An Accelerated Fixed-Point Algorithm with an Inertial Technique for a Countable Family of $G$-Nonexpansive Mappings Applied to Image Recovery

Kobkoon Janngam $^1$ and Rattanakorn Wattanataweekul $^{2,*}$

1 Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand; kobkoon_jan@cmu.ac.th
2 Department of Mathematics, Statistics and Computer, Faculty of Science, Ubon Ratchathani University, Ubon Ratchathani 34190, Thailand
* Correspondence: rattanakorn.w@ubu.ac.th

Abstract: Many authors have proposed fixed-point algorithms for obtaining a fixed point of $G$-nonexpansive mappings without using inertial techniques. To improve convergence behavior, some accelerated fixed-point methods have been introduced. The main aim of this paper is to use a coordinate affine structure to create an accelerated fixed-point algorithm with an inertial technique for a countable family of $G$-nonexpansive mappings in a Hilbert space with a symmetric directed graph $G$ and prove the weak convergence theorem of the proposed algorithm. As an application, we apply our proposed algorithm to solve image restoration and convex minimization problems. The numerical experiments show that our algorithm is more efficient than FBA, FISTA, Ishikawa iteration, S-iteration, Noor iteration and SP-iteration.

Keywords: convex minimization; coordinate affine; $G$-nonexpansive; image restoration problem; inertial techniques; weak convergence

1. Introduction

Let $H$ be a real Hilbert space with the norm $\| \cdot \|$ and $C$ be a nonempty closed convex subset of $H$. A mapping $T : C \to C$ is said to be nonexpansive if it satisfies the following symmetric contractive-type condition:

$$\|Tx - Ty\| \leq \|x - y\|,$$

for all $x, y \in C$; see [1].

The notation of the set of all fixed points of $T$ is $F(T) := \{ x \in C : x = Tx \}$.

Many mathematicians have studied iterative schemes for finding the approximate fixed-point theorem of nonexpansive mappings over many years; see [2,3]. One of these is the Picard iteration process, which is well known and popular. Picard’s iteration process is defined by

$$x_{n+1} = Tx_n,$$

where $n \geq 1$ and an initial point $x_1$ is randomly selected.

The iterative process of Picard has been developed extensively by many mathematicians, as follows:

Mann iteration process [4] is defined by

$$x_{n+1} = (1 - \rho_n)x_n + \rho_nTx_n,$$  \hspace{1cm} (1)

where $n \geq 1$ and an initial point $x_1$ is randomly selected and $\{\rho_n\}$ is a sequence in $[0, 1]$. 

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Ishikawa iteration process [5] is defined by
\[
\begin{align*}
    y_n &= (1 - \xi_n)x_n + \xi_n Tx_n, \\
    x_{n+1} &= (1 - \rho_n)x_n + \rho_n Ty_n,
\end{align*}
\]
where \( n \geq 1 \) and an initial point \( x_1 \) is randomly selected and \( \{\xi_n\}, \{\rho_n\} \) are sequences in \([0,1]\).

S-iteration process [6] is defined by
\[
\begin{align*}
    y_n &= (1 - \xi_n)x_n + \xi_n Tx_n, \\
    x_{n+1} &= (1 - \rho_n)Tx_n + \rho_n Ty_n,
\end{align*}
\]
where \( n \geq 1 \) and an initial point \( x_1 \) is randomly selected and \( \{\xi_n\}, \{\rho_n\} \) are sequences in \([0,1]\). We know that the S-iteration process (3) is independent of Mann and Ishikawa iterative schemes and converges quicker than both; see [6].

Noor iteration process [7] is defined by
\[
\begin{align*}
    z_n &= (1 - \eta_n)x_n + \eta_n Tx_n, \\
    y_n &= (1 - \xi_n)x_n + \xi_n Tz_n, \\
    x_{n+1} &= (1 - \rho_n)x_n + \rho_n Ty_n,
\end{align*}
\]
where \( n \geq 1 \) and an initial point \( x_1 \) is randomly selected and \( \{\eta_n\}, \{\xi_n\}, \{\rho_n\} \) are sequences in \([0,1]\). We can see that Mann and Ishikawa iterations are special cases of the Noor iteration.

SP-iteration process [8] is defined by
\[
\begin{align*}
    z_n &= (1 - \eta_n)x_n + \eta_n Tx_n, \\
    y_n &= (1 - \xi_n)z_n + \xi_n Ty_n, \\
    x_{n+1} &= (1 - \rho_n)y_n + \rho_n Ty_n,
\end{align*}
\]
where \( n \geq 1 \) and an initial point \( x_1 \) is randomly selected and \( \{\eta_n\}, \{\xi_n\}, \{\rho_n\} \) are sequences in \([0,1]\). We know that Mann, Ishikawa, Noor and SP-iterations are equivalent and the SP-iteration converges faster than the other; see [8].

The fixed-point theory is a rapidly growing field of research because of its many applications. It has been found that a self-map on a set admits a fixed point under specific conditions. One of the recent generalizations is due to Jachymski.

Jachymski [9] proved some generalizations of the Banach contraction principle in a complete metric space endowed with a directed graph using a combination of fixed-point theory and graph theory. In Banach spaces with a graph, Aleomraninejad et al. [10] proposed an iterative scheme for G-contraction and G-nonexpansive mappings. G-monotone nonexpansive multivalued mappings on hyperbolic metric spaces endowed with graphs were defined by Alfuraidan and Khamis [11]. On a Banach space with a directed graph, Alfuraidan [12] showed the existence of fixed points of monotone nonexpansive mappings. For G-nonexpansive mappings in Hilbert spaces with a graph, Tammee et al. [13] demonstrated Browder’s convergence theorem and a strong convergence theorem of the Halpern iterative scheme. The convergence theorem of the three-step iteration approach for solving general variational inequality problems was investigated by Noor [7]. According to [14–17], the three-step iterative method gives better numerical results than the one-step and two-step approximate iterative methods. For approximating common fixed points of a finite family of G-nonexpansive mappings, Suantai et al. [18] combined the shrinking projection with the parallel monotone hybrid method. Additionally, they used a graph to derive a strong convergence theorem in Hilbert spaces under certain conditions and applied it to signal recovery. There is also research related to the application of some fixed-point theorem on the directed graph representations of some chemical compounds; see [19,20].

Several fixed-point algorithms have been introduced by many authors [7,9–18] for finding a fixed point of G-nonexpansive mappings with no inertial technique. Among these algorithms, we need those algorithms that are efficient for solving the problem. So, some
accelerated fixed-point algorithms have been introduced to improve convergence behavior; see [21–28]. Inspired by these works mentioned above, we employed a coordinate affine structure to define an accelerated fixed-point algorithm with an inertial technique for a countable family of $G$-nonexpansive mappings applied to image restoration and convex minimization problems.

This paper is divided into four sections. The first section is the introduction. In Section 2, we recall the basic concepts of mathematics, definitions, and lemmas that will be used to prove the main results. In Section 3, we prove a weak convergence theorem of an iterative scheme with the inertial step for finding a common fixed point of a countable family of $G$-nonexpansive mappings. Furthermore, we apply our proposed method for solving image restoration and convex minimization problems; see Section 4.

2. Preliminaries

The basic concepts of mathematics, definitions, and lemmas discussed in this section are all important and useful in proving our main results.

Let $X$ be a real normed space and $C$ be a nonempty subset of $X$. Let $\Delta = \{(u,u) : u \in C\}$, where $\Delta$ stands for the diagonal of the Cartesian product $C \times C$. Consider a directed graph $G$ in which the set $V(G)$ of its vertices corresponds to $C$, and the set $E(G)$ of its edges contains all loops, that is $E(G) \supseteq \Delta$. Assume that $G$ does not have parallel edges. Then, $G = (V(G), E(G))$. The conversion of a graph $G$ is denoted by $G^{-1}$. Thus, we have

$$E(G^{-1}) = \{(u,v) \in C \times C : (v,u) \in E(G)\}.$$

A graph $G$ is said to be symmetric if $(x,y) \in E(G)$; we have $(y,x) \in E(G)$.

A graph $G$ is said to be transitive if for any $u,v,w \in V(G)$ such that $(u,v), (v,w) \in E(G)$; then, $(u,w) \in E(G)$.

Recall that a graph $G$ is connected if there is a path between any two vertices of the graph $G$. Readers might refer to [29] for additional information on some basic graph concepts.

We say that a mapping $T : C \rightarrow C$ is said to be $G$-contraction [9] if $T$ is edge preserving, i.e., $(Tu, Tv) \in E(G)$ for all $(u,v) \in E(G)$, and there exists $\rho \in [0,1)$ such that

$$\|Tu - Tv\| \leq \rho\|u - v\|$$

for all $(u,v) \in E(G)$, where $\rho$ is called a contraction factor. If $T$ is edge preserving, and

$$\|Tu - Tv\| \leq \|u - v\|$$

for all $(u,v) \in E(G)$, then $T$ is said to be $G$-nonexpansive; see [13].

A mapping $T : C \rightarrow C$ is called $G$-demiclosed at 0 if for any sequence $\{u_n\} \subseteq C$, $(u_n, u_{n+1}) \in E(G)$, $u_n \rightharpoonup u$ and $Tu_n \rightarrow 0$; then, $Tu = 0$.

To prove our main result, we need to introduce the concept of the coordinate affine of the graph $G = (V(G), E(G))$. For any $\alpha, \beta \in \mathbb{R}$ with $\alpha + \beta = 1$, we say that $E(G)$ is said to be left coordinate affine if

$$\alpha(x,y) + \beta(u,y) \in E(G)$$

for all $(x,y), (u,y) \in E(G)$. Similar to this, $E(G)$ is said to be right coordinate affine if

$$\alpha(x,y) + \beta(x,z) \in E(G)$$

for all $(x,y), (x,z) \in E(G)$.

If $E(G)$ is both left and right coordinate affine, then $E(G)$ is said to be coordinate affine.

The following lemmas are the fundamental results for proving our main theorem; see also [21,30,31].
Lemma 1 ([30]). Let \( \{u_n\}, \{w_n\} \) and \( \{\theta_n\} \subset \mathbb{R}^+ \) such that
\[
v_{n+1} \leq (1 + \theta_n)v_n + w_n,
\]
where \( n \in \mathbb{N} \). If \( \sum_{n=1}^{\infty} \theta_n < \infty \) and \( \sum_{n=1}^{\infty} w_n < \infty \), then \( \lim_{n \to \infty} v_n \) exists.

Lemma 2 ([31]). For a real Hilbert space \( H \), the following results hold:
(i) For any \( p \in H \) and \( \gamma \in [0, 1] \),
\[
\|\gamma u + (1 - \gamma)v\|^2 = \gamma\|u\|^2 + (1 - \gamma)\|v\|^2 - \gamma(1 - \gamma)\|u - v\|^2.
\]
(ii) For any \( u, v \in H \),
\[
\|u \pm v\|^2 = \|u\|^2 \pm 2\langle u, v \rangle + \|v\|^2.
\]

Lemma 3 ([21]). Let \( \{v_n\} \) and \( \{\mu_n\} \subset \mathbb{R}^+ \) such that
\[
v_{n+1} \leq (1 + \mu_n)v_n + \mu_nv_{n-1},
\]
where \( n \in \mathbb{N} \). Then,
\[
v_{n+1} \leq M \cdot \prod_{j=1}^{n} (1 + 2\mu_j),
\]
where \( M = \max\{v_1, v_2\} \). Furthermore, if \( \sum_{n=1}^{\infty} \mu_n < \infty \), then \( \{v_n\} \) is bounded.

Let \( \{u_n\} \) be a sequence in \( X \). We write \( u_n \rightharpoonup u \) to indicate that a sequence \( \{u_n\} \) converges weakly to a point \( u \in H \). Similarly, \( u_n \rightarrow u \) will symbolize the strong convergence. For \( v \in C \), if there is a subsequence \( \{u_{n_k}\} \) of \( \{u_n\} \) such that \( u_{n_k} \rightharpoonup v \), then \( v \) is called a weak cluster point of \( \{u_n\} \). Let \( \omega_w(u_n) \) be the set of all weak cluster points of \( \{u_n\} \).

The following lemma was proved by Moudafi and Al-Shemas; see [32].

Lemma 4 ([32]). Let \( \{u_n\} \) be a sequence in a real Hilbert space \( H \) such that there exists \( \emptyset \neq \Lambda \subset H \) satisfying:
(i) For any \( p \in \Lambda, \lim_{n \to \infty} \|u_n - p\| \) exists.
(ii) Any weak cluster point of \( \{u_n\} \in \Lambda \).
Then, there exists \( x^* \in \Lambda \) such that \( u_n \rightharpoonup x^* \).

Let \( \{T_n\} \) and \( \psi \) be families of nonexpansive mappings of \( C \) into itself such that \( \emptyset \neq F(\psi) \subset \cap_{n=1}^{\infty} F(T_n) \), where \( F(\psi) \) is the set of all common fixed points of each \( T \in \psi \). A sequence \( \{T_n\} \) satisfies the NST-condition (I) with \( \psi \) if, for any bounded sequence \( \{u_n\} \) in \( C \),
\[
\lim_{n \to \infty} \|T_nu_n - u_n\| = 0 \text{ implies } \lim_{n \to \infty} \|Tu_n - u_n\| = 0,
\]
for all \( T \in \psi \); see [33]. If \( \psi = \{T\} \), then \( \{T_n\} \) satisfies the NST-condition (I) with \( T \).

The forward–backward operator of lower semi-continuous and convex functions of \( f, g : \mathbb{R}^n \to (-\infty, +\infty] \) has the following definition:
A forward–backward operator \( T \) is defined by \( T := \text{prox}_{\lambda g}(I - \lambda \nabla f) \) for \( \lambda > 0 \), where \( \nabla f \) is the gradient operator of function \( f \) and \( \text{prox}_{\lambda g}x := \text{argmin}_{y \in \mathbb{H}} \left\{ g(y) + \frac{1}{2\lambda} \|y - x\|^2 \right\} \) (see [34,35]). Moreau [36] defined the operator \( \text{prox}_{\lambda g} \) as the proximity operator with respect to \( g \) and function \( \lambda \). Whenever \( \lambda \in (0, 2/L) \), we know that \( T \) is a nonexpansive mapping and \( L \) is a Lipschitz constant of \( \nabla f \). We have the following remark for the definition of the proximity operator; see [37].

Remark 1. Let \( g : \mathbb{R}^n \to \mathbb{R} \) be given by \( g(x) = \lambda \|x\|_1 \). The proximity operator of \( g \) is evaluated by the following formula
\[
\text{prox}_{\lambda \|\cdot\|_1}(x) = (\text{sign}(x_i)\max(|x_i| - \lambda, 0))^n_{i=1},
\]
where \( x = (x_1, x_2, \ldots, x_n) \) and \( ||x||_1 = \sum_{i=1}^{n} |x_i| \).

The following lemma was proved by Bassaban et al.; see [22].

**Lemma 5.** Let \( H \) be a real Hilbert space and \( T \) be the forward–backward operator of \( f \) and \( g \), where \( g \) is a proper lower semi-continuous convex function from \( H \) into \( \mathbb{R} \cup \{\infty\} \), and \( f \) is a convex differentiable function from \( H \) into \( \mathbb{R} \) with gradient \( \nabla f \) being \( L \)-Lipschitz constant for some \( L > 0 \). If \( \{T_n\} \) is the forward–backward operator of \( f \) and \( g \) such that \( a_n \to a \) with \( a_n \in (0, 2/L) \), then \( \{T_n\} \) satisfies the NST-condition (I) with \( T \).

### 3. Main Results

In this section, we obtain a useful proposition and a weak convergence theorem of our proposed algorithm by using the inertial technique.

Let \( C \) be a nonempty closed and convex subset of a real Hilbert space \( H \) with a directed graph \( G = (V(G), E(G)) \) such that \( V(G) = C \). Let \( \{T_n\} \) be a family of \( G \)-nonexpansive mappings of \( C \) into itself such that \( \emptyset \neq \bigcap_{n=1}^{\infty} F(T_n) \).

The following proposition is useful for our main theorem.

**Proposition 1.** Let \( x^* \in \bigcap_{n=1}^{\infty} F(T_n) \) and \( x_0, x_1 \in C \) be such that \( (x_0, x^*), (x_1, x^*) \in E(G) \). Let \( \{x_n\} \) be a sequence generated by Algorithm 1. Suppose \( E(G) \) is symmetric, transitive and left coordinate affine. Then, \( (x_n, x^*), (y_n, x^*), (z_n, x^*), (x_n, x_{n+1}) \in E(G) \) for all \( n \in \mathbb{N} \).

**Algorithm 1 (MSPA) A modified SP-algorithm**

1. **Initial.** Take \( x_0, x_1 \in C \) are arbitrary and \( n = 1, \alpha_n \in [a, b] \subset (0, 1), \beta_n \in (0, 1), \theta_n \geq 0 \) and \( \sum_{n=1}^{\infty} \theta_n < \infty \) where \( \theta_n \) is called an inertial step size.

2. **Step 1.** \( y_n, z_n \) and \( x_{n+1} \) are computed by

\[
\begin{align*}
    y_n &= x_n + \theta_n (x_n - x_{n-1}), \\
    z_n &= (1 - \beta_n)y_n + \beta_n T_ny_n, \\
    x_{n+1} &= (1 - \alpha_n)z_n + \alpha_n T_nz_n,
\end{align*}
\]

Then, \( n := n + 1 \) and go to Step 1.

**Proof.** We shall prove the results by using mathematical induction. From Algorithm 1, we obtain

\[
(y_1, x^*) = (x_1 + \theta_1 (x_1 - x_0), x^*)
\]

\[
= ((1 + \theta_1)x_1 - \theta_1 x_0, x^*)
\]

\[
= (1 + \theta_1)(x_1, x^*) - \theta_1 (x_0, x^*).
\]

Since \((x_0, x^*), (x_1, x^*) \in E(G) \) and \( E(G) \) is left coordinate affine, we obtain \((y_1, x^*) \in E(G) \) and

\[
(z_1, x^*) = ((1 - \beta_1)y_1 + \beta_1 T_1y_1, x^*)
\]

\[
= (1 - \beta_1)(y_1, x^*) + \beta_1 (T_1y_1, x^*).
\]

Since \((y_1, x^*) \in E(G) \) and \( T_n \) is edge preserving, we obtain \((z_1, x^*) \in E(G) \). Next, suppose that

\[
(x_k, x^*), (y_k, x^*) \text{ and } (z_k, x^*) \in E(G) \quad (6)
\]

for \( k \in \mathbb{N} \). We shall show that \((x_{k+1}, x^*), (y_{k+1}, x^*) \) and \((z_{k+1}, x^*) \in E(G) \). By Algorithm 1, we obtain
Theorem 1. Suppose that \( \{ F \} \) is a sequence of \( \alpha \)-nonexpansive mappings with \( \alpha \) < 1. Let \( x^* \) be a fixed point of \( F \). Then, for each \( n \), we have

\[
(x_{k+1}, x^*) = ((1 - \alpha_k)z_k + \alpha_k T_kz_k, x^*) = (1 - \alpha_k)(z_k, x^*) + \alpha_k (T_kz_k, x^*),
\]

and

\[
(y_{k+1}, x^*) = (x_{k+1} + \theta_{k+1}(x_{k+1} - x_k), x^*) = (1 + \theta_{k+1}x_{k+1} - \theta_{k+1}x_k, x^*)
\]

and

\[
(z_{k+1}, x^*) = ((1 - \beta_{k+1})y_{k+1} + \beta_{k+1}T_{k+1}y_{k+1}, x^*) = (1 - \beta_{k+1})(y_{k+1}, x^*) + \beta_{k+1} (T_{k+1}y_{k+1}, x^*).
\]

Since \( E(G) \) is left coordinate affine, \( T_n \) is edge preserving and from (6)–(9), we obtain \( (x_{k+1}, x^*), (y_{k+1}, x^*) \) and \( (z_{k+1}, x^*) \in E(G) \). By mathematical induction, we conclude that \( (x_n, x^*), (y_n, x^*), (z_n, x^*) \in E(G) \) for all \( n \in \mathbb{N} \). Since \( E(G) \) is symmetric, we obtain \( (x^*, x_{n+1}) \in E(G) \). Since \( (x_n, x^*), (x^*, x_{n+1}) \in E(G) \) and \( E(G) \) is transitive, we obtain \( (x_n, x_{n+1}) \in E(G) \). The proof is now complete. \( \Box \)

In the following theorem, we prove the weak convergence of \( G \)-nonexpansive mapping by using Algorithm 1.

Theorem 1. Let \( C \) be a nonempty closed and convex subset of a real Hilbert space \( H \) with a directed graph \( G = (V(G), E(G)) \) with \( V(G) = C \) and \( E(G) \) is symmetric, transitive and left coordinate affine. Let \( x_0, x_1 \in C \) and \( \{ x_n \} \) be a sequence in \( H \) defined by Algorithm 1. Suppose that \( \{ T_n \} \) satisfies the NST-condition (I) with \( T \) such that \( \emptyset \neq F(T) \subset \bigcap_{n=1}^{\infty} F(T_n) \) and \( (x_0, x^*), (x_1, x^*) \in E(G) \) for all \( x^* \in \bigcap_{n=1}^{\infty} F(T_n) \). Then, \( \{ x_n \} \) converges weakly to a point in \( F(T) \).

Proof. Let \( x^* \in \bigcap_{n=1}^{\infty} F(T_n) \). By the definitions of \( y_n \) and \( z_n \), we obtain

\[
\| y_n - x^* \| = \| x_n + \theta_n (x_n - x_{n-1}) - x^* \|
\]

and

\[
\| z_n - x^* \| = \| (1 - \beta_n) y_n - x^* + \beta_n T_n y_n - \beta_n x^* \|
\]

By the definition of \( x_{n+1} \) and (11), we obtain

\[
\| x_{n+1} - x^* \| = \| (1 - \alpha_n) z_n - x^* + \alpha_n T_n z_n - \alpha_n x^* \|
\]
From (10)–(12), we obtain
\[
\|x_{n+1} - x^*\| \leq \|x_n - x^*\| + \theta_n \|x_n - x_{n-1}\|
\]
\[
\leq (1 + \theta_n)\|x_n - x^*\| + \theta_n \|x_{n-1} - x^*\|.
\] (13)

So, we obtain \(\|x_{n+1} - x^*\| \leq M \cdot \prod_{i=1}^{n} (1 + 2\theta_i)\), where \(M = \max\{\|x_1 - x^*\|, \|x_2 - x^*\|\}\) from Lemma 3. Thus, \(\{x_n\}\) is bounded because \(\sum_{n=1}^{\infty} \theta_n < \infty\). Then,
\[
\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty.
\] (14)

Note that \(\{x_n\}\) being bounded implies that \(\{y_n\}\) and \(\{z_n\}\) are also bounded. By Lemma 1 and (13), we find that \(\lim_{n \to \infty} \|x_n - x^*\|\) exists. Then, we let \(\lim_{n \to \infty} \|x_n - x^*\| = a\). From the boundedness of \(\{y_n\}\) and (12), we obtain
\[
\liminf_{n \to \infty} \|y_n - x^*\| \geq a.
\] (15)

By (10) and (14), we obtain
\[
\limsup_{n \to \infty} \|y_n - x^*\| \leq a.
\] (16)

From (15) and (16), it follows that
\[
\lim_{n \to \infty} \|y_n - x^*\| = a.
\] (17)

Similarly, from (11), (12), (17) and the boundedness of \(\{z_n\}\), we obtain
\[
\limsup_{n \to \infty} \|z_n - x^*\| \leq a \quad \text{and} \quad \liminf_{n \to \infty} \|z_n - x^*\| \geq a.
\] (18)

From (18), we obtain that \(\lim_{n \to \infty} \|z_n - x^*\| = a\). It follows that \(\lim_{n \to \infty} \|z_n - x^*\|\) exists. By the definition of \(x_{n+1}\) and Lemma 2 (i), we obtain
\[
\|x_{n+1} - x^*\|^2 = \|(1 - \alpha_n)(z_n - x^*) + \alpha (T_n z_n - x^*)\|^2
\]
\[
= (1 - \alpha_n)\|z_n - x^*\|^2 + \alpha \|T_n z_n - x^*\|^2 - (1 - \alpha_n)\alpha_n \|z_n - T_n z_n\|^2
\]
\[
\leq (1 - \alpha_n)\|z_n - x^*\|^2 + \alpha \|z_n - x^*\|^2 - (1 - \alpha_n)\alpha_n \|z_n - T_n z_n\|^2
\]
\[
= \|z_n - x^*\|^2 - (1 - \alpha_n)\alpha_n \|z_n - T_n z_n\|^2
\]
\[
\leq (\|x_n - x^*\| + \theta_n \|x_n - x_{n-1}\|)^2 - (1 - \alpha_n)\alpha_n \|z_n - T_n z_n\|^2
\]
\[
= \|x_n - x^*\|^2 + 2\theta_n \|x_n - x^*\| \|x_n - x_{n-1}\| + \theta_n^2 \|x_n - x_{n-1}\|^2
\]
\[
- (1 - \alpha_n)\alpha_n \|z_n - T_n z_n\|^2.
\] (19)

From (14) and (19), we obtain
\[
\|z_n - T_n z_n\| \to 0.
\] (20)

Since
\[
\|x_{n+1} - z_n\| = \|(1 - \alpha_n) z_n + \alpha_n T_n z_n - z_n\| = \alpha_n \|T_n z_n - z_n\|
\]
and from (20), it follows that
\[
\|x_{n+1} - z_n\| \to 0.
\] (21)

Since \(\{z_n\}\) is bounded, (20), and \(\{T_n\}\) satisfies the NST-condition (I) with \(T\), we obtain that \(\|z_n - T z_n\| \to 0\). Let \(\omega_w(z_n)\) be the set of all weak cluster points of \(\{z_n\}\). Then,
\( \omega_w \in F(T) \) by the demicloseness of \( I - T \) at 0. By Lemma 3, we conclude that there exists \( x^* \in F(T) \) such that \( z_n \rightarrow x^* \) and it follows from (21) that \( x_n \rightarrow x^* \). The proof is now complete. \( \Box \)

4. Applications

In this section, we are interested in applying our proposed method for solving a convex minimization problem. Furthermore, we also compared the convergence behavior of our proposed algorithm with the others and give some applications to solve the image restoration problem.

4.1. Convex Minimization Problems

Our proposed method will be used to solve a convex minimization problem of the sum of two convex and lower semicontinuous functions \( f, g : \mathbb{R}^n \rightarrow (-\infty, +\infty) \). So, we consider the following convex minimization problem: \( \min (f(x) + g(x)) \ x \in \mathbb{R}^n \). It is well known that \( x^* \) is a minimizer of (22) if and only if \( x^* = Tx^* \), where \( T = \text{prox}_{g}(I - \rho \nabla f) \); see Proposition 3.1 (iii) [35]. It is also known that \( T \) is nonexpansive if \( \rho \in (0, 2/L) \) when \( L \) is a Lipschitz constant of \( \nabla f \). Over the past two decades, several algorithms have been introduced for solving the problem (22). A simple and classical algorithm is the forward–backward algorithm (FBA), which was introduced by Lions, P.L. and B. Mercier [23].

The forward–backward algorithm (FBA) is defined by

\[
\begin{aligned}
\{ y_n &= x_n - \gamma \nabla f x_n, \\
x_{n+1} &= x_n + \rho_n (\nabla g y_n - x_n),
\end{aligned}
\]  

(22)

where \( n \geq 1, x_0 \in H \) and \( L \) is a Lipschitz constant of \( \nabla f, \gamma \in (0, 2/L), \delta = 2 - (\gamma L/2) \) and \( \{\rho_n\} \) is a sequence in \([0, \delta]\) such that \( \sum_{n \in \mathbb{N}} \rho_n (\delta - \rho_n) = +\infty \). A technique for improving speed and giving a better convergence behavior of the algorithms was introduced firstly by Polyak [38] by adding an inertial step. Since then, many authors have employed the inertial technique to accelerate their algorithms for various kinds of problems; see [21, 22, 24–28]. The performance of FBA can be improved using an iterative method with the inertial steps described below.

A fast iterative shrinkage-thresholding algorithm (FISTA) [27] is defined by

\[
\begin{aligned}
\{ y_n &= x_n - \gamma \nabla f x_n, \\
t_{n+1} &= \frac{1 + \sqrt{1 + 4t_n^2}}{2}, \\
\theta_n &= \frac{t_{n+1}}{t_n}, \\
x_{n+1} &= y_n + \theta_n (y_n - y_{n-1}),
\end{aligned}
\]  

(23)

where \( n \geq 1, t_1 = 1, x_1 = y_0 \in \mathbb{R}^n, T := \text{prox}_{\frac{1}{2}} (I - \frac{1}{2} \nabla f) \) and \( \theta_n \) is the inertial step size. The FISTA was suggested by Beck and Teboulle [27]. They proved the convergence rate of the FISTA and applied the FISTA to the image restoration problem [27]. The inertial step size \( \theta_n \) of the FISTA was firstly introduced by Nesterov [39].

A new accelerated proximal gradient algorithm (nAGA) [28] is defined by

\[
\begin{aligned}
\{ y_n &= x_n + \mu_n (x_n - x_{n-1}), \\
x_{n+1} &= T_n [(1 - \rho_n) y_n + \rho_n T_n y_n],
\end{aligned}
\]  

(24)

where \( n \geq 1, T_n \) is the forward–backward operator of \( f \) and \( g \) with \( \omega_n \in (0, 2/L) \) and \( \{\mu_n\}, \{\rho_n\} \) are sequences in \((0, 1)\) and \( \frac{\|x_n - x_{n-1}\|}{\mu_n} \rightarrow 0 \). The nAGA was introduced for proving a convergence theorem by Verma and Shukla [28]. The nonsmooth convex minimization problem with sparsity, including regularizers, was solved using this method for the multitask learning framework.

The convergence of Algorithm 2 is obtained using the convergence result of Algorithm 1, as shown in the following theorem.
Algorithm 2 (FBMSPA) A forward–backward modified SP-algorithm

1. **Initial.** Take \( x_0, x_1 \in C \) are arbitrary and \( n = 1, \alpha_n \in [a, b] \subset (0, 1), \beta_n \in (0, 1), \theta_n \geq 0 \) and \( \sum_{n=1}^{\infty} \theta_n < \infty. \)

2. **Step 1.** \( y_n, z_n \) and \( x_{n+1} \) are computed by

\[
\begin{align*}
    y_n &= x_n + \theta_n(x_n - x_{n-1}), \\
    z_n &= (1 - \beta_n)y_n + \beta_n \text{prox}_{\alpha_n g}(I - a_n \nabla f)y_n, \\
    x_{n+1} &= (1 - a_n)z_n + a_n \text{prox}_{\alpha_n g}(I - a_n \nabla f)z_n.
\end{align*}
\]

Then, \( n := n + 1 \) and go to Step 1.

**Theorem 2.** For \( f, g : \mathbb{R}^n \to (-\infty, \infty) \), \( g \) is a convex function and \( f \) is a smooth convex function with a gradient having a Lipschitz constant \( L \). Let \( a_n \in (0, 2/L) \) be such that \( \{a_n\} \) converges to \( a \) and let \( T := \text{prox}_{ag}(I - a \nabla f) \) and \( T_n := \text{prox}_{a_n g}(I - a_n \nabla f) \) and let \( \{x_n\} \) be a sequence generated by Algorithm 2, where \( \beta_n, \alpha_n \) and \( \theta_n \) are the same as in Algorithm 1. Then, the following holds:

(i) \( \|x_{n+1} - x^*\| \leq M \cdot \prod_{i=1}^{n} (1 + 2\theta_i) \), where \( M = \max \{\|x_1 - x^*\|, \|x_2 - x^*\|\} \) and \( x^* \in \text{Argmin}(f + g) \);

(ii) \( \{x_n\} \) converges weakly to a point in \( \text{Argmin}(f + g) \).

**Proof.** We know that \( T \) and \( \{T_n\} \) are nonexpansive operators, and \( F(T) = \bigcap_{n=1}^{\infty} F(T_n) = \text{Argmin}(f + g) \) for all \( n \); see Proposition 26.1 in \([34]\). By Lemma 5, we find that \( \{T_n\} \) satisfies the NST-condition (i) with \( T \). From Theorem 1, we obtain the required result directly by putting \( C = \mathbb{R}^n \times \mathbb{R}^m \), the complete graph, on \( \mathbb{R}^n \).

4.2. The Image Restoration Problem

We can describe the image restoration problem as a simple linear model

\[
Bx = c + u,
\]

where \( B \in \mathbb{R}^{m \times n} \) and \( c \in \mathbb{R}^{m \times 1} \) are known, \( u \) is an additive noise vector, and \( x \) is the “true” image. In image restoration problems, the blurred image is represented by \( c \), and \( x \in \mathbb{R}^{n \times 1} \) is the unknown true image. In these problems, the blur operator is described by the matrix \( B \). The problem of finding the original image \( x^* \in \mathbb{R}^{n \times 1} \) from the noisy image and observed blurred is called an image restoration problem. There are several methods that have been proposed for finding the solution of problem \((25)\); see, for instance, \([40–43]\). A new method for the estimation a solution of \((25)\), called the least absolute shrinkage and selection operator (LASSO), was proposed by Tibshirani \([44]\) as follows:

\[
\min_x \left\{ \|Bx - c\|_2^2 + \lambda \|x\|_1 \right\},
\]

where \( \lambda > 0 \) is called a regularization parameter and \( \| \cdot \|_1 \) is an l1-norm defined by \( \|x\|_1 = \sum_{i=1}^{n} |x_i| \). The LASSO can also be applied to solve image and regression problems \([27,44]\), etc.

Due to the size of the matrix \( B \) and \( x \) along with their members, the model \((26)\) has the computational cost of the multiplication \( Bx \) and \( \|x\|_1 \) for solving the RGB image restoration problem. In order to solve this issue, many mathematicians in this field have used the 2-D fast Fourier transform for true RGB image transformation. Therefore, the model \((26)\) was slightly modified using the 2-D fast Fourier transform as follows:

\[
\min_x \left\{ \|Bx - C\|_2^2 + \lambda \|Wx\|_1 \right\}
\]

where \( \lambda \) is a positive regularization parameter, \( R \) is the blurring matrix, \( W \) is the 2-D fast Fourier transform, \( B \) is the blurring operation with \( B = RW \) and \( C \in \mathbb{R}^{m \times n} \) is the observed blurred and noisy image of size \( m \times n \).
We apply Algorithm 2 to solve the image restoration problem (27) by using Theorem 2 when \( f(x) = \|Bx - C\|_2^2 \) and \( g(x) = \lambda \|Wx\|_1 \). Then, we compare Algorithm 2’s deblurring to that of FISTA and FBA. In this experiment, we consider the true RGB images, Suan Dok temple and Aranyawiwek temple of size 500\(^2\), as the original images. We blur the images with a Gaussian blur of size 9\(^2\) and \( \sigma = 4 \), where \( \sigma \) is the standard deviation. To evaluate the performance of these methods, we utilize the peak signal-to-noise ratio (PSNR) [45] to measure the efficiency of these methods when PSNR\((x_n)\) is defined by

\[
PSNR(x_n) = 10\log_{10}\left( \frac{255^2}{MSE} \right),
\]

where a monotic image with 8 bits/pixel has a maximum gray level of 255 and \( MSE = \frac{1}{N} \| x_n - x^* \|_2^2 = \frac{1}{N} \sum_{i=1}^{N} |x_n(i) - x^*(i)|^2 \), \( x_n(i) \) and \( x^*(i) \) are the i-th samples in image \( x_n \) and \( x^* \), respectively. \( N \) is the number of image samples and \( x^* \) is the original image. We can see that a higher PSNR indicates better a deblurring image quality. For these experiments, we set \( \lambda = 5 \times 10^{-5} \) and the original image was the blurred image. The Lipchitz constant \( L \) is calculated using the matrix \( B^*B \) as the maximum eigenvalues.

The parameters of Algorithm 2, FISTA, FBA, Ishikawa iteration, S-iteration, Noor iteration and SP-iteration are the same as in Table 1.

Table 1. Methods and their setting controls.

<table>
<thead>
<tr>
<th>Methods</th>
<th>Setting</th>
</tr>
</thead>
<tbody>
<tr>
<td>Algorithm 2</td>
<td>( \alpha_n = 0.9, \beta_n = 0.5, c = 1/L, \theta_n = n/(n + 1) ) if ( 1 \leq n \leq 500 ), and ( 1/2^n ) otherwise</td>
</tr>
<tr>
<td>FISTA</td>
<td>( t_1 = 1, t_{n+1} = (1 + \sqrt{1 + 4t_n^2})/2, \theta_n = (t_n - 1)/t_{n+1} )</td>
</tr>
<tr>
<td>FBA</td>
<td>( \rho_n = 0.9, \gamma = 1/L )</td>
</tr>
<tr>
<td>Ishikawa iteration</td>
<td>( \rho_n = 0.9, \zeta_n = 0.5, \eta_n = 0.5, c = 1/L )</td>
</tr>
<tr>
<td>S-iteration</td>
<td>( \rho_n = 0.9, \zeta_n = 0.5, \eta_n = 0.5, c = 1/L )</td>
</tr>
<tr>
<td>Noor iteration</td>
<td>( \rho_n = 0.9, \zeta_n = 0.5, \eta_n = 0.5, c = 1/L )</td>
</tr>
<tr>
<td>SP-iteration</td>
<td>( \rho_n = 0.9, \zeta_n = 0.5, \eta_n = 0.5, c = 1/L )</td>
</tr>
</tbody>
</table>

Note that all of the parameters in Table 1 satisfy the convergence theorems for each method. The convergence of the sequence \( \{x_n\} \) generated by Algorithm 2 to the original image \( x^* \) is guaranteed by Theorem 2. However, the PSNR value is used to measure the convergence behavior of this sequence. It is known that PSNR is a suitable measurement for image restoration problems.

The following experiments show the efficacy of the blurring results of Suan Dok and Aranyaviwek temples at the 500th iteration of Algorithms 2, FISTA, FBA, Ishikawa iteration, S-iteration, Noor iteration and SP-iteration using PSNR as our measurement, shown in tables and figures as follows.

It is observed from Figures 1 and 2 that the graph of PSNR of Algorithm 2 is higher than that of FISTA FBA, Ishikawa iteration, S-iteration, Noor iteration and SP-iteration which shows that Algorithm 2 gives a better performance than the others.

The efficiency of each algorithm for image restoration is shown in Tables 2–5 for different number of iterations. The value of PSNR of Algorithm 2 is shown to be higher than that of FISTA, FBA, Ishikawa iteration, S-iteration, Noor iteration and SP-iteration. Thus, Algorithm 2 has a better convergence behavior than the others.

We show the original images, blurred images, and deblurred images by Algorithm 2, FISTA, FBA, Ishikawa iteration, S-iteration, Noor iteration and SP-iteration for Suan Dok (Figure 3) and Aranyawiwek temples (Figure 4).
Figure 1. The graphs of PSNR of each algorithm for Suan Dok temple.

Figure 2. The graphs of PSNR of each algorithm for Aranyawiwek temple.
Table 2. The values of PSNR for Algorithm 2, FISTA, FBA of Suan Dok temple.

<table>
<thead>
<tr>
<th>No. Iterations</th>
<th>Algorithm 2</th>
<th>FISTA</th>
<th>FBA</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20.41801</td>
<td>20.36432</td>
<td>20.27827</td>
</tr>
<tr>
<td>5</td>
<td>21.56154</td>
<td>21.13340</td>
<td>20.64981</td>
</tr>
<tr>
<td>10</td>
<td>22.81140</td>
<td>22.00081</td>
<td>20.96027</td>
</tr>
<tr>
<td>25</td>
<td>24.54825</td>
<td>23.73266</td>
<td>21.56257</td>
</tr>
<tr>
<td>100</td>
<td>27.80053</td>
<td>26.71268</td>
<td>22.93002</td>
</tr>
<tr>
<td>250</td>
<td>30.21461</td>
<td>29.28515</td>
<td>23.92280</td>
</tr>
<tr>
<td>500</td>
<td>31.57117</td>
<td>31.21182</td>
<td>24.66522</td>
</tr>
</tbody>
</table>

Table 3. The values of PSNR for Ishikawa iteration, S-iteration, Noor iteration and SP-iteration of Suan Dok temple.

<table>
<thead>
<tr>
<th>No. Iterations</th>
<th>Ishikawa Iteration</th>
<th>S-Iteration</th>
<th>Noor Iteration</th>
<th>SP-Iteration</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20.41010</td>
<td>20.42585</td>
<td>20.43611</td>
<td>20.47630</td>
</tr>
<tr>
<td>25</td>
<td>22.44491</td>
<td>22.51817</td>
<td>22.59948</td>
<td>22.79284</td>
</tr>
<tr>
<td>100</td>
<td>23.98112</td>
<td>24.05696</td>
<td>24.14345</td>
<td>24.33880</td>
</tr>
<tr>
<td>250</td>
<td>24.97654</td>
<td>25.05383</td>
<td>25.14335</td>
<td>25.43583</td>
</tr>
<tr>
<td>500</td>
<td>25.75882</td>
<td>25.84223</td>
<td>25.93954</td>
<td>26.16025</td>
</tr>
</tbody>
</table>

Table 4. The values of PSNR for Algorithm 2, FISTA and FBA of Aranyawiwek temple.

<table>
<thead>
<tr>
<th>No. Iterations</th>
<th>Algorithm 2</th>
<th>FISTA</th>
<th>FBA</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20.62485</td>
<td>20.57077</td>
<td>20.48543</td>
</tr>
<tr>
<td>5</td>
<td>21.85350</td>
<td>21.37734</td>
<td>20.86196</td>
</tr>
<tr>
<td>10</td>
<td>23.31840</td>
<td>22.35583</td>
<td>21.19050</td>
</tr>
<tr>
<td>25</td>
<td>25.29317</td>
<td>24.39293</td>
<td>21.85570</td>
</tr>
<tr>
<td>100</td>
<td>28.86437</td>
<td>27.75046</td>
<td>23.44804</td>
</tr>
<tr>
<td>250</td>
<td>31.32694</td>
<td>30.48999</td>
<td>24.60734</td>
</tr>
<tr>
<td>500</td>
<td>32.66988</td>
<td>32.43108</td>
<td>25.45769</td>
</tr>
</tbody>
</table>

Table 5. The values of PSNR for Algorithm 2, Ishikawa iteration, S-iteration, Noor iteration and SP-iteration of Aranyawiwek temple.

<table>
<thead>
<tr>
<th>No. Iterations</th>
<th>Ishikawa Iteration</th>
<th>S-Iteration</th>
<th>Noor Iteration</th>
<th>SP-Iteration</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20.61695</td>
<td>20.63272</td>
<td>20.64371</td>
<td>20.65791</td>
</tr>
<tr>
<td>25</td>
<td>22.87547</td>
<td>22.96182</td>
<td>22.057648</td>
<td>22.27264</td>
</tr>
<tr>
<td>100</td>
<td>24.64759</td>
<td>24.76190</td>
<td>24.86123</td>
<td>25.06857</td>
</tr>
<tr>
<td>500</td>
<td>26.96400</td>
<td>26.78725</td>
<td>26.89572</td>
<td>27.11590</td>
</tr>
</tbody>
</table>
Figure 3. Results for Suan Dok temple’s deblurring image.

Figure 4. Results for Aranyawiwek temples’s deblurring image.
5. Conclusions

In this study, we used a coordinate affine structure to propose an accelerated fixed-point algorithm with an inertial technique for a countable family of $G$-nonexpansive mappings in a Hilbert space with a symmetric directed graph $G$. Moreover, we proved the weak convergence theorem of the proposed algorithm under some suitable conditions. Then, we compared the convergence behavior of our proposed algorithm with FISTA, FBA, Ishikawa iteration, S-iteration, Noor iteration and SP-iteration. We also applied our results to image restoration and convex minimization problems. We found that Algorithm 2 gave the best results out of all of them.

Author Contributions: Conceptualization, R.W.; Formal analysis, K.J. and R.W.; Investigation, K.J.; Methodology, R.W.; Supervision, R.W.; Validation, R.W.; Writing—original draft, K.J.; Writing—review and editing, R.W. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by Fundamental Fund 2022, Chiang Mai university and Ubon Ratchathani University.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The first author was supported by Fundamental Fund 2022, Chiang Mai university, Thailand. The second author would like to thank Ubon Ratchathani University.

Conflicts of Interest: The authors declare no conflict of interest.

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