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Comparing Quantum Gravity Models: String Theory, Loop Quantum Gravity, and Entanglement Gravity versus $SU(\infty)$-QGR

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Abstract: In a previous article we proposed a new model for quantum gravity (QGR) and cosmology, dubbed $SU(\infty)$-QGR. One of the axioms of this model is that Hilbert spaces of the Universe and its subsystems represent the $SU(\infty)$ symmetry group. In this framework, the classical spacetime is interpreted as being the parameter space characterizing states of the $SU(\infty)$ representing Hilbert spaces. Using quantum uncertainty relations, it is shown that the parameter space—the spacetime—has a 3+1 dimensional Lorentzian geometry. Here, after a review of $SU(\infty)$-QGR, including a demonstration that its classical limit is Einstein gravity, we compare it with several QGR proposals, including: string and M-theories, loop quantum gravity and related models, and QGR proposals inspired by the holographic principle and quantum entanglement. The purpose is to find their common and analogous features, even if they apparently seem to have different roles and interpretations. The hope is that this exercise provides a better understanding of gravity as a universal quantum force and clarifies the physical nature of the spacetime. We identify several common features among the studied models: the importance of 2D structures; the algebraic decomposition to tensor products; the special role of the $SU(2)$ group in their formulation; the necessity of a quantum time as a relational observable. We discuss how these features can be considered as analogous in different models. We also show that they arise in $SU(\infty)$-QGR without fine-tuning, additional assumptions, or restrictions.

Keywords: quantum gravity; quantum cosmology; symmetry

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1. Introduction and Results

Several fundamental questions about gravity and spacetime are not still answered by general relativity or by various attempts to find a consistent quantum description for gravitational interaction. The most daunting of these issues is the dimension of spacetime, which is usually considered to be the observed (3+1) without any explanation for its origin. Moreover, general relativity and Einstein gravity do not specify the nature of spacetime, except that it is curved in the presence of matter and energy. Most QGR models treat spacetime as a physical entity, which despite being coupled to matter, has an independent existence. Indeed, often quantization of gravitational interaction, which is necessary in a Universe with quantum matter [1], is interpreted as an inevitability of a quantized spacetime. There are, however, multiple pieces of evidence against this conclusion:

- It has been demonstrated [2] that Einstein's equation can be obtained from the second law of thermodynamics and the holographic principle—that is, the proportionality of entropy inside a null (light-like) surface to its area rather than volume [3–6]. Holographic behavior has been also observed in many-body systems with negligible gravity [7,8]. These observations confirm the conclusion of [2] that Einstein's equation should be considered an equation of state. This interpretation and universality of gravitational interaction imply that what is perceived as `space` and its geometrical properties, such as distance and curvature, represent the state of its matter content. Thus, it seems that spacetime and matter are inseparable aspects of the same physical reality/entity.

- Even without the holographic principle, the fact that the energy–momentum tensor of matter—the source of gravitational interaction—depends on the spacetime metric means that spacetime and matter are more intertwined than, for instance, bosonic gauge fields and their matter source in Yang–Mills models.
- In quantum field theory (QFT) spacetime or its dual energy–momentum mode space (but not both at the same time) are used as `indices` to keep track of the continuum of matter and radiation. The fact that in a quantum realm the classical vacuum—the apparently empty space between particles—can be described as a sea of virtual—off-shell—quantum states [9] means that we could completely neglect the physical space—the perceived 3-dimensional space. This would be possible if we could identify, tag, and order all real and virtual particles, for instance, by using the strength of their mutual quantum entanglement [10–14] or interaction strength [15]. In this view, the classical Einstein equation could be interpreted as an equation of state, which dynamically modifies parameter (index) space according to variations of interactions and entanglement between particles, and with respect to a relational quantum clock [16].
- It is useful to remind that in most QGR models the dimension of spacetime is considered as a parameter and little attempt is made to explain why it has the observed value.

In the last decade or so progress in quantum information theory has motivated the construction of QGR models which are not based on the quantization of a classical theory. They are sometimes called `Quantum First` models in the literature [17]. In addition, progress in quantum information has highlighted the crucial role of the division of the Universe in parts—subsystems—and thereby, the necessity for a proper mathematical definition of what can be considered as a distinguishable quantum (sub)system. This concept has special importance for gravity, because as far as we know from general relativity, it is a universal force, coupling everything to the rest of the Universe. Indeed, we will see later in this work that some QGR models struggle to find a naturally factorized—tensor product—Hilbert space in which each factor can be considered as presenting the Hilbert space of a subsystem. In QFTs without gravity, subsystems are particles /fields or their collections. A priori, the same concept can be applied to QGR. However, in the strong coupling limit of QGR spacetime/gravity and matter may be indistinguishable. Therefore, it is necessary to have a physically and mathematically well defined description of what may be called a distinguishable subsystem of the Universe. We also remind that the tensor product of Hilbert spaces of subsystems is not only important for gravitational interaction, but also for meaningful definition of locality, quantum clocks, quantum information flow and relative entropy, renormalization flow, and holographic properties of states. None of these concepts would make sense without mathematical and physical notion of distinguishable subsystems.

In [18,19] we proposed a model for a quantum Universe which can be placed in the quantum first category. Here we call this model $SU(\infty)$-QGR. It is briefly reviewed in Section 2. $SU(\infty)$-QGR is a fundamentally quantum model, in the sense that its axioms come from quantum physics and its formulation is not a quantized version of a classical model. It does not include in its foundation, neither explicitly nor implicitly, a background spacetime or ingredients from Einstein general relativity, such as an entropy–area relation. It is shown that both spacetime and Einstein's equation emerge from quantum properties. The physical space is identified with the space of indices parameterizing Hilbert and Fock spaces of the Universe and their subsystems. Einstein's equation presents the projection of relational evolution of subsystems on the parameter space—the only observable when experiments do not have sufficient sensitivity to observe the quantum field of gravitational interaction.

In other quantum first proposals usually a background spacetime is implicitly present in their axioms. Examples of such models are those described in [12,15]. There is also implicit assumption of a physical space in models based on the holographic principle—a hypothesis inspired by semi-classical general relativity [20,21], such as [13,14]. Indeed, it is obvious that holography without a geometrical space is meaningless. By contrast, in $SU(\infty)$-QGR the classical physical space and time genuinely emerge from quantum structure of the model and the assumption that any physical entity must be inside the Universe.

Quantum first QGRs and other modern approaches to QGR at first sight seem very different from each other. However, the history of science is full of cases where seemingly different theories and interpretations were finally turned up to present the same physical concept viewed from different perspectives. The best examples are Schrödinger's wave mechanics and Heisenberg's matrix mechanics approaches to quantum mechanics, which were later proved to be equivalent. For this reason, any new theory should look for what it has in common with other relevant models, and what new concepts or interpretations it is proposing. Such verification is particularly necessary for new QGR proposals, because it has been under intensive investigation for close to a century. Moreover, given the fact that at present none of the proposals is fully satisfactory or has observational support, a better understanding of common aspects of different QGR candidates may provide a direction and path to further developments, and eventually to the true model, unless all the proposals are completely irrelevant.

In this work we compare $SU(\infty)$-QGR with some of popular approaches to QGR, namely: symplectic models, including loop quantum gravity (LQG) and related models; string theory and its closely related matrix models (M-theory) and Anti-de Sitter–Conformal Field Theory (AdS/CFT) duality—more generally gauge-gravity duality; and models based on the holographic principle and quantum entanglement. Although our purpose is to find similarities and analogous features of these models, this investigation also clarifies their principle differences, which may be equally useful for further theoretical development, and eventually for discriminating or constraining them by experiments and observations.

We do not consider more traditional approaches, such as canonical quantization [22–24] (see, e.g., [25,26] for reviews) and the ADM 3+1 method [27]. After decades of research, it is now clear that they do not lead to a consistent and renormalizable theory. Other models omitted here are QGR models based on non-commutative spacetimes; models based on the quantum history interpretation of quantum mechanics; and causal sets. These models are based on postulates that fundamentally deviate from those of $SU(\infty)$-QGR, and their comparison with the latter is meaningless.

We begin by presenting a summary of the results of this work in Section 1.1. Its purpose is to provide a quick overlook of comparison results. Therefore, if there are unclear points, the reader should refer to Section 3 for more explanation. In Section 2, and its subsections we briefly review $SU(\infty)$-QGR and show that the common features of QGR models arise naturally and without fine-tuning or additions of new assumptions to the initial axioms. Details of comparisons between models are discussed in Section 3. For each model we first briefly remind its main features. Then, we compare them with those of $SU(\infty)$-QGR. It is obvious that detailed and technical descriptions of models and their variants, about which in some cases thousands of papers and numerous text books are written, is out of the scope of the present work. The aim of short reminders here is to introduce features and notation useful for comparisons with $SU(\infty)$-QGR. Section 3.1 reviews several background independent QGR models, including the Ponzano–Regge model and LQG. Quantum first models are reviewed and compared with $SU(\infty)$-QGR in Section 3.2. We compare string and M-theories, and gauge-gravity duality conjecture with $SU(\infty)$-QGR in Section 3.3. A short outline is given in Section 4.

1.1. Summary of Comparison Results

From comparison of the $SU(\infty)$-QGR proposal with some of other approaches to QGR, we recognized a series of similar aspects, symmetries, and structures, which despite their different roles and interpretations in different models, can be considered as analogous and common. Here we should emphasize that what we call `similarity` or `analogy` should not be interpreted as one-to-one correspondence. For instance, decomposition of $SU(\infty)$ to $SU(2)$ factors in $SU(\infty)$-QGR is not the same operation as discretizing space to tetrahedra weighed by spins on their edges. Nonetheless, they have analogous mathematical descriptions—in this case a spin network. If the QGR models reviewed in this work contain at least some of the features and properties of the true theory, they should be, most probably, reflected in these common or analogous characteristics.

The common features that we found in the models investigated in Section 3 are summarized in the following subsections.

1.1.1. Presence of 2-Dimensional Spaces or Structures in the Construction of Models

In some QGR models 2D spaces are used to construct a quantized space. They are either 2D boundaries of a symplectic representation of the 3D physical space, consisting of connected tetrahedra with nonzero curvature at vertices, or 2D worldsheets/membranes embedded in a multi-dimensional space. These structures are treated as fundamental objects of the models—analogous to particles in QFTs without gravity—and despite significant differences in their interpretation in different models, they have a crucial role in the generation of what is perceived as spacetime and gravity. In these models, 2D structures are usually postulated and considered as physical entities. In this respect, $SU(\infty)$-QGR is an exception, because diffeo-surfaces emerge from axioms and symmetries, and are considered as properties rather than being physical objects. Notice that the issue of what makes an abstract entity a `physical` object is rather philosophical. In practice, in mathematical formulations of physical phenomena, all entities are abstract but related to what can be measured. Thus, in this sense they can be considered as `physical`.

The extended nature of 2D structures has a crucial role in making QGR models renormalizable and in preventing singularities. In $SU(\infty)$-QGR this property is reflected in the fact that by definition a diffeo-surface cannot be shrunk to a point; otherwise, $SU(\infty)$ symmetry would be represented trivially. Another rationale behind the emergence of 2D structures in QGR models is the algebraic relation between diffeomorphism of 2D surfaces (2D gravity), Virasoro algebra, and Kac–Moody algebra of 2D conformal transformations. The fact that Virasoro algebra is a subalgebra of $su(\infty)$ algebra establishes their connection with the symmetry of gravitational sector in $SU(\infty)$-QGR.

1.1.2. Decomposition to an Algebraic Tensor Product

The Universe is a composite system, and by definition, the Hilbert space of composite quantum systems is decomposed to a tensor product of the Hilbert spaces of their subsystems [28]. Therefore, it is normal that an algebraic tensor product structure emerges, in one way or another, in the construction of QGR models. However, the most crucial tensor product structures in background independent models, such as LQG and related models, are spin-networks associated to the symplectic geometry and quantization of space. Specifically, their Hilbert space consists of all embedding of spin-weighted graphs—spin networks—generating the symplectic geometry states - the classical physical space can be considered as being the outcome of measurements. For this reason, tensor products in spin networks cannot be interpreted as division of space into separable subsystems. This aspect is also shared by quantum first models based on the entanglement.

In string and matrix theories, tensor products emerge as decomposition to compactified and non-compactified fields or as special configurations of string condensate in the form of D-branes (Moyal–Weyl solutions). This process can be interpreted as regarding spacetime and particles/matter fields as separate subsystems. Of course one may consider ensemble of strings, or more generally membranes or their presentation as large matrices as subsystems. However, these structures live in a higher dimensional space. In perturbative formulation, this space has to be flat, and it is not clear how strings/membranes interactions can generate gravity. In non-perturbative M-theory and matrix formulation, strings/membranes are `frozen` in a brane condensate in order to explain the observed (3+1)D spacetime. Their quantum fluctuations are treated similar to fields in QFTs, in which `particles` (modes) are fundamental subsystems. In these models, the fundamental $D = 10$ dimensional background space is static and unobservable.

1.1.3. $SU(2)$ Group and Spin Network

$SU(2)$ symmetry and/or its representations have a special role in most QGR models. In particular, they intervene in the construction of quantized geometry, because $SU(2) \cong SO(3)$ is the coordinate rotation symmetry of the physical space. Exceptions are string theory and $SU(\infty)$-QGR. Although $SU(2)$ group and its representations are extensively used in the formulation of $SU(\infty)$-QGR, it remains a purely mathematical utility without prior connection to the structure of classical spacetime.

1.1.4. A Hidden or Explicit $SU(\infty)$ Symmetry

In models based on the symplectic construction of space the number of cells—usually tetrahedra—has to be considered to go to infinity to obtain a continuum at large distance scales—low energies. As these cells are indistinguishable from each others, the Hilbert space and dynamics of these models is invariant under $SU(\infty)$ group defined on $\mathbb{R}$, rather than $\mathbb{C}$ considered in $SU(\infty)$-QGR.

String-gauge duality (M-theory) conjecture [29–31] identifies Yang–Mills models with large number of colors N_c with the string states. For $N_c \to \infty$, states of the Yang–Mills theory represent $SU(\infty)$ symmetry. Indeed, in matrix model implementation of string-gauge duality conjecture, the fundamental objects are $N \times N|_{N\to\infty}$ matrices. The $SO(D)$ symmetry of the fundamental $D = 10$ dimensional space according to string theory can be interpreted as a special case of internal symmetry G in $SU(\infty)$-QGR. However, despite these similarities, interpretation of matrices in M-theory and $SU(\infty)$-QGR are very different. In addition, matrix models do not explore $SU(\infty)$ symmetry and only use large matrices as representation of strings worldsheets or membranes in a special state of the fundamental background spacetime. Indeed, a large random square matrix can be interpreted as a symplectic presentation of a surface such that the column and row indices tag vertices of the symplectic structure. By contrast, in $SU(\infty)$-QGR no special configuration is necessary to explain the observed (3+1)D spacetime. The tensor product $SU(\infty) \times G$ provides mathematical requirements for division to subsystems [28] and grantees the existence of a perturbative expansion for both gravitational and matter sectors, without any constraint on the finite rank internal symmetry G. On the other hand, as $SU(\infty) \times G \cong SU(\infty)$, the model has also a non-perturbative limit.

1.1.5. Emergence of Time and Evolution as Relative and Relational Phenomenon

A relational clock and its associated time parameter are necessary in many QGR approaches, including $SU(\infty)$-QGR, but not in string theory in which time, space, and matter are treated in a same manner and are included in the foundation of the model.

The common properties summarized in this section demonstrate that despite their apparent differences, QGR candidates have many shared aspects. In particular, these features arise naturally and straightforwardly in $SU(\infty)$-QGR. Nonetheless, this model is a new proposal, and much more must be done and understood about it before it can be considered as a genuine contender for a consistent and testable quantum gravity model. Specifically, its explanation and predictions for phenomena in which QGR may be important such as the puzzle of black hole information loss (see the Outline Section 4 for a longer list), and its predictions for future experiments seeking the detection of decoherence by quantum gravitational interactions should be investigated.

2. A Brief Review of $SU(\infty)$-QGR

In this section, we briefly summarize axioms, structure, and constituents of $SU(\infty)$-QGR. Only mathematical formulations necessary for comparisons with other QGR models are presented here.

2.1. Axioms and Algebra

The $SU(\infty)$-QGR is based on three well motivated assumptions:

1. Quantum mechanics is valid at all scales and applies to every entity, including the Universe as a whole;
2. Every quantum system is described by its symmetries and its Hilbert space represents them;
3. The Universe has infinite number of independent degrees of freedom, that is, mutually commuting observables.

In these axioms and through this article the `Universe` means the ensemble of everything causally or through its quantum correlations observable. Independent quantum observables correspond to mutually commuting hermitian operators applied to the Hilbert space, and their subspace is homomorphic to the Cartan subspace of the symmetry group of the quantum system [32].

These axioms might seem trivial and generic. Here we briefly argue that they are not:

Axiom 1 is not trivial because some QGR models extend or restrict quantum mechanics and/or QFT in order to accommodate QGR; see Section 3.2.1 for a brief review of some of these models. As there is no spacetime in the above axioms, we also remind that QFT is not a model by itself and does not necessarily need to be defined in a spacetime. It is a formulation of quantum mechanics suitable for studying many-body systems parameterized by continuous variables, such as a Lorentz invariant spacetime.

Axiom 2 is added to the above list because in postulates of quantum mechanics, as defined by Dirac [33] and von Neumann [34], the Hilbert space is an abstract Banach space and no relation to symmetries is explicitly mentioned. Axioms of quantum mechanics with symmetry as a foundational concept are described in [32]. Of course, in practice the Hilbert space is chosen such that it represents symmetries of the quantum system. However, this is due to the fact that the choice of Hilbert space is motivated by the configuration space of the classical limit and its symmetries. If we want to construct a fundamentally quantum model without referring to a corresponding classical system, we must specify how the Hilbert space should be defined.

Axiom 3 defines the symmetry of the system—the Universe—which as explained above is the basis for determining other properties of the system. Of course, QFTs by definition have infinite number of observables/degrees of freedom, one or more at each point of the spacetime. However, in $SU(\infty)$-QGR there is no spacetime and the model is constructed as an abstractly and is defined exclusively by its symmetry and its representation by the Hilbert space. Therefore, this axiom is essential and far from being trivial.

In the hindsight, the simplicity of these axioms is their advantage, and in the following subsections we briefly review what can be concluded from these apparently generic assumptions. Recalling that at present we do not have any observed evidence of quantum gravity, sophisticated and designed axioms of some QGR models look rather imaginative, and one wonders why nature should have selected them among many other possibilities.

2.2. Representation of the $SU(\infty)$ Group and Hilbert Space

Axiom 3 of the model means that the Hilbert space of the Universe $\mathcal{H}_U$ is infinite dimensional and represents $SU(\infty)$ symmetry group, that is:

$$\mathcal{B}[\mathcal{H}_u] \cong SU(\infty) \tag{1}$$

where the sign $\cong$ means homomorphism and $\mathcal{B}[\mathcal{H}_u]$ is the space of bounded linear operators acting on $\mathcal{H}_U$. Generators $\hat{L}_{lm},\ l \geqslant 0,\ |m| \leqslant l$ of $\mathcal{B}[\mathcal{H}_u]$ satisfy the Lie algebra:

$$[\hat{L}_{lm}, \hat{L}_{l'm'}] = i f^{l''m''}_{lm,l'm'} \hat{L}_{l''m''} \tag{2}$$

where structure coefficients $f_{lm,l'm'}^{l''m''}$ can be determined using properties of spherical harmonic functions; see, e.g., [35] for more details. The reason for this property is that $SU(\infty)$ can be decomposed to tensor products of $SU(2)$:

$$\hat{L}_{lm} = \mathcal{R} \sum_{i_\alpha=1,\,2,\,3,\alpha=1,\cdots,l} a_{i_1,\cdots i_l}^{(m)} \sigma_{i_1} \cdots \sigma_{i_l}, \qquad (l,m) \mid l = 0,\cdots,\infty; -l \leqslant m \leqslant +l \quad (3)$$

where σ_{i_α}s are $N \to \infty$ representations of Pauli matrices [35] and $\mathcal{R}$ is a normalization constant. Coefficients $a^{(m)}$ are determined from expansion of spherical harmonic functions with respect to spherical description of Cartesian coordinates [35].

The model is quantized using dual of its Hilbert space $\mathcal{H}_U^*$ and its space of bounded linear operators $\mathcal{B}[\mathcal{H}_U^*]$:

$$[\hat{L}_a, \hat{J}_b] = -i\delta_{ab}\hbar, \qquad \hat{J}_a \in \mathcal{B}[\mathcal{H}_U^*], \quad (4)$$

where $\hbar$ is the Planck constant.

It is known that $SU(\infty)$ is homomorphic to the area preserving diffeomorphism of compact 2D surfaces [35–39]. From now on we use the shorthand name `diffeo-surface` for the surfaces whose area preserving diffeomorphism is homomorphic to $SU(\infty)$ of interest. Diffeo-surfaces with different genus correspond to non-equivalent (non-isometric) representations of $SU(\infty)$ [38,39]. These surfaces, and thereby $\mathcal{B}[\mathcal{H}_U] \cong SU(\infty)$, are parameterized by two angular parameters $(\theta,\ \phi)$. On the other hand, $su(\infty)$ algebra is homomorphic to Poisson bracket of spherical harmonic functions, which for $\hbar = 1$ and dimensionless operators can be written as:

$$\hat{L}_{lm} = i\left(\frac{\partial Y_{lm}}{\partial \cos\theta}\frac{\partial}{\partial \phi} - \frac{\partial Y_{lm}}{\partial \phi}\frac{\partial}{\partial \cos\theta}\right) = i\sqrt{|g^{(2)}|}\,\epsilon^{\mu\nu}(\partial_\mu Y_{lm})\partial_\nu, \quad \mu,\ \nu \in \{\theta,\phi\} \quad (5)$$

$$\hat{L}_{lm} Y_{l'm'} = -i\{Y_{lm},\ Y_{l'm'}\} = -i f_{lm,l'm'}^{l''m''} Y_{l''m''} \quad (6)$$

$$\{\mathtt{f},\ \mathtt{g}\} \equiv \frac{\partial \mathtt{f}}{\partial \cos\theta}\frac{\partial \mathtt{g}}{\partial \phi} - \frac{\partial \mathtt{f}}{\partial \phi}\frac{\partial \mathtt{g}}{\partial \cos\theta}, \quad \forall\, \mathtt{f},\ \mathtt{g} \quad (7)$$

In this representation of $\mathcal{B}[\mathcal{H}_U]$, vectors of the Hilbert space $\mathcal{H}_U$ are complex functions of the angular parameters $(\theta,\ \phi)$. If $\hat{L}_{lm}$ (or equivalently $\hat{J}_{lm}$) operators are normalized by a constant factor proportional to $\frac{i\hbar}{cM_P}$, where M_P is a mass scale—presumably Planck mass—the right-hand side commutation relation (2) becomes zero for $\hbar \to 0$ or $M_P \to \infty$ and the algebra of observables becomes Abelian, as in the classical mechanics. Thus, only when $\hbar \neq 0$ and $M_P < \infty$ the model presents a quantum system. This property establishes an inherent relationship between gravity and quantumness, as suggested in [40].

$SU(2)$ in $SU(\infty)$-QGR

The symmetry group $SU(2) \cong SO(3)$ has a special place in many QGR models, because it corresponds to the rotation symmetry of the physical space. Its importance in $SU(\infty)$-QGR is, however, for another reason: it is used for Cartan decomposition [41] of $SU(\infty)$ and descriptions of its representations [35–39]. In particular, it allows one to expand members of the $SU(\infty)$ group as linear functionals of spherical harmonic functions, analogous to an infinite spin chain. In this way, generators of $SU(\infty)$ can be characterized by spin quantum numbers (l,m). In analogy with the decomposition of Fourier modes of fields in QFT, this representation is more suitable for applications such as constructing states and solving field equations, than the abstract complex functions of two angular parameters (θ,ϕ). Nonetheless, one can easily transform one representation to the other; see, e.g., appendices in [18] for details of such transformation.

We should emphasize that despite the importance of $SU(2)$ for $SU(\infty)$-QGR, it is not anything more than a mathematical tool. In fact, using the relation:

$$SU(N) \supseteq SU(N-K) \otimes SU(K) \quad (8)$$

the group $SU(\infty)$ can be decomposed to tensor products of any $SU(K)$, $K < \infty$ by repeated application of (8). Decomposition (3) corresponds to the case of $K = 2$. It is the smallest non-Abelian special unitary group which can be used in the Cartan decomposition of $SU(\infty)$.

2.3. Subsystems of the Universe

In [18], it is shown that the quantum Universe, as defined in the previous section, is static and trivial. This is not a surprise, because there is no time parameter or subsystem which plays the role of a quantum clock. On the other hand, according to a corollary in a description of quantum mechanics in which symmetry is considered to be foundational [32,42], this quantum system must inevitably be decomposable to subsystem. To this end, the Hilbert space must be factorized such that subsystems satisfy conditions defined in [28]. They include, among other things, factorization of the system's symmetry group and its representations. Using properties of $SU(\infty)$ [19], in particular its multiplication [39]:

$$(SU(\infty))^n \cong SU(\infty) \quad \forall\, n > 0 \tag{9}$$

in [18,19], it is demonstrated that Hilbert spaces of subsystems have the general form of:

$$\mathcal{B}[\mathcal{H}_s] \cong SU(\infty) \times G \tag{10}$$

where $\mathcal{H}_s$ indicates the Hilbert space of a subsystem and G is a finite rank symmetry group. The presence of internal symmetries in the Standard Model of particle physics is the main motivation for the existence of G. Other motivations are discussed in [18]. Like any other quantum system, observables of the subsystem defined by (10) are hermitian members of $\mathcal{B}[\mathcal{H}_s]$.

The Hilbert space of all subsystems is the tensor product of representations of the symmetry of subsystems (10). Using (9), the Hilbert space of the ensemble of subsystems is:

$$\mathcal{B}[\bigotimes_s \mathcal{H}_s] \cong \Big(SU(\infty) \times G\Big)^{N\to\infty} \cong SU(\infty) \times G^{N\to\infty} \tag{11}$$

As $G^{N\to\infty} \cong SU(\infty)$, (11) is consistent with (1). Moreover, (11) shows that $SU(\infty)$ factor of the Hilbert space can be considered as common to all subsystems. Thus, it has a role analogous to that of classical spacetime for all entities in the Universe.

2.3.1. Parameter Space of Subsystems

In addition to the emergence of an `internal` symmetry, the division of this quantum Universe induces a size or more precisely an area scale. Indeed, although the preserved area of one diffeo-surface is irrelevant for its diffeomorphism as representation of $SU(\infty)$ group, it becomes important when parameter spaces of multiple systems with this symmetry, including the Universe as a whole, are compared. This is analogous to comparing finite intervals on a line with each other. An infinite line alone is scale invariant. However, lengths of finite intervals can be compared with each others. This operation induces a length scale for the finite intervals, and thereby for the whole line. Therefore, after division into subsystems, the parameter space of $SU(\infty)$ part of their Hilbert spaces will depend on a third dimensionful parameter that we call r. It is measured with respect to a reference subsystem. Diffeo-surfaces of subsystems can be considered to be embedded in this 3D space. Notice that quantum state of a subsystem does not necessarily have a fixed r, and can be a superposition of pointer states with fixed r.

Finally, to make the above setup dynamical, the relational dynamics and evolution à la Page and Wootter [16] or similar methods—see, e.g., [43] for a review—can be introduced by selecting one of the subsystems as a quantum clock. Variations of states of other subsystems are compared with the variation of state of the clock and parameterized by a time parameter t. We interpret this 4D parameter space, which is homomorphic to $\mathbb{R}^{(4)}$ as the classical spacetime shared by all subsystems of the Universe. In addition, the Hilbert space of every subsystem has a factor representing its internal symmetry G, as shown in Equation (10). As $SU(\infty)$ and G are considered to be orthogonal, their parameter spaces and actions on the states are separable. Specifically, G transformations are performed `locally` to states $|t, r, \theta, \phi; a\rangle$, where index a represents G symmetry. These properties are similar to those of a Yang–Mills gauge field defined on the classical spacetime. Thus, considering this analogy, we identify the parameter space of the $SU(\infty)$ symmetry with the classical spacetime.

2.4. Relation to Classical Geometry and Einstein Equation

Using the Mandalestam–Tam uncertainty relation [44], a quantity proportional to the quantum fidelity of two close states ρ and $\rho_1 = \rho + d\rho$ of subsystems (except reference and clock) can be defined [18]:

$$ds^2 \equiv Q(\rho, \hat{H}) dt^2 = \text{tr}(\sqrt{d\rho}\sqrt{d\rho}^\dagger), \qquad Q(\rho, \hat{H}) \equiv \frac{1}{2}|\text{tr}([\sqrt{\rho}, \hat{H}]^2)| \tag{12}$$

where $\hat{H}$ is a Hamiltonian operator that generates the evolution of subsystems with respect to a selected quantum clock and its associated time parameter t, in a time interval dt. Notice that here we have assumed that internal symmetry states of ρ and ρ_1 are the same. We also remind that integrating out reference and clock subsystems makes the states of other subsystem mixed, and they should be treated as open quantum systems [45].

The infinitesimal quantity ds is a scalar of both the Hilbert space of subsystems and its parameter space. Due to the similarity of ds to affine separation in Riemann geometry in the rest frame of subsystems, we can identify the two quantities up to an irrelevant normalization constant. Then, in an arbitrary reference frame of the parameter space, ds can be expanded as:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{13}$$

where x^μ is a point in the parameter space and $g_{\mu\nu}$ is metric of the parameter space at x^μ. Using the Mandalestam–Tam inequality, in [18] it is proved that the signature of metric $g_{\mu\nu}$ must be negative. Notice that the presence of a trace operator in the right-hand side of (12) means that its left-hand side is independent of the reference frame of the parameter space. This can be proved by expanding operators ρ and $d\rho$ in an arbitrary basis $|t, r, \theta, \phi\rangle$ of the Hilbert space and calculating the trace in (12)—tracing amounts to integration over parameters (t, r, θ, ϕ). Thus, ds is independent of spacetime parametrization, and coordinates x^μ in (13) should be considered as representative or average parameters of the quantum state ρ. In general relativity, integration over the affine displacement ds generates the world line of the system in the spacetime. Quantum systems do not follow a path in the classical phase space. Nonetheless, the world line generated by integration of ds defined in (12) or (13) for quantum subsystems can be interpreted as a projection of the averaged path of the state in the Hilbert space into its parameter space—the spacetime.

These properties of $SU(\infty)$-QGR show that we have to find a pure quantum definition for an extreme object such as a black hole, because its general relativity definition through its metric is highly degenerate and does not define its quantum state. Moreover, spacetime singularities of classical black holes may be irrelevant when they are considered as a many-body quantum state. Equations (12) and (13) are obtained from uncertainty principle [18]. Thus, similar to quantum mechanics, the singularity of metric (13) can be interpreted as an indistinguishability of states, or in other words infinite uncertainty. In any case, this important topic needs more investigation once a proper quantum definition of a black hole is found. For instance, this can be a quantum gravitational bounded state of the Lagrangian functional (21) obtained in Section 2.5.2.

Notice that here t and r are considered as classical values. More generally, measurements performed on the clock and on the subsystems relative to a reference to determine r do not need to be projective. Such cases are intensively studied in the literature for general quantum systems [43,45], and we leave their application to $SU(\infty)$-QGR to future works.

Lorentz Invariance of the Parameter Space

A remark about Lorentz invariance of the parameter space is in order. In (13) this property is manifest. However, given the different origins of time, distance, and angular coordinates in $SU(\infty)$-QGR and their interpretation as classical spacetime, the question arises of whether their $\mathbb{R}^{(3+1)}$ space is Lorentz invariant. The answer to this question is positive for the following reasons:

- Choices of a quantum clock and a reference subsystem for comparisons between diffeo-surfaces are arbitrary. Changes in these choices amount to changing the corresponding parameters.
- Division into subsystems is not rigid and may change with a change of clock and reference subsystem such that they respect necessary conditions defined in [28]. Thus, changing t and r in general lead to modification of $SU(\infty)$ parameters (θ, ϕ), and each of the new parameters (t', r', θ', ϕ') would be a function of old parameters (t, r, θ, ϕ).
- By definition, the ensemble of subsystems must generate the static 2D Universe irrespective of how subsystems are defined and parameterized. This condition imposes Lorentz and diffeomorphism invariance on the parameter space—the spacetime.

2.5. Evolution

In this section, we briefly review dynamics of the Universe as a whole and after its division to subsystems. We begin by constructing a symmetry-invariant functional for the whole Universe and then its modification when subsystems are taken into account.

2.5.1. The Whole Universe

According to the symmetry description of quantum mechanics foundations [32], the Universe as a whole is static and in a sort of equilibrium state. Specifically, using properties of $SU(N)$ groups, a $SU(\infty)$-invariant functional consisting of elements of $\mathcal{H}_U$ and $\mathcal{B}[\mathcal{H}_U]$ has the following form:

$$\mathcal{L}_U = \int d^2\Omega \bigg[\frac{1}{2}\sum_{a,\, b} L^*_a(\theta,\phi) L_b(\theta,\phi) \mathrm{tr}(\hat{L}_a \hat{L}_b) + \frac{1}{2}\sum_a \Big(L_a(\theta,\phi)\mathrm{tr}(\hat{L}_a \rho) + C.C\Big)\bigg] \quad (14)$$

$$d^2\Omega \equiv \sqrt{|g^{(2)}|}\, d(\cos\theta) d\phi \quad (15)$$

where $a = (l, m)$ or (θ', ϕ'), as explained in [18]. C-number amplitudes L_a determine the contribution of $SU(\infty)$ generators $\hat{L}^a$ in the dynamics. We remind that integration over angular coordinates of the diffeo-surface is part of the tracing operation, because in (5) generators of the $SU(\infty)$ symmetry are defined independently at each point of the diffeo-surface. Notice that the functional $\mathcal{L}_U$ includes only the lowest order of invariant traces of $SU(\infty)$. Higher order invariants trace over multiplication of several generators. However, their values are not independent of those used in (14) and structure coefficients. Using only the lowest order nonzero traces makes $\mathcal{L}_U$ equivalent to the classical Lagrangian in QFT, despite the fact that $SU(\infty)$-QGR is not related to a classical model. In analogy with QFT, a path integral can generate higher order terms.

By definition, the whole Universe is in a pure state, because there is nothing outside it, which could have been possibly traced out. Therefore, its density matrix can be written as $\rho = |\Psi\rangle\langle\Psi|$. In [18], it is explicitly shown that, as expected, applying variational principle with respect to amplitudes L_a and components of the density matrix ρ to $\mathcal{L}_U$ leads to a vacuum state as the equilibrium solution.

The action $\mathcal{L}_U$ is a formal description. In particular, it does not clarify how amplitudes L_as and density operator ρ should change under application of $SU(\infty)$ group and reparametrization of diffeo-surface to preserve $\mathcal{L}_U$. This subject is described in detail in [19]. As its results are crucial for the interpretation of the model as QGR and establishment of its relation with classical gravity, here we review them in more details.

We first remind that the surface element $d^2\Omega$ in (14) is invariant under reparametrization of angular coordinates. Thus, each term in the integrand must be reparametrization invariant. Moreover, for $SU(N)$ groups $\text{tr}(\hat{L}_a \hat{L}_b) = C_a \delta_{ab}$, where C_a is a constant. Therefore, phases of amplitudes L_a in the first term of (14) are irrelevant. In addition, in the second term ρ is hermitian, and without loss of generality generators $\hat{L}_a$ can be chosen to be hermitian too. Thus, phases of L_as are irrelevant and L_as can be considered to be real `fields` defined on the 2D diffeo-surface.

Amplitudes L_a must be invariant under translation $\theta \rightarrow \theta + \theta_0$, $\phi \rightarrow \phi + \phi_0$ for arbitrary constant shift of the coordinates at origin by θ_0 and ϕ_0, and rigid rotation of the frame. This means that L_a must have a differential form with respect to coordinates (θ, ϕ). Considering, in addition, the invariance under non-commutative $SU(\infty)$ symmetry, we find that the first term in (14) should have the form of a 2D Yang–Mills Lagrangian for $SU(\infty)$. Thus, $\mathcal{L}_U$ can written as [18]:

$$\mathcal{L}_U = \int d^2\Omega \left[\frac{1}{2}\,\text{tr}(F^{\mu\nu}F_{\mu\nu}) + \frac{1}{2}\text{tr}(\not{D}\rho) \right], \qquad \mu,\nu \in \cos\theta, \phi \tag{16}$$

$$F_{\mu\nu} \equiv F^a_{\mu\nu}\hat{L}^a \equiv [D_\mu, D_\nu], \quad D_\mu = (\partial_\mu - \Gamma_\mu)\mathbb{1} - \sum_a A^a_\mu \hat{L}^a, \tag{17}$$

$$F^a_{\mu\nu}F^{\mu\nu}_a = L^*_a L^a, \quad \forall a. \tag{18}$$

where D_μ is a 2D covariant and gauge preserving derivative with an appropriate 2D connection Γ_μ (in $F^{\mu\nu}$ the connection will be canceled). Exact expression of the differential operator $\not{D}$ depends on the representation of 2D Euclidean group by the state $|\Psi\rangle$. For a scalar-type state $\not{D} = \overleftarrow{D}_\mu \overrightarrow{D}^\mu$ and for a spinor-type state $\not{D} = i\sigma^0\sigma^i e^\mu_i \overleftrightarrow{D}_\mu$, where σ^i, $i = \{1,2\}$ are two of the $N \rightarrow \infty$-dimensional representation of Pauli matrices, and σ^0 is the third Pauli matrix; e^μ_i's are zweibeins (analogous to vierbein in 2D). We remind that in 2D spaces the 2-form $F^a_{\mu\nu}$ has only one independent nonzero component. Therefore, the numbers of degrees of freedom in the two sides of (18) are the same. For the same reason, the dual field $\tilde{F}^{\mu\nu}$ is the same as $F^{\mu\nu}$ up to a sign.

In Yang–Mills models, the field strength $F_{\mu\nu}$ is a gauge invariant measurable. Moreover, in (16) variation of L^as and thereby $F_{\mu\nu}$ can be compensated by a diffeomorphism transformation of the compact 2D surface, i.e., the variation of $g^{\mu\nu}$. On the other hand, up to a global rescaling of the area of the diffeo-surface, this transformation can be considered as the application of $SU(\infty)$ under which $F_{\mu\nu}$ is invariant. As we discussed before, the area of diffeo-surface of the whole Universe is not measurable. In this sense the first term in (16) is topological and can be identified, up to an irrelevant normalization constant, with the topological Euler characteristic class of the compact 2D diffeo-surface:

$$\int d^2\Omega\, \text{tr}(F^{\mu\nu}F_{\mu\nu}) \propto \int d^2\Omega\, \mathcal{R}^{(2)} = 4\pi\chi(\mathcal{M}) \tag{19}$$

where $\mathcal{R}^{(2)}$ is the 2D Ricci scalar of the parameter space compact manifold $\mathcal{M}$—the diffeo-surface. The Euler characteristic $\chi(\mathcal{M}) = 2 - \mathcal{G}(\mathcal{M})$ of compact 2D Riemann surfaces depends only on their genus $\mathcal{G}$.

The topological nature of action (16), expressed in (19) as its relation with the Euler characteristic, is not surprising, because a single indivisible quantum system is trivial [32]. In a geometrical view, if `local` details of the Universe are not distinguishable, only its global—topological—properties may characterize its states. In the present model, the relevant global property is the topology of the diffeo-surface, which determines non-homomorphic representations of the $SU(\infty)$ symmetry [38,39] of the Universe. In other words, the only possible difference between whole Universes is the representation of the $SU(\infty)$ symmetry realized by their Hilbert spaces.

Equation (19) establishes the relation between $SU(\infty)$-QGR and classical gravity. Specifically, it shows that if quantum operators $F^a_{\mu\nu}\hat{L}^a$ cannot be distinguished or observed, their overall effects are observed as variations in the geometry of the parameter space. We can also interpret the right-hand side of (19) as the projection of `dynamics` of the quantum Universe onto its parameter space.

2.5.2. Evolution of Subsystems

When the Universe is divided into subsystems and a reference subsystem and a quantum clock are selected, it is still possible to construct a $SU(\infty)$ invariant action functional similar to (16), but it will depend on two additional parameters r and t. They reflect the fact that $SU(\infty)$ symmetry is now respected not only by the whole Universe, but also by its subsystems, which have acquired a new relative observable r, interpreted as the `distance` from reference subsystem, and their relative evolution is measured by a clock parameter t. In addition, a full action must include terms invariant under the internal symmetry group of subsystems G. The formal description of this functional is [18]:

$$\mathcal{L}_{U_s} = \frac{1}{4\pi L_P^4} \int d^4x \sqrt{-g} \left[\frac{1}{4} \left(\sum_{l,m,l',m'} \mathrm{tr}(L^*_{lm}(x) L_{l'm'}(x) \hat{L}_{lm} \hat{L}_{l'm'}) + \sum_{l,m,a} \mathrm{tr}(L_{lm}(x) T_a(x) \hat{L}_{lm} \otimes \hat{T}_a) + \sum_{lm} L_{lm}\, \mathrm{tr}(\hat{L}_{lm} \otimes \mathbb{1}_G\, \rho_s(x)) + \sum_{a,b} \mathrm{tr}(T^*_a(x) T_b(x) \hat{T}_a \hat{T}_b) \right) + \frac{1}{2} \sum_a T_a\, \mathrm{tr}(\mathbb{1}_{SU(\infty)} \otimes \hat{T}_a\, \rho_s(x)) \right]. \tag{20}$$

where T^a's are generators of the finite rank internal symmetry G of subsystems, and $L_P \equiv \sqrt{\hbar G_N / c^2}$ is the Planck length. The functional $\mathcal{L}_{U_s}$ is normalization such that amplitudes L_{lm} and T_a are dimensionless. Moreover, we have used Cartesian frame for coordinates of the 4D parameter space—the spacetime—and explicitly shown the dimensionful coupling constant of $SU(\infty)$ symmetry. Following the same line of argument given for (14) about the invariance of parameter space under coordinate transformations, invariance under $SU(\infty)$ transformations, and a demonstration that the resulting action has the form of a Yang–Mills model with $SU(\infty)$ symmetry, we conclude that (20) has the form of a Yang–Mills model for $SU(\infty) \times G$ symmetry in the $\mathbb{R}^{(3+1)}$ curved parameter space—spacetime—of the subsystems:

$$\mathcal{L}_{U_s} = \int d^4x \left[\frac{1}{4} \mathrm{tr}(F^{\mu\nu} F_{\mu\nu}) + \frac{1}{4} \mathrm{tr}(G^{\mu\nu} G_{\mu\nu}) + \frac{\mathcal{M}}{2} \mathrm{tr}(\not{D} \rho_s) \right], \quad \mu, \nu \in 0, 1, 2, 3 \tag{21}$$

$$F_{\mu\nu} \equiv F^{lm}_{\mu\nu} \hat{L}^{lm} \equiv [D_\mu, D_\nu], \quad D_\mu = \partial_\mu - \Gamma_\mu - \sum_{lm} A^{lm}_\mu \hat{L}^{lm}, \quad F^{lm}_{\mu\nu} F^{\mu\nu}_{lm} = L^*_{lm} L^{lm}. \tag{22}$$

$$G_{\mu\nu} \equiv G^a_{\mu\nu} \hat{T}^a \equiv [D'_\mu, D'_\nu], \quad D'_\mu = \partial_\mu - \Gamma_\mu - \sum_a B^a_\mu \hat{T}^a, \quad G^a_{\mu\nu} G^{\mu\nu}_a = T^*_a T^a. \tag{23}$$

The dimensionful constant $\mathcal{M} \propto M_P^n$ and its dimension depends on the representation of the Lorentz group of the parameter space—spacetime—realized by subsystems states ρ_s. The expression for $\not{D}$ would be similar to the examples given in Section 2.5.1 with an additional field term for G symmetry. Equations (12) and (13) show how the metric of the parameter space is related to quantum states of the subsystems obtained from action (23).

2.5.3. Classical Limit

When experiments are not sensitive to quantum field strength $F^{\mu\nu}$ of the $SU(\infty)$ symmetry, only its effect on the geometry of the (3+1)D parameter space—the spacetime—described by (12) and (13) would be observable. Using (19), in Appendix A we show that in the classical limit, the pure $SU(\infty)$ term in (21) can be approximated by the 4D Ricci scalar $R^{(4)}$, whose integration over the 4D parameter space is no longer topological:

$$\int d^4x\, \mathrm{tr}(F^{\mu\nu} F_{\mu\nu}) \xrightarrow[\text{limit}]{\text{classical}} \propto \int d^4x R^{(4)} \tag{24}$$

This last step finalizes our demonstration that in $SU(\infty)$-QGR, Einstein's equation is a property of the parameter space characterizing the underneath quantum states of the Universe and its subsystems - matter. It confirms that Einstein's equation should be considered as an equation of state [2] and its quantization, as well as the quantization of spacetime, are meaningless. Moreover, as the quantum gravity interaction has the form of a Yang–Mills model, its effect at high energies should resemble additional gauge interactions on a curved spacetime. Therefore, it is also meaningless to talk about quantum corrections to the Einstein equation. In any case, it is well established that any change in Einstein's equation can be considered either as a change in the geometry part or in the matter part. They correspond to formulation in Jordan and Einstein frame, respectively.

We should remark that in the above formulation, it is assumed that multiple copies of the quantum clock are available for estimating the average value of an observable used to define the time parameter t. In other words, the clock is tomographically complete. This is not a necessity, and time and/or relative distance may be quantified by non-projective measurements. We leave the investigation of such general case to future works. Furthermore, we do not discuss the origin of dark energy/cosmological constant in this framework here, because it may depend on the quantum aspects of clock and reference, and the fact that after their selection the Universe must be considered as an open quantum system.

2.5.4. Spin-1 Quantum Gravitational Interaction

A note is in order about the finding that in $SU(\infty)$-QGR quantum gravity is a Yang–Mills model. This means that its mediator quantum field is a vector—in the parameter space—rather than the observed spin-2 graviton field of the classical Einstein gravity. Nonetheless, the relation (24) shows that there is no contradiction between the two observations. This is analogous to the predictions of the early models for strong interaction before discovery of the QCD model. Due to the strong coupling at low energies and confinement of constituent partons, the observations seemed to show a nonlocal and geometrically extended interaction analogous to a string. We now know that this phenomenological interpretation is wrong, and the confusion is caused by non-perturbative nature of the QCD interaction at energy scales lower than Λ_{QCD}. In the same manner, the deformation of spacetime, which in general relativity is interpreted as gravity, is generated by relative variations of quantum states of all constituents of the Universe, and the local metric and curvature of the parameter space—spacetime—present their `average` effect.

2.6. Summary of $SU(\infty)$-QGR Model and Its Properties

We conclude this section by summarizing the $SU(\infty)$-QGR model and what has been found, so far, about QGR in this framework:

- Assuming that Hilbert spaces of the Universe and its subsystems represent $SU(\infty)$ symmetry, we showed that the Hilbert space of the Universe as a whole can be parameterized by two continuous parameters. When the Universe is divided into subsystems presenting a finite rank symmetry group G, and a quantum reference subsystem and a quantum clock are chosen, two additional parameters arise: a relative distant and a relative time à la Page and Wootter or equivalent proposals.
- We interpreted the above 4D parameter space as the classical spacetime and demonstrated that its signature must be negative—i.e., it has a Lorentzian metric. Moreover, as the spacetime is a parameter space, its quantization is meaningless.
- The coordinate independent affine parameter of the spacetime is related to the variations in the quantum states of the subsystems.
- We defined two symmetry invariant functionals over the Hilbert space of the Universe as a whole, and over those of its subsystems. They play the role of action functional for the evolution of the Universe and its subsystems, respectively.
- The action for the subdivided Universe has the form of Yang–Mills gauge theories on the parameter space for both $SU(\infty)$ and subsystem specific (`internal`) finite rank

G symmetry. Thus, similar to other fundamental forces, at the quantum level the mediator boson of gravity is spin-1.

- We showed that the action functional for the whole Universe is static. Moreover, its purely $SU(\infty)$ Yang–Mills part is topological and proportional to the Euler constant. Thereby, it is proportional to integral over the 2D Ricci scalar curvature. The constant of the proportionality is not an observable.
- When the Universe is divided into subsystems, in the classical limit when the quantum Yang–Mills vector field of the $SU(\infty)$ symmetry cannot be detected, the purely $SU(\infty)$ Yang–Mills part of the action functional will be proportional to the 4D Ricci scalar curvature. Therefore, the classical limit of $SU(\infty)$-QGR is the Einstein gravity and the observed spin-2 graviton is an effective classical field.
- This important prediction should be testable with future quantum experiments, for instance, those seeking decoherence or entanglement initiated by quantum gravity.

3. Comparison with Other Quantum Gravity Proposals

In this section, we compare $SU(\infty)$-QGR with LQG and related models, string theories and related models, AdS/CFT conjecture, and several quantum first models. This list of models and their citations are far from covering all the QGR proposals and literature about them. In particular, because of fundamental differences between $SU(\infty)$-QGR and models such as non-commutative spacetimes, causal sets, and quantum gravity models based on the quantum histories, they are not compared. Nonetheless, some of these proposals are briefly mentioned because of their connections with models reviewed here.

For each model, we first remind its main assumptions and results, only for the purpose of fixing notations necessary for comparison of the model with $SU(\infty)$-QGR. We should emphasize that for the sake of briefness, various new ideas and methods added to the original construction of these models are not explored here. Given the fact that some of these proposals have been intensively under investigation for decades, a detailed review of them and comparisons with $SU(\infty)$-QGR need a more extended investigation than this article. Moreover, our goal here is finding common features of the models rather than assessing their performance. For these reasons, only the most foundational aspects and results of the models are considered and compared with those of $SU(\infty)$-QGR. We should also remind that $SU(\infty)$-QGR is a recent and under development proposal and its properties are not fully investigated. For this reason, its comparison with other QGR models is limited to what is known about it. Notably, its application to various QGR related phenomena is left to future works.

As we discussed in the Introduction, due to the close relation between gravity and geometry of spacetime in the classical general relativity and Einstein gravity, finding a quantum model for gravitational interaction has been usually considered to be equivalent to quantization of spacetime as a physical entity. A notable difference between $SU(\infty)$-QGR and other QGR models is the absence of a quantized background or quantized spacetime. This unique feature becomes fundamental when one tries to compare this model with other QGR proposals. Indeed, a direct comparison cannot be made. Therefore, the goal of this work is limited to investigating whether there are comparable or analogous features in these models. For instance, the $SU(2)$ group is present in the construction of many QGR models, including $SU(\infty)$-QGR. Our aim is to clarify the origins of these sorts of similarities, and investigate whether they are superficial and unrelated, or despite their apparent differences, they reflect deeper relations between models.

3.1. Background Independent Models

Following the failure of coordinate-dependent canonical quantization of the Einstein–Hilbert equation [22,23,46] (see, e.g., [25] for a review) and ADM (3+1)D description of Einstein's equation and its quantization [27], in 1961 Tullio Regge proposed a discrete but coordinate independent description of Einstein gravity [47]. This model is the basis of most background independent QGR models. For this reason, we briefly review it here.

3.1.1. Regge Discrete Geometry

According to this model, a curved two or higher dimensional space can be approximately considered as flat everywhere except on the triangulated 2D surfaces—two simplexes. In particular, 3D or (2+1)D curved spaces can be approximated by sticking together tetrahedra with Euclidean or Lorentzian geometry in their bulks. The deficit angle of a vertex in the bulk of space is $\varepsilon = 2\pi - \sum_f \theta_f$, where θ_f is the angle of triangle (face) f attached to a vertex v; see, e.g., [48] for a review of Regge calculus. For vertices sitting on the boundary of the symplectic surface, the deficit angle is $\varepsilon = \pi - \sum_f \theta_f$. The discretized gravity Regge action is:

$$S_{Regge} = \sum_e l_e \varepsilon_e \tag{25}$$

where index e runs over all edges, l_e is length of the edge e, and ε_e is the deficit angle of the vertex opposite to it. In Regge action, tetrahedra edges can take any positive real value.

3.1.2. Ponzano–Regge 3D QGR

In 1968, Ponzano and Regge proposed a 3D discretized quantum geometry model [49] based on the Regge action S_{Regge}. They showed that if in (25) l_es are chosen to be quantized spin, that is $l_e = j_e,\ j_e \in \{0, 1/2, 1, 3/2, \cdots\}$ and j_e's of each face satisfy triangle rule:

$$|j_1 - j_2| \leqslant j_3 \leqslant j_1 + j_2; \tag{26}$$

their 6j symbol will be nonzero and approximately equal to the cosine of Regge action.

The partition function of the Ponzano–Regge QGR is constructed from multiplication of the positive exponent of the cosine of Regge action for all tetrahedra, weighted, and summed over all configurations of spins:

$$\mathcal{Z}_{PR} = \lim_{N\to\infty} \sum_{j\leqslant N} \Lambda^{N_0}(N) \prod_{e\in S_1} (-1)^{2j_e}(2j_e+1) \prod_{t\in S_3} (-1)^{-\sum\limits_{e=1,\cdots 6} j_e} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \tag{27}$$

Ponzano–Regge discrete quantum gravity was the first evidence of a close relationship between gravity in 3D space or (2+1)D spacetime and representations of $SU(2)$ group. This relationship was later confirmed by the introduction of Ashtekar variables [50] in the framework of (3+1)D ADM formulation for quantization of gravity. In fact, as we explain in the following sections, the concept of triangulation and assigning spins to edges of triangles arises, in one way or another, in several other QGR models as well.

3.1.3. Ashtekar Variables and Loop Quantum Gravity

Loop quantum gravity (LQG) [51,52] can be considered as a continuum limit of symplectic QGR models [53]. It uses ADM (3+1)D formalism with background-independent Ashtekar variables [50]. They consist of a spin connection 2-form $\omega^a_i(x)$, defined on the product of a 3D Euclidean manifold and a $SU(2)$ group manifold—more precisely a $SU(2)$ bundle on a 3D Euclidean manifold—and triads E^i_a, such that $E^i_a E^b_j = \delta^i_j \delta^b_a$, where $i = 1, 2, 3$ is the coordinate index of the Euclidean space and $a = 1, 2, 3$ indicates generators of the $SU(2)$ symmetry group. They replace coordinates and metric as dynamical variables. In the quantized model, their dual variables are, respectively, E^i_a and gauge field $A^a_i = \omega^a_i + \gamma K^a_i$, where $K^a_i \equiv K_{ij} E^{ja} / \sqrt{|h|}$, K_{ij} is the extrinsic curvature tensor of the 3D space, h is determinant of the metric of physical 3D space, and γ is the Immirzi constant [54].

3.1.4. $SU(2)$ Symmetry, Degeneracies, and Observables in LQG

Although a metric, and thereby coordinates, are apparently present in the definition of Ashtekar variables, they do not affect geometry of space and its quantization. The reason is that space curvature is described by the $SO(3) \cong SU(2)$ transformation of a rigid frame, rather than a deformation of the metric. Specifically, the rigid frame rotates when it is transported across the curved space manifold. On the other hand, the freedom of the choice of rigid frame at each point of the 3D manifold means that its $SO(3) \cong SU(2)$ symmetry is a gauge symmetry. Thus, A_i^a and E_a^i include more degrees of freedom than $g_{\mu\nu}$ in the (3+1)D classical gravity. This is evident from counting the number of components of these fields.

To eliminate degeneracies, observables of LQG and the spin network (or foam) [55–57]—its discretized version—are quantized topological quantities generated by Wilson loops [58]. This is why the model is called `Loop` QGR, and one of its most remarkable predictions is the quantization of area [58]. This feature establishes the relation between LQG formulation using continuous Ashtekar variables, a spin network as its approximation, and the symplectic geometry of Ponzano–Regge: Quantized surfaces have non-trivial $SU(2)$ holonomy, and triangulated 3D space à la Regge becomes a manageable approximation, including essential properties of a quantized curved space with a meaningful continuum limit.

3.1.5. Analogies between the Foundations of LQG and Related Models and $SU(\infty)$-QGR

In $SU(\infty)$-QGR, conserved areas of diffeo-surfaces and their comparisons induce an area (length) scale in the model, without necessity of quantizing the physical space. Moreover, E_i^a fields are analogous to amplitudes $L_{l,m}$ in $SU(\infty)$-QGR. In fact, in [19] we show that in order to be invariant under coordinate transformations of the parameter space, these amplitudes must be differential operators in the parameter space. Indices (l,m) are analogous to the `internal` $SU(2)$ symmetry of triads. However, in contrast to Ashtekar variables, their values are obtained from ensembles of representations of $SU(2)$ factors in the decomposition of $SU(\infty)$ in Equation (3). This property is similar to Ponzano–Regge and spin networks, where edges of tetrahedra are weighed by spins. However, in $SU(\infty)$-QGR, both l and m quantum numbers of $SU(2)$ representations are involved in the action of the model, and they are not constrained. The reason is that in contrast to LQG and Ponzano–Regge models, in $SU(\infty)$-QGR the Hilbert space does not represent a real space geometry.

3.1.6. Hilbert Spaces of LQG and Related Models

6j symbols consist of summations of weighted multiplications of four Wigner 3j symbols. In turn, 3j symbols are proportional to the Clebsch–Gordan coefficients $\langle j_1, m_1; j_2, m_2 | j_3, m_3 \rangle$, where $j_1,\ j_2,\ j_3$ respect the triangle condition (26). Therefore, each term in the partition function of the Ponzano–Regge model (27) is proportional to the projection of an N-spin to a one-spin state constrained by the triangle relation (26) between `adjacent` spin states.

Considering the expansion (3) of $SU(\infty)$ group, it is clear that the Ponzano–Regge partition function $\mathcal{Z}_{PR}$ includes special configurations of a quantum system, whose Hilbert space represents $SU(\infty)$ symmetry, namely, states that can be arranged as tetrahedra in a 3D space. This observation can be extended to other models based on a symplectic representation of space, such as LQG, spin network, and Group Field Theories (GFT). Indeed, reference [59] describes the explicit construction of the Hilbert space of a single tetrahedron in LQG/spin network by associating $SU(2)$ operators to edges of the tetrahedron. The state of a unit cell of space—sometimes called an `atom of space`—is generated by application of these operators to a vacuum state, such that the projection (amplitude) of the total spin of the tetrahedron is equal to its associated 6j symbol. This procedure can be extended to ensemble of $N \to \infty$ tetrahedra content of space, which can be also considered as spin-weighted graphs [60]. Thus, we conclude that state generator operators, and thereby Hilbert spaces of discrete QGR (DQGR) models such as Ponzano–Regge and LQG models, which we collectively call $\mathcal{H}_{DQGR}$, are subspaces of the Hilbert space of a quantum system with $SU(\infty)$ symmetry, such as $SU(\infty)$-QGR.

3.1.7. Kinematical and Physical Hilbert Spaces and Reality Conditions

It is useful to remind that the fundamental representation of $SU(2)$, as well as 3j and 6j symbols are in general defined on the field of complex numbers $\mathbb{C}$. By contrast, a partition function or path integral over geometries of the physical space or spacetime, which should approach Einstein gravity in the limit of $\hbar \rightarrow 0$, must have a real value [60,61]. Moreover, due to the degeneracies discussed in Section 3.1.4, the Hilbert spaces $\mathcal{H}_{DQGR}$ of LQG and related models are not `physical`, but `kinematical` [60]. The Hilbert space of physical states $\mathcal{H}_{phys}$ containing quantized background independent geometries is a subspace of $\mathcal{H}_{DQGR}$; that is, $\mathcal{H}_{DQGR} \supset \mathcal{H}_{phys}$. However, it is in general difficult to construct $\mathcal{H}_{phys}$ explicitly [60]. In addition, demonstration of diffeomorphism and Lorentz invariance of the physical states is not straightforward, and one might expect violations of Lorentz invariance in QGR models with discretized space [62]. Indeed, the diffeomorphism invariance of DQGR is explicitly shown only for special cases [63,64].

Even DQGR/LQG models that preserve the Lorentz invariance dispersion relation of gravitational waves [65] and electromagnetic radiation [66] may deviate from general relativity. However, both of these deviations are stringently constrained [67–69]. Moreover, the Immirzi parameter may affect the interactions of fermions [70], and thereby induces a `fifth force`-type effect on matter. This effect is also constrained by various tests of gravity [71].

Complexities analogous to nonphysical states in the formulation of LQG and related models do not arise in $SU(\infty)$-QGR, because parameters defining its Hilbert space, namely, (t, r, θ, ϕ), are real. Moreover, by construction, their redefinitions—in other words, diffeomorphism of the parameter space—corresponds to changing the Hilbert space's basis by application of a unitary transformation. On the other hand, such unitary operators are members of $SU(\infty)$ symmetry group of the subdivided quantum Universe and preserve the action (21). We also notice that although $SU(2)$ symmetry is used in the construction of the Ponzano–Regge model, LQG, spin networks, GFT, and $SU(\infty)$-QGR, in practice all of them, except $SU(\infty)$-QGR, use only the Casimir operator of $SU(2)$. The reason is that the $SU(2)$ symmetry in background-independent models is related to the rotation symmetry of the physical space. On the other hand, eigen states m of the azimuthal projection of a spin induce a preferred direction—a frame—which background-independent models want to avoid. Therefore, these models effectively use only the Casimir operator, which is frame independent.

3.1.8. Time and Matter in LQG

Similar to $SU(\infty)$-QGR, in LQG and related models time must be considered as a relational observable. One way of making the model dynamic is to consider time as the classical affine parameter of histories [72,73] or path integrals in the quantized physical space [74]. Although in such setups Lorentz and diffeomorphism invariance is not trivial, it may be achievable [63,64,74].

Describing time by histories needs a `historian`—a reference subsystem with respect to which histories are defined. However, construction of background independent QGR models does not clarify how to satisfy necessary conditions for division of a quantum system [28]. In fact, kinematical Hilbert space H_{DQGR} seems to be inseparable [60]. Specifically, the division of Hilbert space to orthogonal blocks of subsystems needs an additional symmetry. The $SU(2)$ symmetry in these models cannot used for this purpose, because it is inherently related to the construction of space and gravitational interaction. We might consider tetrahedra as the most fundamental `atomic` subsystems [75]. However, to discriminate one tetrahedron as a reference, there must be a selection criterion—another symmetry and its observables (charges). This issue is directly related to the fact that LQG and related models do not consider matter fields—a symmetry orthogonal to space—in their foundations. Although a time parameter and matter fields can be easily added to the Lagrangian of gravity sector, described with respect to Ashtekar variables and their duals (see, e.g., [76]), the foundational issue of time definition in LQG and related models is not fully solved. Attempts to solve this problem, for instance, through quantization of phase space [77–79], indeed include matter and/or symmetries orthogonal to the diffeomorphism symmetry.

3.1.9. The Non-Perturbative Characteristic of LQG and Related Models

The origin of subsystem definition issue in background independent models is their non-perturbative approach to QGR. Division into subsystems needs a criterion for breaking the Hilbert space or its parameter space into distinguishable sectors. Such an operation implies the possibility of a perturbative description of the system at some scale. However, in LQG and related models, in the absence of matter there is no natural covariant rule for a quantum gravitational perturbative expansion. This observation clarifies why there is no inherent way to include matter in these models. In fact, division into subsystems, emergence of a quantum clock, inclusion of matter in the foundation of the model, and the existence of both perturbative and non-perturbative regimes are related concepts. In $SU(\infty)$-QGR they are naturally implemented in the construction of the model through special properties of the $SU(\infty)$ symmetry.

3.1.10. Outline of Comparison between Background Independent Models and $SU(\infty)$-QGR

In conclusion, although $SU(2)$ symmetry plays an essential role in the construction of background independent models and $SU(\infty)$-QGR, its roles and `raisons d'être` in these models are very different. Notably, in LQG, GFT, and other symplectic QGR models it is strictly related to the assumption of a physical 3D quantum space. Nonetheless, spin network realization of LQG can be considered as a subspace of the Hilbert space of $SU(\infty)$-QGR.

Both background independent models and $SU(\infty)$-QGR rely on the definition of a relative time or histories, which need division of the Universe into subsystems. In $SU(\infty)$-QGR, this concept is built in the construction of the model. It provides the necessary ingredients for assignment of a quantum subsystem as a clock and inclusion of matter fields in the model.

3.2. *Quantum Approaches to QGR*

Inherently quantum approaches—called `Quantum First` by some authors [15]—are relatively recent arrivals into the jungle of QGR proposals, and $SU(\infty)$-QGR can be classified in this group. For this reason it is crucial to investigate its similarities with and differences from other models in this category.

A shared characteristic of quantum first models is the absence of a classical spacetime as a foundational concept in their axioms—or at least this is the claim. Consequently, it has to emerge down the road from more primary properties and structures of an abstract quantum system. It is useful to remind that the concept of an emergent spacetime is not limited to these models. The possibility that spacetime may not be a fundamental entity is also considered by other QGR candidates as well; see e.g., [80–82]. Specifically, it is suggested that a quantum Lorentz invariant spacetime orthogonal to internal gauge symmetries may emerge in QGR models based on the extension of the Poincaré group and gauge symmetries [80,83,84]. The idea of spacetime emergence is also explored by models in which, in one way or another, thermodynamics and quantum gravity are unified [85–87]. These models seem to have little common aspects with $SU(\infty)$-QGR, and we do not discuss them further here.

In the absence of any hint about the quantum nature of gravity, for instance, its Hilbert space, and its relationships with classical gravity and other interactions, quantum first models usually use priors inspired from semi-classical gravity, in particular from properties of semi-classical physics of black holes. Based on these priors, two categories of quantum first models other than $SU(\infty)$-QGR can be distinguished:

- Models that consider locality and causality as indispensable for QGR: some of these models require modifications of standard quantum mechanics;
- Models inspired by black hole entropy and its relationship with the Yang–Mills and AdS/CFT duality.

3.2.1. Modified Quantum Mechanics and Locality

`Locality` is considered to be crucial for describing black holes, their thermodynamics [3,20], and its puzzles [21,88]. More generally, causality and observed finite speed of information propagation in both classical general relativity and QFT imply some degree of locality in any interaction, including QGR. For these reasons, locality and its close relationship with the definition of subsystems as `localized` entities in the Universe have been the motivation of authors of [15,89,90] for proposing a `generalized` quantum mechanics. Specifically, a history description of quantum mechanics [91,92] is generalized in [89] to define `coarse-grained` histories as a bundle of `fine-grained` histories (path integrals). They replace the Hilbert space of quantum mechanics, which in a QGR framework corresponds to a spacelike surface during an infinitesimal time interval, defined with respect to a reference clock. In addition, in this modified quantum mechanics, projection operators to eigen states of position are time-dependent, and during each time interval they project states to a different set of histories. In turn, sets of histories present subspaces of the bundle. Presumably, in this model not only the state of a system, but also its whole Hilbert space, changes with time.

Inspired by generalized quantum mechanics, [90] proposes an alternative way to implement locality in what is called `universal` quantum mechanics. In analogy with the bundle space of [89], it extends the space of physical states to provide additional labeling, such as `in and out` states in curved spacetimes [93]. In addition, these labels can be interpreted as time or labels of states in a multiverse, as needed. Physical states can be considered as `local` in this extended state space.

These models and other QGR proposals based on the quantum histories (see, e.g., [94,95] and references therein) have few common features with $SU(\infty)$-QGR, which is strictly based on the highly tested standard quantum mechanics. The reason for having reviewed them here is their roles in the development of further models with some similarities with $SU(\infty)$-QGR, which we will review in the following subsections.

3.2.2. QGR from Locality and Causality

The localization of quantum mechanics in [89] does not specify an explicit implementation procedure. Nonetheless, motivated by this model, [15,17,96,97] propose a road-map for realization of this concept in what they call local quantum field theories (LQFT). In these QFT models, observables convoy quantum information only locally. Here we call the corresponding QGR proposal LQFT-QGR.

In quantum systems with infinite degrees of freedom, such as in QFTs, the spacetime sector of the Hilbert space cannot be factorized to disconnected (untangled) subspaces without violating causality. Such quantum systems are said to have type III operator algebra in the classification of [98,99]. For this reason, in LQFT-QGR the division into subsystems is performed algebraically. Specifically, it is assumed that for any region of spacetime U there is an extension U_e. Observables $\hat{A}$ and $\bar{\hat{A}}$ are defined such that they have nonzero support, respectively, on U and $\bar{U}_e$, where $\bar{U}_e$ is the complementary space of U_e. Under these conditions, $\hat{A}$ and $\bar{\hat{A}}$ are assumed to be disentangled in a specific vacuum, i.e., there is a vacuum state $|0\rangle$ such that:

$$\langle U_e|\hat{A}\bar{\hat{A}}|U_e\rangle = \langle 0|\hat{A}|0\rangle\langle 0|\bar{\hat{A}}|0\rangle \quad (28)$$

The vacua $|U_e\rangle$ and $|0\rangle$ are related by a Bogoliubov transformation. This definition is considered to provide a sort of localization without factorization of the Hilbert space. However, it is evident that this algebraic structure is not in general diffeomorphism invariant, and observable operators $\{\hat{A}\}$ and $\{\bar{\hat{A}}\}$ must satisfy specific conditions to retain their invariance and physical meaning [97]. Seeking such operators, [96,97] found that in analogy with gauge invariant Wilson loops in Yang–Mills theories, diffeomorphism invariant operators $\Phi_\Gamma \in \{\hat{A}\}$ are nonlocal structures, which depend only on the spacetime connection [96]. Specifically:

$$\Phi_\Gamma(x) = \phi(x^\mu + V^\mu_\Gamma) \quad (29)$$

where Γ is a path running from point x of the spacetime to infinity, and V^μ_Γ is the integral of an expression depending on the metric of spacetime along the path Γ. An explicit expression for V^μ_Γ is obtained for the weak coupling limit of semi-classical gravity in [97].

3.2.3. Comparison of LQFT-QGR with $SU(\infty)$-QGR

In LQFT-QGR two essential concepts for QGR, namely, division of the Universe into subsystems and carriers of quantum information are considered to be the same. In this respect, the model is similar to $SU(\infty)$-QGR; that is, carriers of information are matter and radiation fields having internal symmetries, which are orthogonal to the diffeomorphism symmetry—a necessary criteria for the division of the Universe into subsystems. However, the two models are conceptually very different. In LQFT-QGR, subsystems are somehow localized in spacetime. By contrast, in $SU(\infty)$-QGR spacetime is not a quantizable entity and no locality condition is imposed on subsystems (particles). In fact, in $SU(\infty)$-QGR locality and causality are not postulated. As we discussed in Section 2.4, they arise from quantum uncertainties. Moreover, the relation between the Riemannian metric and evolution of quantum state of the content of the Universe in (13), shows that in agreement with the quantum mechanical observations, in QGR locality is in general an approximation.

LQFT-QGR and $SU(\infty)$-QGR share the absence of a classical dynamics in their foundation. Moreover, both models are types of QFT on curved spacetimes, which play the role of a parameter space. Their difference is in the definition of observable fields: LQFT-QGR constrains field operators to realize special algebraic structures and a sort of locality, whereas in $SU(\infty)$-QGR both gravity and matter sectors are quantum fields similar to QFTs without gravity. In addition, in $SU(\infty)$-QGR spacetime genuinely emerges, whereas in LQFT-QGR it is implicitly postulated and is present in the foundations of the model. Although in contrast to many other QGR proposals, in LQFT-QGR spacetime per se is not quantized, the model offers no explanation for its origin, dimension, properties of the metric, and its fundamental relationship with quantum fields.

Type III Algebra in LQFT-QGR and $SU(\infty)$-QGR

Operators indexed or parameterized by $\mathbb{R}^n$ cannot be divided into subsets associated with limited regions of the indices, if the whole algebra has to be invariant under diffeomorphism [98,99]. That is why a symmetry orthogonal to diffeomorphism is necessary for `tagging` and fulfilling conditions necessary for defining quantum subsystems [28].

As QFTs, both LQFT-QGR and $SU(\infty)$-QGR are type III quantum systems. In $SU(\infty)$-QGR, the inseparability of continuous operators is reflected in the common $SU(\infty)$ symmetry of all subsystems, including the Universe as a whole, and the need for a factorized finite rank `internal` symmetry. By contrast, LQFT-QGR considers strict locality as a foundational concept and tries to use nontrivial topological structures as a replacement for internal symmetries in order to tag and identify subsystems. So far, quantum field solutions with such property are obtained only in the weak coupling regime of semi-classical gravity [97], and at present there is no evidence that such algebraic structures can exist in the general setup of QFTs.

Although topological structures are observed in condensed matter, they are extremely fragile. By contrast, symmetry breaking or emergence, as requested in $SU(\infty)$-QGR, is widespread in nature. We also notice that topological structures proposed by LQFT-QGR are different from those used in LQG as observables. In LQG, Wilson loops do exist because of axioms and construction of the model. By contrast, to a large extent the existence of localized operators in LQFT-QGR are conjectured. In particular, the model explored in [97] for such structures is semi-classical and includes the perturbative Einstein equation, which is non-renormalizable. Therefore, the model cannot be considered as a genuine QGR.

3.2.4. QGR and Emergent Spacetime from Entropy and Holography

Another set of conjectures used for getting insight into QGR without considering an underlying classical dynamics are the holographic principle [3–5] and gauge-gravity duality conjecture [31,100], especially in the form of AdS/CFT duality; see Section 3.3.3 for more details. Notice that this conjecture should not be confused with models that try to quantize gravity by extending the gauge group of the Standard Model, such that it includes Lorentz and Poincaré symmetries [80,83,84].

Motivation for the holography conjecture [3–5] is the proportionality of semi-classical black hole entropy to the area of its horizon, rather than to its volume [20,21]. According to holography conjecture, there is an upper limit on the amount of quantum information contained inside the bulk of a region of spacetime with a lightlike boundary [3]. It is proportional to the area of its boundary and is maximal for black holes [20,21]. This conjecture is not limited to gravitational systems, and similar behavior is observed in other many-body quantum systems, if a suitable null (lightlike) boundary surface can be defined [6]. In particular, the entanglement entropy of some low dimensional many-body quantum systems at critical point, that is, when the system is scale invariant and behaves conformally, is calculable analytically, and the results show that they follow the holographic principle [4,7,8].

AdS/CFT duality conjecture [29,101] posits that the quantum properties of the boundary of a spacetime region in the limit that it can be approximated by a conformal QFT can be related to the geometry, and thereby to QGR in the bulk, if at classical limit the bulk has an AdS geometry.

Inspired by these conjectures, [12] considers two quantum systems with a quantum CFT living on their common boundary. Then, it establishes an analogy between the reduction of entanglement entropy and exchanged quantum information between the two systems, when their boundary is shrunk, and the reduction of their gravitational interaction, when the distance between their centers of mass is increased. To understand this analogy, imagine squeezing a rubber bar in the middle. More the bar is squeezed, more the material is pushed to the two ends. Moreover, the surface connecting the two sides becomes smaller until the bridge breaks and the two parts get separated. Of course, this analogy is very far from being a QGR model. Nonetheless, it has motivated the construction of QGR models using entanglement entropy as the origin of what is classically perceived as geometrical distance.

3.2.5. Entanglement-Based Models (EBM) of Quantum Gravity

A more systematic approach to the construction of a spacetime from the entropy–area law is proposed in [13,14], where spacetime metric and geometry emerge from tensor decomposition of the Hilbert space of the Universe into entangled subspaces. This model is based on several axioms; see [14] for the complete list. They include:

1. A preferred tensor decomposition of the Hilbert space $\mathcal{H}$ [of the Universe], where each factor $\mathcal{H}_i$ presents the Hilbert space of a point or a small space around a point of space:
$$\mathcal{H} = \bigotimes_i \mathcal{H}_i \tag{30}$$
2. There is what is called redundancy constrained (RC) states for each subset of the Hilbert spaces $B \subset \mathcal{H}$, considered to be a subspace of the physical space. Its entropy is defined as:
$$S(B) \equiv \frac{1}{2} \sum_{i \in B, j \in \bar{B}} I(i:j) \tag{31}$$
$$I(i:j) \equiv S(i) + S(j) - S(i \cup j) \tag{32}$$

where $I(i:j)$ is the mutual information of subsystems i and j. This construction replaces the area law axiom considered in [12,13].

3. It is assumed that the system is in an `entanglement equilibrium` state, when subsystems are in RC states. Under small perturbations the entropy of B is assumed to be conserved. This means that the total entropy is conserved. Moreover, when states deviate from RC, their entropy can be decomposed to the entropy of a fiducial RC state and a subleading component, interpreted as an effective field theory. The two components cancel each other to preserve the total entropy.

It is clear that axiom 1 is constructed such that the Hilbert space $\mathcal{H}$ presents physical space. Thus, we conclude that similar to LQFT-QGR, in this model the space does not really emerge, but its existence is postulated. Moreover, we notice that the definition of subsystems is loose and does not explicitly respect necessary conditions [28]. It is why axiom 1 explicitly states that factorization is static and somehow is `preferred`. However, the model does not specify what are the criteria for its selection.

Axiom 3 replaces the action and variation principle that in classical mechanics and QFT models lead to dynamics and field equations, respectively. In addition, according to this axiom, RC states can be considered as a `background` around which a perturbation is performed. Indeed, the model does not consider highly non-RC states and studies only the case of weak gravity interaction [14].

The structure described by above axioms can be considered as an information graph, whose vertices are factors of the Hilbert space, and its edges are weighted by mutual information $I(i:j)$ of subsystems corresponding to factors of the Hilbert space. This graph is analogous to discrete geometry in the Ponzano–Regge model, spin network, and LQG.

To complete the geometrical interpretation, the area of information graph or its subgraphs must be related to entanglement information. In [12,13], this connection is established by assuming the holographic principle. However, when RC structure is assumed [14], according to one of the axioms of the model (axiom 3 in [14]), the area associated with a subspace B of the space is:

$$\mathcal{A}(B,\bar{B}) = \frac{G_N}{2} I(B:\bar{B}) \tag{33}$$

where G_N is the Newton constant (for $\hbar = 1$ and $c = 1$) and $\bar{B}$ is the complementary of B. Although the area $\mathcal{A}$ associated with a subspace of the Hilbert space is not the boundary of a bulk space, the inspiration from the holographic principle is evident. This axiom and Radon transform are used to describe the area as a function of the entropy of factors $\hat{H}_i \forall i$ of the Hilbert space and to define a background metric. Perturbations of this metric is interpreted as the perturbation of the quantum state of the physical space.

Additionally, variations in the entanglement graph geometry are used as a `clock` with which a Hamiltonian and an operator analogous to energy–momentum can be associated. The latter can be considered as an effective field theory generating subleading entropy of states, which are perturbatively deviated from RC states. Finally, by comparing this formulation with general relativity and by using Radon transform, reference [14] argues that Einstein's equation can be concluded.

3.2.6. Comparison of EBM with $SU(\infty)$-QGR

We found that EBM is more similar to $SU(\infty)$-QGR—in spirit rather than construction—than other models. Here we briefly highlight their common features.

`Factorization of the Hilbert Space and Division to Subsystems`

The importance of the division of Hilbert space into factors presenting subsystems is crucial in both models. Many QGR proposals do not consider this issue explicitly or struggle to conclude it from their axioms. However, as remarked earlier, in EBM the division is considered to be rigid and `preferred`. This is in strict opposition to the approach of $SU(\infty)$-QGR. The reason behind the special factorization is again the absence of a concrete criterion to discriminate between factors-subsystems.

We notice that the issue of how to divide the Universe and its Hilbert space to quantum subsystems generally arises in the quantum approach to QGR usually due to the foundational requirements [28,32] for such operation. In other proposals difficulties related to this task have various reasons and model makers use different schemes to deal with this crucial matter. For instance, they introduce topological structures—as in LQG and LQFT-QGR; or simply consider a fixed decomposition without addressing its origin, as in EBM.

$SU(\infty)$-QGR assumes an orthogonal finite rank symmetry—presumably from symmetry breaking or emerging—to fulfill the general conditions for division of a quantum system into subsystems, according to [28]. Although the nature and origin of this symmetry are not specified in the construction of $SU(\infty)$-QGR, properties of $SU(\infty)$ symmetry, notably Equations (8) and (9), facilitate the interpretation of the Universe as a many-body quantum system, in which based on our knowledge from condensed matter, a symmetry of the form (10) can arise relatively easily. More importantly, in $SU(\infty)$-QGR the finite rank symmetry is associated with matter. In this way, matter and space become intertwined and inseparable. This is not the case in EBM, LQFT-QGR, or LQG and related models.

`Geometry and Classical Gravity`

Another common aspect of EBM and $SU(\infty)$-QGR is the explicit dependence of the space geometry on the quantum state—through entanglement entropy in EBM and through fidelity in $SU(\infty)$-QGR. However, emergence, construction, and physical meaning of the space in the two models are very different. In EBM of [13,14], factors of the Hilbert space are considered to present points or regions of the physical space, and the information graph is interpreted as a symplectic geometry, which in the continuum limit can be considered as a quantized space. Therefore, although the existence of a physical space is not explicitly mentioned in the axioms of EBM, it is implicitly behind the factorization of the Hilbert space. By contrast, in $SU(\infty)$-QGR, what is perceived as the physical space genuinely emerges as the parameter space of $SU(\infty)$ representations.

A consequence of these differences is that $SU(\infty)$-QGR has an explicit explanation for the dimension of spacetime, whereas in EBM, it is an unspecified stochastic parameter. In fact, the information graph can be embedded in any space with dimension $d \geqslant 2$. Notice that the relation between area of a subgraph (subsystem) and its entanglement entropy with its complementary in (33) do not restrict the graph to be planar—not even locally. A priori, every vertex—that is every factorized subsystem of the Hilbert space—can be entangled with all other subsystems. In [13], it is assumed that the number of subsystems entangled with a vertex—corresponding to the number of edges attached to it—is limited. Nonetheless, their number can be large and rules for the construction of graphs do not constrain their mutual angle. Thus, in contrast to Ponzano–Regge and LQG, in which spins assigned to edges of the symplectic space must satisfy triangle constraint at each vertex, the information graph in EBM can be embedded in a multi-dimensional space. For these reasons, d is considered as a stochastic parameter determined from averaging over geometries of many information graphs [13]. However, the spacetime dimension is a fundamental quantity, which affects many observables in particle physics and cosmology at all energy scales. To date, no evidence of an extra/infra or stochastic dimension has been detected.

In $SU(\infty)$-QGR the relationship between affine parameter, metric, and quantum fidelity in equation (12) naturally relates ensemble of parameters (not just distance or area) to quantum states of the subsystems. In both EBM and $SU(\infty)$-QGR, Einstein's equation remains classical and is obtained from relationship between quantities with underlying quantum origin.

`Analogy between Distance and Entanglement`

In both models, an area quantity emerges and it is crucial for their interpretation as QGR. In $SU(\infty)$-QGR it emerges from a comparison of the preserved areas of diffeo-surfaces of subsystems with an arbitrary reference subsystem. In EBM it is postulated in (33), where a dimensionful area/distance parameter is mandatory. Although the way a scale emerges in these models is very different, in both cases it is related to the division of the Universe into subsystems. Indeed, in EBM entanglement and its associated entropy are meaningful only when multiple quantum systems are present. In $SU(\infty)$-QGR division to subsystems is necessary to make the conserved area of diffeo-surfaces relevant and measurable.

In addition to differences in the manner that a dimensionful scale arises in these models, there is another important difference. In $SU(\infty)$-QGR the area is related to geometry of the compact parameter space of representations of $SU(\infty)$ symmetry of subsystems. Thus, it is a well defined and unique measurable for each subsystem relative to a reference. By contrast, quantification of entanglement and relative quantum information is not unique and various definitions, e.g., von Neumann or Rényi entropy can be used, and each of them has its own merit and applications. EBM models of [12–14] do not specify which one of these entropies should be used or what is rationale for preferring one to others, or whether different definitions should be interpreted as different choices of coordinates.

3.3. String Theory, M-Theory, and AdS/CFT Duality in Three or More Dimensions

String theory and related models are without any doubt the most intensively studied QGR proposals. Although some of quantum first models are inspired by (super)string theories and AdS/CFT duality conjecture, string theories are not, properly speaking, quantum first. Their perturbative formulation is a quantized 2D sigma model, originally proposed for describing strong nuclear interaction [102]. Non-perturbative formulation of string models, also called M-theory, and its realization as a matrix model, has the form of a (super) Yang–Mills QFT.

In recent decades, new approaches to string theories are extensively studied in the literature, and various concepts and structures are added to their initial construction. Their list includes: D-branes states [103]; string condensation and its relation with p- and D-brane solutions [103–106], which are important for the conjectured non-perturbative formulation of string models, also called the M-theory; additionally, AdS/CFT duality [29,107], which is the simplest case of gauge-gravity conjecture [30,31,108] and is closely related to M-theory and matrix models. Nonetheless, the basic structure of (super)string theories and their properties continue to be considered as foundational and established knowledge for development of these more advanced theories. In particular, M-theory uses the 10D or (9+1)D Euclidean or Minkowski spacetimes, respectively, which is the fundamental dimension of spacetime in perturbative superstring theories. Similarly, the first evidence of AdS/CFT correspondence was discovered for D3 brane models in a 10D compactified $AdS_5 \times S_5$ background spacetime [109]. Thus, due to the importance of perturbative formulation of string models, in this section we first briefly remind their findings and how they compare with $SU(\infty)$-QGR. Then, we review and compare M-theory and its matrix realization, and AdS/CFT conjecture.

3.3.1. Perturbative String Theories and Their Comparison with $SU(\infty)$-QGR

As extended literature and textbooks on string theory related subjects such as [102,110] is available, we do not review these models in detail and only remind their most important properties used for comparison with $SU(\infty)$-QGR wherever they are necessary.

`2D Surfaces in String Theory and` $SU(\infty)$`-QGR`

Overlooking all the complexities of string and superstring theories, they can be summarized as 2D quantum gravity of a conformal quantum sigma model. Here quantum gravity means summation over all possible geometries of their 2D worldsheet—more generally a membrane. In this view of string theories, their most evident common feature with $SU(\infty)$-QGR is the crucial role of 2D surfaces and their diffeomorphism. As we will see in more detail in Section 3.3.2, even in non-perturbative approaches such as M-theory and p- and D-brane models, the 2D surfaces do not lose their crucial role and are implicitly present and represented by large matrices.

On the other hand, roles, properties, and interpretation of 2D surfaces in these theories and in $SU(\infty)$-QGR are profoundly different. In string theories, 2D worldsheets of strings, or more generally, membranes, are quantized, and the summation of their geometries is interpreted as the path integral of 2D quantum gravity. By contrast, diffeo-surfaces in $SU(\infty)$-QGR are not independent physical entities, neither are they quantized. They are associated to quantum states of the Universe and its content—subsystems—in the same way that in QFT a charge or spin is associated to a particle, but it is not the particle. Deformations of diffeo-surfaces do not correspond to different (quantum)-gravitational states, but rather represent members of the symmetry group of shared by all quantum subsystems, including the whole Universe. Another crucial difference between these model is the fundamental role of subsystems and environment in the foundation of $SU(\infty)$-QGR, whereas in string theories they are not explicitly involved in its formulation.

`String Sigma Model`

In a sting sigma model, both bosonic and fermionic quantum fields live on the 2D worldsheets [102,110]. In superstring models they are interpreted as coordinates of an $n-$dimensional quantum spacetime and their supersymmetric counterparts, respectively. One can equally interpret the worldsheet of a string as a 2D extended membrane embedded or emerged in an $n-$dimensional spacetime. Additionally, string theories are in general 2D Conformal Field Theories (CFT). This means that they are invariant under rescaling of both worldsheet 2D coordinates and local rescaling of the fields. This double conformality is a necessary condition for eliminating central charge and anomalies, which arise when these models are quantized [110]. Cancellation of these unwanted elements limits the spacetime (target space) dimension to $n = 26$ for bosonic strings or to $n = 10$ in superstring models. More generally, the sigma model can be any CFT with Kac–Moody algebra having the same number of degrees of freedom as bosonic and supersymmetric models. As mentioned earlier, the value of fundamental—rather than observed—spacetime dimension obtained from sigma model formulation of superstrings is taken for granted in further developments of these models. Interestingly, an $n = 1$ model is also a consistent quantum model [102]. The single (super)field in such model cannot be interpreted as a background spacetime, but this case is studied as a decoupled sector in matrix formulation of string theory [100].

In the framework of $SU(\infty)$-QGR, the string setup—without quantization of 2D worldsheet/membrane—can be considered as a special state for a quantum system with $SU(\infty)$ symmetry. The sigma model of strings—without constraints arising from conformal symmetry and quantization—can be interpreted as special states for subsystems with an internal symmetry G. Quantization of such a state in the framework of (super)string theory restricts the internal symmetry G to groups allowed by the cancellation of anomalies. For instance, G may be identified with: a 10D sigma model of superstring models and/or its $SO(32)$ or $E8$ internal symmetry; symmetries of the corresponding low energy $\mathcal{N} = 4$ supergravity in 11D; or symmetries of compactified coordinates or quantum fluctuations of D-brane solutions in M-theory. The origin of these similarities can be traced back to the Virasoro algebra of string fluctuations, which is a subalgebra of surface-preserving diffeomorphism (SDiff) of a torus i.e., $SDiff(T^2)$ algebra and the fact that the latter is a representation of $SU(\infty)$ group [37,111,112].

One of the main advantages of string theory to canonical QGR is its renormalizabilty and the absence of UV singularity, owing to the extended nature of strings. Although renormalization of $SU(\infty)$-QGR is not yet studied in detail, from its Yang–Mills action we expect that it should be renormalizable. Moreover, UV singularity should not arise, because the distance between subsystems can be related to the relative area of their diffeo-surfaces. The latter by definition cannot shrink to a point, equivalent to zero distance; otherwise, $SU(\infty)$ would be represented trivially. This feature should play the role of a built-in ultraviolet cut-off without introducing any fixed scale.

`Curved Spacetime and Gravity in String Theory`

In the sigma model formulation of string theories, quantization is consistent only when the geometry of the field (target) space—interpreted as fundamental spacetime—is flat. This means that only metric perturbations around this Minkowski background are quantum mechanically meaningful. The issue of a curved target (field) spacetime in string theory does not have an analogy in $SU(\infty)$-QGR. Nonetheless, solutions proposed to overcome this problem and their role in non-perturbative formulation of string theory in the framework of M-theory can be compared with $SU(\infty)$-QGR.

One way of studying perturbative strings in curved spacetimes is introducing a string gas. In particular, This approach is used for the purpose of describing cosmological perturbations in the framework of string theory [113]. However, the inherently intertwined nature of spacetime and strings may make it impossible to consider their evolution separately. There are, nonetheless, exceptions. AdS/CFT duality conjecture, discussed in more detail in Section 3.3.3, is proved for AdS_3 space, and is considered as to be the evidence of consistency of perturbative string theory, at least in some curved background spaces.

Another way to overcome the issue of curved target space is considering special configurations/solutions for the dynamics of strings in the target space. These solutions usually include localization of perturbative string modes. For instance, the extremity of an open string can be restricted to move on a $p < D$ dimensional subspace of the target space, called a p-brane or more general solutions in the form of D-branes [103]. The induced geometry on p/D-branes can be curved. In this framework, the observed (3+1) dimensional spacetime can be a brane in D-dimensional fundamental target space. In the same manner, D-branes can be formed from condensation of closed strings, but they may be unstable [104–106]. D0 branes are another class of interesting configurations of Yang–Mills gauge fields in the target space. They correspond to coordinate independent configurations, which may change with time [100] or be static [114]. These models are studied in the framework of gauge-gravity duality conjecture [100] and M-theory and have more common aspects with $SU(\infty)$-QGR than perturbative string models. For this reason we review them in more details in the next subsection.

3.3.2. M-Theory and Matrix Theories

M-theory and matrix models are developed as candidates for the non-perturbative formulation of string theory; see, e.g., [115–117] for reviews. Using various concepts, including large N expansion of perturbative QFTs [118] and holography principle, it is conjectured that non-perturbative type II string theories can be described as $U(N)$ supersymmetric Yang–Mills theories and present quantum states of type IIA strings in D0 background [100,119].

The BFSS [100] matrix model—called also D1+0 brane—is a 10D super Yang Mills model, obtained from compactification of one dimension of the 11D supergravity effective field theory of string theory at low energies. It is reduced to 1+0 dimension by assuming that all the fields in the model are independent of nine spatial coordinates. The dependence on the last coordinate is removed in what is called D0 brane or IKKT matrix model [114]. It is demonstrated that IKKT corresponds to a high temperature—slow variation—limit of BFSS when the Euclidean time is treated as the inverse of temperature [120].

The action of IKKT model is defined as:

$$S_{IKKT}[X] = \frac{1}{g^2} \operatorname{Tr}\left(\frac{1}{4}[X^a, X^b][X^c, X^d]\eta_{ac}\eta_{bd} - \frac{i}{2}\bar{\psi}_\alpha(\mathcal{C}\sigma^a_{\alpha\beta}[A_a, \psi_\beta]\right) \tag{34}$$

where X^a, $a = 0, \cdots, 9$ and ψ_α, $\alpha = 1, \cdots, 16$ are hermitian $N \times N$ matrices representing $SO(D = 10)$ (Euclidean) or $SO(D - 1, 1)$ (Minkowski); η_{ab} is the metric of a flat Minkowski or Euclidean 10D space; σ^as are 16×16 Pauli matrices for D=10 space; and $\mathcal{C}$ is charge conjugate operator of the same dimensionality. The action $S_{IKKT}[X]$ is similar to that of a type IIB superstring in Schild gauge [121].

It is assumed that $N \to \infty$ such that $Ng^2 \equiv \bar{g}^2 < \infty$. Therefore, X^a and ψ_α are, respectively, bosonic and fermionic $N \to \infty$ dimensional representations of non-Abelian $SO(D)$ (or SO(D-1,1)). Although the original motivation for this model has been the string theory, it can be studied without referring to the latter, and is also studied in $D \neq 10$ [117,120].

Using the variation principle, one can obtain `field` equations for X^a and ψ_α. In particular, considering only bosonic Yang–Mills sector, the field equation for X^a is:

$$[X^a, [X^b, X^c]]\eta_{ac} = 0 \tag{35}$$

Given the finite rank of the Yang–Mills symmetry group of the model and the large dimensionality of its representation by X^a, Equation (35) has many solutions. Aside from trivial commuting matrices, solutions having the form:

$$[Y^\mu, Y^\nu] = i\theta^{\mu\nu}\mathbb{1}, \qquad \mu,\ \nu = 0, \cdots 2n-1,\ 2n \leqslant D \tag{36}$$

where $\theta^{\mu\nu}$ is an anti-hermitian constant matrix $\theta^{\nu\mu} = -\theta^{*\mu\nu}$, provide reductions of the space dimensions and a quantized non-commutative geometry [122] for these branes. Specifically, in case of the Moyal–Weyl solution, the background 10D space is static with $Y^\mu = \bar{Y}^\mu$, $\mu = 0, \cdots 2n-1$ and $Y^i = 0, i = 2n, \cdots, D-1$. To define quantum fluctuations, Y^a matrices are decomposed as:

$$Y^a = \bar{Y}^a + (A^\mu, 0) + (0, \phi^i) \tag{37}$$

Although A^μ can be considered as a $U(1)$ gauge group, it actually belongs to the gravity sector and cannot be identified as $U(1)$ symmetry of the Standard Model. Matter fields in matrix models can arise, such as in string theory by compactifying $D-2n$ fields. See, e.g., [115] for a review. For Moyal–Weyl-type solutions considering k coinciding branes, see, e.g., [103,117,123] for a review. Assuming quantum superposition of fluctuations of k branes as defined in (37), they are locally invariant under $SU(k)$ symmetry. Thus, in this approach A^μ and ϕ^i can be expanded as:

$$A^\mu = -\theta^{\mu\nu} A^b_\mu(\bar{Y}) T^b, \qquad \phi^i = \phi^i_b(\bar{Y}) T^b \tag{38}$$

where T^bs are generators of adjoint representation of $SU(k)$. As indicated earlier, due to the fundamental non-commutation nature of Y^a, this construction cannot accommodate a $U(1)$ symmetry. In the same manner, fermion fields can be constructed, but they would be in adjoint representation of $SU(k)$, because in 10D fundamental space they are supersymmetric partners of coordinates X^a, and cannot be identified with matter. For $n = 2$, this setup leads to a quantum non-commutative $\mathbb{R}^4_\theta \subset \mathbb{R}^D$ space identified as the physical spacetime.

In contrast to Randall–Sundrum-type brane models, in matrix theories, quantum fluctuations of geometry and matter do not propagate to the 10D bulk. Therefore, matrix theories, and more generally models based on the condensation of string modes to branes, are not dependent on the 10D background geometry. This solves the problem of string formulation in curved spaces discussed in Section 3.3.1. However, in matrix models, only fluctuations of background (target) 10D space are observable. Therefore, the role of unobservable static (in some models) of the 10D fundamental background/target space is not clear. Moreover, D-branes may decay [104–106] and the stability of overall setup is not certain. In any case, these field/string solutions are special configurations, many of them are plausible [124], and it is not clear why nature should prefer the one corresponding to our Universe.

Finally, the low energy effective action of matrix models is a modified version of Einstein Equation [117,125,126], which is stringently constrained, specifically with gravitational waves [127].

`Comparison of Matrix Models with` $SU(\infty)$`-QGR`

There are many similarities between matrix models and $SU(\infty)$-QGR, but also significant differences. In both models the fundamental objects are $N \to \infty$ matrices. Matrix models are pure (super)Yang–Mills models, and X^a and ψ^i in (34) are $N \times N$ matrices in the adjoint representation of an `internal` finite rank symmetry. By contrast, $SU(\infty)$-QGR is constructed from a Hilbert space and includes both square and column matrices as primary entities, in adjoint and fundamental representations of both $SU(\infty)$ and internal symmetries.

In matrix theory, the large dimension of matrices are inspired by the large `color` and loop number limit of QFT [118], conjectured to present a strong coupling regime. In $SU(\infty)$-QGR, the motivation is rather cosmological and based on the observed large number of degrees of freedom in the Universe. These apparently different motivations converge to each other, because for a perturbative estimation of observables, for instance an S-matrix, up to a given degree of precision, one has to take into account more loops, virtual particles, and their degrees of freedom for stronger couplings. The assumption of $SU(\infty)$-QGR that every subsystem of the Universe represents infinite degrees of freedom is an explicit realization of the above concept.

$SU(\infty)$-QGR uses the above axiom as a foundation for constructing other aspects of the model. Moreover, based on the observed spontaneous breaking and emergence of symmetries in many-body systems—see e.g., [128,129]—it defines subsystems according to the well established criteria in quantum information theory [28]. The formulation of the model, especially in what concerns the universal quantum gravitational interaction, is completely independent of `internal` symmetries of subsystems (particles or fields), which are not constrained by the model. In fact, considering that when all constituents interact with each other, the symmetry is $G \to SU(\infty)$, one expects that many other smaller rank symmetries should have a nonzero probability arise in intermediate states, where the number of effectively coupled or entangled subsystems is finite but large.

By contrast, in matrix models, the large dimensionality of matrices remains a background concept and does not directly intervene in the construction of symmetries and dynamics. The latter are to a large extent inspired by or concluded from superstring theory. Indeed, large matrices in these models present a membrane or worldsheet of a string [35,111,112]. Despite the fact that matrix models can be considered as stand-alone, and what they consider as fundamental spacetime can have other dimension than 10 of superstring models (see, e.g., [120]), BFSS and IKKT models and their variants are mostly constructed and studied in 10D Euclidean or Minkowski space. In any case, even without referring to string theory, the presence of extra-dimensions in matrix models is inevitable for the introduction of matter and other interactions than gravity. They are considered to be the quantum fluctuations of a D0 condensate [117] (and references therein) or compactified dimensions [115] (and references therein). However, similar to many QGR proposals and in contrast to $SU(\infty)$-QGR, matrix models do not provide any explanation for the observed dimension of spacetime.

In summary, both M-theory (matrix models) and $SU(\infty)$-QGR emphasize the importance of $SU(\infty)$ in a quantum description of gravity. However, they diverge on many details. In particular, in matrix models, $SU(\infty)$ symmetry is not explored. The manner in which internal symmetries arise and are constrained in these models are different. Finally, the large dimension of fundamental spacetime—the target space—in M-theory does not have an analogue in $SU(\infty)$-QGR.

3.3.3. Anti-de Sitter–Conformal Field Theory (AdS-CFT) Duality

According to AdS-CFT duality conjecture [29], and more generally gauge-gravity duality [30,31] in M-theory, there is a one to one correspondence between quantum states of a suitable quantum CFT living on the boundary of a region of the spacetime and supergravity (string theory) in its AdS bulk.

This conjecture is closely related to the holographic principle, but there is not yet a general proof for it, except in (2+1)D spaces [101]. Specifically, consider a conformal field theory on the (1+1)D space $\mathbb{R} \times S^1$ boundary of an AdS_3 spacetime. Define two complementary subsystems A and B divided along the $\mathbb{R}$ axis of the bulk (see Figure 1 of [101]). The Hilbert space of the quantum CFT is factorized to $\hat{H}_A \otimes \hat{H}_B$, and entanglement entropy between A and B is defined as $S(A) \equiv -\mathrm{tr}(\rho_A \ln \rho_A)$, where ρ_A is the density matrix of A when the state of B is traced out. Notice that the geometrical division to A and B, and factorization of their Hilbert space is in general valid only in a given frame because, as we discussed in Section 3.2.2, QFTs are type III and cannot be restricted to an arbitrary region of spacetime without violating Lorentz invariance.

It is proved [101] that the static entanglement entropy between the two subsystems at a constant time t is proportional to the length of the geodesic (null) curve passing inside the AdS_3 and joining the 2-point cross-section on the constant t, S^1 boundary. More generally, for an AdS_{d+2} spacetime, the entanglement entropy is conjectured to be:

$$S(A) = \frac{\text{Area of}\gamma_A}{4G_N^{d+2}} \tag{39}$$

where γ_A is the $d-$dimensional minimal (geodesic) boundary surface and G_N is the Newton constant. Additionally, it is shown that $S(A) \to 0$ only when the size of the system goes to infinity [7,8]. This case corresponds to when the two subsystems are infinitely separate from each other.

We notice that the definition of subsystems in [29,101] is geometric. This is an important point, because as we discussed in Section 3.2.3, QFTs have type III algebra and Lorentz invariant quantum subsystems cannot be defined by division of their support spacetime. Thus, A and B are not, properly speaking, subsystems and are not diffeomorphism invariant. It is not clear whether and how this issue affects the AdS/CFT duality conjecture, especially in higher dimensional spaces for which a proof is not available.

For $d = 2$, the AdS $\cong R \times R \times S^d$ geometry is homomorphic from the simplest geometry of parameter space in $SU(\infty)$-QGR after division of the Universe into subsystems. For this case, relation with a CFT on the boundary in the framework of $SU(\infty)$-QGR can be understood as the following: For the whole Universe or an approximately isolated subsystem, the size of the diffeo-surface of its $SU(\infty)$ symmetry is approximately irrelevant for its observables. This property can be interpreted as an approximate conformal symmetry—that is, scaling invariance of the parameter space of the system and its pullback into the Hilbert space. Considering an external quantum clock, at a given time the parameter space of such an isolated subsystem is approximately 2D, and its quantum dynamics is approximately a 2D CFT. Its operators generate a Virasoro algebra, which is a subalgebra of $SDiff(T^{(2)}) \cong SU(\infty)$ [37,111,112]. Invariance by scaling means that any arbitrary diffeomorphism is equivalent to a surface preserving one.

4. Outline

The comparison of several popular QGR models with $SU(\infty)$-QGR in this work found a number of common or analogous features between them. The results of this exercise, summarized in Section 1.1, highlight the origins of these properties and shows that they either arise in $SU(\infty)$-QGR from its axioms or can be concluded from them. Giving simple axioms of $SU(\infty)$-QGR and systematic and natural emergence of common features in this model, it may help to clarify some of puzzling properties of other QGR proposals.

As $SU(\infty)$-QGR is a new model, its comparison with other QGR candidates presented here is limited to their construction, rather than predictions for various physical phenomena, in which QGR may be involved. Future works should concentrate on the applications of this model. Examples of problems to be studied are:

- Hawking radiation;
- Information loss paradox of black holes;
- Particle physics at Planck scale;
- Spacetime singularities;

Topology of parameter space of the model identified as the classical spacetime and whether it can be changed dynamically.

Regarding particle physics, as mentioned in Section 2.5.2, in the framework of $SU(\infty)$-QGR we should expect more unbroken gauge symmetries G at high energies. Therefore, a crucial task is to determine how internal symmetries vary with energy scale Λ. Particle physics experiments show that $G|_{\Lambda\sim 1\text{TeV}} = SU(3) \times SU(2) \times U(1)$, i.e., the Standard Model symmetry. However, a signature of interactions at higher energies may be smeared by the physics at lower energies [130]. Nonetheless, if many-body high energy states behave similar to their low energy analogues, their analogy may help to find the best criteria for detecting signatures of `phase transitions` due to symmetry transition at high energies in (astro-)particle physics experiments or cosmological observations.

In addition to gauge symmetries, C, P, and CP violations responsible for many aspects of low energy particle physics, in particular, matter–antimatter asymmetry, may have gravitational origins. However, we should remind that in the 2D parameter space of the whole Universe, all spinors are Weyl type—i.e., there is no matter–antimatter discrimination. Therefore, the observed asymmetry necessarily arises after division of the Universe into subsystems, and may or may not have a gravitational origin.

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Appendix A. Classical Limit of the $SU(\infty)$ Yang–Mills Model of Subsystems

We first review some of properties of curvatures of (pseudo)Riemannian manifolds. For any Riemannian or pseudo-Riemannian manifold $(\mathcal{M}, g)$ of dimensions $d \geqslant 2$ and metric g equipped with a Levi–Civita connection ∇, the Riemann curvature (1,3) tensor at point $p \in \mathcal{M}$ is defined as:

$$R_p(X,Y)Z = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \tag{A1}$$

Vector fields $X,\ Y,\ Z \in T\mathcal{M}_p$, where $T\mathcal{M}_p$ is the tangent space of $\mathcal{M}$ at p. When $X,\ Y,\ Z$ are chosen to be $\partial_i \equiv \partial/\partial x^i$ basis of the tangent space for coordinates x^i, $i = 0, \cdots d-1$, one recovers the usual coordinate-dependent definition of the Riemann curvature tensor (we drop p because it corresponds to the point with coordinates x^i):

$$R(\partial_i, \partial_j)\partial_k = R^l_{kij}, \tag{A2}$$

$$R_{ijkl} \equiv R(\partial_i, \partial_j, \partial_k, \partial_l) \equiv \left\langle R(\partial_i, \partial_j)\partial_k, \partial_l \right\rangle = g_{ml} R^m_{kij} \tag{A3}$$

Using the notation defined in (A3) for (0,4) Riemann curvature tensor, the sectional curvature $K(\Pi) = K(X,Y)$ with respect to a 2D plane $\Pi \subset T\mathcal{M}_p$ containing two vectors $X, Y \in T\mathcal{M}_p$ at $p \in \mathcal{M}$ is defined as:

$$K(\Pi) \equiv K(X,Y) = \frac{R_p(X,Y,X,Y)}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2} \tag{A4}$$

Notice that $K(\Pi)$ is independent of the choice of X and Y, and depends only on the plane passing through them. It can be shown that $\langle R_p(X,Y)Z, W)\rangle$ is expandable with respect to sectional curvatures [131]. Using relations between different curvature tensors of a Riemannian manifold, the Ricci scalar at point $p \in \mathcal{M}$ is defined as [132]:

$$R(p) = \sum_{i \neq j} R_p(e_i, e_j, e_i, e_j) = \sum_{i \neq j} K_p(e_i, e_j) \tag{A5}$$

where $e_i,\ i = 0, \cdots, d-1$ is an orthonormal basis of $T\mathcal{M}_p$. From (A5), we conclude that there is only one sectional curvature at each point of a 2D surface, and it is equal to its Ricci scalar curvature.

In order to extend the relation (19) between $SU(\infty)$ Yang–Mills action, and the Ricci scalar curvature of its diffeo-surface of an isolated quantum system with $SU(\infty)$ symmetry, to a large number of such systems when the Universe is divided into subsystems, we have to integrate their contributions. In Section 2.3.1 we showed that the parameter space of subsystems is (3+1)D dimensional. Applying definitions of curvature tensors to this parameter space, each sectional curvature in (A5) can be interpreted as the Ricci curvature of the diffeo-surface of a subsystem. The summation in the right-hand side of (A5) and integration over the volume of the parameter space in the action functional amount to taking into account all subsystems of the Universe. Therefore, the right-hand side of (24) corresponds to the right-hand side of (19) when the Universe is divided into subsystems. In the same way, the left-hand side of (24) corresponds to the left-hand side of (19) when the contributions of subsystems are calculated separately. However, (24) is valid only in the classical limit, because as discussed in Section 2.4, after the selection of the reference and clock, the ensemble of remaining subsystems should be considered as an open quantum system. Therefore, (24) is an approximation, valid only in the classical limit where the reference and the clock can be considered as classical.

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