by a $\pi/4$ rotation of the (v,u) coordinate system around the origin. In fact, they are the sum and difference of the one-dimensional free fall Hamiltonian in the ξ and ζ directions. Hence, $\{F_\sigma, H\} = 0$. In reference [10], it was claimed that at the quantum level, F_σ is due to separability in *translated parabolic* coordinates. But so far, no proof has appeared in print.

(c) As H is the sum of a free-motion Hamiltonian in the v-direction, and the Hamiltonian is that of a one-dimensional free fall in the u-direction, it is then clear that F_τ, which is the difference of these two Hamiltonians, should verify $\{F_\tau, H\} = 0$. Since the total energy is conserved as $H = E$, it may be simpler to write $F_\tau = (p_v^2 - E)$. □

Remark 2. *We notice that v-parity is also a dynamical symmetry and a* discrete *canonical transformation since $v \to -v$ (which also implies $p_v \to -p_v$) leaves H as well as $\{v, p_v\} = 1$ and all other Poisson brackets invariant.*

2.5. The Free Fall Problem as a Special Limiting Case of the Kepler–Coulomb Problem

In this section, we show that the free fall problem can emerge from a special limit of the Kepler–Coulomb (KC) problem or the problem of the inverse distance potential, occurring either in gravitational or electrical interaction. This problem is known to be super-integrable and in two dimensions, and the set of its three integrals of motion makes up the components of the so-called Runge–Lenz vector [11]. Here, we adopt the notations of [12], with the particle mass set equal to one.

The idea is simple. The inverse distance potential κ/r arising from a source of strength κ is rotation-symmetric, where r is the distance from the source to the observation site. If the source recedes to infinity in the u-direction, the potential at the observation site tends toward zero. But one may compensate this potential decrease by taking a source strength $\kappa(r)$, which increases with the distance. Thus, we may choose an increasing functional dependence so that in the limit of infinite source-observation separation, a linear potential appears.

In our Cartesian coordinate system, if the source of strength κ is placed at the coordinate origin, the Hamiltonian of the inverse distance problem is the following:

$$H_{KC} = \frac{1}{2}(p_v^2 + p_u^2) - \frac{\kappa}{\sqrt{u^2+v^2}}, \quad (19)$$

and the three integrals of motion are given by [12]:

$$\begin{aligned} L_w &= (vp_u - up_v), \\ K_v &= (vp_u - up_v)p_u - v\frac{\kappa}{\sqrt{u^2+v^2}}, \\ K_u &= -(vp_u - up_v)p_v - u\frac{\kappa}{\sqrt{u^2+v^2}}, \end{aligned} \quad (20)$$

where L_w is the angular momentum around Ow, which is orthogonal to the plane Ovu.

They verify the Poisson bracket relations of the dynamical symmetry algebra $\mathfrak{A}_{KC}$

$$\{L_w, K_v\} = K_u, \quad \{K_u, L_w\} = K_v, \quad \{K_v, K_u\} = -2H_{KC}\, L_w. \quad (21)$$

Now, if the source is no longer at the origin O but situated on the Ou axis at a distance l from the origin, the Hamiltonian and the components of the Runge–Lenz vector have new expressions H'_{KC} and (L'_w, K'_v, K'_u), which are deduced from the previous expressions in which u is replaced by $(u+l)$ and κ by $\kappa(l)$:

$$H'_{KC} = \frac{1}{2}(p_v^2 + p_u^2) - \frac{\kappa(l)}{\sqrt{(u+l)^2 + v^2}}, \tag{22}$$

and

$$\begin{aligned} L'_w &= (vp_u - up_v) - lp_v, \\ K'_v &= (vp_u - up_v - lp_v)p_u - v\frac{\kappa(l)}{\sqrt{(u+l)^2 + v^2}}, \\ K'_u &= -(vp_u - up_v - lp_v)p_v - (u+l)\frac{\kappa(l)}{\sqrt{(u+l)^2 + v^2}}. \end{aligned} \tag{23}$$

Now, as $l \to \infty$, the asymptotic behaviour of the inverse distance potential is as follows:

$$\frac{\kappa(l)}{\sqrt{(u+l)^2 + v^2}} \sim \frac{\kappa(l)}{l}\left(1 - \frac{u}{l} + \mathcal{O}(\frac{1}{l^2})\right). \tag{24}$$

Hence, if the source strength increases as $\kappa(l) = l^2$, then the inverse distance potential reaches the limiting form of a linear potential in u:

$$-\frac{\kappa(l)}{\sqrt{(u+l)^2 + v^2}} \to -l + u + \mathcal{O}(\frac{1}{l}). \tag{25}$$

In this limit, the inverse distance potential problem tends toward the free fall problem up to a negative infinite constant:

$$H'_{KC} \sim H - l + \mathcal{O}\left(\frac{1}{l}\right), \tag{26}$$

and the components of the Runge–Lenz vector take the following asymptotic forms:

$$\begin{aligned} L'_w &\sim -lp_v + (vp_u - up_v), \\ K'_v &\sim -l(p_u p_v + v) + (vp_u - up_v)p_u + uv - v\mathcal{O}(\frac{1}{l}), \\ K'_u &\sim -l^2 + lp_v^2 - \left[(vp_u - up_v)p_v - u^2\right] + \mathcal{O}(\frac{1}{l}), \end{aligned} \tag{27}$$

Theorem 1. *The dynamical symmetry algebra of the free fall problem $\mathfrak{A}_{FF}$ is a contraction of the dynamical symmetry algebra of the Kepler–Coulomb problem $\mathfrak{A}_{KC}$.*

Proof. We now rewrite the Poisson brackets of the Kepler–Coulomb problem when the potential source is at a large distance l from the origin, and then replace the generators (L'_w, K'_v, K'_u) by their asymptotic expansions for $l \to \infty$. Therefore:

(a) $\{L'_w, K'_v\} = K'_u$ becomes

$$\{(vp_u - up_v) - l\,F_\rho, -l\,F_\sigma + (vp_u - up_v)p_u + uv - v\mathcal{O}(\frac{1}{l})\} =$$

$$-l^2 + l\,(F_\tau + H) - \left[(vp_u - up_v)p_v - u^2\right] + \mathcal{O}(\frac{1}{l}). \tag{28}$$

Extracting the leading order in l^2 on the left-hand side and on the right-hand side, we obtain $\{F_\rho, F_\sigma\} = -I$, as expected; see Equation (17).

(b) $\{K'_u, L'_w\} = K'_v$ becomes

$$\{-l^2 + l\,(F_\tau + H) - \Big[(vp_u - up_v)p_v - u^2\Big] + \mathcal{O}(\frac{1}{l}), (vp_u - up_v) - l\,F_\rho\} =$$

$$-\,l\,F_\sigma + (vp_u - up_v)p_u + uv - v\mathcal{O}(\frac{1}{l}). \tag{29}$$

Collecting terms of leading order in l^2 on both sides of this equation, we get $\{F_\tau, F_\rho\} = 0$ since there is no term in l^2 on the right-hand side.

(c) $\{K'_v, K'_u\} = -2H'_{KC}\,L'_w$ becomes

$$\{-l\,F_\sigma + (vp_u - up_v)p_u + uv - v\mathcal{O}(\frac{1}{l}), -l^2 + l\,(F_\tau - H) - \Big[(vp_u - up_v)p_v - u^2\Big] + \mathcal{O}(\frac{1}{l})\} =$$

$$-\,2\left(H - l + \mathcal{O}\left(\frac{1}{l}\right)\right)\big((vp_u - up_v) - l\,F_\rho\big). \tag{30}$$

Equating terms of order d^2 on both sides of this equation yields precisely $\{F_\sigma, F_\tau\} = 2F_\rho$. Hence, we reproduce all the Poisson brackets of $\mathfrak{A}_{FF}$. $\square$

2.6. A "Higher" Order Integral of the Motion

We now raise the question whether there exists a "higher" order integral of the motion as a construct of the dynamical symmetry algebra generators. What comes to mind is a weighted sum of squares of the $(F_\rho, F_\sigma, F_\tau)$, an object similar to the square of the angular momentum in a rotation algebra. Instead of a tedious systematic search, a more astute way of finding such an integral of the motion would start by observing that the symmetry algebra of the Kepler–Coulomb problem does have a non-trivial limit when the source strength is turned off, i.e., $\kappa = 0$. Then, the generators take the following form:

$$L_w = (vp_u - up_v), \quad K_v(0) = (vp_u - up_v)p_u, \quad K_u(0) = -(vp_u - up_v)p_v. \tag{31}$$

They fulfil the same Poisson bracket relations as those with $\kappa \neq 0$:

$$\{L_w, K_v(0)\} = K_u(0), \quad \{K_u(0), L_w\} = K_v(0), \quad \{K_v(0), K_u(0)\} = -2H_0\,L_w, \tag{32}$$

where H_0 is the Hamiltonian for free particle motion in two dimensions.

Now, if we allow a linear potential to "*grow*" in the u-direction, rotational symmetry disappears. Hence, L_w must be discarded as a possible integral of motion in the presence of a linear potential u. Thus, from the two remaining $(K_v(0), K_u(0))$, only one can survive under a modified form as a "higher" order integral of the motion, because of Poisson 's theorem in Section 2.1.

Let F^2 be this hypothetical "higher" integral of the motion. As the introduced linear potential is in the u direction, we may assume F^2 to be of the simple form:

$$F^2 = K_u(0) + h(v, u, p_v, p_u), \tag{33}$$

where $h(v, u, p_v, p_u)$ is an unknown function in phase space.

Proposition 7. *There exists a second order integral of the motion F^2 given by the following equation:*

$$F^2 = K_u(0) - \frac{1}{2}v^2 = -(vp_u - up_v)p_v - \frac{1}{2}v^2. \tag{34}$$

Proof. The function $h(v, u, p_v, p_u)$ is determined by the condition $\{F^2, H\} = 0$. Since from explicit computation one gets the following:

$$\left\{F^2, H\right\} = \frac{\partial h}{\partial v}p_v + \frac{\partial h}{\partial u}p_u + \frac{\partial h}{\partial p_u} + vp_v = 0, \tag{35}$$

it is obvious that one should require that $h(v, u, p_v, p_u) = -\frac{1}{2}v^2$. □

Remark 3. *The same search procedure for another quadratic integral of the motion does not work with $K_v(0)$ because the commutativity of the Poisson bracket with H leads to impossible conditions having to be satisfied.*

Proposition 8. *The expression of F^2 in terms of the generators (F_r, F_s, F_t) is as follows:*

$$F^2 = HF_\rho^2 - \frac{1}{2}F_\sigma^2 - \frac{1}{2}(F_\tau + H)^2, \tag{36}$$

Proof. Substitute the expressions $(H, F_\rho, F_\sigma, F_\tau)$ into the expression of F^2. □

Corollary 1. *The Poisson brackets of F^2 with the generators $(F_\rho, F_\sigma, F_\tau)$ are easily obtained:*

$$\left\{F^2, F_\rho\right\} = -F_\sigma, \quad \left\{F^2, F_\sigma\right\} = 2F_\rho F_\tau, \quad \left\{F^2, F_\tau\right\} = -2F_\rho F_\sigma. \tag{37}$$

Proposition 9. *The infinitesimal transformation generated by F^2 with parameter z is canonical.*

Proof. The infinitesimal canonical transform generated by F^2 with the parameter z is given by the following equations:

$$\begin{aligned} V &= v + (vp_u - 2up_v)\,\Delta z + \dots \\ U &= u + vp_v\,\Delta z + \dots \\ P_v &= p_v - (p_v p_u + v)\,\Delta z + \dots \\ P_u &= p_u + p_v^2\,\Delta z \dots \end{aligned} \tag{38}$$

We can check that the Hamiltonian remains invariant if the terms of order $(\Delta z)^2$ are ignored:

$$H(v', u', p_{v'}, p_{u'}) = H(v, u, p_v, p_u) + \left((p_v p_u + v)^2 + p_v^4\right)(\Delta z)^2 + \dots \tag{39}$$

Next, we can verify that the six canonical Poisson brackets are preserved, but the details are not presented here. Moreover, it does possess the additive abelian group property with respect to z, i.e., $S(z) \cdot S(z') = S(z + z')$, as can be checked explicitly. □

2.7. Passage to Parabolic Coordinates and the Physical Meaning of F^2

The issue here is to understand how F^2 arises as a dynamical symmetry. The comprehensive work of Miller et al. [13] on quantum separability has revealed that parabolic coordinates do play a central role. Following this indication, we make a passage to parabolic coordinates (x, y) from our Cartesian coordinates (v, u), as defined by $u = \frac{1}{2}(x^2 - y^2), \quad v = xy$. The Lagrangian L then changes to a new expression:

$$L = \frac{1}{2}(\dot{u}^2 + \dot{v}^2) - u = \frac{1}{2}(x^2 + y^2)(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}(x^2 - y^2). \tag{40}$$

From the following definitions of conjugate momenta:

$$\begin{aligned} p_u &= \frac{\partial L}{\partial \dot{u}} = \dot{u}, \qquad\qquad p_v = \frac{\partial L}{\partial \dot{v}} = \dot{v}, \\ p_x &= \frac{\partial L}{\partial \dot{x}} = (x^2+y^2)\dot{x}, \quad p_y = \frac{\partial L}{\partial \dot{y}} = (x^2+y^2)\dot{y}, \end{aligned} \tag{41}$$

we deduce a relation between the (v,u) and the (x,y) momenta:

$$p_u = \frac{xp_x - yp_y}{(x^2+y^2)}, \qquad p_v = \frac{yp_x + xp_y}{(x^2+y^2)}. \tag{42}$$

This allows for the acquisition of new expressions of H and F^2 in parabolic coordinates:

$$H = \frac{1}{2}\frac{p_x^2+p_y^2}{(x^2+y^2)} + \frac{1}{2}(x^2-y^2), \tag{43}$$

$$F^2 = \frac{x^2y^2}{2(x^2+y^2)}\left\{\left(\frac{p_x^2}{x^2}+x^2\right) - \left(\frac{p_y^2}{y^2}-y^2\right)\right\}. \tag{44}$$

As such, these expressions do not show any obvious x and y variable separation. The three integrals of motion $(F_\rho, F_\sigma, F_\tau)$ also do not display any obvious separation into an x part and a y part when re-expressed in the parabolic coordinates. Next, we recall that the initial conditions (v_0, u_0, p_v^0, p_v^0) fully determine the four integrals of motion $(H, F_\rho, F_\sigma, F_\tau)$. Hence, F^2 has a fixed value because it is a construct of these four integrals of motion.

Proposition 10. *For $H = E$, F^2 takes a constant value in the range of energy values $E_+(E)$ of a confining quartic oscillator with an angular frequency square $(-E)$.*

Proof. As total energy is conserved, $H = E$ implies that the following relation must be verified for all $(x,y) \neq (0,0)$:

$$\frac{1}{(x^2+y^2)}\left\{\left(\frac{1}{2}p_x^2 - Ex^2 + \frac{1}{2}x^4\right) + \left(\frac{1}{2}p_y^2 - Ey^2 - \frac{1}{2}y^4\right)\right\} = 0. \tag{45}$$

This means that for a given E, the sum of the Hamiltonians of a confining quartic x-oscillator and a non-confining quartic y-oscillator must be equal to zero for all $(x,y) \neq (0,0)$. Since these one dimensional quartic oscillators are time-independent, their respective Hamiltonians have fixed values, i.e.:

$$\left(\frac{1}{2}p_x^2 - Ex^2 + \frac{1}{2}x^4\right) = E_+(E), \quad \left(\frac{1}{2}p_y^2 - Ey^2 - \frac{1}{2}y^4\right) = E_-(E), \tag{46}$$

however, subjected to the condition $E_+(E) + E_-(E) = 0$. Note that for a given E, $E_+(E)$ takes all real values above the minimum value of the quartic x-oscillator Hamiltonian polynomial in phase space.

On the other hand, F^2 may be rewritten as follows:

$$\begin{aligned} F^2 &= \frac{x^2y^2}{(x^2+y^2)}\left\{\frac{1}{x^2}\left(\frac{1}{2}p_x^2 - Ex^2 + \frac{1}{2}x^4\right) + \frac{1}{y^2}\left(\frac{1}{2}p_y^2 - Ey^2 - \frac{1}{2}y^4\right)\right\} \\ &= \frac{x^2y^2}{(x^2+y^2)}\left(\frac{E_+(E)}{x^2} - \frac{E_-(E)}{y^2}\right). \end{aligned} \tag{47}$$

Since $E_+(E) + E_-(E) = 0$, one gets $F^2 = E_+(E)$. This is due to this *new* aspect of separation of variables called the Stäckel separation of variables. □

2.8. Third Order Integrals of the Motion

In [13], it is shown that third order super-integrable systems separable in parabolic coordinates admit second order integrals, while third order integrals are reducible, i.e., they are Poisson brackets of second order integrals of motion. Here, we have:

$$F^{3\sigma} = \left\{F_\sigma, F^2\right\} = p_v^3 - 2up_v - p_v p_u^2, \quad F^{3\tau} = \left\{F_\tau, F^2\right\} = -(p_u p_v^2 + 2vp_v). \tag{48}$$

However, these third order reducible integrals of motion do not generate one parameter infinitesimal canonical transformations because they do not leave H functionally invariant and are consequently uninteresting.

3. Schrödinger Quantization of the Two-Dimensional Free Fall Problem

3.1. Quantization

The Hamiltonian form of the free fall problem lends itself nicely to its quantization. The classical canonical variables are put into one-to-one correspondence with their quantum counterparts, as self-adjoint operators in a Hilbert space of states $\mathfrak{H}$:

$$(v, p_v, u, p_u) \;\leftrightarrow\; (\widehat{Q}_v, \widehat{P}_v, \widehat{Q}_u, \widehat{P}_u). \tag{49}$$

They build a direct product of v and u Heisenberg algebras (here $\hbar = 1$ for ease of writing):

$$[\widehat{Q}_v, \widehat{P}_v] = i, \quad [\widehat{Q}_u, \widehat{P}_u] = i, \quad \text{and} \quad [\widehat{O}_v, \widehat{O}_u] = 0, \tag{50}$$

where $\widehat{O}_v$ (respectively $\widehat{O}_u$) means $(\widehat{Q}_v, \widehat{P}_v)$ (respectively $\widehat{O}_u = (\widehat{Q}_u, \widehat{P}_u)$).

The quantum Hamiltonian, the dynamical symmetry algebra generators, and the quadratic integral of motion are then given by the following equation:

$$\widehat{H} = \frac{1}{2}\left(\widehat{P}_v^2 + \widehat{P}_u^2\right) + \widehat{Q}_u, \quad \widehat{F}_\rho = \widehat{P}_v, \quad \widehat{F}_\sigma = \widehat{P}_v\widehat{P}_u + \widehat{Q}_v, \quad \widehat{F}_\rho = \widehat{P}_v^2. \tag{51}$$

They fulfil the quantum commutation relations deduced from their Poisson brackets classical counterparts:

$$[\widehat{F}_\rho, \widehat{F}_\sigma] = -iI, \quad [\widehat{F}_\sigma, \widehat{F}_\tau] = 2i\widehat{F}_\rho, \quad [\widehat{F}_\tau, \widehat{F}_\rho] = 0, \tag{52}$$

where I is the identity operator. The quadratic integral of motion $\widehat{F}^2$ is as follows:

$$\widehat{F}^2 = \widehat{Q}_u\widehat{P}_v^2 - \frac{1}{2}(\widehat{P}_v\widehat{Q}_v + \widehat{Q}_v\widehat{P}_v)\widehat{P}_u - \frac{1}{2}\widehat{Q}_v^2. \tag{53}$$

3.2. Schrödinger Representation

It is convenient to work with the Schrödinger coordinate representation. The Hilbert space $\mathfrak{H}$ of states $|\psi\rangle \in \mathfrak{H}$ is represented by the square integrable functions $\psi(v, u) = \langle v, u|\psi\rangle$, where $|v, u\rangle$ for $(v, u) \in \mathbb{R}^2$ is a continuous set of complete and total set in $\mathfrak{H}$. The canonical dynamical operators by differential operators in (v, u) are as follows:

$$(\widehat{Q}_v, \widehat{P}_v, \widehat{Q}_u, \widehat{P}_u) = \left(v, -i\frac{\partial}{\partial v}, u, -i\frac{\partial}{\partial u}\right). \tag{54}$$

Consequently, the differential operator representation of the Hamiltonian, the dynamical symmetry generators, and the quadratic integral of motion are as follows:

$$
\begin{aligned}
\widehat{H} &= -\frac{1}{2}\left(\frac{\partial^2}{\partial v^2}+\frac{\partial^2}{\partial u^2}\right)+u, \\
\widehat{F}_\rho &= -i\frac{\partial}{\partial v}, \widehat{F}_\sigma = \left(-\frac{\partial^2}{\partial v\,\partial u}+v\right), \widehat{F}_\tau = -\left(\frac{\partial^2}{\partial v^2}+\widehat{H}\right), \\
\widehat{F}^2 &= \left(v\frac{\partial^2}{\partial v\,\partial u}-u\frac{\partial^2}{\partial v^2}+\frac{1}{2}\frac{\partial}{\partial u}-\frac{1}{2}v^2\right).
\end{aligned}
\tag{55}
$$

All Poisson brackets from the classical $(H, F_\rho, F_\sigma, F_\tau, F^2)$ now become quantum commutators between $(\widehat{H}, \widehat{F}_\rho, \widehat{F}_\sigma, \widehat{F}_\tau, \widehat{F}^2)$, as can be easily checked. Let $\widehat{\mathfrak{A}}_{FF}$ be the algebra generated by $(\widehat{H}, \widehat{F}_\rho, \widehat{F}_\sigma, \widehat{F}_\tau)$.

Corollary 2. *$\widehat{F}^2$ can be expressed in terms of $(\widehat{H}, \widehat{F}_\rho, \widehat{F}_\sigma, \widehat{F}_\tau)$ as*

$$
\widehat{F}^2 = \widehat{H}\,\widehat{F}_\rho^2 - \frac{1}{2}\widehat{F}_\sigma^2 - \frac{1}{2}\left(\widehat{F}_\tau + \widehat{H}\right)^2. \tag{56}
$$

Proof. Use the expressions in Equation (55) and substitute in (56). □

As usual, the space of relevant wave functions $\mathcal{L}^2(v,u)$ are generated by the eigenfunctions of the stationary Hamiltonian operator $\widehat{H}$. At this step, to determine this functional space, boundary conditions should be specified. As pointed out in the previous section on the classical mechanics of free fall motion, we are concerned with global motion along parabolic trajectories in $\mathbb{R}^2$, with concavity turned downward and not with the bouncing motion on a horizontal line v =constant or the billiard motion inside a two-dimensional box. Both have overly restrictive boundary conditions, rending the wave functions uninteresting. Therefore, only stationary wave functions with free boundary conditions on the v-axis and on the negative u-axis solutions of $\widehat{H}\,\psi_E(v,u) = E\,\psi_E(v,u)$ are considered here.

Since $\widehat{H}$ is separable in the v and u parts, elementary solutions are of the product form:

$$
e^{ikv}\,\mathrm{Ai}\left(\gamma u + \frac{k^2}{\gamma^2} - \gamma E\right), \tag{57}
$$

where $\gamma^3 = 2$ and $\mathrm{Ai}(x)$ is the first Airy function which decreases at $x \to \infty$; see [14].

Hence, the spectrum of $\widehat{H}$ is real and continuous. An arbitrary stationary eigenfunction $\psi_E(u,v)$ with an eigenvalue E is given by an integral on k:

$$
\psi_E(u,v) = \int_{\mathbb{R}} dk\, e^{ikv}\,\widetilde{\psi}_E(k)\,\mathrm{Ai}\left(\gamma u + \frac{k^2}{\gamma^2} - \gamma E\right), \tag{58}
$$

where $\widetilde{\psi}_E(k)$, yet to be determined, appears as a density amplitude representing the relative distribution of the v-motion with respect to the u-motion parts for a fixed E in $\psi_E(u,v)$. Both $\psi_E(u,v)$ and $\widetilde{\psi}_E(k)$ describe two aspects of the same quantum state $|\psi_E\rangle$.

Proposition 11. *As $\psi_E(u,v)$ represents a probability amplitude in $\mathcal{L}^2(v,u)$, $\widetilde{\psi}_E(k)$ is a square integrable function on $\mathbb{R}$, i.e., $\widetilde{\psi}_E(k) \in \mathcal{L}^2(k)$.*

Proof. Let us compute the overlap integral $\langle\psi_E|\psi_{E'}\rangle$ between two eigenstates of energies E and E':

$$
\langle\psi_E|\psi_{E'}\rangle = \int_{\mathbb{R}^2}\frac{du\,dv}{2\pi}\,\psi_E^*(v,u)\,\psi_{E'}(v,u) = \int_{\mathbb{R}^2}\frac{du\,dv}{2\pi}
$$

$$
\int_{\mathbb{R}} dk\, e^{-ikv}\,\widetilde{\psi}_E^*(k)\,\mathrm{Ai}\left(\gamma u + \frac{k^2}{\gamma^2} - \gamma E\right) \int_{\mathbb{R}} dk'\, e^{ik'v}\,\widetilde{\psi}_{E'}(k')\,\mathrm{Ai}\left(\gamma u + \frac{k'^2}{\gamma^2} - \gamma E'\right). \tag{59}
$$

The 2π factor is just for convenience. Integration over dv yields $2\pi\,\delta(k-k')$. Then, after k'-integration, one may consider the remaining u-integration.

This u-integral is readily given in [14] by equation 3.108 on page 57 as follows:

$$
\begin{aligned}
&\frac{1}{|\alpha\,\beta|}\int_{-\infty}^{\infty} du\,\mathrm{Ai}\left(\frac{u+a}{\alpha}\right)\mathrm{Ai}\left(\frac{u+b}{\beta}\right)\times \\
&= \quad \delta(a-b), \qquad\qquad\qquad \text{if} \quad \beta=\alpha, \\
&= \quad \frac{1}{|\beta^3-\alpha^3|^{\frac{1}{3}}}\,\mathrm{Ai}\left(\frac{b-a}{|\beta^3-\alpha^3|^{\frac{1}{3}}}\right), \quad \text{if} \quad \beta>\alpha.
\end{aligned} \tag{60}
$$

Hence, we have a relation between the inner product in $\mathcal{L}^2(v,u)$ and the inner product in $\mathcal{L}^2(k)$:

$$
\int_{\mathbb{R}^2}\frac{du\,dv}{2\pi}\,\psi_E^*(v,u)\,\psi_{E'}(v,u) = \gamma^{-2}\,\delta(E-E')\int_{\mathbb{R}} dk\,\widetilde{\psi}_E^*(k)\,\widetilde{\psi}_{E'}(k). \tag{61}
$$

It expresses the completeness of the eigenfunctions $\psi_E(v,u)$, provided that:

$$
\int_{\mathbb{R}} dk\,\widetilde{\psi}_E^*(k)\,\widetilde{\psi}_E(k) < \infty. \tag{62}
$$

Hence, $\widetilde{\psi}_E(k)\in\mathcal{L}^2(k)$. $\square$

Theorem 2. *The integral mapping* $\psi_E(v,u)\in\ \mathcal{L}^2(v,u)\ \rightarrow\ \widetilde{\psi}_E(k)\in\mathcal{L}^2(k)$ *is invertible*, i.e.,

$$
\widetilde{\psi}_{E'}(k') = \int_{\mathbb{R}} dE\int_{\mathbb{R}^2} du\,dv\ \psi_E(u,v)\,e^{-ik'v}\,Ai\left(\gamma u+\frac{k'^2}{\gamma^2}-\gamma E'\right). \tag{63}
$$

Proof. By integrating both sides of Equation (58) with the following equation:

$$
\int_{\mathbb{R}^2}\frac{du\,dv}{2\pi}\ e^{-ik'v}\,\mathrm{Ai}\left(\gamma u+\frac{k'^2}{\gamma^2}-\gamma E'\right), \tag{64}
$$

we get the following integral on the right-hand side:

$$
\int_{\mathbb{R}^2}\frac{du\,dv}{2\pi}\ e^{-ik'v}\,\mathrm{Ai}\left(\gamma u+\frac{k'^2}{\gamma^2}-\gamma E'\right)\int_{\mathbb{R}} dk\,e^{ikv}\,\widetilde{\psi}_E(k)\,\mathrm{Ai}\left(\gamma u+\frac{k^2}{\gamma^2}-\gamma E\right). \tag{65}
$$

Assuming Fubini's theorem hypothesis, we can exchange integrations and the $\frac{dv}{2\pi}$-integration would yield $\delta(k-k')$. Then, after integration on dk, the right-hand side becomes:

$$
\widetilde{\psi}_E(k')\int_{\mathbb{R}}\frac{du}{2\pi}\,\mathrm{Ai}\left(\gamma u+\frac{k^2}{\gamma^2}-\gamma E\right)\mathrm{Ai}\left(\gamma u+\frac{k'^2}{\gamma^2}-\gamma E'\right), \tag{66}
$$

which, upon application of the integration formula (60), yields the following:

$$
\int_{\mathbb{R}^2}\frac{du\,dv}{2\pi}\ \psi_E(u,v)\,e^{-ik'v}\,\mathrm{Ai}\left(\gamma u+\frac{k'^2}{\gamma^2}-\gamma E'\right) = \widetilde{\psi}_E(k')\,\gamma^{-2}\,\delta(E-E'). \tag{67}
$$

A last integration on dE on both sides of this equation gives the final result:

$$
\widetilde{\psi}_{E'}(k') = \gamma^2\int_{\mathbb{R}} dE\int_{\mathbb{R}^2} du\,dv\ \psi_E(v,u)\,e^{-ik'v}\,\mathrm{Ai}\left(\gamma u+\frac{k'^2}{\gamma^2}-\gamma E'\right). \tag{68}
$$

Hence, the integral mapping $\psi_E(v,u)\ \rightarrow\ \widetilde{\psi}_E(k)$ is *invertible*, provided that a summation on E is performed or uses complete data. In this respect, it may be called the

Fourier-Airy Transform. These two amplitudes describe the same physics in two different contexts. □

3.3. Representation of the Dynamical Symmetry Algebra $\widehat{\mathfrak{A}}_{FF}$ by Klink's Algebra

Proposition 12. *The action of the $\widehat{\mathfrak{A}}_{FF}$ generators on the wave function $\psi_E(v,u)$ is easily transferred to the wave function $\widetilde{\psi}_E(k)$ according to the following:*

$$\begin{aligned}
\widehat{H}\psi_E(v,u) &= E\int_{\mathbb{R}} dk\,\widetilde{\psi}_E(k)\,e^{ikv}\,Ai\Big(\gamma u+\frac{k^2}{\gamma^2}-\gamma E\Big),\\
\widehat{F}_\rho\psi_E(v,u) &= \int_{\mathbb{R}} dk\,(k\widetilde{\psi}_E(k))\,e^{ikv}\,Ai\Big(\gamma u+\frac{k^2}{\gamma^2}-\gamma E\Big),\\
\widehat{F}_\sigma\psi_E(v,u) &= \int_{\mathbb{R}} dk\,\Big(i\frac{d}{dk}\Big)\widetilde{\psi}_E(k)\,e^{ikv}\,Ai\Big(\gamma u+\frac{k^2}{\gamma^2}-\gamma E\Big),\\
\widehat{F}_\tau\psi_E(v,u) &= \int_{\mathbb{R}} dk\,(k^2-E)\widetilde{\psi}_E(k)\,e^{ikv}\,Ai\Big(\gamma u+\frac{k^2}{\gamma^2}-\gamma E\Big).
\end{aligned}\tag{69}$$

Proof. The action of $\widehat{H}$ is E since $\psi_E(v,u)$ is an eigenfunction of $\widehat{H}$.

As $\widehat{F}_\rho$ and $\widehat{F}_\tau$ are represented in $\mathcal{L}^2(v,u)$ by v-derivatives acting under the integral sign on e^{ikv}, we successively get k and k^2 acting on $\widetilde{\psi}_E(k)$.

For the action of $\widehat{F}_\sigma=\Big(-\frac{\partial^2}{\partial u\,\partial v}+v\Big)$, we observe that:

$$v\,\psi_E(v,u)=\int_{\mathbb{R}} dk\,\widetilde{\psi}_E(k)\,\Big(-i\frac{\partial}{\partial k}e^{ikv}\Big)\,\mathrm{Ai}\Big(\gamma u+\frac{k^2}{\gamma^2}-\gamma E\Big).\tag{70}$$

Since the Airy function vanishes for $k=\pm\infty$, we perform a partial integration in k to get the following:

$$v\,\psi_E(v,u)=$$
$$\int_{\mathbb{R}} dk\,\Big(i\frac{\partial}{\partial k}\widetilde{\psi}_E(k)\Big)\,e^{ikv}\,\mathrm{Ai}\Big(\gamma u+\frac{k^2}{\gamma^2}-\gamma E\Big)+\int_{\mathbb{R}} dk\,\widetilde{\psi}_E(k)\,e^{ikv}\,\mathrm{Ai}'\Big(\gamma u+\frac{k^2}{\gamma^2}-\gamma E\Big)i\gamma k.\tag{71}$$

where $\mathrm{Ai}'(x)$ is the derivative of $\mathrm{Ai}(x)$. We then observe that the second integral is as follows:

$$\frac{\partial^2}{\partial u\,\partial v}\psi_E(v,u).\tag{72}$$

Hence, the action of $\widehat{F}_\sigma$ on $\psi_E(v,u)$ is replaced by $i\frac{\partial}{\partial k}\widetilde{\psi}_E(k)$ in $\mathcal{L}^2(k)$. □

Corollary 3. *As a consequence, the action of $\widehat{F}^2$ on $\psi_E(v,u)$ is translated into action on $\widetilde{\psi}_E(k)$ as the action of the one-dimensional confining quartic anharmonic oscillator Hamiltonian in the k variable on the wave function $\widetilde{\psi}_E(k)$:*

$$\widehat{F}^2\,\psi_E(v,u)=-\int_{\mathbb{R}} dk\,\Big\{\Big(-\frac{1}{2}\frac{d^2}{dk^2}-Ek^2+\frac{1}{2}k^4\Big)\widetilde{\psi}_E(k)\Big\}\,e^{ikv}\,\mathrm{Ai}\Big(\gamma u+\frac{k^2}{\gamma^2}-\gamma E\Big)\tag{73}$$

Proof. Use the expression of $\widehat{F}^2$ in terms of the following dynamical symmetry algebra generators $(\widehat{F}_\rho,\widehat{F}_\sigma,\widehat{F}_\tau)$:

$$\widehat{F}^2=(\widehat{H}\,\widehat{F}_\rho^2-\frac{1}{2}\widehat{F}_\sigma^2-\frac{1}{2}(\widehat{F}_\tau+\widehat{H})^2),\tag{74}$$

and the previous proposition. □

Corollary 4. *The mapping $\psi_E(v,u)\ \to\ \widetilde{\psi}_E(k)$ induces an isomorphism between the two-variable algebra generated by $(\widehat{F}_\rho,\widehat{F}_\sigma,\widehat{F}_\tau)$, and the one-variable algebra generated by $(k,i\frac{d}{dk},k^2)$, which is known as Klink's algebra for quartic anharmonic oscillator and for which the quadratic integral of motion $\widehat{F}^2$ is the Schrödinger Hamiltonian of this quartic anharmonic oscillator [15].*

Proof. The commutators of $(\widehat{F}_\rho, \widehat{F}_\sigma, \widehat{F}_\tau)$ are isomorphic to those of $(k, i\frac{d}{dk}, k^2)$ by computational checking. □

This relation shows how the one-dimensional confining quartic anharmonic oscillator is linked to the two-dimensional free fall. Note that this relation cannot be established if one had started with the one-dimensional free fall problem. In classical physics, the presence of quartic anharmonic oscillators in the free fall problem arises only when parabolic coordinates are introduced. This is not the case here. There is still another curious relation between the two systems discovered by Voros in [16].

3.4. Finite Quantum Canonical Transforms

Proposition 13. *The finite quantum unitary transforms generated by $(\widehat{F}_\rho, \widehat{F}_\sigma, \widehat{F}_\tau)$ are expressed as follows:*

$$\widehat{U}_\rho = \exp -i\rho\, \widehat{F}_\rho, \quad \widehat{U}_\sigma = \exp -i\sigma\, \widehat{F}_\sigma, \quad \widehat{U}_\tau = \exp -i\tau\, \widehat{F}_\tau. \tag{75}$$

They produce the quantum version of the classical Iwai–Rew transform (see Equation (8)).

Proof. The proof is trivial and involves the use of the Baker–Hausdorff–Campbell formula:

$$e^{\widehat{Y}}\widehat{X}e^{\widehat{Y}} = \widehat{X} + [\widehat{Y}, \widehat{X}] + \frac{1}{2!}[\widehat{Y}, +[\widehat{Y}, \widehat{X}]] + \ldots$$

where $(\widehat{X}, \widehat{Y})$ are operators. Because of the commutation relations of $\widehat{\mathfrak{A}}_{FF}$, the computation of this Baker–Hausdorff–Campbell on $(\widehat{Q}_v, \widehat{P}_v, \widehat{Q}_u, \widehat{P}_u)$ yields only a few terms, whose coefficients are precisely those in Equation (8). □

Proposition 14. *The action of $(\widehat{U}_\rho, \widehat{U}_\sigma, \widehat{U}_\tau)$ on the wave function $\psi_E(v, u)$ may be represented by the following integral transforms:*

$$\begin{aligned}
\widehat{U}_\rho \psi_E(v,u) &= \int_{\mathbb{R}} dk \left(e^{-ik\rho}\widetilde{\psi}_E(k)\right) e^{ikv}\, \mathrm{Ai}\left(\gamma(u-E) + \frac{k^2}{\gamma^2}\right), \\
\widehat{U}_\sigma \psi_E(v,u) &= \int_{\mathbb{R}} dk\, \widetilde{\psi}_E(k+\sigma)\, e^{ikv}\, \mathrm{Ai}\left(\gamma(u-E) + \frac{k^2}{\gamma^2}\right), \\
\widehat{U}_\tau \psi_E(v,u) &= \int_{\mathbb{R}} dk \left(e^{-i\tau(k^2-E)}\, \widetilde{\psi}_E(k)\right) e^{ikv}\, \mathrm{Ai}\left(\gamma(u-E) + \frac{k^2}{\gamma^2}\right).
\end{aligned} \tag{76}$$

Proof. The proof is straightforward. It uses the action of each generator given by Equation (69) and then exponentiates it as action on $\widetilde{\psi}_E(k)$. This leads to unitary factors for $\widehat{U}_\rho$ and $\widehat{U}_\tau$ as well as a shift in the argument of $\widetilde{\psi}_E(k)$ for $\widehat{U}_\sigma$. This is an alternative form to the one obtained by Iwai–Rew [4]. □

3.5. Quantum Integrals of Motion and Consequences on the Schrödinger Wave Functions

In this subsection, we study the nature of the quantum integrals of motion. In particular, when a manifest separation of variables occurs in an operator $\widehat{O}(v, u) = \widehat{O}(v) + \widehat{O}(u)$, the operator $\widehat{F}(v, u) = \widehat{O}(v) - \widehat{O}(u)$ automatically commutes with $\widehat{O}(v, u)$.

(a1)$\widehat{F}_\rho$ exists since $[\widehat{H}, \widehat{F}_\rho] = 0$ because ρ is a cyclic variable, as in the classical case.

(a2) $\widehat{F}_\tau$ is due to the separation in the (v, u) Cartesian variables. Hence,

$$\widehat{H} = \left(-\frac{1}{2}\frac{\partial^2}{\partial v^2}\right) + \left(-\frac{1}{2}\frac{\partial^2}{\partial u^2} + u\right), \quad \text{and} \quad \widehat{F}_\tau = \left(-\frac{1}{2}\frac{\partial^2}{\partial v^2}\right) - \left(-\frac{1}{2}\frac{\partial^2}{\partial u^2} + u\right) \tag{77}$$

commute.

(b1) $\widehat{F}_\sigma$ is due to the separation in $\pi/4$ rotated Cartesian variables (ξ, ζ). With the following change in variables:

$$(v, u) = \frac{1}{2}((\xi + \zeta), (\xi - \zeta)), \quad (\xi, \zeta) = ((u + v), (u - v)), \tag{78}$$

$(\widehat{H}, \widehat{F}_\sigma)$ becomes $(\widehat{H}(\xi,\zeta), \widehat{F}_\sigma(\xi,\zeta))$ given by the following equations:

$$\left(\left(-\frac{\partial^2}{\partial\xi^2}+\frac{1}{2}\xi\right)+\left(-\frac{\partial^2}{\partial\zeta^2}+\frac{1}{2}\zeta\right),\left(-\frac{\partial^2}{\partial\xi^2}+\frac{1}{2}\xi\right)-\left(-\frac{\partial^2}{\partial\zeta^2}+\frac{1}{2}\zeta\right)\right) \tag{79}$$

which commute.

(b2) However, there was an unproven claim of separation of variables in displaced parabolic coordinates [10] as a justification for the existence of $(\widehat{F}_\sigma)$. Hereafter, we provide an argument that may explain this claim. Consider the intermediate space (k,u), obtained after a partial v-Fourier transform from the (v,u) space. The Schrödinger equation in the (k,u) space for stationary states of the eigenvalue E admits solutions in the form of:

$$\mathrm{Ai}\left(\gamma u+\frac{k^2}{\gamma^2}-\gamma E\right). \tag{80}$$

This solution has a constant value C on the parabolas of equation $\left(u+\frac{1}{2}k^2-E\right)=C$, where C is a shift of the parabolas. Consequently, the tangential derivative of any function of (k,u) along these parabolas is zero. This tangential derivative is just $\mathbf{t}\cdot\nabla$, where $\nabla=\left(\frac{\partial}{\partial k},\frac{\partial}{\partial u}\right)$ is the gradient operator and $\mathbf{t}=(1,-k)$ is the tangent vector to the parabola. Thus, we have the following:

$$\left(-k\frac{\partial}{\partial u}+\frac{\partial}{\partial k}\right)\mathrm{Ai}\left(\gamma u+\frac{k^2}{\gamma^2}-\gamma E\right)=0. \tag{81}$$

Going back to the (v,u) space by the k-Fourier inverse transform, this tangential derivative reappears as a $(v,u)-$partial differential operator $\left(\frac{\partial^2}{\partial u\,\partial v}-v\right)$, which is just $-\widehat{F}_\sigma$. Therefore, the claim of reference [13] is only valid in the (k,u)-space.

(c) $\widehat{F}^2$ is due to a special form of separation of variables (Stäckel separation of variables), when one changes from Cartesian (v,u) to parabolic (x,y) coordinates by the following formulas: $(u,v)=\left(\frac{x^2-y^2}{2},xy\right)$. After working out the expressions of the (x,y) partial derivatives, we end up with new expressions in (x,y) for $(\widehat{H},\widehat{F}^2)$:

$$\begin{aligned}\widehat{H}(x,y) &= \frac{1}{x^2+y^2}\left\{\left(-\frac{1}{2}\frac{\partial^2}{\partial x^2}+\frac{1}{2}x^4\right)+\left(-\frac{1}{2}\frac{\partial^2}{\partial y^2}-\frac{1}{2}y^4\right)\right\},\\ \widehat{F}^2(x,y) &= \frac{(-1)}{x^2+y^2}\left\{y^2\left(-\frac{1}{2}\frac{\partial^2}{\partial x^2}+\frac{1}{2}x^4\right)-x^2\left(-\frac{1}{2}\frac{\partial^2}{\partial y^2}-\frac{1}{2}y^4\right)\right\}.\end{aligned} \tag{82}$$

These expressions can also be obtained from the quantization of classical expressions by replacing (p_x,p_y) with $\left(-i\frac{\partial}{\partial x},-i\frac{\partial}{\partial y}\right)$. The meaning of this Stäckel separation of variables is given in the following proposition.

Proposition 15. *As the total energy is conserved as $\widehat{H}=E$, the eigenfunctions of $\widehat{H}$ in parabolic coordinates are products of Schrödinger eigenfunctions of confining and non-confining quartic oscillators in the form of $\psi_E(v,u)\sim\overline{\psi}_E(x,y)=\overline{\psi}_E^{(+)}(x)\overline{\psi}_E^{(-)}(y)$. Then, $\widehat{F}^2\overline{\psi}_E^{(+)}(x)\overline{\psi}_E^{(-)}(y)=-E\overline{\psi}_E^{(+)}(x)\overline{\psi}_E^{(-)}(y)$.*

Proof. Using the expression of $\widehat{H}$ in parabolic coordinates given by Equation (82) and calling the corresponding wave function $\overline{\psi}_E(x,y)$, we can transform the stationary Schrödinger equation $\widehat{H}\overline{\psi}_E(x,y)=E\overline{\psi}_E(x,y)$ for all $(x,y)\neq(0,0)$ into the following condition:

$$\left\{\left(-\frac{1}{2}\frac{\partial^2}{\partial x^2}-Ex^2+\frac{1}{2}x^4\right)+\left(-\frac{1}{2}\frac{\partial^2}{\partial y^2}-Ey^2-\frac{1}{2}y^4\right)\right\}\overline{\psi}_E(x,y)=0. \tag{83}$$

Since this condition is to be satisfied for all $(x, y) \neq (0, 0)$, $\overline{\psi}_E(x, y)$ should be a product of eigenfunctions of the two separate Schrödinger operators in x and in y, with opposite eigenvalues. For the confining quartic potential in x, it is known that this Schrödinger operator has discrete non-degenerate point spectrum with the eigenvalues $E = \epsilon_n$, with $n \in \mathbb{N}$, which are bounded below. On the other hand, the Schrödinger operator with non-confining quartic potential in y has a continuous, real, non-degenerate, scattering-type spectrum. Thus, $\psi_E(v, u) \sim \overline{\psi}_E(x, y)$ must be of the product form (up to a multiplicative constant):

$$\overline{\psi}_E(x, y) = \overline{\psi}_E^{(+)}(x)\, \overline{\psi}_E^{(-)}(x),$$

verifying the stationary Schrödinger equations:

$$\begin{aligned} \left(-\frac{1}{2}\frac{\partial^2}{\partial x^2} - Ex^2 + \frac{1}{2}x^4\right)\overline{\psi}_E^{(+)}(x) &= \mathcal{H}_{osc}^{(+)}(x)\overline{\psi}_E^{(+)}(x) = +E\,\overline{\psi}_E^{(+)}(x), \\ \left(-\frac{1}{2}\frac{\partial^2}{\partial y^2} - Ey^2 - \frac{1}{2}y^4\right)\overline{\psi}_E^{(-)}(y) &= \mathcal{H}_{osc}^{(-)}(y)\overline{\psi}_E^{(-)}(y) = -E\,\overline{\psi}_E^{(-)}(y). \end{aligned} \tag{84}$$

Then, after rewriting the expression of $\widehat{F}^2$ under the following form:

$$\widehat{F}^2 = \frac{(-1)}{x^2 + y^2}\left\{y^2\left(-\frac{1}{2}\frac{\partial^2}{\partial x^2} - Ex^2 + \frac{1}{2}x^4\right) - x^2\left(-\frac{1}{2}\frac{\partial^2}{\partial y^2} - Ey^2 - \frac{1}{2}y^4\right)\right\}, \tag{85}$$

we see that

$$\widehat{F}^2\overline{\psi}_E(x, y) = \widehat{F}^2\overline{\psi}_E^{(+)}(x)\,\overline{\psi}_E^{(-)}(y) = -E\,\overline{\psi}_E^{(+)}(x)\,\overline{\psi}_E^{(-)}(y). \tag{86}$$

Hence, for $\widehat{H} = E$, we have necessarily $-E$ as an eigenvalue of $\widehat{F}^2$, where $E = \epsilon_n$ is any eigenvalue of the confining quartic potential $+\frac{1}{2}x^4$. □

3.6. Integral Relation for Schrödinger Eigenfunctions of Quartic Oscillators

The passage to parabolic coordinates has a remarkable consequence on the eigenfunctions of quantum quartic oscillators, as stated in the following theorem.

Theorem 3. *The eigenfunctions of quantum confining and non-confining quartic oscillators verify the following integral relation:*

$$\overline{\psi}_E^{(+)}(x)\,\overline{\psi}_E^{(-)}(y) = \mu_E \int_{\mathbb{R}} dk\; \overline{\psi}_E^{(+)}(k)\, e^{ikxy}\, \mathrm{Ai}\left(\frac{x^2 - y^2 + k^2 - 2E}{\gamma^2}\right). \tag{87}$$

Proof. Recast the expression of $\psi_E(v, u)$ in parabolic coordinates to get the following:

$$\overline{\psi}_E^{(+)}(x)\,\overline{\psi}_E^{(-)}(y) = \mu_E \int_{\mathbb{R}} dk\; \widetilde{\psi}_E(k)\, e^{ikxy}\, \mathrm{Ai}\left(\frac{x^2 - y^2 + k^2 - 2E}{\gamma^2}\right), \tag{88}$$

where μ_E takes care of the fact that $\psi_E(v, u) \sim \overline{\psi}_E(x, y)$ up to a multiplicative constant. Let $\widehat{F}^2$ operate on both sides of this equation. On the left-hand side, we get $-E\overline{\psi}_E^{(+)}(x)\,\overline{\psi}_E^{(-)}(y)$ according to Equation (86). On the right-hand side, we obtain:

$$-\mu_E \int_{\mathbb{R}} dk\, \left(-\frac{1}{2}\frac{d^2}{dk^2} - Ek^2 + \frac{1}{2}k^4\right)\widetilde{\psi}_E(k)\, e^{ikxy}\, \mathrm{Ai}\left(\frac{x^2 - y^2 + k^2 - 2E}{\gamma^2}\right), \tag{89}$$

according to Equation (73). Hence, to have consistency between the two sides, one must require that $\widetilde{\psi}_E(k) = \overline{\psi}_E^{(+)}(k)$. Therefore, it follows that the eigenfunctions of the confining and non-confining quartic oscillators should verify the integral relation above. □

Since the confining quartic oscillator is of dominant physical interest as a non-trivial field theory model in zero space dimension, an integral identity for its eigenfunctions can be derived from the previous identity via a "Wick-rotation" [17].

Corollary 5. *The eigenfunctions $\overline{\psi}_E^{(+)}(x)$ of the confining quartic oscillator fulfil the following integral identity:*

$$\overline{\psi}_E^{(+)}(x)\,\overline{\psi}_E^{(+)}(x') = \mu_E \int_{\mathbb{R}} dk\; \overline{\psi}_E^{(+)}(k) \begin{pmatrix} \cosh xx'k \\ \sinh xx'k \end{pmatrix} \mathrm{Ai}\left(\frac{x^2 + x'^2 + k^2 - 2E}{\gamma^2}\right). \tag{90}$$

Proof. Since

$$\mathcal{H}_{osc}^{(-)}(ix') = -\mathcal{H}_{osc}^{(+)}(x'), \quad \text{and} \quad \overline{\psi}_E^{(-)}(ix') = \overline{\psi}_E^{(+)}(x'), \tag{91}$$

replace y with $y = ix'$ in Equation (88). Then, as parity is a good quantum number for $\mathcal{H}_{osc}^{(+)}$, the factor e^{-kxy} in the integrand should be replaced by either $\sinh xx'k$ or $\cosh xx'k$, according to the parity of the state of energy $E = \epsilon_n$ for $n \in \mathbb{N}$. This last integral identity was discovered long ago via the Weyl quantization of anharmonic oscillators [18] and recently rediscovered by [19]. □

4. An Application to Wave Propagation in Duct of Varying Section

In 2003, B J Forbes et al. [20] observed that the Webster equation describing the excess pressure $p(u,t)$ in a fluid flowing in a duct with a slowly varying circular cross-sectional area $S(u)$:

$$\frac{1}{c^2}\frac{\partial^2 p(u,t)}{\partial t^2} = \frac{1}{S(u)}\frac{\partial}{\partial u}\left(S(u)\,\frac{\partial p(u,t)}{\partial u}\right), \tag{92}$$

where u is the space coordinate along the axis of the duct, t the time, and c is the constant wave speed in the fluid, which can be turned into a Klein–Gordon equation with a potential $V(u)$ for a wave function $\psi(u,t)$:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial u^2} + V(u)\right)\psi(u,t) = 0, \tag{93}$$

if $\psi(u,t) = p(u,t)\,\sqrt{S(u)}$ and

$$V(u) = \frac{1}{\sqrt{S(u)}}\,\frac{d^2\sqrt{S(u)}}{du^2}. \tag{94}$$

Hence, if we require that the potential to be linear $V(u) = (u-d)$, where d is an arbitrary distance, Equation (94) shows that the duct section should vary as $S(u) = \mathrm{Ai}(u-d)$, and consequently, Equation (93) is obtained from the two-dimensional free fall Schrödinger equation by "Wick-rotation" $v = ict$ [17] (without a factor $1/2$ in the partial derivatives). This Klein–Gordon Equation (93) admits a dynamical symmetry algebra which is the Wick rotated $\widehat{\mathfrak{A}}_{FF}$. Note that the free fall $\widehat{F}_\tau$ integral of motion now becomes the Klein–Gordon operator, and conversely, the free fall Hamiltonian operator is now an integral of the motion for the Klein–Gordon operator. The quadratic integral operator of the Klein–Gordon problem is now the Wick rotated free fall $\widehat{F}^2$ operator. In this case, one directly gets the integral identity for the Schrödinger eigenfunctions of the confining quartic anharmonic oscillator; see Equation (90).

The general solution of this Klein–Gordon wave equation $\phi_d(t,u)$ is formally analogous to $\psi_E(v,u)$; it is given by the ω-integral for $t > 0$:

$$\phi_d(t,u) = \int_{\mathbb{R}} d\omega\, f(\omega)\, \mathrm{Ai}\left(\gamma(u-d) + \frac{\omega^2}{c^2\gamma^2}\right), \tag{95}$$

with the initial conditions $\phi_d(0,u)$ and $\dot{\phi}_d(0,u) = \frac{d}{dt}\phi_d(t,u)|_{t=0}$.

Theorem 4. *The angular frequency distribution $f(\omega)$ of $\phi_d(t,u)$ is fully determined by the initial conditions.*

Proof. The proof uses the integral given by Equation (60) and also by correctly integrating the delta function $\delta(\omega^2 - \omega'^2)$:

$$\begin{aligned} \frac{1}{2}(f(\omega') + f(-\omega')) &= \frac{|\omega'|}{c^2} \int_{\mathbb{R}} du\, \phi_d(0,u)\, \mathrm{Ai}\Big((u-d) + \frac{\omega'^2}{c^2}\Big), \\ \frac{1}{2}(-f(\omega') + f(-\omega')) &= \frac{\mathrm{sgn}\omega'}{c^2} \int_{\mathbb{R}} du\, \dot{\phi}_d(0,u)\, \mathrm{Ai}\Big((u-d) + \frac{\omega'^2}{c^2}\Big) \end{aligned} \tag{96}$$

where $\mathrm{sgn}\omega = \frac{\omega}{|\omega|}$ is the sign of ω. This fully determines $f(\omega)$. □

5. Conclusions and Perspectives

In this paper, a complete account of the dynamical symmetries of the two-dimensional free fall is provided both classically and quantum mechanically. The results may open the way towards the construction of the representation of its dynamical algebra. This is a challenging task which makes it necessary to understand the *zonal* character of the integral relation fulfilled by the eigenfunctions of the confining quartic oscillator [21]. An extension to dimensions higher than two, which may reveal new unexpected features of this simple physical problem as it did in two dimensions, is foreseen as future work.

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References

1. Halliday, D.; Resnick, R.; Walker, J. *Fundamentals of Physics*; John Wiley and Sons, Inc.: New York, NY, USA, 2005; pp. 24–26.
2. Wheeler, N. Classical/Quantum Dynamics in a Uniform Gravitational Field: An Unobstructed Free Fall. Available online: https://www.reed.edu/physics/faculty/wheeler/documents/Quantum%20Mechanics/Miscellaneous%20Essays/Quantum%20Bouncer/E4.%20Free%20Fall.pdf (accessed on 27 October 2021).
3. Engelhardt, W. Free fall in gravitational theory. *Phys. Essays* **2017**, *30*, 294. [CrossRef]
4. Iwai, T.; Rew, S.G. Classical and quantum symmetry groups of a free-fall particle. *J. Math. Phys.* **1985**, *26*, 55–61. [CrossRef]
5. Barut, A.O.; Kleinert, H. Transition probabilities of the Hydrogen atom from noncompact dynamical groups. *Phys. Rev.* **1967**, *156*, 1541–1545. [CrossRef]
6. Alliluev, S.P. On the relation between "accidental" degeneracy and "hidden" symmetry of a system. *Sov. Phys. JETP* **1958**, *6*, 156–159.
7. Whittaker, E.T. *A Treatrise on the Analytical Dynamics of Particles and Rigid Bodies*, 4th ed.; Cambridge University Press: Cambridge, UK, 1959.
8. Fordy, A.P. Quantum super-integrable systems as exactly solvable systems. *Symmetry Integr. Geom. Methods Appl.* **2007**, *3*, 025.
9. Jauch, J.M.; Hill, E.L. On the problem of degeneracy in Quantum Mechanics. *Phys. Rev.* **1940**, *57*, 641–645. [CrossRef]
10. Miller, W.; Kalnins, E.G.; Kress, J.M.; Pogosyan, G.S. Infinite-order Symmetries for Quantum Separable Systems. *Phys. At. Nucl.* **2005**, *68*, 1756–1763. [CrossRef]
11. Englefield, M.J. *Group Theory and the Coulomb Problem*;Wiley-Interscience: New York, NY, USA, 1972.
12. Wheeler, N. *Classical/Quantum Theory of the 2-Dimensional Hydrogen*; Lecture Notes Reed College: Portland Oregon, OR, USA, 1999. Available online: www.reed.edu/physics/faculty/wheeler/documents/QuantumMechanics/Miscellaneous (accessed on 26 October 2021).
13. Popper, I.; Post, S.; Winternitz, P. Third order super-integrable systems separable in parabolic coordinates. *J. Math. Phys.* **2012**, *53*, 062105. [CrossRef]
14. Vallée, O.; Soares, M. *Airy Functions and Applications to Physics*; World Scientific: London, UK, 2004.
15. Klink, W.H. Nilpotent groups and anharmonic oscillators. In *Noncompact Lie Groups and Some of Their Applications*; Tanner, E.A., Wilson, R., Eds.; Kluwer Academic Publishers: Norwell, MA, USA, 1994; pp. 301–313.
16. Voros, A. Airy function—Exact WBK results for potential of odd degree. *J. Phys. A Math. Gen.* **1999**, *32*, 1301–1311. [CrossRef]
17. Wick, G.C. Properties of Bethe-Salpeter Wave Functions. *Phys. Rev.* **1954**, *96*, 1124–1134. [CrossRef]
18. Truong, T.T. Weyl quantization of anharmonic oscillators. *J. Math. Phys.* **1975**, *16*, 1034–1043. [CrossRef]
19. Hallnas, M.; Langmann, E. A product formula for the eigenfunctions of a quartic oscillator. *J. Math. Anal. Appl.* **2015**, *426*, 1012–1025. [CrossRef]

20. Forbes, B.J.; Pike, E.R.; Sharp, D.B. The acoustical Klein-Gordon equation: The wave-mechanical step and barrier potential functions. *J. Acoust. Soc. Am.* **2003**, *114*, 1291–1302. [CrossRef] [PubMed]
21. Berezin, F.A.; Gelfand, I.M. Some remarks on the theory of spherical functions. *Tr. Mosk. Mat. Obs.* **1956**, *5*, 311–351.