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On the Bertrand Offsets of Timelike Ruled Surfaces in Minkowski 3-Space

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Abstract: This work examines some classical results of Bertrand curves for timelike ruled and developable surfaces using the E. Study map. This provides the ability to define two timelike ruled surfaces which are offset in the sense of Bertrand. It is shown that every timelike ruled surface has a Bertrand offset if and only if an equation should be satisfied between the dual geodesic curvatures. Some new results and theorems related to the developability of the Bertrand offsets of timelike ruled surfaces are also obtained.

Keywords: dual angle of pitch; Bertrand offsets; binormal surface

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1. Introduction

In Euclidean 3-space, the trajectories of oriented lines embedded in a moving rigid body are generally ruled surfaces. The geometry of ruled surfaces has been widely applied in Computer-Aided Manufacturing (CAM), Computer-Aided Geometric Design (CAGD), geometric modeling and kinematics [1–3]. From past to present, both offsets surfaces and ruled surfaces have been examined in Euclidean and non-Euclidean spaces: Ravani and Ku [4] generalized the theory of Bertrand curves for Bertrand ruled surface-offsets based on line geometry. They showed that a ruled surface can have an infinity of Bertrand offsets, in the same way as a planar curve can have an infinity of Bertrand mates. Based on this study, Küçük and Gürsoy provided some characterizations of Bertrand offsets of trajectory ruled surfaces in terms of the relationships between the projection areas for the spherical images of Bertrand offsets and their integral invariants [5]. In [6], Kasap and Kuruoglu obtained the relationships between Integral invariants of the pairs of the Bertrand ruled surface in Euclidean 3-space. In [7], Kasap and Kuruoglu initiated the study of Bertrand offsets of ruled surfaces in Minkowski 3-space. The involute-evolute offsets of a ruled surface is defined by Kasap et al. in [8]. Orbay et al., in [9], initiated the study of Mannheim offsets of the ruled surface. Onder and Ugurlu obtained the relationships between invariants of Mannheim offsets of timelike ruled surfaces, and they gave the conditions for these surface offsets to be developable [10]. These offset surfaces are defined using the geodesic Frenet frame which was given by [8]. Based on the involute-evolute offsets of ruled surfaces in [10], Senturk and Yuce calculated integral invariants of these offsets with respect to the geodesic Frenet frame [11]. Important contributions to the Bertrand offsets of these ruled surfaces have been studied in [12–15].

In this paper, a generalization of the well known theory of Bertrand curves is presented for timelike ruled surfaces in Minkowski space. Using lines instead of points, two timelike ruled surfaces which are offset in the sense of Bertrand are defined. In particular, we investigate how to construct the timelike Bertrand offset from a timelike ruled surface with vanishing dual geodesic curvature. Meanwhile, a timelike developable surface can have a timelike developable Bertrand offset if a linear equation holds between the curvature and torsion of its edge of regression.

2. Basic Concepts

We begin with requisite concepts on the dual numbers, dual Lorentzian vectors and E. Study map (see [1–3,16–18]): An oriented (non-null) line in Minkowski 3-space can be defined by a point $\boldsymbol{\alpha} \in L$ and a normalized direction vector $\mathbf{x}$ of L, that is, $\langle \mathbf{x}, \mathbf{x} \rangle = \pm 1$. To acquire components for L, one forms the moment vector $\mathbf{x}^* = \boldsymbol{\alpha} \times \boldsymbol{x}$ with respect to the origin point in $\mathbb{E}_1^3$. If $\boldsymbol{\alpha}$ is replaced by any point $\boldsymbol{\beta} = \boldsymbol{\alpha} + t\mathbf{x}$, $t \in \mathbb{R}$ on L, this shows that x^* is independent of $\boldsymbol{\alpha}$ on L. The two vectors $\mathbf{x}$ and $\mathbf{x}^*$ are not independent of one another; they satisfy the following:

$$\langle \mathbf{x}, \mathbf{x} \rangle = \pm 1, \ \ \langle \mathbf{x}^*, \mathbf{x} \rangle = 0.$$

The six components x_i, $x_i^* (i = 1, 2, 3)$ of $\mathbf{x}$ and $\mathbf{x}^*$ are named the normalized Plücker coordinates of the line L. Thus, the two vectors, $\mathbf{x}$ and $\mathbf{x}^*$, locate the oriented line L.

A dual number $\widehat{x}$ is a number $x + \varepsilon x^*$, where x, x^* in $\mathbb{R}$ and ε is a dual unit with the property that $\varepsilon^2 = 0$. Then the set:

$$\mathbb{D}^3 = \{\widehat{\mathbf{x}} := \mathbf{x} + \varepsilon \mathbf{x}^* = (\widehat{x}_1, \widehat{x}_2, \widehat{x}_3)\},$$

together with the Lorentzian inner product,

$$\langle \widehat{\mathbf{x}}, \widehat{\mathbf{y}} \rangle = \widehat{x}_1 \widehat{y}_1 + \widehat{x}_2 \widehat{y}_2 - \widehat{x}_3 \widehat{y}_3,$$

forms the so-named dual Lorentzian 3-space $\mathbb{D}_1^3$. This yields:

$$\left. \begin{array}{l} \widehat{\mathbf{f}}_1 \times \widehat{\mathbf{f}}_2 = -\widehat{\mathbf{f}}_3, \ \widehat{\mathbf{f}}_3 \times \widehat{\mathbf{f}}_1 = \widehat{\mathbf{f}}_2, \ \widehat{\mathbf{f}}_2 \times \widehat{\mathbf{f}}_3 = \widehat{\mathbf{f}}_1, \\ \langle \widehat{\mathbf{f}}_1, \widehat{\mathbf{f}}_1 \rangle = \langle \widehat{\mathbf{f}}_2, \widehat{\mathbf{f}}_2 \rangle = -\langle \widehat{\mathbf{f}}_3, \widehat{\mathbf{f}}_3 \rangle = 1, \end{array} \right\}$$

where $\widehat{\mathbf{f}}_1$, $\widehat{\mathbf{f}}_2$, and $\widehat{\mathbf{f}}_3$, are the dual base at the origin point $\mathbf{O}(0, 0, 0)$ of the dual Lorentzian 3-space $\mathbb{D}_1^3$. Thereby a point $\widehat{x} = (\widehat{x}_1, \widehat{x}_2, \widehat{x}_3)^t$ has dual coordinates $\widehat{x}_i = (x_i + \varepsilon x_i^*) \in \mathbb{D}$. If $\mathbf{x} \neq \mathbf{0}$ the norm $\|\widehat{\mathbf{x}}\|$ of $\widehat{\mathbf{x}} = \mathbf{x} + \varepsilon \mathbf{x}^*$ is described by

$$\|\widehat{\mathbf{x}}\| = \sqrt{|\langle \widehat{\mathbf{x}}, \widehat{\mathbf{x}} \rangle|} = \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}(1 + \varepsilon \frac{\langle \mathbf{x}, \mathbf{x}^* \rangle}{\langle \mathbf{x}, \mathbf{x} \rangle}),$$

then the vector $\widehat{\mathbf{x}}$ is called a spacelike (resp. timelike) dual unit vector if $\langle \mathbf{x}, \mathbf{x} \rangle = 1$(resp. $\langle \mathbf{x}, \mathbf{x} \rangle = -1$). It is clear that:

$$\langle \widehat{\mathbf{x}}, \widehat{\mathbf{x}} \rangle = \pm 1 \Longleftrightarrow \langle \mathbf{x}, \mathbf{x} \rangle = \pm 1, \ \langle \mathbf{x}, \mathbf{x}^* \rangle = 0.$$

The hyperbolic, and Lorentzian (de Sitter space) dual unit spheres with the center $\mathbf{O}$ are:

$$\mathbb{H}_+^2 = \left\{ \widehat{\mathbf{x}} \in \mathbb{D}_1^3 \mid \widehat{x}_1^2 + \widehat{x}_2^2 - \widehat{x}_3^2 = -1 \right\},$$

and

$$\mathbb{S}_1^2 = \left\{ \widehat{\mathbf{x}} \in \mathbb{D}_1^3 \mid \widehat{x}_1^2 + \widehat{x}_2^2 - \widehat{x}_3^2 = 1 \right\},$$

respectively. Via this we have the following map (E. Study's map): The dual unit spheres are shaped as a pair of conjugate hyperboloids. The common asymptotic cone represents the set of null (lightlike) lines, the ring shaped hyperboloid represents the set of spacelike lines, and the oval shaped hyperboloid forms the set of timelike lines, opposite points of each hyperboloid represent the pair of opposite vectors on a line (see Figure 1).

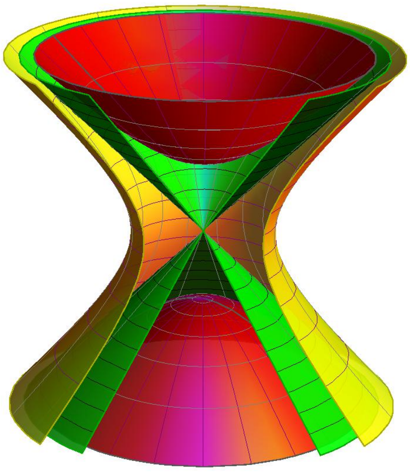

Figure 1. The dual hyperbolic and dual Lorentzian unit spheres.

3. Bertrand Offsets of Timelike Ruled Surfaces

In the Minkowski 3-space, a one-parameter Lorentzian spatial motion of a spacelike oriented line forms a timelike or spacelike ruled surface. We assume that the surfaces in this paper are all timelike ruled surfaces, and we denote the surface with (X). Applying to the E. Study map, a spacelike differentiable curve,

$$t \in \mathbb{R} \mapsto \widehat{\mathbf{x}}(t) \in \mathbb{S}_1^2,$$

represents a timelike ruled surface (X) in Minkowski 3-space $\mathbb{E}_1^3$. $\widehat{\mathbf{x}}(t)$ are identified with the rulings of the surface and from now on we do not distinguish between a ruled surface and its representing dual curve. The spacelike vector,

$$\widehat{\mathbf{t}}(t) = \mathbf{t} + \varepsilon \mathbf{t}^* = \frac{d\widehat{\mathbf{x}}(t)}{dt} \left\| \frac{d\widehat{\mathbf{x}}(t)}{dt} \right\|^{-1},$$

is the dual unit tangent vector on $\widehat{\mathbf{x}}(t)$. Introducing the timelike dual unit vector $\widehat{\mathbf{g}}(t) = \mathbf{g}(t) + \varepsilon \mathbf{g}^*(t) = -\widehat{\mathbf{x}} \times \widehat{\mathbf{t}}$ we have the moving frame $\{\widehat{\mathbf{x}}(t), \widehat{\mathbf{t}}(t), \widehat{\mathbf{g}}(t)\}$ on $\widehat{\mathbf{x}}(t)$ called Blaschke frame. Then,

$$\left.\begin{array}{c} \langle \widehat{\mathbf{x}}, \widehat{\mathbf{x}} \rangle = \langle \widehat{\mathbf{t}}, \widehat{\mathbf{t}} \rangle = -\langle \widehat{\mathbf{g}}, \widehat{\mathbf{g}} \rangle = 1, \\ \widehat{\mathbf{g}} = -\widehat{\mathbf{x}} \times \widehat{\mathbf{t}}, \widehat{\mathbf{x}} = \widehat{\mathbf{t}} \times \widehat{\mathbf{g}}, \widehat{\mathbf{t}} = \widehat{\mathbf{g}} \times \widehat{\mathbf{x}}. \end{array}\right\}$$

The Blaschke formula is [16]:

$$\frac{d}{dt}\begin{pmatrix} \widehat{\mathbf{x}} \\ \widehat{\mathbf{t}} \\ \widehat{\mathbf{g}} \end{pmatrix} = \begin{pmatrix} 0 & \widehat{p} & 0 \\ -\widehat{p} & 0 & \widehat{q} \\ 0 & \widehat{q} & 0 \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{x}} \\ \widehat{\mathbf{t}} \\ \widehat{\mathbf{g}} \end{pmatrix}, \tag{1}$$

where

$$\widehat{p}(t) = p(t) + \varepsilon p^*(t) = \left\| \frac{d\widehat{\mathbf{x}}(t)}{dt} \right\|, \ \widehat{q} = q + \varepsilon q^* = -\det(\widehat{\mathbf{x}}, \frac{d\widehat{\mathbf{x}}(t)}{dt}, \frac{d^2\widehat{\mathbf{x}}(t)}{dt^2}), \tag{2}$$

are named the Blaschke invariants of the spacelike dual curve $\widehat{\mathbf{x}}(t)$. The dual unit vectors $\widehat{\mathbf{x}}$, $\widehat{\mathbf{t}}$, and $\widehat{\mathbf{g}}$ corresponding to three concurrent mutually orthogonal oriented lines in $\mathbb{E}_1^3$ and they intersected at a point $\mathbf{c}$ on $\widehat{\mathbf{x}}$ named the striction (or central) point. The trajectory of the central points is the striction curve on (X).

The dual arc-length $\widehat{s}$ of $\widehat{\mathbf{x}}(t)$ is defined by:

$$d\widehat{s} = ds + \varepsilon ds^* = \left\| \frac{d\widehat{\mathbf{x}}(t)}{dt} \right\| dt = \widehat{p}(t)dt. \tag{3}$$

The distribution parameter of (X) is:

$$\mu(t) := \frac{ds^*}{ds} = \frac{p^*(t)}{p(t)}. \tag{4}$$

From Equations (1) and (3) we also obtain [16]:

$$\begin{pmatrix} \widehat{\mathbf{x}}' \\ \widehat{\mathbf{t}}' \\ \widehat{\mathbf{g}}' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & \widehat{\gamma} \\ 0 & \widehat{\gamma} & 0 \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{x}} \\ \widehat{\mathbf{t}} \\ \widehat{\mathbf{g}} \end{pmatrix} = \widehat{\omega} \times \begin{pmatrix} \widehat{\mathbf{x}} \\ \widehat{\mathbf{t}} \\ \widehat{\mathbf{g}} \end{pmatrix}; \ ('= \frac{d}{d\widehat{s}}), \tag{5}$$

where $\widehat{\omega} = \boldsymbol{\omega} + \varepsilon\boldsymbol{\omega}^* = \widehat{\chi}\widehat{\mathbf{x}} + \widehat{\mathbf{g}}$ is the Darboux vector, and $\widehat{\gamma}(\widehat{s}) := \gamma + \varepsilon\gamma^*$ is the dual geodesic curvature of $\widehat{\mathbf{x}}(\widehat{s})$ on $\mathbb{H}^2_+$. The tangent direction of $\mathbf{c}(s)$, is given by [16]:

$$\frac{d\mathbf{c}}{ds} = \Gamma(s)\mathbf{x} - \mu(s)\mathbf{g}, \tag{6}$$

which is a spacelike (resp. a timelike) curve if $|\mu|\langle|\Gamma|$ (resp. $|\mu| > |\Gamma|$. The functions $\gamma(s)$, $\Gamma(s)$ and $\mu(s)$ are named the curvature (construction) functions of the ruled surface. These functions described as follows: γ is the geodesic curvature of the spherical image spacelike curve $\mathbf{x} = \mathbf{x}(s)$; Γ describes the angle between the ruling of (X) and the tangent to the striction curve; and μ is its distribution parameter at the ruling. These functions locate an approach for establishing timelike ruled surfaces with a given spacelike or timelike striction curve by the equation:

$$(X) : \mathbf{y}(s, v) = \int_0^s (\Gamma(s)\mathbf{x}(s) - \mu(s)\mathbf{g}(s))ds + v\mathbf{x}(s). \tag{7}$$

The unit normal vector field $\mathbf{e}(s, v)$ at any point is:

$$\mathbf{e}(s, v) = \frac{\frac{\partial \mathbf{y}(s,v)}{\partial s} \times \frac{\partial \mathbf{y}(s,v)}{\partial v}}{\left\| \frac{\partial \mathbf{y}(s,v)}{\partial s} \times \frac{\partial \mathbf{y}(s,v)}{\partial v} \right\|} = \pm \frac{\mu\mathbf{t} - v\mathbf{g}}{\sqrt{\mu^2 - v^2}}, \tag{8}$$

which is the spacelike central normal at the striction point ($v = 0$). Let ϕ be the angel between the unit normal spacelike vector $\mathbf{e}$ and the central normal $\mathbf{t}$, then,

$$\mathbf{e}(s, v) = \cosh\phi\mathbf{t} - \sinh\phi\mathbf{g}.$$

It is clear that:

$$\tan\phi = \frac{v}{\mu}.$$

This result is a Minkowski version of the well known Chasles Theorem [1–3]. Under the assumption that $|\widehat{\gamma}|\langle 1$, we also specify the timelike Disteli-axis:

$$\widehat{\mathbf{b}}(\widehat{s}) := \mathbf{b} + \varepsilon\mathbf{b}^* = \frac{\widehat{\omega}}{\|\widehat{\omega}\|} = \frac{\widehat{\chi}\widehat{\mathbf{x}} + \widehat{\mathbf{g}}}{\sqrt{1 - \widehat{\gamma}^2}} = \sinh\widehat{\psi}\widehat{\mathbf{x}} + \cosh\widehat{\psi}\widehat{\mathbf{g}}, \tag{9}$$

where $\widehat{\psi} = \psi + \varepsilon\psi^*$ is the dual radius of curvature between $\widehat{\mathbf{b}}$ and $\widehat{\mathbf{x}}$. The dual geodesic curvature $\widehat{\gamma}(\widehat{s})$ in terms of Γ, μ and γ is [16]:

$$\widehat{\gamma}(\widehat{s}) = \gamma + \varepsilon(\Gamma - \gamma\mu) = \tanh\widehat{\psi}. \tag{10}$$

We also have:

$$\left.\begin{array}{l}\widehat{\gamma}(\widehat{s}) = \gamma + \varepsilon(\Gamma - \gamma\mu) = \tanh\psi + \varepsilon\psi^{*}(1-\tanh^{2}\psi),\\ \widehat{\kappa}(\widehat{s}) := \kappa + \varepsilon\kappa^{*} = \sqrt{1-\widehat{\gamma}^{2}} = \frac{1}{\cosh\widehat{\psi}},\\ \widehat{\tau}(\widehat{s}) := \tau + \varepsilon\tau^{*} = \pm\widehat{\psi}' = \pm\frac{\widehat{\gamma}'}{1-\widehat{\gamma}^{2}},\end{array}\right\} \tag{11}$$

where $\widehat{\kappa}(\widehat{s})$ is the dual curvature, and $\widehat{\tau}(\widehat{s})$ is the dual torsion of the spacelike dual curve $\widehat{\mathbf{x}}(\widehat{s}) \in \mathbb{S}_1^2$.

Proposition 1. *If the dual geodesic curvature function $\widehat{\gamma}(\widehat{s})$ is constant, $\widehat{\mathbf{x}}(\widehat{s})$ is a dual circle on $\mathbb{S}_1^2$.*

Proof. From Equation (11), we can find that $\widehat{\gamma}(\widehat{s})$ = constant yields that $\widehat{\tau}(\widehat{s}) = 0$, and $\widehat{\kappa}(\widehat{s})$ is constant, which implies $\widehat{\mathbf{x}}(\widehat{s})$ is a spacelike dual circle on $\mathbb{S}_1^2$. □

Definition 1. *A non-developable timelike ruled surface (X) is defined as a constant Disteli-axis timelike ruled surface if its dual geodesic curvature $\widehat{\gamma}(\widehat{s})$ is constant.*

According to the E. Study map, the constant Disteli-axis timelike ruled surface (X) is formed by a one-parameter helical motion with constant pitch h about the timelike Disteli-axis $\widehat{\mathbf{b}}$, by the oriented spacelike line $\widehat{\mathbf{x}}$ situated at a Lorentzian constant distance ψ^{*} and a Lorentzian constant angle ψ relative to the timelike Disteli-axis $\widehat{\mathbf{b}}$. Moreover, if $\widehat{\gamma}(\widehat{s}) = 0$, then $\widehat{\mathbf{x}}(\widehat{s})$ is a spacelike great dual circle on $\mathbb{S}_1^2$, that is,

$$\widehat{c} = \{\widehat{\mathbf{x}} \in \mathbb{S}_1^2 \mid \langle \widehat{\mathbf{x}}, \widehat{\mathbf{b}} \rangle = 0, \text{ with } \langle \widehat{\mathbf{b}}, \widehat{\mathbf{b}} \rangle = -1\}.$$

In this case, all the rulings of (X) intersected orthogonally with the timelike Disteli-axis $\widehat{\mathbf{b}}$, that is, $\psi = \psi^{*} = 0$. Thus, we have $\widehat{\gamma}(\widehat{s}) = 0 \Leftrightarrow (X)$ is a timelike helicoidal surface. The constant Disteli-axis rule is essential to the curvature theory of ruled surfaces. We will therefore examine its properties in some detail later.

Now, we present a kinematic interpretation of $\widehat{\gamma}(\widehat{s})$ as follows: If $\mathbf{x}(\widehat{s}) = \widehat{\mathbf{x}}(\widehat{s} + 2\pi)$, then $\mathbf{x}(\widehat{s})$ is a closed curve on $\mathbb{S}_1^2$. According to the E. Study map, this curve corresponds to (X)-closed timelike ruled surface in $\mathbb{E}_1^3$. We define a timelike dual unit vector $\widehat{\mathbf{u}}$ rigidly linked with the Blaschke frame $\{\widehat{\mathbf{x}}(\widehat{s}), \widehat{\mathbf{t}}(\widehat{s}), \widehat{\mathbf{g}}(\widehat{s})\}$ such that the timelike oriented line corresponding to $\widehat{\mathbf{u}}$ generates a spacelike developable ruled surface (spacelike torse) along the orthogonal trajectory of the (X)-closed timelike ruled surface. Then, the timelike dual unit vector $\widehat{\mathbf{u}}$ can be represented as:

$$\widehat{\mathbf{u}}(\widehat{s}) = \sinh\widehat{\beta}(\widehat{s})\widehat{\mathbf{t}}(\widehat{s}) + \cosh\widehat{\beta}(\widehat{s})\widehat{\mathbf{g}}(\widehat{s}), \text{ with } \widehat{\beta} = \beta + \varepsilon\beta^{*},$$

from which we obtain:

$$\widehat{\mathbf{u}}'(\widehat{s}) = (\widehat{\beta}' + \widehat{\gamma})\widehat{\mathbf{u}}^{\perp} - \sinh\widehat{\beta}\widehat{\mathbf{x}}(\widehat{s}).$$

Then we call the total change of $\widehat{\beta}(\widehat{s})$ the dual angle of the pitch of the (X)-closed timelike ruled surface, that is:

$$\oint d\widehat{\beta} = -\oint \widehat{\gamma} d\widehat{s} = -\oint \langle \widehat{\mathbf{t}}, \widehat{\mathbf{g}}' \rangle d\widehat{s}. \tag{12}$$

It is separated into the real and dual parts as:

$$\oint d\beta = -\oint \gamma(s)ds, \text{ and } \oint d\beta^{*} = -\oint \Gamma(s)ds. \tag{13}$$

Hence, we arrive therefore at the conclusion that:
(1) The angle pitch of (X)-closed timelike ruled surface is:

$$\lambda_x(s) = -\oint \gamma(s)ds = -\oint \langle \mathbf{t}, \frac{d\mathbf{g}}{ds} \rangle ds; \tag{14}$$

(2) The pitch of the (X)-closed timelike ruled surface is:

$$L_x(s) = -\oint \Gamma(s)ds = -\oint \langle \mathbf{x}, \frac{d\mathbf{c}}{ds} \rangle ds. \tag{15}$$

The pitch L_x and the angle of pitch λ_x are integral invariants of the $(\overline{X})$-closed timelike ruled surface. Then,

$$\Lambda_x(\widehat{s}) = \lambda_x + \varepsilon L_x = \oint d\widehat{\beta} = -\oint \langle \widehat{\mathbf{t}}, \widehat{\mathbf{g}}' \rangle d\widehat{s}, \tag{16}$$

is the Minkowski version of the dual angle of pitch defined in [5,12,19].

Corollary 1. *Any timelike ruled surface (X) is a timelike helicoidal surface if and only if its dual angle of pitch $\widehat{\Lambda}(\widehat{s})$ is identically zero.*

Notice that in Equation (7): (a) When $\mu(s) = 0$ ($\frac{d\mathbf{c}}{ds} \parallel \mathbf{x}$), the Blaschke frame $\{\mathbf{x}(s), \mathbf{t}(s), \mathbf{g}(s)\}$ is the Serret–Frenet frame, that is, $\zeta_1 = \mathbf{x}(s)$, $\zeta_2 = \mathbf{t}(s)$, $\zeta_3 = \mathbf{g}(s)$. Then (X) is a timelike tangential developable ruled surface (timelike tangential surface for short). Let u be the arc length parameter of $\mathbf{c}(s)$ and $\{\zeta_1(u), \zeta_2(u), \zeta_3(u)\}$ is the usual moving Serret-Frenet frame of $\mathbf{c}(s)$. Then,

$$\frac{d}{du}\begin{pmatrix} \zeta_1(u) \\ \zeta_2(u) \\ \zeta_3(u) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(u) & 0 \\ -\kappa(u) & 0 & \tau(u) \\ 0 & \tau(u) & 0 \end{pmatrix}\begin{pmatrix} \zeta_1(u) \\ \zeta_2(u) \\ \zeta_3(u) \end{pmatrix},$$

where $\kappa(u)$ and $\tau(u)$ are the natural curvature and torsion of the striction curve $\mathbf{c}(u)$, respectively;

$$\kappa(u) = \frac{1}{\Gamma(u)},\ \tau(u) = \frac{\gamma(u)}{\Gamma(u)}, \text{ with } \Gamma(u) \neq 0.$$

Therefore, the curvature function $\Gamma(u)$ is the radius of curvature of the spacelike striction curve $\mathbf{c}(u)$. We arrive therefore at the conclusion that the spacelike striction curve $\mathbf{c}(u)$ is the edge of regression of (X). Based on [20], we summarize this result in the following:

Theorem 1. *Any timelike ruled surface (X) with the curvature function,*

$$\Gamma(u) = a\cosh\int_0^u \tau(u)du - b\sinh\int_0^u \tau(u)du;\ \tau(u) \neq 0,$$

with real constants $(a, b) \neq (0, 0)$, is a timelike tangential surface of a spacelike curve lies on a Lorentzian sphere with radius $\sqrt{a^2 - b^2} > 0$.

If $\Gamma(s) = 0$, then the striction curve is tangent to $\mathbf{g}$; it is normal to the ruling through $\mathbf{c}(s)$. In this case, (X) is a timelike binormal ruled surface. Similarly, we find:

$$\frac{d}{du}\begin{pmatrix} \zeta_1(u) \\ \zeta_2(u) \\ \zeta_3(u) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(u) & 0 \\ \kappa(u) & 0 & -\tau(u) \\ 0 & \tau(u) & 0 \end{pmatrix}\begin{pmatrix} \zeta_1(u) \\ \zeta_2(u) \\ \zeta_3(u) \end{pmatrix},$$

where $\kappa(u)$ and $\tau(u)$ are the natural curvature and torsion of the striction curve $\mathbf{c}(u)$, respectively;

$$\kappa(u) = \frac{\gamma(u)}{\mu(u)},\ \tau(u) = \frac{1}{\mu(u)}, \text{ with } \mu(u) \neq 0.$$

Therefore, the curvature function $\mu(u)$ is the radius of the torsion of the spacelike striction curve $\mathbf{c}(s)$. Similarly, we summarize this result in the following:

Theorem 2. *Any timelike ruled surface (X) with the curvature function,*

$$\frac{\mu(u)}{\gamma(u)} = a\cos\int_0^u \tau(u)du + b\sin\int_0^u \tau(u)du;\ \tau(u) \neq 0,$$

with real constants $(a,b) \neq (0,0)$, is a timelike binormal surface of a timelike curve lying on a Lorentzian sphere with radius $\sqrt{a^2+b^2}$.

Definition 2. *Let (X) and $(\overline{X})$ be two timelike ruled surfaces in $\mathbb{E}_1^3$; $(\overline{X})$ is said to be Bertrand offsets of (X) if there exists a one-to-one correspondence between their rulings such that both surfaces have a common spacelike central normal at the striction points of their corresponding rulings.*

Let us consider a timelike ruled surface $(\overline{X})$ represented by a spacelike dual unit vector,

$$\widehat{\overline{\mathbf{x}}}(\widehat{s}) := \overline{\mathbf{x}} + \varepsilon\overline{\mathbf{x}}^* = \widehat{x}_1\widehat{\mathbf{x}} + \widehat{x}_2\widehat{\mathbf{t}} + \widehat{x}_3\widehat{\mathbf{g}}, \tag{17}$$

where $\widehat{x}_i = \widehat{x}_i(\widehat{s}),\ (i = 1,\ 2,\ 3)$ are its dual coordinate functions. Then,

$$\widehat{x}_1^2 + \widehat{x}_2^2 - \widehat{x}_3^2 = 1. \tag{18}$$

Differentiating Equation (17), (3) and with the aid of Equation (5), we find:

$$\left.\begin{array}{l} \widehat{\overline{\mathbf{x}}}' = (\widehat{x}_1' - \widehat{x}_2)\widehat{\mathbf{x}} + (\widehat{x}_2' + \widehat{x}_1 + \widehat{\gamma}\widehat{x}_3)\widehat{\mathbf{t}} + (\widehat{x}_3' + \widehat{\gamma}\widehat{x}_2)\widehat{\mathbf{g}}, \\ \widehat{x}_1\widehat{x}_1' + \widehat{x}_2\widehat{x}_2' - \widehat{x}_3\widehat{x}_3' = 0. \end{array}\right\} \tag{19}$$

If we suppose that the spacelike dual curves $\widehat{\overline{\mathbf{x}}} = \widehat{\overline{\mathbf{x}}}(\widehat{s})$ and $\widehat{x}=\widehat{x}(\widehat{s})$ are Bertrand offsets, that is, $\widehat{\mathbf{t}}=\widehat{\overline{\mathbf{t}}}$, then we have:

$$\widehat{x}_1' - \widehat{x}_2 = 0,\ \widehat{x}_2' + \widehat{x}_1 + \widehat{\gamma}\widehat{x}_3 = \langle \widehat{\overline{\mathbf{x}}}', \widehat{\mathbf{t}}\rangle,\ \widehat{x}_3' + \widehat{\gamma}\widehat{x}_2 = 0. \tag{20}$$

Substituting Equation (20) into the second equation of (19) and simplifying it, we obtain:

$$\widehat{x}_2 = 0. \tag{21}$$

From Equations (19) and (21) we obtain the equations:

$$\widehat{x}_1' = 0,\ \widehat{x}_1 + \widehat{\gamma}\widehat{x}_3 = \langle \widehat{\overline{\mathbf{x}}}', \widehat{\mathbf{t}}\rangle,\ \ \widehat{x}_3' = 0 \Rightarrow \widehat{x}_1 = \widehat{c}_1,\ \widehat{x}_3 = \widehat{c}_3 \in \mathbb{D}, \tag{22}$$

where $\widehat{c}_1$ and $\widehat{c}_3$ are dual constants of integrations. Therefore, we can find a hyperbolic constant dual angle $\widehat{\vartheta} = \vartheta + \varepsilon\vartheta^*$ such that $\widehat{c}_1 = \cosh\widehat{\vartheta}$ and $\widehat{c}_3 = \sinh\widehat{\vartheta}$. Thus, as a result the following Theorem can be given:

Theorem 3. *The offset hyperbolic dual angle formed by the generating timelike lines of a non-developable timelike ruled surface and its timelike Bertrand offset at corresponding central points remains constant.*

It is obvious from the above notations that a timelike ruled surface, generally, has a double infinity of timelike Bertrand offsets. Every timelike Bertrand offset can be generated by a hyperbolic constant linear offset $\vartheta^* \in \mathbb{R}$ and a hyperbolic constant angular offset $\vartheta \in \mathbb{R}$. Any two timelike surfaces of this family of timelike ruled surfaces are reciprocal of one another; if $(\overline{X})$ is a timelike Bertrand offset of (X), then (X) is also a timelike Bertrand offset of $(\overline{X})$. Thereby, Equation (17) becomes:

$$\widehat{\overline{\mathbf{x}}}(\widehat{s}) = \cosh\widehat{\vartheta}\widehat{\mathbf{x}}(\widehat{s}) + \sinh\widehat{\vartheta}\widehat{\mathbf{g}}(\widehat{s}). \tag{23}$$

In the view of the reality that for a ruled surface and its Bertrand offset the central normals coincide, it follows from the above theorem that the central tangents of the two timelike ruled surfaces also have the same constant dual angle at the corresponding points on the two striction curves. The relationship between the Blaschke frame of a timelike ruled surface and that of its Bertrand offset can therefore be written as:

$$\begin{pmatrix} \widehat{\overline{\mathbf{x}}}(\widehat{s}) \\ \widehat{\overline{\mathbf{t}}}(\widehat{s}) \\ \widehat{\overline{\mathbf{g}}}(\widehat{s}) \end{pmatrix} = \begin{pmatrix} \cosh\widehat{\vartheta} & 0 & \sinh\widehat{\vartheta} \\ 0 & 1 & 0 \\ \sinh\widehat{\vartheta} & 0 & \cosh\widehat{\vartheta} \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{x}}(\widehat{s}) \\ \widehat{\mathbf{t}}(\widehat{s}) \\ \widehat{\mathbf{g}}(\widehat{s}) \end{pmatrix}. \tag{24}$$

Notice that the above equation is exactly the Minkowski version in as of the Euclidean case (see [4,5,12]). If $\vartheta = 0$ (resp. $\vartheta^* = 0$), the timelike Bertrand offsets are called oriented (resp. coincident) offsets. With the aid of Equations (16) and (24) the dual angle of the pitch of $(\overline{X})$-closed timelike ruled surface is given as:

$$\Lambda_{\overline{x}} = \Lambda_x \cosh\widehat{\vartheta} + \Lambda_g \sinh\widehat{\vartheta}. \tag{25}$$

This is a new characterization of Bertrand offsets of closed timelike ruled surfaces in terms of their dual invariants. Therefore, the following theorem may be given.

Theorem 4. *The non-developable timelike ruled surfaces $(\overline{X})$ and (X) form a Bertrand offset if and only if the Equation (25) is satisfied.*

Corollary 2. *The Bertrand offset $(\overline{X})$ of a timelike helicoidal surface, generally, does not have to be a timelike helicoidal surface and can be a regular timelike ruled surface.*

From the real and dual parts of Equation (25), the following—

$$\left.\begin{aligned} \lambda_{\overline{x}} &= \lambda_x \cosh\vartheta + \lambda_g \sinh\vartheta, \\ L_{\overline{x}} &= (L_x - \vartheta^*\lambda_g)\cosh\vartheta + (L_g - \vartheta^*\lambda_x)\sinh\vartheta - -- \end{aligned}\right\} \tag{26}$$

are obtained. This is a Lorentzian version of Holditch's theorem [19–22]. Let $\widehat{\overline{s}}$ be the dual arc length of $\widehat{\overline{\mathbf{x}}}(\widehat{s}) \in \mathbb{S}_1^2$, then:

$$\frac{d}{d\widehat{\overline{s}}} \begin{pmatrix} \widehat{\overline{\mathbf{x}}}(\widehat{s}) \\ \widehat{\overline{\mathbf{t}}}(\widehat{s}) \\ \widehat{\overline{\mathbf{g}}}(\widehat{s}) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & \widehat{\overline{\gamma}} \\ 0 & \widehat{\overline{\gamma}} & 0 \end{pmatrix} \begin{pmatrix} \widehat{\overline{\mathbf{x}}}(\widehat{s}) \\ \widehat{\overline{\mathbf{t}}}(\widehat{s}) \\ \widehat{\overline{\mathbf{g}}}(\widehat{s}) \end{pmatrix}, \tag{27}$$

where

$$1 = (\cosh\widehat{\vartheta} + \widehat{\gamma}\sinh\widehat{\vartheta})\frac{d\widehat{s}}{d\widehat{\overline{s}}},\ \widehat{\overline{\gamma}} = (\widehat{\gamma}\cosh\widehat{\vartheta} + \sinh\widehat{\vartheta})\frac{d\widehat{s}}{d\widehat{\overline{s}}}. \tag{28}$$

From Equation (28), by eliminating $\frac{d\widehat{s}}{d\widehat{\overline{s}}}$, we obtain:

$$(\widehat{\overline{\gamma}} - \widehat{\gamma})\cosh\widehat{\vartheta} + (\widehat{\overline{\gamma}}\widehat{\gamma} - 1)\sinh\widehat{\vartheta} = 0. \tag{29}$$

This is another characterization of Bertrand offsets of timelike ruled surfaces in terms of their dual geodesic curvatures. Therefore, the following theorem can be given.

Theorem 5. *The non-developable timelike ruled surfaces $(\overline{X})$, and (X) form a Bertrand offset if and only if Equation (29) is satisfied.*

Corollary 3. *The Bertrand offset of a constant Disteli-axis timelike ruled surface is a constant Disteli-axis timelike ruled surface too.*

On the other hand, for the timelike ruled surface $(\overline{X})$, let $\overline{\mathbf{e}}(\overline{s}, v)$ be the spacelike unit normal of an arbitrary point. Hence, as in Equation (8), we have:

$$\overline{\mathbf{e}}(\overline{s}, v) = \pm \frac{\overline{\mu}\,\overline{\mathbf{t}} - v\overline{\mathbf{g}}}{\sqrt{\overline{\mu}^2 - v^2}}, \tag{30}$$

where $\overline{\mu}$ is the distribution parameter of $(\overline{X})$. It is clear from Equations (8) and (30) that the spacelike normal to a timelike ruled surface and its timelike Bertrand offsets are not the same. This means that the Bertrand offsets of a timelike ruled surface are, generally, not parallel offsets. Now, it seems natural to pose the following question: Under what position a timelike ruled surface and its timelike Bertrand offsets are parallel offsets? The answer is affirmative and can be declared as follows:

Theorem 6. *Two non-developable timelike ruled surfaces (X) and $(\overline{X})$ are parallel offsets if and only if: (i) $\mu = \overline{\mu}$; and (ii) each axis of the Blaschke frame of (X) is collinear with the conformable axis of $(\overline{X})$.*

Proof. Assume that (X) and $(\overline{X})$ are parallel offsets, or $\overline{\mathbf{e}}(\overline{s}, v) \times \mathbf{e}(s, v) = \mathbf{0}$. Then, we have:

$$-v^2 \sinh \vartheta \mathbf{t} + v(\overline{\mu} - \mu \cosh \vartheta)\mathbf{x} - v\mu \sinh \vartheta \mathbf{g} = \mathbf{0}.$$

The above equation must hold true for any value $v \neq 0$, which leads to $\vartheta = 0$ and $\mu = \overline{\mu}$. $\square$

Corollary 4. *Two developable timelike ruled surfaces (X) and $(\overline{X})$ are parallel offsets if and only if each axis of the Blaschke frame of (X) is collinear with the corresponding axis of $(\overline{X})$.*

Example 1. *In what follows, we will construct the constant Disteli-axis timelike ruled surface (X) Since $\widehat{\gamma}(\widehat{s})$ is constant, from Equation (5) we obtain the ODE $\widehat{\mathbf{t}}'' + \widehat{\kappa}^2\widehat{\mathbf{t}} = \mathbf{0}$. Without loss of generality, we may assume $\widehat{\mathbf{t}}(0) = (0, 1, 0)$; the general solution of the ODE becomes:*

$$\widehat{\mathbf{t}}(\widehat{s}) = \left(\widehat{b}_1 \sin(\widehat{\kappa}\widehat{s}), \cos(\widehat{\kappa}\widehat{s}) + \widehat{b}_2 \sin(\widehat{\kappa}\widehat{s}), \widehat{b}_3 \sin(\widehat{\kappa}\widehat{s})\right),$$

where $\widehat{b}_1$, $\widehat{b}_2$, and $\widehat{b}_3$ are some dual constants satisfying $\widehat{b}_1^2 - \widehat{b}_3^2 = 1$, and $\widehat{b}_2 = 0$. From this, we can obtain:

$$\widehat{\mathbf{x}}(\widehat{s}) = \left(-\widehat{b}_1 \frac{1}{\widehat{\kappa}} \cos(\widehat{\kappa}\widehat{s}) + \widehat{d}_1, \frac{1}{\widehat{\kappa}} \sin(\widehat{\kappa}\widehat{s}), -\widehat{b}_3 \frac{1}{\widehat{\kappa}} \cos(\widehat{\kappa}\widehat{s}) + \widehat{d}_3\right),$$

where $\widehat{d}_1$, and $\widehat{d}_3$ are some dual constants satisfying $\widehat{b}_3\widehat{d}_3 - \widehat{b}_1\widehat{d}_1 = 0$, and $\widehat{d}_1^2 - \widehat{d}_3^2 = \widehat{\rho}^2 - 1$; where $\widehat{\rho}\widehat{\kappa} = 1$. If we adopt the dual coordinates transformation such that:

$$\begin{pmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \\ \widetilde{x}_3 \end{pmatrix} = \begin{pmatrix} \widehat{b}_1 & 0 & -\widehat{b}_3 \\ 0 & 1 & 0 \\ -\widehat{b}_3 & 0 & \widehat{b}_1 \end{pmatrix} \begin{pmatrix} \widehat{x}_1 \\ \widehat{x}_2 \\ \widehat{x}_3 \end{pmatrix},$$

then $\widehat{\mathbf{x}}(\widehat{s})$ *turns into*

$$\widehat{\mathbf{x}}(\widehat{s}) = \Big(-\cosh\widehat{\psi}\cos(\widehat{\kappa}\widehat{s}), \cosh\widehat{\psi}\sin(\widehat{\kappa}\widehat{s}), \widehat{d}\Big), \tag{31}$$

for a dual constant $\widehat{d} = \widehat{b}_1\widehat{d}_3 - \widehat{b}_3\widehat{d}_1$, *with* $\widehat{d} = \pm\sinh\widehat{\psi}$. *Notice that* $\widehat{\mathbf{x}}(\widehat{s})$ *does not depend on the choice of the lower sign or upper sign of* $\mp$. *Therefore through the paper we choice upper sign, that is*

$$\widehat{\mathbf{x}}(\widehat{\varphi}) = (-\cosh\widehat{\psi}\cos\widehat{\varphi}, \cosh\widehat{\psi}\sin\widehat{\varphi}, \sinh\widehat{\psi}), \tag{32}$$

where $\widehat{\varphi} = \widehat{\kappa}\widehat{s}$. *It is a spacelike spherical curve with the dual curvature* $\widehat{\kappa} = \sqrt{1-\widehat{\gamma}^2}$ *on the Lorentzian dual unit sphere* $\mathbb{S}^2_1$. *Let* $\widehat{\varphi} = \varphi(1+\varepsilon h)$, *h denoting the pitch of the helical motion, then Equation (31) represents a timelike ruled surface. Thus, the Blaschke frame is found as:*

$$\begin{pmatrix} \widehat{\mathbf{x}} \\ \widehat{\mathbf{t}} \\ \widehat{\mathbf{g}} \end{pmatrix} = \begin{pmatrix} -\cosh\widehat{\psi}\cos\widehat{\varphi} & \cosh\widehat{\psi}\sin\widehat{\varphi} & \sinh\widehat{\psi} \\ \sin\widehat{\varphi} & \cos\widehat{\varphi} & 0 \\ \sinh\widehat{\psi}\cos\widehat{\varphi} & -\sinh\widehat{\psi}\sin\widehat{\varphi} & -\cosh\widehat{\psi} \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{f}}_1 \\ \widehat{\mathbf{f}}_2 \\ \widehat{\mathbf{f}}_3 \end{pmatrix}. \tag{33}$$

It is easily seen from Equation (32) that:

$$\left.\begin{array}{c} \widehat{p}(\varphi) = (1+\varepsilon h)\cosh\widehat{\psi}, \widehat{q}(\varphi) = (1+\varepsilon h)\sinh\widehat{\psi}, \\ \\ d\widehat{s} = \widehat{p}(\varphi)d\varphi, \widehat{\gamma}(\varphi) =: \frac{\widehat{q}(\varphi)}{\widehat{p}(\varphi)} = \tanh\widehat{\psi}. \end{array}\right\} \tag{34}$$

From the real and dual parts of Equation (34), we find

$$\mu = \psi^*\tanh\psi + h, \text{ and } \Gamma = \psi^* - h\tanh\psi. \tag{35}$$

Further, the Disteli-axis is:

$$\widehat{\mathbf{b}} = \sinh\widehat{\psi}\widehat{\mathbf{x}} + \cosh\widehat{\psi}\widehat{\mathbf{g}} = -\widehat{\mathbf{f}}_3. \tag{36}$$

This means that the axis of the helical motion is the constant timelike Disteli-axis $\widehat{\mathbf{b}}$.

On the other hand, the base curve can be obtained as—if we separate Equation (32) into real and dual parts we reach:

$$\mathbf{x}(\varphi) = (-\cosh\psi\cos\varphi, \cosh\psi\sin\varphi, \sinh\psi), \tag{37}$$

and

$$\mathbf{x}^*(\varphi) = \begin{pmatrix} x_1^* \\ x_2^* \\ x_3^* \end{pmatrix} = \begin{pmatrix} \varphi^*\sin\varphi\cosh\psi - \psi^*\sinh\psi\cos\varphi \\ \varphi^*\cos\varphi\cosh\psi + \psi^*\sinh\psi\sin\varphi \\ \psi^*\cosh\psi \end{pmatrix}. \tag{38}$$

Let $\boldsymbol{\alpha}(\alpha_1, \alpha_2, \alpha_3)$ *denote a point on* $\widehat{\mathbf{x}}$. *Since* $\boldsymbol{\alpha}\times\mathbf{x} = \mathbf{x}^*$ *we have the system of linear equations in* α_1, α_2, *and* α_3*:*

$$\left.\begin{array}{r} \alpha_2\sinh\psi - \alpha_3\cosh\psi\sin\varphi = x_1^*, \\ -\alpha_1\sinh\psi - \alpha_3\cosh\psi\cos\varphi = x_2^*, \\ -\alpha_2\cosh\psi\cos\varphi - \alpha_1\cosh\psi\sin\varphi = x_3^*. \end{array}\right\}$$

The matrix of coefficients of unknowns α_1, α_2, *and* α_3 *is:*

$$\begin{pmatrix} 0 & \sinh\psi & -\cosh\psi\sin\varphi \\ -\sinh\psi & 0 & -\cosh\psi\cos\varphi \\ -\cosh\psi\sin\varphi & -\cosh\psi\cos\varphi & 0 \end{pmatrix},$$

and therefore its rank is 2 with $\varphi \neq p\pi$ (p is an integer), and $\psi \neq 0$. In addition the rank of the augmented matrix,

$$\begin{pmatrix} 0 & \sinh\psi & -\cosh\psi\sin\varphi & x_1^* \\ -\sinh\psi & 0 & -\cosh\psi\cos\varphi & x_2^* \\ -\cosh\psi\sin\varphi & -\cosh\psi\cos\varphi & 0 & x_3^* \end{pmatrix},$$

is 2. Hence this system has infinitely many solutions represented with

$$\begin{gathered} \alpha_1 = -\psi^*\sin\varphi - (\varphi^* + \alpha_3)\coth\psi\cos\varphi, \\ \alpha_2 = -\psi^*\cos\varphi + (\varphi^* + \alpha_3)\coth\psi\sin\varphi, \\ \alpha_1\sin\varphi + \alpha_2\cos\varphi = -\psi^*. \end{gathered} \tag{39}$$

Since α_3 is taken at random, then we may take $\varphi^* + \alpha_3 = 0$. In this case, Equation (39) reduces to:

$$\alpha_1 = -\psi^*\sin\varphi,\ \alpha_2 = -\psi^*\cos\varphi,\ \alpha_3 = -\varphi^*.$$

We now simply find the base curve as:

$$\boldsymbol{\alpha}(\varphi) = (-\psi^*\sin\varphi, -\psi^*\cos\varphi, -h\varphi).$$

It can be shown that $\langle \boldsymbol{\alpha}', \mathbf{x}' \rangle = 0$; $(' = \frac{d}{d\varphi})$ so the base curve of (X) is its striction curve. Thus we have the timelike ruled surface (X)

$$(X):\ \mathbf{y}(\varphi, v)) = \begin{pmatrix} -\psi^*\sin\varphi - v\cosh\psi\cos\varphi \\ -\psi^*\cos\varphi + v\cosh\psi\sin\varphi \\ -h\varphi - v\sinh\psi \end{pmatrix}. \tag{40}$$

If $(x,\ y,\ z)$ are the coordinates of $\mathbf{y}$, then:

$$\left.\begin{aligned} x &= -\psi^*\sin\varphi - v\cosh\psi\cos\varphi, \\ y &= -\psi^*\cos\varphi + v\cosh\psi\sin\varphi, \\ z &= -h\varphi - v\sinh\psi, \end{aligned}\right\}$$

or

$$(X):\ \frac{x^2}{\psi^{*2}} + \frac{y^2}{\psi^{*2}} - \frac{Z^2}{\chi^2} = 1, \tag{41}$$

where $\chi = \psi^*\coth\psi$, and $Z = z + h\varphi$. The parameters h, ψ and ψ^* can control the shape of the surface (X). Thus, Equation (41) is a 3-parameter family of Lorentzian unit spheres . The intersection of each Lorentzian unit sphere and the corresponding spacelike plane $z = -h\varphi$ is a one-parameter family of Euclidean circles $x^2 + y^2 = \psi^{*2}$ Therefore, the envelope of Equation (41) is a one-parameter family of Lorentzian cylinders.

Remark 1. *(a) If $h = 0$, then the striction curve is a circle, and (X) is a Lorentzian unit sphere (Figure 2); $\psi = 1.4$, $\psi^* = 2$, $-3 \leq v \leq 3$, and $0 \leq \varphi \leq 2\pi$; (b) if $\widehat{\gamma} = 0$ ($\psi = \psi^* = 0$, $h = 1$), then the striction curve is the timelike Disteli-axis, and (X) is a timelike helicoidal surface; $-3 \leq v \leq 3$, and $0 \leq \varphi \leq 2\pi$ (Figure 3).*

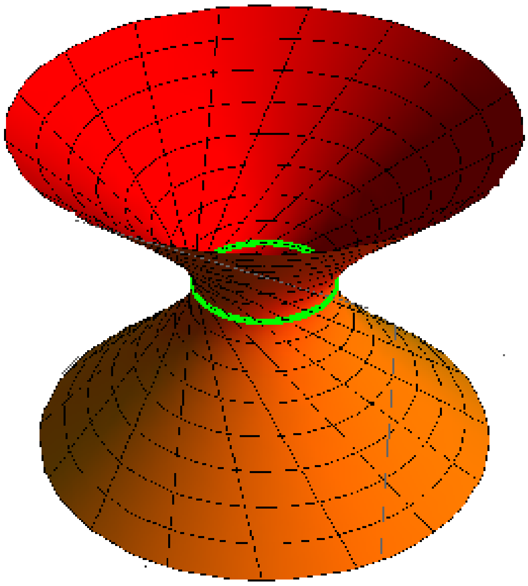

Figure 2. Lorentzian sphere.

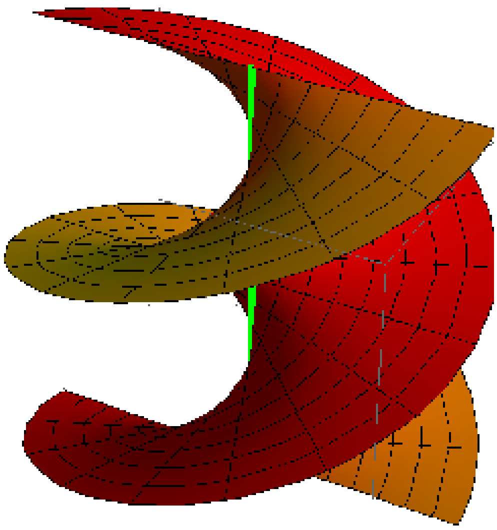

Figure 3. Timelike helicoid.

Corollary 5. *The Bertrand offset of a timelike helicoidal surface, generally, does not have to be a timelike helicoidal surface and can be a regular timelike ruled surface.*

Example 2. *In this example, we verify the idea of Corollary 5. In view of Equations (3), (29) and (33) we have:* $\widehat{\gamma} = 0$ $(\psi = \psi^* = 0) \Leftrightarrow \widehat{\overline{\gamma}} = \coth\widehat{\vartheta}$, *and*

$$\widehat{\overline{\mathbf{x}}}(\widehat{\varphi}) = \Big(-\cosh\widehat{\vartheta}\cos\widehat{\varphi}, \cosh\widehat{\vartheta}\sin\widehat{\varphi}, \sinh\widehat{\vartheta}\Big). \tag{42}$$

The equation of the striction curve of $(\overline{\mathrm{X}})$*, in terms of* (X)*, can therefore be written as:*

$$\overline{\boldsymbol{\alpha}}(\varphi) := \boldsymbol{\alpha}(\varphi) + \vartheta^* \mathbf{t}(\varphi) = (0, 0, -h\varphi) + \vartheta^*(\sin\varphi, \cos\varphi, 0). \tag{43}$$

By a similar procedure as in Equation (40), we have:

$$\left.\begin{array}{l}\overline{x} = \vartheta^* \sin\varphi - v\cosh\vartheta\cos\varphi,\\ \overline{y} = \vartheta^* \cos\varphi + v\cosh\vartheta\sin\varphi,\\ \overline{z} = -h\varphi + v\sinh\vartheta,\end{array}\right\}$$

or

$$(\overline{X}) : \frac{\overline{x}^2}{\vartheta^{*2}} + \frac{\overline{y}^2}{\vartheta^{*2}} - \frac{\overline{Z}^2}{\overline{\chi}^2} = 1, \tag{44}$$

where $\overline{\chi} = \vartheta^* \coth\vartheta$, and $Z = z + h\varphi$. The parameters h, ϑ and ϑ^* can control the shape of the surface $(\overline{X})$. Hence, $(\overline{X})$ has the same geometrical properties as in Equation (41). The graphs of the timelike Bertrand offset $(\overline{X})$, and (X) with its Bertrand offset $(\overline{X})$ are shown in Figures 4 and 5 , respectively; $\vartheta = 1.7$, $\vartheta^* = h = 1$, $-3 \leq v \leq 3$, and $0 \leq \varphi \leq 2\pi$.

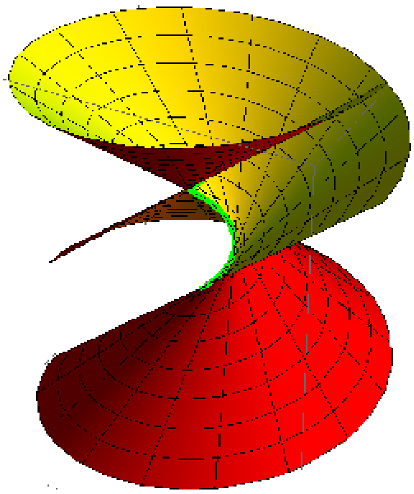

Figure 4. Bertrand offset of a timelike helicoid.

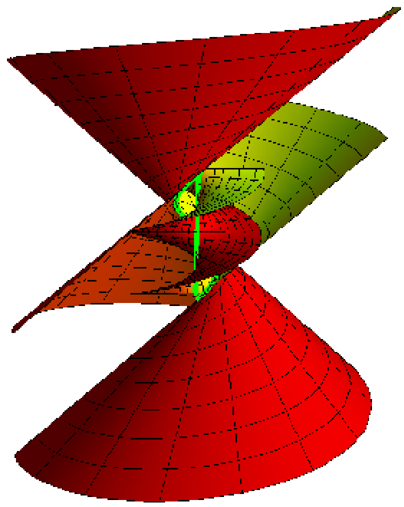

Figure 5. A timelike helicoid and its Bertrand offset.

Properties of the Striction Curves

In this subsection, we discuss the properties of the striction curves of the Bertrand offsets. With the aid of Definition 2, the striction curve $\overline{\mathbf{c}}(\overline{s})$ of $(\overline{X})$ is obtained as:

$$\overline{\mathbf{c}}(\overline{s}) = \mathbf{c}(s) + \vartheta^* \mathbf{t}(s), \tag{45}$$

from which we obtain:

$$\frac{d\overline{\mathbf{c}}(\overline{s})}{d\overline{s}}\frac{d\overline{s}}{ds}=(-\Gamma+\vartheta^{*})\mathbf{x}+(\mu+\gamma\vartheta^{*})\mathbf{g}, \tag{46}$$

whereas, as in Equation (11), it is:

$$\frac{d\overline{\mathbf{c}}(\overline{s})}{d\overline{s}}=-\overline{\Gamma}(\overline{s})\overline{\mathbf{x}}(\overline{s})+\overline{\mu}(\overline{s})\overline{\mathbf{g}}(\overline{s}). \tag{47}$$

Thus, from Equations (46) and (47), we have:

$$\frac{d\overline{s}}{ds}=\frac{-\Gamma+\vartheta^{*}}{-\overline{\Gamma}\cosh\vartheta+\overline{\mu}\sinh\vartheta}=\frac{\mu+\gamma\vartheta^{*}}{-\overline{\Gamma}\sinh\vartheta+\overline{\mu}\cosh\vartheta}. \tag{48}$$

If $\mu=0$, that is, (X) is the timelike tangent ruled surface of a given timelike space curve of class three, thus, from Equation (48), it follows that:

$$\overline{\mu}=\overline{\Gamma}\frac{\gamma\vartheta^{*}\cosh\vartheta+(\Gamma-\vartheta^{*})\sinh\vartheta}{\gamma\vartheta^{*}\sinh\vartheta+(\Gamma-\vartheta^{*})\cosh\vartheta}. \tag{49}$$

Thus the Bertrand offset of a timelike tangential is not a timelike tangential, that is, $\overline{\mu}(s)\neq 0$. If $(\overline{X})$ is also a timelike tangential, that is, $\overline{\mu}(s)=0$. In this case, we obtain the relation:

$$(1-\vartheta^{*}\kappa(u))\sinh\vartheta+\vartheta^{*}\tau(u)\cosh\vartheta=0. \tag{50}$$

Corollary 6. *If (X) and $(\overline{X})$ are two timelike tangential Bertrand offsets then their striction curves are timelike Bertrand curves.*

Furthermore, from Equation (50) the offset distance ϑ^{*} is:

$$\vartheta^{*}=\frac{\sinh\vartheta}{\kappa(u)\sinh\vartheta-\tau(u)\cosh\vartheta}. \tag{51}$$

Hence, if the timelike tangential offset of a timelike tangential surface is an oriented offset, it is then a coincident offset. In the case of a timelike plane curve, $\tau(u)=0$, which, in view of Equation (50), implies that $\kappa(u)$ is constant. In this case $\overline{\mathbf{c}}(u)=\overline{\mathbf{c}}(u)$, and $\mathbf{c}(u)$ are closed self-mated and Equation (45) becomes:

$$\overline{\mathbf{c}}(u)=\mathbf{c}(u)+\vartheta^{*}\zeta_2(u). \tag{52}$$

From the fact that $\|\zeta_2\|^2=1$, we get:

$$\|\overline{\mathbf{c}}(u)-\mathbf{c}(u)\|^2=\vartheta^{*2}>0.$$

Corollary 7. *Every closed self-mated timelike tangential surface has a constant width.*

When $\Gamma=0$, the surface (X) is a timelike binormal ruled surface of its timelike striction curve. From Equation (48), it follows that:

$$\overline{\Gamma}=\overline{\mu}\frac{\vartheta^{*}\cosh\vartheta-(\gamma\vartheta^{*}+\mu)\sinh\vartheta}{\vartheta^{*}\sinh\vartheta-(\gamma\vartheta^{*}+\mu)\cosh\vartheta}. \tag{53}$$

Thus, the Bertrand offset of a timelike binormal is not a timelike binormal, that is, $\overline{\Gamma}(s)\neq 0$. Furthermore, if the timelike Bertrand offset $(\overline{X})$ is also a timelike binormal, then we have:

$$\tau(u)\vartheta^{*}\cosh\vartheta-(\kappa(u)\vartheta^{*}+1)\sinh\vartheta=0.$$

In similar arguments, we can produce the corresponding results for a timelike tangential; we omit the details here.

4. Conclusions

In this paper, a generalization of Bertrand offsets of curves for timelike ruled surfaces has been developed. Interestingly, there are many similarities between the theory of Bertrand curves and the theory of Bertrand offsets for timelike ruled surfaces. For instance, a timelike ruled surface can have an infinity of Bertrand offsets in the same way as a plane curve can have an infinity of Bertrand mates. Furthermore, recently, the application of singularity theory, submanifolds theory and harmonic quasiconformal mappings and so forth has attracted great interest. For future research, we will engage with the new ideas that Gaussian and mean curvatures of these Bertrand offsets can be calculated, when the Weingarten map for the Bertrand offsets spacelike ruled surfaces is defined. We will also consider integrating the study of singularity theory and submanifolds theory and so forth, presented in [23–42], with the results of this paper to explore new methods to find more results and theorems related to symmetric properties about this topic.

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