symmetry

MDPI

Article

Z_L-Completions for Z_L-Semigroups

Shuhua Su [1,*], Qingguo Li [2] and Qi Li [3]

[1] School of Mathematics, Physics and Statistics, Shanghai University of Engineering Science, Songjiang, Shanghai 201620, China
[2] School of Mathematics, Hunan University, Changsha 410082, China; liqingguoli@aliyun.com
[3] Fuzhou Teachers' College, East China University of Technology, Fuzhou 344000, China; qli@ecit.cn
* Correspondence: sushuhua913@163.com

Abstract: In this paper, we generalize a common completion pattern of ordered semigroups to the fuzzy setting. Based on a standard L-completion Z_L, we introduce the notion of a Z_L-semigroup as a generalization of an L-ordered semigroup, where L is a complete residuated lattice. For this asymmetric mathematical structure, we define a Z_L-completion of it to be a complete residuated L-ordered semigroup together with a join-dense L-ordered semigroup embedding satisfying the universal property. We prove that: (1) For every compositive Z_L, the category $\mathbf{CS}_L$ of complete residuated L-ordered semigroups is a reflective subcategory of the category $\mathbf{S}_{Z_L}$ of Z_L-semigroups; (2) for an arbitrary Z_L, there is an adjunction between $\mathbf{S}_{Z_L}$ and the category $\mathbf{S}^{\rightarrow}_{Z_L}\mathbf{E}$ of weakly Z_L-continuous L-ordered semigroup embeddings of Z_L-semigroups. By appropriate specialization of Z_L, the results can be applied to the DM_L-completion, certain completions associated with fuzzy subset systems, etc.

Keywords: fuzzy subset selection; fuzzy order; symmetry and asymmetry; Z_L-semigroup; Z_L-continuous mapping; Z_L-completion; complete residuated lattice

Citation: Su, S.; Li, Q.; Li, Q. Z_L-Completions for Z_L-Semigroups. *Symmetry* **2022**, *14*, 578. https://doi.org/10.3390/sym14030578

Academic Editors: Yiming Tang, Yong Zhang, Zhaohong Deng and Xiaohui Yuan

Received: 19 February 2022
Accepted: 14 March 2022
Published: 15 March 2022

1. Introduction

Complete residuated lattices are important symmetric algebraic structures for fuzzy logic, probabilistic logic and linear logic. As the table of truth values, it has been extensively applied to fuzzy order (or L-order, Q-order) theory originated by Zadeh [1]. A fuzzy order is a kind of fuzzy relation with reflexivity, transitivity and asymmetry. Now, fuzzy order theory has been widely used in various branches of mathematics and computer sciences. For crisp order structures as models in theoretical computer science, some form of completeness should be required, such as the DM-completion, the ideal completion, the join-completion, the Z_Γ-completion, and so on (see [2–13]).

The fruitful results about completions of crisp order structures have inspired some scholars to study the completions in the fuzzy setting. Wagner [14], Bělohlávek [15,16] and Xie [17] generalized the DM-completion to the multi-valued framework from different perspectives. The interesting thing is that the DM-completions above are exactly the same when $\Omega = L$ is a frame. Subsequently, Wang and Zhao [18] obtained a category characterization of the DM-completion of fuzzy posets by constructing the join-completion of fuzzy posets. Moreover, they also solved the problem of whether the join-completions of fuzzy posets are entirely determined by the consistent fuzzy closure operators. Based on the work of [18], Su and Li [19] constructed the Δ_1-completion and discussed how it relates to formal L-contexts over fuzzy posets. In general, the completion of fuzzy posets can be composed of some special fuzzy subset selection. Ma [20], Rao [21] and Zhao [22] systematically studied the fuzzy subset system and used it to construct the Z_L-continuity and Z_L-algebra of fuzzy posets. Following this line, Su et al. [23] established the Z_L-completions of fuzzy posets and then constructed a minimum Z_L-completion.

In order to enhance the applicability of ordered semigroups as mathematical tools, the combination of a fuzzy order and an algebraic system has appeared in recent years. Šešlja [24], Hao [25], Wang [26] and Borzooei [27] fused fuzzy order and algebraic structure in different ways and put forward new concepts such as fuzzy ordered group, fuzzy ordered semigroup and fuzzy lattice ordered group. Most of the above studies focus on the discussion of fuzzy order structure, in which algebraic structure only plays a secondary role. Then, Huang et al. [28] established various fuzzy ideals on fuzzy ordered semigroups from the perspective of algebraic structure as the main and fuzzy order structure as the auxiliary. In addition, they also established the category characterization of fuzzy ordered semigroups, which further improved fuzzy set theory of ordered algebra. Following this viewpoint, Su et al. [29] constructed an organic combination of fuzzy poset and semihypergroup, which is called an L-ordered L-semihypergroup. Then, they established three types of L-hyperideals and intra-regularity on it.

Motivated by studies for fuzzy ordered semigroups and completions of fuzzy posets, Zhao et al. [30] proposed the concept of L-quantale as a generalization of a crisp quantale. For each fuzzy ordered semigroup, they defined the L-quantale completion to be an L-quantale together with a special embedding. In particular, they discussed the relationship between fuzzy ordered semigroups and L-quantales from the categorical perspective. Following this line, Xia and Zhao [31] further studied the L-quantale completion and obtained the minimum L-quantale completion of every fuzzy ordered semigroup.

In the crisp setting, Erné and Reichman [32] introduced $\mathcal{Y}$-semigroups with ordered semigroups as a special case, and established a standard completion $\mathcal{Y}$ for a $\mathcal{Y}$-semigroup. This completion provided the common pattern for various completions of ordered semigroups. It is then natural to ask whether one can extend the theory of standard completions to the setting of L-ordered semigroups. In this paper, we answer this question by defining a new completion for fuzzy ordered semigroups, called Z_L-completion. It turns out that many kinds of completions, such as the DM-completion and L-quantale completion, are a special case of the Z_L-completion.

This paper is organized as follows: Section 2 lists some notions and results in [16,25,26,30,33–38]. In Section 3, based on an L-subset selection, a standard L-completion Z_L of an L-ordered semigroup and (weakly) Z_L-continuous mapping preserving structures are introduced. In particular, some examples of compositive standard completions are presented. In Section 4, a Z_L-semigroup and its Z_L-completion with the university property are built. Moreover, it is shown that for every Z_L-semigroup, the standard L-completion Z_L with a join-dense L-ordered semigroup embedding is a Z_L-completion. In Section 5, for different cases of Z_L, the corresponding categorical characterization of a Z_L-completion is established. Finally, we present conclusions in Section 6.

2. Preliminaries

This section reviews some basic concepts and results on fuzzy posets (see [16,33–38]). Throughout this paper, L denotes a complete residuated lattice $(L, \wedge, \vee, *, \rightarrow, 0, 1)$ [39], where L is a complete lattice with the bottom 0 and the top 1, and $* : L \times L \longrightarrow L$ is a commutative, associative binary operator such that (1) $*$ is monotone on each argument; (2) $*$ has a right adjoint $\rightarrow$, that is, $a * b \leq c \Leftrightarrow a \leq b \rightarrow c$ for all $a, b, c \in L$; (3) $1 * a = a$ for all $a \in L$. Such a symmetry algebraic structure is significant in the narrow sense of fuzzy logic (see [15,39]).

An L-subset of X is a mapping from X to L; write L^X for the set $\{A \mid A\ is\ an\ L-subset\ of\ X\}$. For $A, B \in L^X$, define $A \subseteq B$ as $A(x) \leq B(x), \forall x \in X$. We never distinguish between an element $b \in L$ and the constant function $\overline{b} : X \longrightarrow L$ satisfying $\forall x \in X, b(x) = b$, such as $(b * B)(x) = b * B(x)$ and $(b \rightarrow B)(x) = b \rightarrow B(x), \forall x \in X$.

Definition 1 ([16,34])**.** *A fuzzy order (or an L-order) e on a set X is an L-relation such that*

(1) $\forall x \in X, e(x, x) = 1$ *(reflexivity);*
(2) $\forall x, y, z \in X, e(x, y) * e(y, z) \leq e(x, z)$ *(transitivity);*

(3) $\forall x, y \in X, e(x, y) = e(y, x) = 1 \Rightarrow x = y$ (asymmetry).
The pair (X, e) *is called a fuzzy poset (or an L-ordered set).*

For an L-ordered set (X, e), define a crisp order $\leq_e$ on X as follows: $x \leq_e y$ iff $e(x, y) = 1$, and so $(X, \leq_e)$ is a poset.

For a nonempty set X, (L^X, sub) is an L-ordered set, where the inclusion L-order sub [15] defined by $sub(A, B) = \bigwedge_{x \in X} A(x) \rightarrow B(x), \forall A, B \in L^X$.

Proposition 1 ([16,33]). *$\mathcal{T} \subseteq L^X$ is an L-closure system (or a fuzzy closure system) on X iff*

(1) $\forall \mathcal{A} \subseteq \mathcal{T}, \bigwedge \mathcal{A} \in \mathcal{T}$;
(2) $\forall a \in L, A \in \mathcal{T}, a \rightarrow A \in \mathcal{T}$.

For an L-closure operator C on X and an L-closure system $\mathcal{T}$ on X, $\mathcal{T}_C = \{A \in L^X : C(A) = A\}$ is an L-closure system on X and $C_{\mathcal{T}}$ is an L-closure operator on X given by $\forall A \in L^X, x \in X, C_{\mathcal{T}}(A)(x) = \bigwedge_{B \in \mathcal{T}} sub(A, B) \rightarrow B(x)$. Moreover, $C = C_{\mathcal{T}_C}$ and $\mathcal{T} = \mathcal{T}_{C_{\mathcal{T}}}$. In particular, $\forall A \in L^X, B \in \mathcal{T}, sub(A, B) = sub(C(A), B)$.

Definition 2 ([36,38]). *Let (X, e) be an L-ordered set. $x_0 \in X$ is called a join (resp., meet) of A denoted by $x_0 = \sqcup A$ (resp., $x_0 = \sqcap A$), iff*

(1) $\forall x \in X, A(x) \leq e(x, x_0)$ (resp., $A(x) \leq e(x_0, x)$);
(2) $\forall y \in X, \bigwedge\limits_{x \in X} A(x) \rightarrow e(x, y) \leq e(x_0, y)$ (resp., $\bigwedge\limits_{x \in X} A(x) \rightarrow e(y, x) \leq e(y, x_0)$).

An L-ordered set (X, e) is a complete L-lattice iff $\forall A \in L^X, \sqcup A$ exists. $A \in L^X$ is called a lower L-subset if $\forall x, y \in X, A(x) * e(y, x) \leq A(y)$ and $\downarrow A \in L^X$ is defined by $\downarrow A(x) = \bigvee_{x' \in X} A(x') * e(x, x'), \forall x \in X$. As usually, the set of all lower L-subsets of X is denoted by $\mathrm{A}_L X$ and $A \in \mathrm{A}_L X$ iff $A = \downarrow A$. Obviously, $\mathrm{A}_L X = \{\downarrow A \mid A \in L^X\}$ is an L-closure system and so $(\mathrm{A}_L X, sub_X)$ is a complete L-lattice. $A \in L^X$ is called a directed L-subset if $\bigvee_{x \in X} A(x) = 1$ and $A(x) * A(y) \leq \bigvee_{z \in X} A(z) * e(x, z) * e(y, z)$, $\forall x, y \in X$. Let $\mathrm{D}_L X$ denote the set of all directed L-subsets of X. For $x \in X$, a mapping $\iota_x : X \longrightarrow L$ defined by $\forall y \in X$, $\iota_x(y) = e(y, x)$ is called a principal L-ideal of X. Then $\mathrm{M}_L X = \{\iota_x \mid x \in X\} \subseteq \mathrm{A}_L X$. Similarly, for $A \in L^X$, we can obtain a mapping $\iota_A : L^X \longrightarrow L$ defined by $\iota_A(B) = sub(B, A), \forall B \in L^X$.

For each mapping $f : X \rightarrow Y$, we can define $f^{\rightarrow} : L^X \rightarrow L^Y$ and $f^{\leftarrow} : L^Y \rightarrow L^X$ (see [15,35]) as

$$f^{\rightarrow}(A)(y) = \bigvee_{f(x)=y} A(x),\ f^{\leftarrow}(B)(x) = B(f(x)),\ \forall x \in X,\ y \in Y.$$

Definition 3 ([37,38]). *A mapping $f : (X, e_X) \rightarrow (Y, e_Y)$ is called*

(1) *L-order preserving if* $e_X(x, y) \leq e_Y(f(x), f(y)), \forall x, y \in X$;
(2) *L-order embedding if* $e_X(x, y) = e_Y(f(x), f(y)), \forall x, y \in X$;
(3) *join-preserving if* $f(\sqcup A) = \sqcup f^{\rightarrow}(A), \forall A \in L^X$ *with* $\sqcup A$ *existing*;
(4) *join-dense if* $y = \sqcup f^{\rightarrow}(f^{\leftarrow}(\iota_y)), \forall y \in Y$.

For an L-ordered set (X, e), $X' \subseteq X$ is said to be join-dense in X iff for every $x \in X$, there is $A \in L^{X'}$ such that $x = \sqcup i^{\rightarrow}_{X', X}(A)$ iff $x = \sqcup(\iota_x \wedge \chi_{X'}), \forall x \in X$. Therefore, a mapping $f : X \rightarrow Y$ is join-dense iff $f(X)$ is join-dense in Y.

Definition 4 ([35,37]). *If two L-order preserving mappings $f : (X, e_X) \rightarrow (Y, e_Y)$, $g : (Y, e_Y) \rightarrow (X, e_X)$ satisfy $\forall x \in X, y \in Y, e_Y(f(x), y) = e_X(x, g(y))$, then (f, g) is called an L-adjunction between X and Y, where f is called the left L-adjoint of g and dually g the right L-adjoint of f. In this case, a left L-adjoint is also called a residuated mapping and its right L-adjoint a residual mapping.*

Obviously, for each mapping $f : X \to Y$, the pair $(f^{\to}, f^{\leftarrow})$ is an L-adjunction between L^X and L^Y.

Theorem 1 ([35,37])**.** *Let $f : (X, e_X) \to (Y, e_Y)$, $g : (Y, e_Y) \to (X, e_X)$ be two mappings.*

(1) If (X, e_X) is complete, then f has a right L-adjoint iff $f(\sqcup A) = \sqcup f^{\to}(A), \forall A \in L^X$;

(2) If (Y, e_Y) is complete, then g has a left L-adjoint iff $g(\sqcap B) = \sqcap g^{\to}(B), \forall B \in L^Y$.

Clearly, a mapping between complete L-lattices is join-preserving iff it is residuated. For an L-ordered set (X, e) and $A \in L^X$, $A^u, A^l \in L^X$ are defined as follows:

$$A^u(x) = \bigwedge_{x' \in X} A(x') \to e(x', x),\ A^l(x) = \bigwedge_{x' \in X} A(x') \to e(x, x'),\ \forall x \in X.$$

The cut operator $(-)^\delta : L^X \longrightarrow L^X$ is defined by $A^\delta = A^{ul}$. A is called a cut if $A = A^\delta$. Obviously, every cut is always a lower L-subset and the cut operator is an L-closure operator on X. Moreover, $\mathrm{DM}_L X = \{A \in L^X : A = A^\delta\}$ is the Dedekind–MacNeille completion.

Definition 5 ([25,26,30])**.** *A triple $(X, \cdot, e)$ is called an L-ordered semigroup (L-OS for short) provided that*

(1) (X, e) is an L-ordered set;

(2) $(X, \cdot)$ is a semigroup;

(3) $e(x, y) \leq e(z \cdot x, z \cdot y)$ and $e(x, y) \leq e(x \cdot z, y \cdot z), \forall x, y, z \in X$.

Remark 1.

(1) The condition (3) is equivalent to all left translations $\ell_x : X \longrightarrow X,\ x' \mapsto x \cdot x'$ and all right translations $\hbar_x : X \longrightarrow X,\ x' \mapsto x' \cdot x$ are L-order preserving.

(2) An L-OS is called residuated if all left and right translations are residuated.

A mapping $f : (X, \cdot_X) \longrightarrow (X, \cdot_Y)$ is called a semigroup homomorphism if $f(x \cdot_X x') = f(x) \cdot_Y f(x'), \forall x, x' \in X$. Moreover, a semigroup homomorphism $f : (X, \cdot_X, e_X) \longrightarrow (X, \cdot_Y, e_Y)$ is called (1) an L-OS homomorphism if it is L-ordered preserving; (2) an L-OS embedding if it is an L-ordered embedding. Then L-OSs with L-OS homomorphisms as morphisms form a category, denoted by L-**OSG**.

A complete L-OS $(X, \cdot, e)$ means $\sqcup A$ exists for all $A \in L^X$. Let $(X, \cdot, e)$ be an L-OS and $A, B \in L^X$, define a binary operator $\cdot : L^X \times L^X \longrightarrow L^X$, as follows: $(A \cdot B)(x) = \bigvee_{x' \cdot x'' = x} A(x') * B(x''), \forall x \in X$. Then we can check that $(L^X, \cdot, sub_X)$ is a complete L-OS and $\sqcup \mathcal{A} = \bigvee_{A \in L^X} \mathcal{A}(A) * A, \forall \mathcal{A} \in L^{L^X}$.

3. Standard *L*-Completions for *L*-Ordered Semigroups

In the crisp setting, a systematic analysis of standard completions for ordered semigroups has been developed in [32]. In this section, let us discuss such completions under a fuzzy framework.

An L-subset selection (or a fuzzy subset selection) Z_L denotes a function which assigns to every L-OS $(X, \cdot, e)$ a set $\mathrm{Z}_L X$ of L-subsets of X, and Z_L is called (1) a standard L-completion if $\mathrm{M}_L X \subseteq \mathrm{Z}_L X \subseteq \mathrm{A}_L X$ and $\mathrm{Z}_L X$ is an L-closure system; (2) an L-subset system if for each L-ordered preserving mapping $f : X \longrightarrow X'$ and each $A \in \mathrm{Z}_L X$, $f^{\to}(A) \in \mathrm{Z}_L(X')$. Clearly, there is no implication relationship between the two L-subset selections.

$(X, \cdot, e)$ is called Z_L-complete if $\sqcup D$ exists for all $D \in \mathrm{Z}_L X$. It is noteworthy that for any standard L-completion Z_L, $(X, \cdot, e)$ is Z_L-complete iff (X, e) is complete. Moreover, for every $\mathfrak{X} \subseteq L^X$, we may form the system $\mathrm{Z}_L \mathfrak{X}$ and call $\mathfrak{X}$ Z_L-union complete if $\bigcup \mathcal{A} = \bigvee_{A \in \mathfrak{X}} \mathcal{A}(A) * A \in \mathfrak{X}$ for all $\mathcal{A} \in \mathrm{Z}_L \mathfrak{X}$. Of course, for any $\mathrm{Z}_L \mathfrak{X}$, Z_L-union completeness entails Z_L-completeness, but not conversely. $P \subseteq X$ is called a Z_L-complete subsemigroup if $(P, \cdot)$ is a subsemigroup of $(X, \cdot)$ and $\sqcup i^{\to}_{P,X}(D) \in P$ for all $D \in \mathrm{Z}_L(P)$. To simplify the description, Z_L and $\mathrm{Z}_L X$ are referred to as a standard L-completion without distinction.

Example 1. *Each of the following listed L-subset selection is a standard L-completion:*

(1) *the Alexandroff completion* $\mathrm{A}_L X$;

(2) *the Dedekind–MacNeille completion* $\mathrm{DM}_L X$;

(3) *the L-Frink ideal completion* $\mathrm{F}^{\delta}_L X$ *defined by*

$$\mathrm{F}^{\delta}_L X = \{A \in \mathrm{A}_L X \mid sub(F, A) \leq sub(F^{\delta}, A) \ for\ all\ finite\ L-subsets\ F\ of\ X\},$$

where finite L-subset F means that $\{x \in X : F(x) \neq 0\}$ *is finite;*

(4) *the* $\sqcup$*-ideal completion* $\mathrm{J}_L X$ *given by*

$$\mathrm{J}_L X = \{A \in \mathrm{A}_L X \mid sub(D, A) \leq A(\sqcup D) \ for\ all\ D \in L^X\ with\ \sqcup D\ existing\};$$

(5) *the Scott completion* $\mathrm{D}^{\delta}_L X$ *defined by*

$$\mathrm{D}^{\delta}_L X = \{A \in \mathrm{A}_L X \mid sub(D, A) \leq sub(D^{\delta}, A) \ for\ all\ directed\ L-subsets\ D\ of\ X\};$$

(6) *the Scott join-completion* $\mathrm{D}^{\sqcup}_L X$ *given by*

$$\mathrm{D}^{\sqcup}_L X = \{A \in \mathrm{A}_L X \mid sub(D, A) \leq A(\sqcup D) \ for\ all\ directed\ L-subsets\ D\ with\ \sqcup D\ existing\}.$$

In fact, it obtained from (4) by admitting directed L-subsets D only;

(7) *the* $\Gamma_{\star}$*-completion defined by* $\Gamma_{\star} X = \{A^{\star} \mid A \in L^X\}$ *(See [31]);*

(8) *Any L-subset selection* Z_L *gives rise to the following standard L-completions:*

$$\mathrm{Z}^{\sqcup}_L X = \{A = \downarrow A \mid sub(D, A) \leq A(\sqcup D) \ for\ all\ D \in \mathrm{Z}_L X\ with\ \sqcup D\ existing\}$$

and

$$\mathrm{Z}^{\delta}_L X = \{A = \downarrow A \mid sub(D, A) \leq sub(D^{\delta}, A) \ for\ all\ D \in \mathrm{Z}_L X\}.$$

Obviously, $\mathrm{Z}^{\delta}_L X \subseteq \mathrm{Z}^{\sqcup}_L X$ *and for every* Z_L*-complete L-ordered semigroup* $(X, \cdot, e)$*,* $\mathrm{Z}^{\delta}_L X = \mathrm{Z}^{\sqcup}_L X$*.*

Through simple calculation we get the following conclusion:

Proposition 2. *For an L-OS* $(X, \cdot, e)$*,* $\mathrm{A}_L X$ *is the greatest standard L-completion and* $\mathrm{DM}_L X$ *is the least one.*

Proof. It is obviously true that $\mathrm{A}_L X$ is the greatest standard L-completion. So we only need to show that $\mathrm{DM}_L X$ is the least standard L-completion. In fact, for any $x \in X$ and any standard L-completion Z_L with $\mathrm{Z}_L X \subseteq \mathrm{DM}_L X$, then $A^{\delta}(x) = \bigwedge_{s \in X} sub(A, \iota_s) \to \iota_s(x) \geq \bigwedge_{B \in \mathrm{Z}_L X} sub(A, B) \to B(x) = C(A)(x)$ and $A^{\delta}(x) = \bigwedge_{B \in \mathrm{DM}_L X} sub(A, B) \to B(x) \leq \bigwedge_{B \in \mathrm{Z}_L X} sub(A, B) \to B(x) = C(A)(x)$ and so $A^{\delta} = C(A)$. This means that $\mathrm{Z}_L X = \mathrm{DM}_L X$. □

Obviously, for an arbitrary standard L-completion Z_L, $(\mathrm{Z}_L X, \odot, sub)$ is a complete L-OS by [30], where $\odot : L^X \times L^X \longrightarrow L^X$ is defined by $A \odot B = C(A \cdot B) = \bigwedge_{A' \in \mathrm{Z}_L X} sub(A \cdot B, A') \to A'$.

Example 2. *For an L-OS* $(X, \cdot, e)$*,* $\mathrm{D}_L X$*,* $\mathrm{A}_L X$*, and* $\mathrm{P}_L X = L^X$ *are L-subset selections and so* $\mathrm{D}^{\sqcup}_L X$*,* $\mathrm{A}^{\sqcup}_L X$*,* $\mathrm{P}^{\sqcup}_L X$*,* $\mathrm{D}^{\delta}_L X$*,* $\mathrm{A}^{\delta}_L X$*,* $\mathrm{P}^{\delta}_L X$ *are standard L-completions.*

Given an arbitrary standard L-completion Z_L, we call a mapping $f : (X, \cdot_X, e_X) \longrightarrow (Y, \cdot_Y, e_Y)$ weakly Z_L-continuous if $f^{\leftarrow}(\iota_y) \in \mathrm{Z}_L X$ for all $y \in Y$, and we call f Z_L-continuous if $f^{\leftarrow}(B) \in \mathrm{Z}_L X$ for all $B \in \mathrm{Z}_L Y$. Then we can check that for a standard L-completion Z_L, f is Z_L-continuous iff $f^{\rightarrow}(C(A)) \subseteq C(f^{\rightarrow}(A)), \forall A \in L^X$.

Proposition 3. *Let Z_L be an L-subset selection and consider the following conditions on a mapping $f:(X,\cdot_X,e_X)\longrightarrow(X',\cdot_{X'},e_{X'})$:*

(1a) f is Z_L^{δ}-continuous;
(1b) f is weakly Z_L^{δ}-continuous;
(1c) f is L-order preserving and $f^{\rightarrow}(D^{\delta})\subseteq(f^{\rightarrow}(D))^{\delta},\forall D\in\mathrm{Z}_L X$;
(2a) f is $\mathrm{Z}_L^{\sqcup}$-continuous;
(2b) f is weakly $\mathrm{Z}_L^{\sqcup}$-continuous;
(2c) f is L-order preserving and $f(\sqcup D)=\sqcup f^{\rightarrow}(D)$ for all $D\in\mathrm{Z}_L X$ with $\sqcup D$ existing.

Then

$$\begin{array}{c}(1a)\Rightarrow(1b)\Leftrightarrow(1c)\\ \Downarrow\\ (2a)\Rightarrow(2b)\Leftrightarrow(2c).\end{array}$$

If Z_L is an L-subset system then $(Na)\Leftrightarrow(Nb)\Leftrightarrow(Nc)$ $(N=1,2)$. If, in addition, $(X,\cdot_X,e_X)$ is Z_L-complete then all six conditions are equivalent.

Proof. (1a)⇒(1b): It is obvious by $\mathrm{M}_L X'\subseteq\mathrm{Z}_L^{\delta}X'$.

(1b)⇒(1c): Obviously, f is L-order preserving. Now we need to show that $f^{\rightarrow}(D^{\delta})\subseteq(f^{\rightarrow}(D))^{\delta},\forall D\in\mathrm{Z}_L X$. In fact, for any $D\in\mathrm{Z}_L X$ and any $x'\in X'$,

$$\begin{aligned}(f^{\rightarrow}(D))^{\delta}(x')&=\bigwedge_{x''\in X'}sub_{X'}(f^{\rightarrow}(D),\iota_{x''})\rightarrow\iota_{x''}(x')\\&=\bigwedge_{x''\in X'}sub_{X}(D,f^{\leftarrow}(\iota_{x''}))\rightarrow\iota_{x''}(x')\\&\geq\bigwedge_{x''\in X'}sub_{X}(D^{\delta},f^{\leftarrow}(\iota_{x''}))\rightarrow\iota_{x''}(x')\\&=\bigwedge_{x''\in X'}sub_{X'}(f^{\rightarrow}(D^{\delta}),\iota_{x''})\rightarrow\iota_{x''}(x')\\&=(f^{\rightarrow}(D^{\delta}))^{\delta}(x')\geq f^{\rightarrow}(D^{\delta})(x').\end{aligned}$$

(1c)⇒(1b): Suppose that f is L-order preserving and Z_L-join preserving, then for any $x'\in X'$, $f^{\leftarrow}(\iota_{x'})$ is a lower L-subset. Moreover, for any $D\in\mathrm{Z}_L X$,

$$\begin{aligned}sub_X(D,f^{\leftarrow}(\iota_{x'}))=sub_{X'}(f^{\rightarrow}(D),\iota_{x'})&\leq sub_{X'}((f^{\rightarrow}(D))^{\delta},\iota_{x'})\\&\leq sub_{X'}(f^{\rightarrow}(D^{\delta}),\iota_{x'})=sub_X(D^{\delta},f^{\leftarrow}(\iota_{x'})).\end{aligned}$$

Therefore $f^{\leftarrow}(\iota_{x'})\in\mathrm{Z}_L^{\delta}X$.

If Z_L is an L-subset system, then we have (1c)⇒(1a). In fact, for any $B\in\mathrm{Z}_L^{\delta}X$ and any $D\in\mathrm{Z}_L X$, $f^{\leftarrow}(B)$ is a lower L-subset and $f^{\rightarrow}(D)\in\mathrm{Z}_L X'$; moreover,

$$\begin{aligned}sub_X(D,f^{\leftarrow}(B))=sub_{X'}(f^{\rightarrow}(D),B)&\leq sub_{X'}((f^{\rightarrow}(D))^{\delta},B)\\&\leq sub_{X'}(f^{\rightarrow}(D^{\delta}),B)=sub_X(D^{\delta},f^{\leftarrow}(B)).\end{aligned}$$

Therefore $f^{\leftarrow}(B)\in\mathrm{Z}_L^{\delta}X$.

(1b)⇒(2b): It holds by $\mathrm{Z}_L^{\delta}X\subseteq\mathrm{Z}_L^{\sqcup}X$.

(2a)⇒(2b): It is obvious by $\mathrm{M}_L X'\subseteq\mathrm{Z}_L^{\sqcup}X'$.

(2b)⇒(2c): Similar to (1b)⇒(1c), we only need to show that $f(\sqcup D)=\sqcup f^{\rightarrow}(D)$ for all $D\in\mathrm{Z}_L X$ with $\sqcup D$ existing. In fact, for any $x'\in X'$ and any $D\in\mathrm{Z}_L X$ with $\sqcup D$ existing, we have

$$\begin{aligned}e_{X'}(f(\sqcup D),x')=f^{\leftarrow}(\iota_{x'})(\sqcup D)&\geq sub_X(D,f^{\leftarrow}(\iota_{x'}))=sub_{X'}(f^{\rightarrow}(D),\iota_{x'})\\&=\bigwedge_{x''\in X'}f^{\rightarrow}(D)(x'')\rightarrow e_{X'}(x'',x').\end{aligned}$$

Moreover, for any $x \in X$ and any $x' \in X'$, we have $e_{X'}(f(x), f(\sqcup D)) \geq e_X(x, \sqcup D) \geq D(x)$, and so $f^{\rightarrow}(D)(x') = \bigvee_{f(x)=x'} D(x) \leq e_{X'}(x', f(\sqcup D))$. Therefore, $f(\sqcup D) = \sqcup f^{\rightarrow}(D)$.

(2c)⇒(2b): For any $x' \in X'$ and any $D \in \mathrm{Z}_L X$ with $\sqcup D$ existing, $f^{\leftarrow}(\iota_{x'})$ is a lower L-subset and $sub_X(D, f^{\leftarrow}(\iota_{x'})) = sub_{X'}(f^{\rightarrow}(D), \iota_{x'}) = e_{X'}(\sqcup f^{\rightarrow}(D), x') = e_{X'}(f(\sqcup D), x') = f^{\leftarrow}(\iota_{x'})(\sqcup D)$. Therefore $f^{\leftarrow}(\iota_{x'}) \in \mathrm{Z}_L^{\sqcup} X'$ and thus f is weakly $\mathrm{Z}_L^{\sqcup}$-continuous.

Suppose that Z_L is an L-subset system, then (1c)⇒(1a). In fact, for any $B \in \mathrm{Z}_L^{\delta} X$ and any $D \in \mathrm{Z}_L X$, $f^{\leftarrow}(B)$ is a lower L-subset and $f^{\rightarrow}(D) \in \mathrm{Z}_L X'$; moreover,

$$sub_X(D, f^{\leftarrow}(B)) = sub_{X'}(f^{\rightarrow}(D), B) \leq B(\sqcup f^{\rightarrow}(D)) = B(f(\sqcup D) = f^{\leftarrow}(B)(\sqcup D).$$

Therefore, $f^{\leftarrow}(B) \in \mathrm{Z}_L^{\sqcup} X$. Suppose that $(X', \cdot_{X'}, e_{X'})$ is Z_L-complete, then $(1c) \Leftrightarrow (2c)$. In fact, for any $D \in \mathrm{Z}_L X$, $\sqcup D$ and $\sqcup f^{\rightarrow}(D)$ exist. Moreover, $e_{X'}(\sqcup f^{\rightarrow}(D), f(\sqcup D)) = \bigwedge_{x \in X} D(x) \to e_{X'}(f(x), f(\sqcup D)) \geq \bigwedge_{x \in X} D(x) \to e_X(x, \sqcup D) = e_X(\sqcup D, \sqcup D) = 1$ and

$$\begin{aligned} e_{X'}(f(\sqcup D), \sqcup f^{\rightarrow}(D)) &= sub_{X'}(\iota_{f(\sqcup D)}, \iota_{\sqcup f^{\rightarrow}(D)}) = sub_{X'}(\iota_{f(\sqcup D)}, (f^{\rightarrow}(D))^{\delta}) \\ &\geq sub_{X'}(\iota_{f(\sqcup D)}, f^{\rightarrow}(D^{\delta})) = sub_{X'}(\iota_{f(\sqcup D)}, f^{\rightarrow}(\iota_{\sqcup D})) \\ &= f^{\rightarrow}(\iota_{\sqcup D})(f(\sqcup D)) = \bigvee_{f(x)=f(\sqcup D)} e_X(x, \sqcup D) \\ &\geq e_X(\sqcup D, \sqcup D) = 1. \end{aligned}$$

□

Clearly, all identity mappings are Z_L-continuous and the compositions of Z_L-continuous mappings are again Z_L-continuous. Thus all L-OSs with Z_L-continuous L-OS homomorphisms form a category denoted by L-$\mathbf{OSG}_{Z_L}$, and it is a subcategory of L-**OSG**. We call a standard L-completion Z_L compositive if the class of all weakly Z_L-continuous mappings is closed under composition.

Lemma 1.

(1) *The embedding $\eta_X : X \longrightarrow \mathrm{Z}_L X$, $x \mapsto \iota_x$ is weakly Z_L-continuous, and so are all corestrictions $\eta_X : X \longrightarrow \mathfrak{X}$ with $\mathrm{M}_L X \subseteq \mathfrak{X} \subseteq \mathrm{Z}_L X$; $\eta_X : X \longrightarrow \mathfrak{X}$ is Z_L-continuous iff $\bigcup \mathcal{A} \in \mathfrak{X}$ for all $\mathcal{A} \in \mathrm{Z}_L \mathfrak{X}$.*

(2) *An arbitrary mapping $f : X \longrightarrow X'$ is Z_L-continuous iff the composite mapping $\eta_{X'} \circ f : X \longrightarrow \mathrm{Z}_L X'$ is weakly Z_L-continuous.*

Proof. (1) For any $D \in \mathrm{Z}_L X$ and any $x \in X$, $\eta_X^{\leftarrow}(\iota_D)(x) = \iota_D(\eta_X(x)) = sub_X(\iota_x, D) = D(x) \in \mathrm{Z}_L X$, which implies that $\eta_X : X \longrightarrow \mathrm{Z}_L X$ is weakly Z_L-continuous. Similarly, we can check that $\eta_X : X \longrightarrow \mathfrak{X}$ is also weakly Z_L-continuous. Note the following fact: $\forall \mathcal{A} \in \mathrm{Z}_L \mathfrak{X}, x \in X$, $\eta_X^{\leftarrow}(\mathcal{A})(x) = \mathcal{A}(\iota_x) = \bigvee_{A \in \mathfrak{X}} \mathcal{A}(A) * sub_X(\iota_x, A) = \bigvee_{A \in \mathfrak{X}} \mathcal{A}(A) * A(x) = (\bigcup \mathcal{A})(x)$. Therefore, $\eta_X : X \longrightarrow \mathfrak{X}$ is Z_L-continuous iff $\bigcup \mathcal{A} \in \mathfrak{X}$ for all $\mathcal{A} \in \mathrm{Z}_L \mathfrak{X}$.

(2) Suppose that $f : X \longrightarrow X'$ is Z_L-continuous. By (1), we have that for any $D \in \mathrm{Z}_L X'$, $\eta_{X'}^{\leftarrow}(\iota_D) \in \mathrm{Z}_L X'$ and so $(\eta_{X'} \circ f)^{\leftarrow}(\iota_D) = f^{\leftarrow}(\eta_{X'}^{\leftarrow}(\iota_D)) \in \mathrm{Z}_L X$. Therefore, $\eta_{X'} \circ f$ is weakly Z_L-continuous. Conversely, suppose that $\eta_{X'} \circ f$ is weakly Z_L-continuous, then for any $D \in \mathrm{Z}_L X'$ and any $x \in X$, $(\eta_{X'} \circ f)^{\leftarrow}(\iota_D) \in \mathrm{Z}_L X$ and $f^{\leftarrow}(D)(x) = D(f(x)) = sub_{X'}(\iota_{f(x)}, D) = \iota_D((\eta_{X'} \circ f)(x)) = (\eta_{X'} \circ f)^{\leftarrow}(\iota_D)(x)$. Thus f is Z_L-continuous. □

According to Lemma 1 we immediately get the following characterization of Z_L-union completeness and compositeness.

Proposition 4.

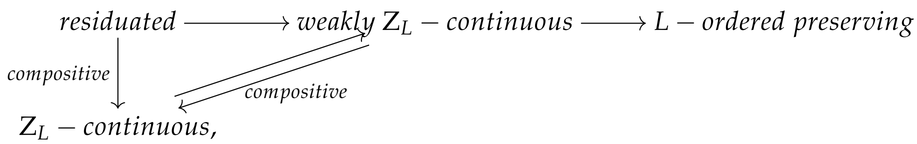
(1) *A standard L-completion Z_L is compositive iff every weakly Z_L-continuous mapping is already Z_L-continuous.*

(2) *A standard L-completion Z_L is Z_L-union complete iff for each L-OS $(X, \cdot, e)$ the embedding η_X is Z_L-continuous.*

(3) *Every compositive standard L-completion* Z_L *is* Z_L*-union complete.*
(4) *For every compositive standard L-completion* Z_L*, every residuated mapping is* Z_L*-continuous.*

Corollary 1. *For an L-subset system* Z_L*, the standard L-completions* Z_L^{δ} *and* $\mathrm{Z}_L^{\sqcup}$ *are compositive.*

From Proposition 4, it is easy to check the following relationships among mappings:



where "compositive" means the standard L-completion Z_L is compositive.

4. Z_L-Completions for Z_L-Semigroups

In this section, we shall consider completions for Z_L-semigroups. In one special case, it is the $\mathcal{Q}$-quantale completion of the $\mathcal{Q}$-ordered semigroup constructed and characterized in [30,31].

Definition 6. *Let* Z_L *be a standard L-completion, we call an L-OS a* Z_L*-semigroup if all left and right translations are* Z_L*-continuous.*

Obviously, when $L = \{0, 1\}$, a Z_L-semigroup is just the $\mathcal{Y}$-semigroup defined in [32].

Example 3.

(1) *An L-OS is an* A_L*-semigroup* $(X, \cdot, e)$ *since* $\forall x \in X, A \in \mathrm{A}_L X$, $\ell_x^{\leftarrow}(A)$ *and* $\hbar_x^{\leftarrow}(A)$ *are still lower L-subsets.*
(2) *A* J_L*-semigroup is characterized by the identities*

$$\ell_x(\sqcup A) = \sqcup \ell_x^{\rightarrow}(A) \text{ and } \hbar_x(\sqcup A) = \hbar_x^{\rightarrow}(A) \qquad (*)$$

for all L-subsets $A \in L^X$ *with* $\sqcup A$ *existing. Moreover, we can check that the equation (*) is equivalent to* $\sqcup A \cdot \sqcup B = \sqcup(A \cdot B)$ *for any* $A, B \in L^X$ *with* $\sqcup A, \sqcup B$ *existing.*
(3) *The example (2) can be generalized in an obvious way. Given an arbitrary L-subset system* Z_L*, it is easy to check that the* $\mathrm{Z}_L^{\sqcup}$*-semigroups are those in which the identities (*) are valid for any L-subset* $A \in \mathrm{Z}_L X$ *with* $\sqcup A$ *existing.*
(4) *Let* (X, e) *be a meet L-semilattice (i.e., each finite L-subset has a meet), then* $(X, \leq_e)$ *is a meet-semilattice and so* $(X, \wedge, e)$ *is an L-OS, where* $\wedge$ *is the meet operation on* $(X, \leq_e)$*. In this case,* $(X, \wedge, e)$ *is a* $\mathrm{D}_L^{\sqcup}$*-semigroup characterized by* $\wedge_x(\sqcup D) = \sqcup \wedge_x^{\rightarrow}(D)$ *for all* $x \in X$ *and all L-ideals D possessing a join. This means that* $(X, \wedge, e)$ *is* $\mathrm{D}_L^{\sqcup}$*-semigroup iff* (X, e) *is meet-continuous (See [40]).*
(5) *Every residuated L-OS is a* Z_L*-semigroup for compositive* Z_L*.*
(6) $(X', \cdot, e)$ *is a* Z_L*-subsemigroup of* $(X, \cdot, e)$ *iff it is an L-ordered subsemigroup of* $(X, \cdot, e)$ *and a* Z_L*-semigroup with respect to the induced L-order.*

Proposition 5. *Let* $(X, \cdot, e)$ *be a* Z_L*-semigroup and* $A, B \in L^X$*. Then*

(1) $A \cdot \mathrm{C}(B) \subseteq A \odot B$ *and* $\mathrm{C}(A) \cdot B \subseteq A \odot B$*;*
(2) $A \odot B = A \odot \mathrm{C}(B) = \mathrm{C}(A) \odot B = \mathrm{C}(A) \odot \mathrm{C}(B)$*; in particular,* $A \odot (B \odot D) = A \odot (B \cdot D) = (A \cdot B) \odot D = (A \odot B) \odot D$*;*
(3) $(\mathrm{Z}_L X, \odot, sub)$ *is a complete residuated L-OS;*
(4) *The natural embedding* $\eta_X : X \longrightarrow \mathrm{Z}_L X,\ x \mapsto \iota_x$ *is a join-dense and weakly* Z_L*-continuous semigroup homomorphism.*

Proof. (1) Let $x \in X$ and $B \in L^X$, $\ell_x^{\rightarrow}(C(B)) \subseteq C(\ell_x^{\rightarrow}(B))$ by the Z_L-continuity of translations. Then for any $A, B \in L^X$,

$$\begin{aligned} sub(A \cdot C(B), C(A \cdot B)) &= \bigwedge_{x \in X} (\bigvee_{s \cdot t = x} A(s) * C(B)(t)) \to C(A \cdot B)(x) \\ &= \bigwedge_{s \in X} A(s) \to sub(\ell_s^{\rightarrow}(C(B)), C(A \cdot B)) \\ &\geq \bigwedge_{s \in X} A(s) \to sub(C(\ell_s^{\rightarrow}(B)), C(A \cdot B)) \\ &\geq \bigwedge_{s \in X} A(s) \to sub(B, \ell_s^{\leftarrow}(A \cdot B)) \\ &= \bigwedge_{s \in X} A(s) \to (\bigwedge_{t \in X} B(t) \to (\bigvee_{s' \cdot t' = s \cdot t} A(s') * B(t'))) \\ &\geq \bigwedge_{s \in X} A(s) \to (\bigwedge_{t \in X} B(t) \to A(s) * B(t)) = 1, \end{aligned}$$

which implies $A \cdot C(B) \subseteq C(A \cdot B)$. Similarly, we have $C(A) \cdot B \subseteq A \odot B$.

(2) By (1), it is easy to check.

(3) It only needs to show that $(Z_L X, \odot, sub)$ is residuated. For $x \in X$ and any $A, B \in Z_L X$, $\ell_x^{\leftarrow}(B) \in Z_L X$ and so $\bigwedge_{x \in X} A(x) \to \ell_x^{\leftarrow}(B) \in Z_L X$ by Proposition 1. Now, we can define two mappings $\ell_A : Z_L X \longrightarrow Z_L X$ and $\bar{\ell}_A : Z_L X \longrightarrow Z_L X$ as follows: $B \mapsto A \odot B$ and $B \mapsto \bigwedge_{x \in X} A(x) \to \ell_x^{\leftarrow}(B)$, respectively. Then we have

$$\begin{aligned} sub(A \odot B, D) = sub(A \cdot B, D) &= \bigwedge_{x \in X} (\bigvee_{s \cdot t = x} A(s) * B(t)) \to D(x) \\ &= \bigwedge_{s,t \in X} A(s) * B(t) \to D(st) = \bigwedge_{s,t \in X} A(s) * B(t) \to \ell_s^{\leftarrow}(D)(t) \\ &= \bigwedge_{s,t \in X} B(t) \to (\bigwedge_{s \in X} A(s) \to \ell_s^{\leftarrow}(D)(t)) \\ &= \bigwedge_{s,t \in X} B(t) \to \bar{\ell}_A(D)(t) = sub(B, \bar{\ell}_A(D)). \end{aligned}$$

This implies that ℓ_A is a residuated mapping. Similarly, $\hbar_A$ is residuated. Therefore, $(Z_L X, \odot, sub)$ is residuated.

(4) Obviously, η_X is weakly Z_L-continuous by Lemma 1 (1). Moreover, for all $A \in Z_L X$ and all $x \in X$,

$$\begin{aligned} \sqcup i^{\rightarrow}_{M_L X, L^X}(i^{\leftarrow}_{M_L X, Z_L X}(\iota_A))(x) &= \bigvee_{B \in L^X} i^{\rightarrow}_{M_L X, L^X}(i^{\leftarrow}_{M_L X, Z_L X}(\iota_A))(B) * B(x) \\ &= \bigvee_{B \in L^X} \bigvee_{i_{M_L X, L^X}(B') = B} i^{\leftarrow}_{M_L X, Z_L X}(\iota_A)(B') * B(x) \\ &= \bigvee_{x' \in X} \iota_A(\iota_{x'}) * \iota_{x'}(x) = A(x), \end{aligned}$$

which implies that $A = \sqcup i^{\rightarrow}_{M_L X, Z_L X}(i^{\leftarrow}_{M_L X, Z_L X}(\iota_A))$. Therefore, η_X is join-dense. Next, we show that η_X is a semigroup homomorphism. In fact, for any $x, x' \in X$,

$$\begin{aligned} sub(\eta_X(x) \odot \eta_X(x'), \eta_X(x \cdot x')) &= \bigwedge_{s,t \in X} \iota_x(s) * \iota_{x'}(t) \to \iota_{x \cdot x'}(s \cdot t) \\ &\geq \bigwedge_{s,t \in X} e(s \cdot x', x \cdot x') * e(s \cdot t, s \cdot x') \to e(s \cdot t, x \cdot x') \\ &\geq \bigwedge_{s,t \in X} e(s \cdot t, x \cdot x') \to e(s \cdot t, x \cdot x') = 1 \end{aligned}$$

and

$$
\begin{aligned}
sub(\eta_X(x \cdot x'), \eta_X(x) \odot \eta_X(x')) &= C(\eta_X(x) \cdot \eta_X(x'))(x \cdot x') \\
&= \bigwedge_{B \in Z_L X} sub(\eta_X(x) \cdot \eta_X(x'), B) \to B(x \cdot x') \\
&= \bigwedge_{B \in Z_L X} \Big(\bigwedge_{s,t \in X} e(s,x) * e(t,x') \to B(s \cdot t) \Big) \to B(x \cdot x') \\
&\geq \bigwedge_{B \in Z_L X} B(x \cdot x') \to B(x \cdot x') = 1.
\end{aligned}
$$

Hence $\eta_X(x) \odot \eta_X(x') = \eta_X(x \cdot x')$. □

Remark 2.

(1) *If $(X, \cdot, e)$ is an L-ordered monoid then so is $(Z_L X, \odot, sub)$. In fact, let $s_0 \in X$ be a unit element and $A \in Z_L X$ then $sub(\iota_{s_0} \odot A, A) = \bigwedge_{s \in X} A(s) \to \hbar_s^{\leftarrow}(A)(s_0) = \bigwedge_{s \in X} A(s) \to A(s_0 \cdot s) = \bigwedge_{s \in X} A(s) \to A(s) = 1$ and $sub(A, \iota_{s_0} \odot A) = \bigwedge_{s \in X} A(s) \to (\bigvee_{t \cdot x = s} \iota_{s_0}(t) * A(x)) \geq \bigwedge_{s \in X} A(s) \to (\iota_{s_0}(s_0) * A(s)) = 1$. Thus it follows that $\iota_{s_0} \odot A = A$ and $A \odot \iota_{s_0} = A$ similarly.*

(2) *$(Z_L X, \odot, sub)$ of commutative Z_L-semigroup $(X, \cdot, e)$ is again commutative. From $(A \cdot B)(x) = \bigvee_{s \cdot t = x} A(s) * B(t) = \bigvee_{t \cdot s = x} B(t) * A(s) = (B \cdot A)(x)$ for any $A, B \in Z_L X$, we have $A \odot B = B \odot A$.*

(3) *For every standard L-completion Z_L, $M_L X$ is join-dense in $Z_L X$.*

(4) *Every residuated L-OS is a J_L-semigroup.*

Definition 7. *Let $(X, \cdot_X, e_X)$ be a Z_L-semigroup. A complete residuated L-OS $(Y, \cdot_Y, e_Y)$ together with a join-dense L-OS embedding $\eta_X : X \longrightarrow Y$ is called a Z_L-completion of $(X, \cdot_X, e_X)$ if $\eta_X : X \longrightarrow Y$ has the following universal property: For any weakly Z_L-continuous semigroup homomorphism $f : X \longrightarrow X'$ mapping into a complete residuated L-OS $(X', \cdot_{X'}, e_{X'})$, there exists a unique join-preserving L-OS homomorphism $\check{f} : Y \longrightarrow X'$ such that $f = \check{f} \circ \eta_X$.*

Clearly, a complete residuated L-OS is a Z_L-completion of itself.

Lemma 2. *Let $f, g : X \longrightarrow Y$ be join-preserving mappings between two Z_L-semigroups $(X, \cdot_X, e_X)$ and $(Y, \cdot_Y, e_Y)$ and X' join-dense in X.*

(1) *If $f(x) = g(x)$ for all $x \in X'$, then $f(x) = g(x), \forall x \in X$;*

(2) *In the case that $(X, \cdot_X, e_X)$ and $(Y, \cdot_Y, e_Y)$ are J_L-semigroups, f is a semigroup homomorphism if $f(s \cdot_X t) = f(s) \cdot_Y f(t), \forall s, t \in X'$.*

Proof. (1) Let $x \in X$, then for any $y \in Y$,

$$
\begin{aligned}
e_Y(f(x), y) &= e_Y(f(\sqcup \iota_x \wedge \chi_{X'}), y) = e_Y(\sqcup f^{\rightarrow}(\iota_x \wedge \chi_{X'}), y) \\
&= \bigvee_{y' \in Y} f^{\rightarrow}(\iota_x \wedge \chi_{X'})(y') \to e_(y', y) = \bigvee_{x' \in X'} \iota_x(x') \to e_Y(f(x'), y) \\
&= \bigvee_{x' \in X'} \iota_x(x') \to e_Y(g(x'), y) = e_Y(g(x), y),
\end{aligned}
$$

which implies that $f(x) = g(x)$ for all $x \in X$.

(2) For $s, t \in X$, let $A = \iota_s \wedge \chi_{X'}$ and $B = \iota_t \wedge \chi_{X'}$. Then $s = \sqcup A$, $t = \sqcup B$, and

$$\begin{aligned}(f^{\rightarrow}(A) \cdot f^{\rightarrow}(B))(y) &= \bigvee_{y_1 \cdot_Y y_2 = y} f^{\rightarrow}(A)(y_1) * f^{\rightarrow}(B)(y_2) \\ &= \bigvee_{y_1 \cdot_Y y_2 = y} (\bigvee_{f(x_1)=y_1} A(x_1)) * (\bigvee_{f(x_2)=y_2} B(x_2)) \\ &= \bigvee_{f(x_1)\cdot_Y f(x_2)=y} A(x_1) * B(x_2) \\ &= \bigvee_{f(x_1 \cdot_X x_2)=y} A(x_1) * B(x_2) = f^{\rightarrow}(A \cdot_X B)(y).\end{aligned}$$

Hence $f(s \cdot_X t) = f(\sqcup A \cdot_X \sqcup B) = f(\sqcup(A \cdot B)) = \sqcup f^{\rightarrow}(A \cdot B) = \sqcup f^{\rightarrow}(A) \cdot_Y \sqcup f^{\rightarrow}(B) = f(\sqcup A) \cdot_Y f(\sqcup B) = f(s) \cdot_Y f(t)$. □

Theorem 2. *For each* Z_L*-semigroup* $(X, \cdot_X, e_X)$*,* $(\mathrm{Z}_L X, \odot, sub)$ *is a* Z_L*-completion of* $(X, \cdot_X, e_X)$*.*

Proof. By Proposition 5 (3) and (4), we only need to show that $\eta_X : X \longrightarrow \mathrm{Z}_L X$ has the universal property. Let $f : X \longrightarrow Y$ be a weakly Z_L-continuous semigroup homomorphism from $(X, \cdot_X, e_X)$ to a complete residuated L-OS $(Y, \cdot_Y, e_Y)$, then for any $A \in \mathrm{Z}_L X$ and $y \in Y$, $\sqcup f^{\rightarrow}(A)$ exists and $f^{\leftarrow}(\iota_y) \in \mathrm{Z}_L X$. Define $\check{f} : \mathrm{Z}_L X \longrightarrow Y$ as $\check{f}(A) = \sqcup f^{\rightarrow}(A)$. Then $e_Y(\check{f}(A), y) = e_Y(\sqcup f^{\rightarrow}(A), y) = sub_Y(f^{\rightarrow}(A), \iota_y) = sub_X(A, f^{\leftarrow}(\iota_y))$. This implies that the mapping $\bar{f} : Y \longrightarrow \mathrm{Z}_L X$, $y \mapsto f^{\leftarrow}(\iota_y)$ is a right adjoint of $\check{f}$ and so $\check{f}$ is residuated. Moreover, for any $x \in X$, $e_Y(\check{f} \circ \eta_X(x), f(x)) = e_Y(\check{f}(\iota_x), f(x)) = sub_X(\iota_x, f^{\leftarrow}(\iota_{f(x)})) = f^{\leftarrow}(\iota_{f(x)}))(x) = 1$ and $e_Y(f(x), \check{f} \circ \eta_X(x)) = e_Y(f(x), \sqcup f^{\rightarrow}(\iota_x)) \geq f^{\rightarrow}(\iota_x)(f(x) = \bigvee_{f(x_1)=f(x)} e_X(x_1, x) \geq e_X(x, x) = 1$; thus $\check{f} \circ \eta_X(x) = f(x)$. The uniqueness of $\check{f}$ can be obtained by Lemma 2 and Remark 2 (3). □

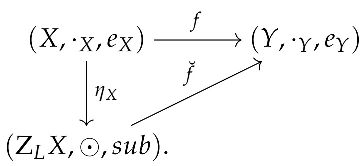
Remark 3.

(1) *In the case of* $\mathrm{Z}_L = DM_L$*, the* Z_L*-completion of a* Z_L*-semigroup is the Dedekind–MacNeille completion.*

(2) *In the case of* $\mathrm{Z}_L = \mathrm{A}_L$*, the* Z_L*-completion of a* Z_L*-semigroup is the Q-quantale completion defined in [31]. Therefore, the Q-quantale completion has the universal property and the* Z_L*-completion can be as a generalization of the Q-quantale completion.*

From the proof of Theorem 2, we can directly reach the following conclusion:

Proposition 6. *We call a weakly* Z_L*-continuous mapping* $f : X \longrightarrow Y$ *between two L-OSs* $(X, \cdot_X, e_X)$ *and* $(Y, \cdot_Y, e_Y)$ *a* Z_L*-arrow if for every* $A \in \mathrm{Z}_L X$*,* $\sqcup f^{\rightarrow}(A)$ *exists. Then a mapping* $f : X \longrightarrow Y$ *between L-OSs* $(X, \cdot_X, e_X)$ *and* $(Y, \cdot_Y, e_Y)$ *is a* Z_L*-arrow iff there exists a unique residuated mapping* $\check{f} : \mathrm{Z}_L X \longrightarrow Y$ *with* $f = \check{f} \circ \eta_X$*, viz.* $\check{f}(A) = \sqcup f^{\rightarrow}(A)$*, i.e., the following commutative diagram:*



5. A Category Characterization of the Z_L-Completion

The Z_L-semigroups together with Z_L-continuous semigroup homomorphisms form a category $\mathbf{S}_{\mathrm{Z}_L}$. For example, $\mathbf{S}_{\mathrm{A}_L}$ is the category of L-OSs and L-OS homomorphisms. Moreover, denote by $\mathbf{CS}_L$ the category of complete residuated L-ordered semigroups together with join-preserving L-OS homomorphisms. Obviously, $\mathbf{CS}_L = Q$-**Quant**, defined in [31]. In this section, we describe the Z_L-completion of a Z_L-semigroup from the perspective of category. Next, we first discuss whether an arbitrary Z_L-continuous semigroup homomorphism $f : X \longrightarrow Y$ may be lifted to a join-preserving semigroup homomorphism between the Z_L-completions $(\mathrm{Z}_L X, \odot, sub_X)$ and $(\mathrm{Z}_L Y, \odot, sub_Y)$.

Lemma 3. *For every Z_L-continuous semigroup homomorphism $f : X \longrightarrow Y$, the mapping $\mathrm{Z}_L f : \mathrm{Z}_L X \longrightarrow \mathrm{Z}_L Y,\ A \mapsto C(f^{\rightarrow}(A))$ is a join-preserving semigroup homomorphism.*

Proof. Since $f : X \longrightarrow Y$ is Z_L-continuous, $f^{\leftarrow}(B') \in \mathrm{Z}_L X$ for any $B' \in \mathrm{Z}_L Y$. So we obtain a mapping $\mathrm{Z}_L^* f : \mathrm{Z}_L Y \longrightarrow \mathrm{Z}_L X, B' \mapsto f^{\leftarrow}(B')$. Then for any $A \in \mathrm{Z}_L X$ and any $B' \in \mathrm{Z}_L Y$,

$$sub_X(A, \mathrm{Z}_L^* f(B')) = sub_Y(f^{\rightarrow}(A), B') = sub_Y(C(f^{\rightarrow}(A)), B') = sub_Y(\mathrm{Z}_L f(A)), B').$$

Hence $\mathrm{Z}_L f$ is residuated and thus is join-preserving by Theorem 1. Moreover, for any $A, B \in \mathrm{Z}_L X$, $\mathrm{Z}_L f(A) \odot \mathrm{Z}_L f(B) = C(f^{\rightarrow}(A)) \odot C(f^{\rightarrow}(B)) = f^{\rightarrow}(A) \odot f^{\rightarrow}(B) = C(f^{\rightarrow}(A) \cdot f^{\rightarrow}(B)) = C(f^{\rightarrow}(A \cdot B))$ and so $\mathrm{Z}_L f(A) \odot \mathrm{Z}_L f(B) \subseteq C(f^{\rightarrow}(A \odot B)) = \mathrm{Z}_L f(A \odot B)$. On the other hand, for all $y \in Y$,

$$\begin{aligned}
\mathrm{Z}_L f(A \odot B)(y) = C(f^{\rightarrow}(A \odot B))(y) &= \bigwedge_{D \in \mathrm{Z}_L Y} sub_Y(f^{\rightarrow}(A \odot B), D) \rightarrow D(y) \\
&= \bigwedge_{D \in \mathrm{Z}_L Y} sub_X(A \odot B, f^{\leftarrow}(D)) \rightarrow D(y) \\
&= \bigwedge_{D \in \mathrm{Z}_L Y} sub_X(C(A \cdot B), C(f^{\leftarrow}(D))) \rightarrow D(y) \\
&\leq \bigwedge_{D \in \mathrm{Z}_L Y} sub_X(A \cdot B, f^{\leftarrow}(D)) \rightarrow D(y) \\
&= \bigwedge_{D \in \mathrm{Z}_L Y} sub_Y(f^{\rightarrow}(A \cdot B), D) \rightarrow D(y) = C(f^{\rightarrow}(A \cdot B)).
\end{aligned}$$

Hence $\mathrm{Z}_L f(A) \odot \mathrm{Z}_L f(B) = \mathrm{Z}_L f(A \odot B)$. This means $\mathrm{Z}_L f$ is a semigroup homomorphism. To sum up, we have that $\mathrm{Z}_L f$ is a join-preserving semigroup homomorphism. □

From this lemma, the following conclusion can be drawn directly:

Corollary 2. *Every standard L-completion Z_L gives rise to a functor between $\mathbf{S}_{\mathrm{Z}_L}$ and $\mathbf{CS}_L$.*

However, it should be pointed out that for a Z_L-continuous mapping f the lifted mapping $\mathrm{Z}_L f$ need not be Z_L-continuous. See the following example:

Example 4. *Define a standard L-completion Z_L by*

$$\mathrm{Z}_L X = \begin{cases} DM_L X & (X, \leq_e) \text{ is a four elements Boolean lattice} \\ DM_L X \bigcup \{0_X\} & \text{otherwise.} \end{cases}$$

Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$ and $L = \{0, 1\}$, define $e_X : X \times X \longrightarrow L$, $e_Y : Y \times Y \longrightarrow L$, $\cdot_X : X \times X \longrightarrow X$ and $\cdot_Y : Y \times Y \longrightarrow Y$ as follows, respectively:

e_X	x_1	x_2
x_1	1	0
x_2	0	1

,

e_Y	y_1	y_2
y_1	1	1
y_2	0	1

,

$\cdot_X$	x_1	x_2
x_1	x_1	x_2
x_2	x_2	x_2

,

$\cdot_Y$	y_1	y_2
y_1	y_1	y_2
y_2	y_2	y_2

.

Then $L^X = \{A_1, A_2, A_3, A_4\}$, $L^Y = \{B_1, B_2, B_3, B_4\}$, where $A_1 = (1,0)$, $A_2 = (0,1)$, $A_3 = (1,1)$, $A_4 = (0,0) = 0_X$, $B_1 = (1,0)$, $B_2 = (0,1)$, $B_3 = (1,1)$, $B_4 = (0,0) = 0_Y$. It is easy to check that $\mathrm{Z}_L X = L^X$ and $\mathrm{Z}_L Y = \{B_1, B_3, B_4\}$. By simple calculations, $(X, \cdot_X, e_X)$ and $(Y, \cdot_Y, e_Y)$ are Z_L-semigroups. Moreover, $(\mathrm{Z}_L X, \leq_{sub_X})$ is a four element Boolean lattice and $(\mathrm{Z}_L Y, \leq_{sub_Y})$ is a three element Boolean lattice and so $\mathrm{Z}_L \mathrm{Z}_L X = DM_L \mathrm{Z}_L X$ and $\mathrm{Z}_L \mathrm{Z}_L Y = DM_L \mathrm{Z}_L Y \bigcup \{0_{\mathrm{Z}_L Y}\}$.

Now, a mapping $f : X \longrightarrow Y$ is defined by $f(x_1) = y_2, f(x_2) = y_1$, then f is obviously Z_L-continuous and $Z_L f$ is residuated. Note that $0_{Z_L Y} \in Z_L Z_L Y$, while $(Z_L f)^{\leftarrow}(0_{Z_L Y}) = 0_{Z_L X} \notin Z_L Z_L X$

Example 4 implies that the functor Z_L does not necessarily induce an endofunctor of $\mathbf{S}_{Z_L}$.

Lemma 4 ([41]). *Let the category $\mathcal{C}$ be the full subcategory of the category $\mathcal{D}$ and $\mathcal{C}$ the reflective subcategory of $\mathcal{D}$, then the inclusion functor $I : \mathcal{C} \longrightarrow \mathcal{D}$ is monadic.*

Proposition 7.

(1) *Every $\mathbf{CS}_L$-morphism $f : X \longrightarrow Y$ satisfies the following commutative diagram:*

$$\begin{array}{ccc} Z_L X & \xrightarrow{Z_L f} & Z_L Y \\ \downarrow \epsilon_X & & \downarrow \epsilon_Y \\ (X, \cdot_X, e_X) & \xrightarrow{f} & (Y, \cdot_Y, e_Y) \end{array},$$

where $\epsilon_X : Z_L X \longrightarrow X$ is defined by $\epsilon_X(A) = \sqcup A$ for all $A \in Z_L X$.

(2) *$\mathbf{CS}_L$ is the full subcategory of the category L-$\mathbf{OSG}_R$ of L-OSs and residuated semigroup homomorphisms.*

(3) *$\mathbf{CS}_L$ is the reflective subcategory of L-$\mathbf{OSG}_R$; moreover, the inclusion functor I from $\mathbf{CS}_L$ to L-$\mathbf{OSG}_R$ is monadic.*

Proof. (1) Let $f : X \longrightarrow Y$ be a $\mathbf{CS}_L$-morphism and $A \in Z_L X$, then $f(\sqcup A) = \sqcup f^{\rightarrow}(A)$ and so

$$\begin{aligned} (f \circ \epsilon_X)(A) &= f(\sqcup A) = \sqcup f^{\rightarrow}(A) = \sqcup C(f^{\rightarrow}(A)) \ (by\ proof\ of\ Lemma\ 3) \\ &= \epsilon_Y(C(f^{\rightarrow}(A))) = \epsilon_Y(Z_L f(A)) = (\epsilon_Y \circ Z_L f)(A). \end{aligned}$$

(2) Clearly, Obj($\mathbf{CS}_L$)$\subseteq$Obj(L-$\mathbf{OSG}_R$). Moreover, for any $(X, e_X), (Y, e_Y)$ $\in$Obj($\mathbf{CS}_L$), $f : X \longrightarrow Y$ $\in$Hom(L-$\mathbf{OSG}_R$) iff f $\in$Hom($\mathbf{CS}_L$) by Theorem 1. Therefore, $\mathbf{CS}_L$ is the full subcategory of L-$\mathbf{OSG}_R$.

(3) By Definition 4.16 of reflective subcategory in [42] and Theorem 2, $\mathbf{CS}_L$ is the reflective subcategory of L-$\mathbf{OSG}_R$ and so the inclusion functor I from $\mathbf{CS}_L$ to L-$\mathbf{OSG}_R$ is monadic by Lemma 4. □

Remark 4. *Obviously, L-$\mathbf{OSG}_R$ is a subcategory of L-$\mathbf{OSG}$ as Obj(L-$\mathbf{OSG}_R$)=Obj(L-$\mathbf{OSG}$) and Hom(L-$\mathbf{OSG}_R$)$\subseteq$ Hom(L-$\mathbf{OSG}$). Then we have that $\mathbf{CS}_L$ is also a reflective subcategory of L-$\mathbf{OSG}$. This is just Proposition 6.19 in [31].*

Theorem 3. *For any standard L-completion Z_L, if every Z_L-arrow into a complete L-OS is Z_L-continuous, then $\mathbf{CS}_L$ is a reflective subcategory of $\mathbf{S}_{Z_L}$ with reflection mappings η_X. In this case, the inclusion functor $I : \mathbf{CS}_L \longrightarrow \mathbf{S}_{Z_L}$ is the right adjoint of the functor $Z_L : \mathbf{S}_{Z_L} \longrightarrow \mathbf{CS}_L$ defined by $Z_L((X, \cdot, e)) = (Z_L X, \odot, sub)$ for any $(X, \cdot, e)$ $\in$Obj($\mathbf{S}_{Z_L}$) and $Z_L(f) = Z_L f$ for any f $\in$Hom($\mathbf{S}_{Z_L}$).*

Proof. Hypothesis and Corollary 2 ensure that $\mathbf{CS}_L$ is a reflective subcategory of $\mathbf{S}_{Z_L}$ with reflection mappings η_X. □

By Proposition 4 and Theorem 3, for every compositive standard L-completion Z_L, $\mathbf{CS}_L$ is a reflective subcategory of $\mathbf{S}_{Z_L}$, but not vice versa. See the following:

Proposition 8. *Let Z_L be an arbitrary standard L-completion such that $Z_L Y = DM_L Y$ for every complete L-OS $(Y, \cdot_Y, e_Y)$. Then $\mathbf{CS}_L$ is a reflective subcategory of $\mathbf{S}_{Z_L}$. However, if $0_X \in Z_L X$*

for at least one Z_L*-semigroup* $(X, \cdot_X, e_X)$ *possessing a least element 0 (i.e.,* $e_X(0, x) = 1, \forall x \in X$*) then* Z_L *is not compositive.*

Proof. Notice that for a complete L-OS $(Y, \cdot_Y, e_Y)$, $\eta_Y : Y \longrightarrow Z_L Y = DM_L Y$ is Z_L-continuous, i.e., $\sqcup(\bigcup \mathcal{A}) \in DM_L Y$ for any $\mathcal{A} \in DM_L DM_L Y$. In fact, $\forall y \in Y$,

$$\begin{aligned} C(\sqcup(\bigcup \mathcal{A}))(y) &= \bigwedge_{A \in DM_L X} sub(\sqcup(\bigcup \mathcal{A}), A) \to A(y) \\ &= \bigwedge_{A \in DM_L X} \Big(\bigwedge_{A' \in DM_L X} \mathcal{A}(A') \to sub(A', A)\Big) \to sub(\iota_y, A) \\ &= \mathcal{A}^{\delta}(\iota_y) = \mathcal{A}(\iota_y) = \bigvee_{A \in DM_L X} \mathcal{A}(A) * sub(\iota_y, A) \\ &= \bigvee_{A \in DM_L X} \mathcal{A}(A) * A(y) = \sqcup(\bigcup \mathcal{A})(y). \end{aligned}$$

Then for every Z_L-arrow f into a complete L-OS $(Y, \cdot_Y, e_Y)$, $\eta_Y \circ f$ is weakly Z_L-continuous and so f is Z_L-continuous by Lemma 1 (2). Therefore, $\mathbf{CS}_L$ is a reflective subcategory of $\mathbf{S}_{Z_L}$. However, if $(X, \cdot_X, e_X)$ has a least element 0 and $0_X \in Z_L X$ then the inclusion mapping $i : \{0\} \longrightarrow X$ is weakly Z_L-continuous but not Z_L-continuous since $i^{\leftarrow}(0_X) = 0_{\{0\}} \notin Z_L\{0\}$. □

Lemma 5. *Let* $f : X \longrightarrow Y$ *be a weakly* Z_L*-continuous L-OS embedding between an L-OS* $(X, \cdot_X, e_X)$ *and a complete residuated L-OS* $(Y, \cdot_Y, e_Y)$*. Then the following three conditions are equivalent:*

(1) f *is join-dense;*

(2) $\check{f} : Z_L Y \longrightarrow Y, B \mapsto \sqcup f^{\rightarrow}(B)$ *is surjective;*

(3) There is a (unique) surjective $\mathbf{CS}_L$*-morphism* $g : Z_L X \longrightarrow Y$ *with* $f = g \circ \eta_X$*.*

Proof. (1)⇒(2): For any $y \in Y$, we have $B = f^{\leftarrow}(\iota_y) \in Z_L Y$ and so $y = \sqcup f^{\rightarrow}(f^{\leftarrow}(\iota_y)) = \check{f}(B)$. Hence $\check{f}$ is surjective.

(2)⇒(3): Obviously.

(3)⇒(1): For any $y \in Y$, there is $A \in Z_L X$ with $y = g(A) = g(\sqcup \eta_X^{\rightarrow}(A)) = \sqcup g^{\rightarrow}(\eta_X^{\rightarrow}(A)) = \sqcup(g \circ \eta_X)^{\rightarrow}(A)) = \sqcup f^{\rightarrow}(A)$. Let $B(y) = f^{\rightarrow}(A)(y), \forall y \in f(X)$, then $B \in L^{f(X)}$ and $i^{\rightarrow}(B)(y') = f^{\rightarrow}(A)(y'), \forall y' \in Y$. Hence $y = \sqcup i^{\rightarrow}(B)$. This means that f is join-dense. □

The category $\mathbf{S}_{\overrightarrow{Z_L}}\mathbf{E}$ of weakly Z_L-continuous L-OS embeddings of Z_L-semigroups is defined as follows: The objects of $\mathbf{S}_{\overrightarrow{Z_L}}\mathbf{E}$ are weakly Z_L-continuous L-OS embeddings of Z_L-semigroups into complete residuated L-OSs and the morphisms of $\mathbf{S}_{\overrightarrow{Z_L}}\mathbf{E}$ are join-preserving semigroup homomorphisms between two weakly Z_L-continuous L-OS embeddings, that is, a morphism between two weakly Z_L-continuous L-OS embeddings $f : X \longrightarrow Y$ and $f' : X' \longrightarrow Y'$ is a join-preserving semigroup homomorphism $h : Y \longrightarrow Y'$ such that there is a Z_L-continuous mapping $h_0 : X \longrightarrow X'$ satisfying $h \circ f = f' \circ h_0$. The objects of a full subcategory $\mathbf{S}_{\overrightarrow{Z_L}}\mathbf{C}$ of $\mathbf{S}_{\overrightarrow{Z_L}}\mathbf{E}$ are the weakly Z_L-continuous and join-dense L-OS embeddings. Then for an arbitrary standard L-completion Z_L, we can define a functor $\overleftarrow{\mathbb{P}} : \mathbf{S}_{\overrightarrow{Z_L}}\mathbf{E} \longrightarrow \mathbf{S}_{Z_L}$ with $\overleftarrow{\mathbb{P}} f =$ domain of f for a $\mathbf{S}_{\overrightarrow{Z_L}}\mathbf{E}$-object f and $\overleftarrow{\mathbb{P}} h = h_0$ for a $\mathbf{S}_{\overrightarrow{Z_L}}\mathbf{E}$-morphism h.

Theorem 4. *Let* Z_L *be an arbitrary standard L-completion.*

(1) $\overleftarrow{\mathbb{P}}$ *has a left adjoint* $\overrightarrow{Z_L} : \mathbf{S}_{Z_L} \longrightarrow \mathbf{S}_{\overrightarrow{Z_L}}\mathbf{E}$ *with* $\overrightarrow{Z_L} X = \eta_X$ *for a* $\mathbf{S}_{Z_L}$*-object X and* $\overrightarrow{Z_L} f = Z_L f$ *for a* $\mathbf{S}_{Z_L}$*-morphism f.*

(2) $\overleftarrow{\mathbb{P}}$ *restricts to a forgetful (i.e., faithful) functor from* $\mathbf{S}_{\overrightarrow{Z_L}}\mathbf{C}$ *to* $\mathbf{S}_{Z_L}$ *which is a right adjoint to the corestriction of the functor* $\overrightarrow{Z_L}$ *to* $\mathbf{S}_{\overrightarrow{Z_L}}\mathbf{C}$*.*

(3) *In each of these adjunctions, the counit ϵ is given by $\epsilon_f : \mathrm{Z}_L X \longrightarrow Y$, $Y \mapsto \sqcup f^{\rightarrow}(A)$ for all weakly Z_L-continuous L-OS embeddings $f : X \longrightarrow Y$.*

Proof. Let $f : X \longrightarrow X'$ be a $\mathbf{S}_{\mathrm{Z}_L}$-morphism and $g : X' \longrightarrow Y'$ a weakly Z_L-continuous L-OS embedding. Then $g \circ f$ is weakly Z_L-continuous, and the mapping $(g \breve{\circ} f)$ is the unique $\mathbf{S}_{\overrightarrow{\mathrm{Z}_L}}\mathbf{E}$-morphism $h : \eta_X \longrightarrow g$ with $h_0 = f$ by Proposition 6. This means that the functor $\overleftarrow{\mathbb{P}}$ has a left adjoint $\overrightarrow{\mathrm{Z}_L}$. Therefore, (1) holds and thus (2) and (3) hold. □

6. Conclusions

In order to provide a common completion pattern for L-OSs, we introduced the notion of standard L-completion Z_L as a generalization of various completions, such as the Dedekind–MacNeille completion, the Alexandroff completion, ect. For L-OSs, the basic notions about (weakly) Z_L-continuous mappings arose naturally. In addition, we built Z_L-semigroups and their Z_L-completion with the universal property. We also present the categorical characterization of Z_L-completion under different cases of Z_L. The work is an extension of the crisp case and an attempt to further study asymmetric fuzzy algebraic structures.

Author Contributions: Investigation, S.S. and Q.L. (Qingguo Li); writing—original draft preparation, S.S.; writing—review and editing, Q.L. (Qingguo Li); supervision, Q.L. (Qingguo Li); project administration, Q.L. (Qi Li). All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the National Natural Science Foundation of China under Grant No. 11861006.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Zadeh, L.A. Similarity relations and fuzzy orderings. *Inf. Sci.* **1971**, *3*, 177–200. [CrossRef]
2. Aravantinos-Sotiropoulos, V. The exact completion for regular categories enriched in posets. *Ann. Pure Appl. Log.* **2022**, *226*, 106885. [CrossRef]
3. Erné, M.; Palko, V. Uniform ideal completion. *Math. Slovaca* **1998**, *48*, 327–335.
4. Frink, O. Ideals in partially ordered sets. *Am. Math. Mon.* **1954**, *61*, 223–234. [CrossRef]
5. Gil-Férez, J.; Spada, L.; Tsinakis, C.; Zhou, H.J. Join-completions of partially ordered algebras. *Ann. Pure Appl. Log.* **2020**, *171*, 102842. [CrossRef]
6. Keimel, K.; Lawson, J.D. *D*-completions and the *d*-topology. *Ann. Pure Appl. Log.* **2009**, *159*, 292–306. [CrossRef]
7. MacNeille, H.M. Partially ordered sets. *Trans. Am. Math. Soc.* **1937**, *42*, 416–460. [CrossRef]
8. Zhang, Z.X.; Li, Q.G. A generalization of the Dedekind–MacNeille completion. *Semigroup Forum* **2018**, *96*, 553–564. [CrossRef]
9. Zhang, Z.X.; Li, Q.G. A uniform approach to completions of posets. *J. Log. Algebr. Methods* **2019**, *106*, 107–116. [CrossRef]
10. Zhang, Z.X.; Li, Q.G.; Zhang, N. A unified method for completions of posets and closure spaces. *Soft Comput.* **2019**, *23*, 10699–10708. [CrossRef]
11. Zhang, Z.X.; Shi, F.G.; Li, Q.G. The R-completion of closure spaces. *Topol. Appl.* **2022**, *305*, 107873. [CrossRef]
12. Zhao, D.S.; Fan, H.T. Dcpo-completion of posets. *Theor. Comput. Sci.* **2010**, *411*, 2167–2173. [CrossRef]
13. Zhao, D.S. Closure spaces and completions of posets. *Semigroup Forum* **2015**, *90*, 545–555. [CrossRef]
14. Wagner, K.R. Solving Recursive Domain Equations with Enriched Categories. Ph.D. Thesis, School of Computer Science, Carnegie Mellon University, Pittsburgh, PA, USA, 1994.
15. Bělohlávek, R. *Fuzzy Relational Systems, Foundations and Principles*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 2002.
16. Bělohlávek, R. Concept lattices and order in fuzzy logic. *Ann. Pure Appl. Log.* **2004**, *128*, 277–298. [CrossRef]
17. Xie, W.X.; Zhang, Q.Y.; Fan, L. The Dedekind–MacNeille completions for fuzzy posets. *Fuzzy Sets Syst.* **2009**, *160*, 2292–2316. [CrossRef]
18. Wang, K.Y.; Zhao, B. Join-completions of *L*-ordered sets. *Fuzzy Sets Syst.* **2012**, *199*, 92–107. [CrossRef]
19. Su, S.H.; Li, Q. The relationship between Δ_1-completions and formal *L*-contexts over an *L*-ordered set. *Procedia Comput. Sci.* **2016**, *102*, 18–25.

20. Ma, N.N.; Zhao, B. Some results on fuzzy Z_L-continuous(algebraic) poset. *Soft Comput.* **2018**, *22*, 4549–4559. [CrossRef]
21. Rao, S.P. Theoretical Research into Bases of Fuzzy Domains and Its Generalization. Ph.D. Thesis, College of Mathematics and Econometrics, Hunan University, Changsha, China, 2014. (In Chinese)
22. Zhao, N.N. Fuzzy Z-Domain and Some Properties of Relevant Categories. Master's Thesis, College of Mathematics and Information Science, Shaanxi Normal University, Xi'an, China, 2011. (In Chinese)
23. Su, S.H.; Yang, S.Q.; Li, Q. The Z_L-completions of fuzzy posets. *J. Intell. Fuzzy Syst.* **2020**, *39*, 1347–1359. [CrossRef]
24. Šešlja, B.; Tepavčević, A.; Udovičić, M. Fuzzy ordered structures and fuzzy lattice ordered groups. *J. Intell. Fuzzy Syst.* **2014**, *27*, 1119–1127. [CrossRef]
25. Hao, J. The Research on Pomonoid Actions and Fuzzy Rough Sets. Ph.D. Thesis, College of Mathematics and Econometrics, Hunan University, Changsha, China, 2012. (In Chinese)
26. Wang, K.Y. Some Researches on Fuzzy Domains and Fuzzy Quantales. Ph.D. Thesis, College of Mathematics and Information Science, Shaanxi Normal University, Xi'an, China, 2012. (In Chinese)
27. Borzooei, R.A.; Dvurečenskij, A.; Zahiri, O. L-ordered and L-lattice ordered groups. *Inf. Sci.* **2014**, *314*, 118–134. [CrossRef]
28. Huang, X.K.; Li, Q.G.; Xiao, Q.M. The L-ordered semigroups based on L-partial orders. *Fuzzy Sets Syst.* **2018**, *339*, 31–50. [CrossRef]
29. Su, S.H.; Liu, F.Y.; Yang, S.Q. The L-ordered L-semihypergroups. *Open Math.* **2020**, *18*, 552–571. [CrossRef]
30. Zhao, B.; Wu, S.P.; Wang, K.Y. Quantale algebras as lattice-valued quantales. *Soft Comput.* **2017**, *21*, 2561–2574. [CrossRef]
31. Xia, C.C.; Zhao, B. A categorical characterization of the least Q-quantale completion of Q-ordered semigroups. *Fuzzy Sets Syst.* **2020**, *382*, 129–141. [CrossRef]
32. Erné, M.; Reichman, J.Z. Completions for partially ordered semigroups. *Semigroup Forum* **1987**, *34*, 253–285. [CrossRef]
33. Bělohlávek, R. Fuzzy closure operators. *J. Math. Anal. Appl.* **2001**, *262*, 473–489. [CrossRef]
34. Fan, L. A new approach to quantitative domain theory. *Elec. Notes Theor. Comput. Sci.* **2001**, *45*, 77–87. [CrossRef]
35. Lai, H.L.; Zhang, D.X. Complete and directed complete Ω-categories. *Theor. Comput. Sci.* **2007**, *388*, 1–24. [CrossRef]
36. Yao, W. Quantitative domains via fuzzy sets: Part I: Continuity of fuzzy directed complete posets. *Fuzzy Sets Syst.* **2010**, *161*, 973–987. [CrossRef]
37. Yao, W.; Lu, L.X. Fuzzy Galois connections on fuzzy posets. *Math. Logic Quart.* **2009**, *55*, 84–91. [CrossRef]
38. Zhang, Q.Y.; Fan, L. Continuity in quantitative domains. *Fuzzy Sets Syst.* **2005**, *154* , 118–131. [CrossRef]
39. Bělohlávek, R. Some properties of residuated lattices. *Czech. Math. J.* **2003**, *53*, 161–171. [CrossRef]
40. Su, S.H.; Li, Q.G.; Chen, H.D. The Meet-continuity of L-semilattices. *Electron. Notes Theor. Comput. Sci.* **2017**, *333*, 123–141. [CrossRef]
41. He, W. *Category Theory*; Science Press of China: Beijing, China, 2006.
42. Adámek, J.; Herrlich, H.G.; Strecker, E. *Abstract and Concrete Categories*; Wiley: New York, NY, USA, 1990.